Análisis Numérico

Examen II: Solución a Sistemas de Ecuaciones Lineales

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Dado el sistema $A\bar{x}=b$ con

$$A=\begin{bmatrix}4 & -2 & 0\\ -2 & 10 & 3\\ 0 & 3 & 5\end{bmatrix};\quad b=\begin{bmatrix}-6\\ 9\\ -2\end{bmatrix}$$

Resolver:

1. Calcular la factorización LU (estándar) de A.
2. Verificar que la matriz es definida positiva.
3. Encontrar la factorización LL^t.
4. Resolver el sistema usando la factorización de Cholesky encontrada.

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1.

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ \frac{2}{4} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{M_1}\underbrace{\begin{bmatrix}4 & -2 & 0\\ -2 & 10 & 3\\ 0 & 3 & 5\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}4 & -2 & 0\\ 0 & 9 & 3\\ 0 & 3 & 5\end{bmatrix}}_{M_1A}$$

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -\frac{1}{3} & 1\end{bmatrix}}_{M_2}\underbrace{\begin{bmatrix}4 & -2 & 0\\ 0 & 9 & 3\\ 0 & 3 & 5\end{bmatrix}}_{M_1A}=\begin{bmatrix}4 & -2 & 0\\ 0 & 9 & 3\\ 0 & 0 & 4\end{bmatrix}$$

$$\boxed{M_2M_1A=U}$$

$$\Rightarrow A = \underbrace{M_1^{-1} M_2^{-1}} U$$

$$A = L \, U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M_1 \qquad\qquad\qquad M_1^{-1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}$$

$$M_2 \qquad\qquad\qquad M_2^{-1}$$

$$L = M_1^{-1} M_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}$$

$$A = L\,U = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{3} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & -2 & 0 \\ 0 & 9 & 3 \\ 0 & 0 & 4 \end{bmatrix}}_{U} = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 10 & 3 \\ 0 & 3 & 5 \end{bmatrix}$$

2. $A = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 10 & 3 \\ 0 & 3 & 5 \end{bmatrix}$

$\begin{matrix} 4 & -2 & 0 \\ -2 & 10 & 3 \end{matrix}$

Para determinar que A sea definida positiva, observamos que A es simétrica, i.e., $A = A^T$.

Falta ver que los determinantes esquinados de A sean positivos.

$\det(4) = 4 > 0$

$\det \begin{vmatrix} 4 & -2 \\ -2 & 10 \end{vmatrix} = 40 - 4 = 36 > 0$

$\det(A) = 4(10)(5) + 0 + 0 - 20 - 36 - 0$

$= 200 - 56 = 144 > 0$

$\therefore$ A es definida positiva.

3. Como A es definida positiva, admite la factorización de Cholesky.

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & -2 & 0 \\ 0 & 9 & 3 \\ 0 & 0 & 4 \end{bmatrix}}_{U}$$

$$L^t = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$DL^t = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 0 \\ 0 & 9 & 3 \\ 0 & 0 & 4 \end{bmatrix} = U$$

$A = LDL^t$.

$$D^{1/2} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \implies A = \underbrace{L D^{1/2}}\underbrace{D^{1/2} L^t}$$
$$= \tilde{L}\,\tilde{L}^t$$

$$\tilde{L} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D^{1/2}} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\tilde{L}^t = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow A = \tilde{L}\tilde{L}^t = \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{\tilde{L}} \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{\tilde{L}^t} = \underbrace{\begin{bmatrix} 4 & -2 & 0 \\ -2 & 10 & 3 \\ 0 & 3 & 5 \end{bmatrix}}_{A}$$

4.

$A\bar{x} = \bar{b} \Rightarrow \tilde{L}\tilde{L}^t\bar{x} = \bar{b}$

Tomando $\tilde{L}^t\bar{x} = \bar{y}$ (2)

$\tilde{L}\bar{y} = \bar{b}$ (1)

Sustituyendo en (1):

$$\underbrace{\begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}}_{\tilde{L}} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\bar{y}} = \begin{bmatrix} -6 \\ 9 \\ -2 \end{bmatrix}$$

(·) $\Rightarrow 2y_1 = -6 \Rightarrow \boxed{y_1 = -3}$

(··) $\Rightarrow -y_1 + 3y_2 = 9$

$\Rightarrow 3 + 3y_2 = 9 \Rightarrow 3y_2 + 6 \Rightarrow \boxed{y_2 = 2}$

(···) $y_2 + 2y_3 = -2$

$\Rightarrow 2 + 2y_3 = -2 \Rightarrow 2y_3 = -4 \Rightarrow \boxed{y_3 = -2}$

$$\Rightarrow \bar{y} = \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}$$

Sustituyendo en 2:

$$\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{\tilde{L}^t} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix}}_{\bar{y}}.$$

•) $2x_3 = -2 \Rightarrow \boxed{x_3 = -1}$

••) $3x_2 + x_3 = 2 \Rightarrow 3x_2 - 1 = 2 \Rightarrow 3x_2 = 3 \Rightarrow \boxed{x_2 = 1}$

•••) $2x_1 - x_2 = -3 \Rightarrow 2x_1 - 1 = -3 \Rightarrow 2x_1 = -2 \Rightarrow \boxed{x_1 = -1}$

$$\Rightarrow \bar{x} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}.$$