Análisis Numérico

Tarea Examen V: Interpolación

Arturo Yitzack Reynoso Sánchez

1. Dados los puntos $x_0 = 0$, $x_1 = 1$, $x_2 = 2$ y $f(x) = \exp(x)$ construir explícitamente los polinomios de Lagrange $L_j(x)$.

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-2)}{(-1)(-2)} = \frac{x^2-3x+2}{2} = \frac{1}{2}x^2 - \frac{3}{2}x + 1.$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = \frac{x^2-2x}{-1} = -x^2+2x$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{x^2-x}{2} = \frac{1}{2}x^2 - \frac{x}{2}$$

y el interpolante de Lagrange para los 3 puntos, está dado como

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$
$$= \left(\frac{1}{2}x^2 - \frac{3}{2}x + 1\right)\cdot 1 + (-x^2+2x)e + \left(\frac{1}{2}x^2 - \frac{x}{2}\right)e^2.$$

2. Dados los puntos $x_0=0$, $x_1=1$, $x_2=2$ y $f(x)=\exp(x)$ construir explícitamente el interpolante de Hermite.

$$H_{2n+1}(x)=\sum_{j=0}^{n} f(x_j)H_j(x)+\sum_{j=0}^{n} f'(x_j)\hat{H}_j(x)$$

donde

$$H_j(x)=[1-2(x-x_j)L_j'(x_j)]L_j^2(x)$$

y $\quad \hat{H}_j(x)=(x-x_j)L_j^2(x)$

con $L_j(x)$ es el j-ésimo polinomio de Lagrange.

$$L_0'(x)=x-\frac{3}{2}$$

$$L_1'(x)=-2x+2$$

$$L_2'(x)=x-\frac{1}{2}$$

$$[L_0(x)]^2=\left(\frac{1}{2}x^2-\frac{3}{2}x+1\right)\left(\frac{1}{2}x^2-\frac{3}{2}x+1\right)$$

$$=\frac{1}{4}x^4-\frac{3}{4}x^3+\frac{1}{2}x^2-\frac{3}{4}x^3-\frac{9}{4}x^2-\frac{3}{2}x+\frac{1}{2}x^2-\frac{3}{2}x+1$$

$$=\frac{1}{4}x^4-\frac{6}{4}x^3-\frac{5}{4}x^2-3x+1$$

$$[L_1(x)]^2=(-x^2+2x)(-x^2+2x)=x^4-2x^3-2x^3+4x^2$$

$$=x^4-4x^3+4x^2$$

$$[L_2(x)]^2=\left(\frac{x^2}{2}-\frac{x}{2}\right)^2=\frac{x^4}{4}-\frac{x^3}{2}+\frac{x^2}{4}$$

$$H_0(x) = \left[1 - 2(x-0)\left(-\tfrac{3}{2}\right)\right]\left(\tfrac{1}{4}x^4 - \tfrac{3}{2}x^3 - \tfrac{5}{4}x^2 - 3x + 1\right)$$

$$= (1+3x)\left(\tfrac{1}{4}x^4 - \tfrac{3}{2}x^3 - \tfrac{5}{4}x^2 - 3x + 1\right)$$

$$= \cancel{\tfrac{1}{4}x^4} - \cancel{\tfrac{3}{2}x^3} - \cancel{\tfrac{5}{4}x^2} - \cancel{3x} + 1 + \tfrac{3}{4}x^5 - \cancel{\tfrac{9}{2}x^4} - \cancel{\tfrac{15}{4}x^3} - \cancel{9x^2} + \cancel{3x}$$

$$= \tfrac{3}{4}x^5 - \tfrac{17}{4}x^4 - \tfrac{21}{4}x^3 - \tfrac{41}{4}x^2 + 1$$

$$\hat{H}_0(x) = (x-0)\left(\tfrac{1}{4}x^4 - \tfrac{3}{2}x^3 - \tfrac{5}{4}x^2 - 3x + 1\right)$$

$$= \tfrac{1}{4}x^5 - \tfrac{3}{2}x^4 - \tfrac{5}{4}x^3 - 3x^2 + x$$

$$H_1(x) = \left[1 - 2(x-1)(0)\right]\left(x^4 - 4x^3 + 4x^2\right)$$

$$= x^4 - 4x^3 + 4x^2$$

$$\hat{H}_1(x) = (x-1)\left(x^4 - 4x^3 + 4x^2\right)$$

$$= \cancel{x^5} - \cancel{4x^4} + \cancel{4x^3} - \cancel{x^4} + \cancel{4x^3} - \cancel{4x^2}$$

$$= x^5 - 5x^4 + 8x^3 - 4x^2$$

$$H_2(x) = \left[1 - 2(x-2)\left(\frac{3}{2}\right)\right]\left(\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4}\right)$$

$$= \left[1 - 3x + 6\right]\left(\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4}\right)$$

$$= [7 - 3x]\left(\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4}\right)$$

$$= \frac{7}{4}x^4 - \frac{7}{2}x^3 + \frac{7}{4}x^2 - \frac{3x^5}{4} + \frac{3x^4}{2} - \frac{3}{4}x^3$$

$$= -\frac{3}{4}x^5 + \frac{13}{4}x^4 - \frac{17}{4}x^3 + \frac{7}{4}x^2$$

$$\hat{H}_2(x) = (x-2)\left(\frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4}\right)$$

$$= \frac{x^5}{4} - \frac{x^4}{2} + \frac{x^3}{4} - \frac{x^4}{2} + x^3 - \frac{x^2}{2}$$

$$= \frac{x^5}{4} - x^4 + \frac{5}{4}x^3 - \frac{x^2}{2}$$

$$H_5(x) = x^5 - \frac{23}{4}x^4 - \frac{13}{2}x^3 - \frac{53}{4}x^2 + x + 1$$

$$+ e\left(x^5 - 4x^4 + 4x^3\right)$$

$$+ e^2\left(-\frac{1}{2}x^5 + \frac{9}{4}x^4 - 3x^3 + \frac{5}{4}x^2\right)$$

3. Dada la siguiente definición:

DEFINICIÓN. Dada una función f definida en $[a,b]$ y el conjunto de puntos $a = x_0 < x_1 < \dots < x_n = b$, el interpolante Spline Cúbico S para f es una función que satisface:

(a) $S(x)$ es un polinomio cúbico, denotado por $S_j(x)$ en el intervalo $[x_j, x_{j+1}]$ para $j = 0, 1, \dots, n-1$.

(b)

(a) $S_j(x_j) = f(x_j)$ para $j = 0, 1, \dots, n$

(b) (c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$

(c) (d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ para $j = 0, 1, \dots, n-2$

(e)

(d) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$

Con condiciones de frontera dadas por:

(a) $S''(x_0) = S''(x_n) = 0$ Llamado Spline Natural.

(b) $S'(x_0) = f'(x_0)$ y $S'(x_n) = f'(x_n)$ Llamado Spline Completo.

Completar el sistema de ecuaciones dado por:

$$h_{i-1} z_{i-1} + 2(h_{i-1} + h_i) z_i + h_i z_{i+1} = \frac{6}{h_i}(y_{i+1} - y_i) - \frac{6}{h_{i-1}}(y_i - y_{i-1})$$

para $1 \le i \le n-1$.

En forma matricial expresado:

$$\begin{bmatrix} h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & h_{n-1} & 2(h_{n-1}+h_{n-2}) & h_{n-2} & 0 \\ 0 & 0 & \cdots & h_{n-2} & 2(h_{n-2}+h_{n-1} & h_{n-1} \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix}$$

donde

$$h_i = x_{i+1} - x_i$$

$$b_i = \frac{6}{h_i}(y_{i+1} - y_i)$$

$$v_i = b_i - b_{i-1}$$

para $1 \le i \le n-1$.

Con las condiciones de frontera dadas por el spline completo.

Tenemos

$$S_i'(x_0) = \frac{z_{i+1}}{2h_i}(x_0 - x_i)^2 - \frac{z_i}{2h_i}(x_{i+1} - x_0)^2 + \frac{y_{i+1}}{h_i} - \frac{z_{i+1}}{6}h_i - \frac{y_i}{h_i} + \frac{z_i}{6}h_i = f'(x_0)$$

$$S_i'(x_n) = \frac{z_{i+1}}{2h_i}(x_n - x_i)^2 - \frac{z_i}{2h_i}(x_{i+1} - x_n)^2 + \frac{y_{i+1}}{h_i} - \frac{z_{i+1}}{6}h_i - \frac{y_i}{h_i} + \frac{z_i}{6}h_i = f'(x_n)$$

$$\Rightarrow S_0'(x_0) = -\frac{z_0}{2h_0}(x_1 - x_0)^2 + \frac{y_1}{h_0} - \frac{z_1}{6}h_0 - \frac{y_0}{h_0} + \frac{z_0}{6}h_0 = f'(x_0) \quad (*)$$

$$\Rightarrow S_n'(x_n) \underset{\substack{\nearrow \\ \text{condición b)}}}{=} S_{n-1}'(x_n) = \frac{z_n}{2h_{n-1}}(x_n - x_{n-1})^2 - \frac{z_{n-1}}{2h_{n-1}}\cancel{(x_n - x_n)^2}^{\,0}$$

$$+ \frac{y_n}{h_{n-1}} - \frac{z_n h_{n-1}}{6} - \frac{y_{n-1}}{h_{n-1}} + \frac{z_{n-1}}{6}h_{n-1} = f'(x_n) \qquad (**)$$

Para $(*)$ simplificamos términos:

$$\Rightarrow z_0\left(\frac{h_0}{6} - \frac{h_0}{2}\right) + z_1\frac{h_0}{6} = f'(x_0) + \frac{1}{h_0}(y_0 - y_1)$$

$$\Rightarrow \frac{2}{6}h_0 z_0 + \frac{h_0}{6}z_1 = -f'(x_0) + \frac{1}{h_0}(y_1 - y_0)$$

$$\Rightarrow 2h_0 z_0 + h_0 z_1 = \frac{6}{h_0}(y_1 - y_0) - 6f'(x_0)$$

$$= b_0 - 6f'(x_0)$$

Para $(**)$ simplificamos términos:

$$\Rightarrow z_n\left(\frac{h_{n-1}}{2} - \frac{h_{n-1}}{6}\right) + z_{n-1}\frac{h_{n-1}}{6} = f'(x_n) - \left(\frac{y_n - y_{n-1}}{h_{n-1}}\right)$$

$$\Rightarrow z_n\left(\frac{2}{6}h_{n-1}\right) + z_{n-1}\frac{h_{n-1}}{6} = f'(x_n) - \left(\frac{y_n - y_{n-1}}{h_{n-1}}\right)$$

$$\Rightarrow 2h_{n-1}z_n + h_{n-1}z_{n-1} = 6f'(x_n) - 6\left(\frac{y_n - y_{n-1}}{h_{n-1}}\right)$$

$$\Rightarrow 2h_{n-1}z_n + h_{n-1}z_{n-1} = 6f'(x_n) - b_i$$

Por lo tanto, completando la matriz del sistema de ecuaciones tenemos una matriz cuadrada de $(n+1)\times(n+1)$:

$$
\begin{bmatrix}
2h_0 & h_0 & 0 & 0 & \cdots & & & 0 \\
h_0 & 2(h_0+h_1) & h_1 & 0 & & & & \\
0 & h_1 & 2(h_1+h_2) & h_2 & & & & \\
\vdots & & & & \ddots & & & \\
0 & \cdots & & h_{n-1} & 2(h_{n-1}+h_{n-2}) & h_{n-2} & 0 & 0 \\
 & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} & \\
 & & & & 0 & & h_{n-1} & 2h_{n-1} \\
0 & & & & & & &
\end{bmatrix}
\begin{bmatrix}
z_0 \\ z_1 \\ \vdots \\ z_{n-1} \\ z_n
\end{bmatrix}
=
\begin{bmatrix}
b_0 - 6f'(x_0) \\ v_1 \\ v_2 \\ \vdots \\ v_{n-2} \\ v_{n-1} \\ 6f'(x_n) - b_1
\end{bmatrix}
$$