Lecture 8: Basic Concepts in Probability

Probability theory makes extensive use of set notation and set operations. We begin this lecture with a brief review of some key concepts.

Set Notation and Set Operations [Bertsekas, 2008]

Recall that a *set* is a collection of objects which are *elements* of the set. If S is a set and x is an element of S we write $x \in S$ where the $\in$ is a symbol for "is an element of". If x is not an element of S we write $x \notin S$ where $\notin$ is a symbol for "is not an element of". A set can have no elements in which case it is called the emptyset, denoted by $\emptyset$. Sets can be specified in a variety of ways. If S contains a finite number of elements, say $x_1, x_2, \ldots, x_n$ we write it as a list of the elements in braces:

$$S = \{x_1, x_2, \ldots, x_n\} \tag{1}$$

Alternatively, we can consider a set of all x that have a certain property P and denote it by

$$S = \{x \mid x \text{ satisfies } P\} \tag{2}$$

where the symbol " $\mid$ " (or sometimes " $:$ ") is read as "such that". Occasionally the condition will include the phrase "for all" which is mathematically written using the symbol "$\forall$". If every element of a set S is also an element of the set T we say that S is a *subset* of T and we write $S \subset T$. If $S \subset T$ and $T \subset S$ the two sets are *equal* and we write $S = T$. It is also useful to consider a *universal set* Ω that contains all objects that could conceivably be of interest in a particular context (in probability theory Ω is the sample space of all possible outcomes of a random experiment). In the following we list some additional set notation and set-related operations:

- The *complement* of a set S with respect to the universal set Ω is $S^c = \{x \in \Omega \mid x \notin S\}$. The complement of a complement is the set itself, $(S^c)^c = S$, and the complement of the universal set is the emptyset, $\Omega^c = \emptyset$.

- The *powerset* of a set S is the set of all possible subsets (including the emptyset) and is denoted $\mathcal{P}(S)$. For example, suppose $S = \{a, b, c\}$, then

$$\mathcal{P}(S) = \{\emptyset, a, b, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\} \tag{3}$$

- The *union* of sets $S_1, \ldots, S_n$ is denoted $S_1 \cup \cdots \cup S_n = \{x \in \Omega \mid x \in S_i \text{ for some } i = 1, \ldots, n\}$. The union of the universal set with any subset is the universal set, $S \cup \Omega = \Omega$ for all $S \in \mathcal{P}(\Omega)$. The union of any set S with the emptyset $\emptyset$ is the set S, $S \cup \emptyset = S$. The union of a set with its complement is the universal set, $S \cup S^c = \Omega$ for all $S \in \mathcal{P}(\Omega)$.

- The *intersection* of sets $S_i, i = 1 \ldots n$ is denoted by $S_1 \cap \cdots \cap S_n = \{x \in \Omega \mid x \in S_i \text{ for all } i = 1, \ldots, n\}$. The intersection of the universal set with a set S is the set itself, $S \cap \Omega = S$. The intersection of a set with its complement is the emptyset, $S \cap S^c = \emptyset$.

- Sets S_1 and S_2 are *disjoint* if their intersection is the empty set, $S_1 \cap S_2 = \emptyset$. The difference $S_1 - S_2$ is the set of elements in S_1 but not in S_2, $S_1 - S_2 = \{x \in \Omega \mid x \in S_1, x \notin S_2\}$.

Many of these relations can be visualized using a Venn diagram as shown below.

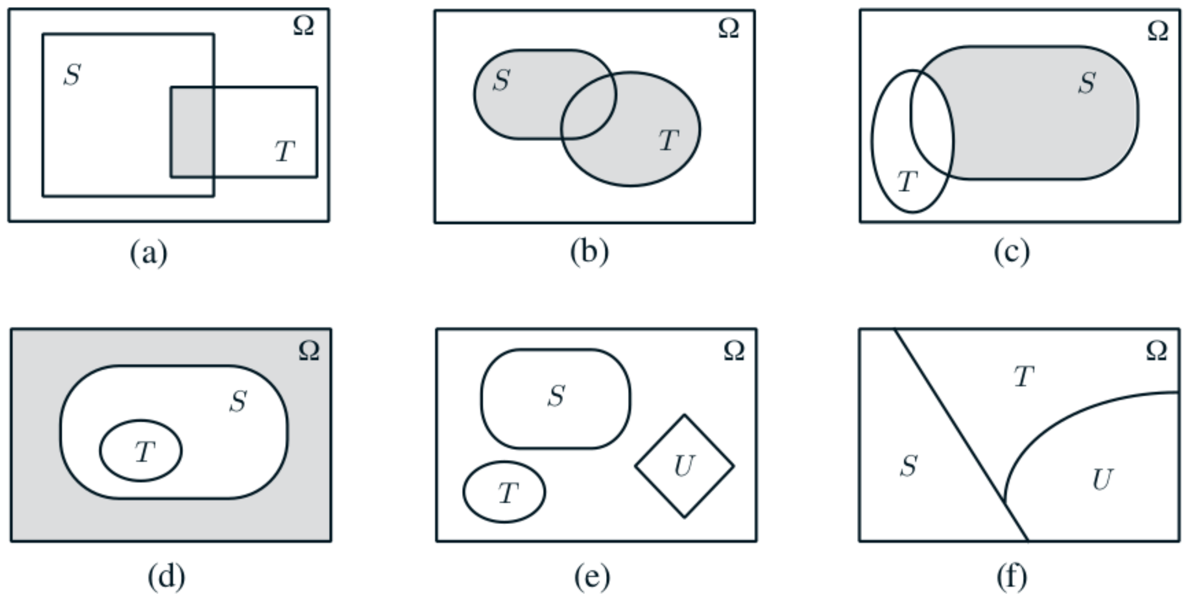


Figure 1.1: Examples of Venn diagrams. (a) The shaded region is $S \cap T$. (b) The shaded region is $S \cup T$. (c) The shaded region is $S \cap T^c$. (d) Here, $T \subset S$. The shaded region is the complement of S. (e) The sets S, T, and U are disjoint. (f) The sets S, T, and U form a partition of the set Ω.

Figure 1: Image Source: [Bertsekas, 2008]

As a direct consequence of the above definitions and set operations we have the following properties concerning algebraic operations with sets (e.g., unions, intersections, complements). Let $S_1, S_2, S_3, \ldots, S_n$ and A all be sets that belong to the same universal set Ω.

- Associative Laws:

$$S_1 \cup (S_2 \cup S_3) = (S_1 \cup S_2) \cup S_3$$
$$S_1 \cap (S_2 \cap S_3) = (S_1 \cap S_2) \cap S_3$$

- Commutative Laws:

$$S_1 \cup S_2 = S_2 \cup S_1$$
$$S_1 \cap S_2 = S_2 \cap S_1$$

- Distributive Laws:

$$A \cap (\cup_{j=1}^{n} S_j) = \cup_{j=1}^{n} (A \cap S_j)$$
$$A \cup (\cap_{j=1}^{n} S_j) = \cap_{j=1}^{n} (A \cup S_j)$$

Probabilistic Models

A probabilsitic model is a mathematical description of an uncertain situation and has the following main components:

- A *random experiment*: a process that produces exactly one outcome, called a sample point ω, each time the process occurs. The set of all possible outcomes is the *sample space* Ω. Each element in the sample space is distinct (the elements are mutually exclusive and collectively exhaustive).

- A *probability law*: a rule, satisfying the axioms to be described next, that assigns a nonnegative number $P(A)$ to each subset of sample points. A collection of experiment outcomes (samples) $A \subseteq \mathcal{F}$ is called an *event* and $\mathcal{F}$ is the *event space*. For example, a valid event space is the power set of the sample space, $\mathcal{F} = \mathcal{P}(\Omega)$.

A *simple event* contains only one sample point $A = \omega \in \Omega$, whereas a *composite event* contains multiple sample points, e.g., $B = \omega_1 \cup \omega_2 \subseteq \Omega$. (Note we also sometimes write the union of sample points as the event $B = \{\omega_1, \omega_2\}$.) These concepts are illustrated in the diagram below:

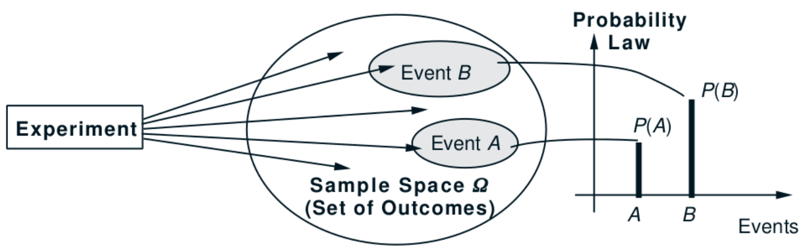


Figure 2: Image Source: [Bertsekas, 2008]

The probability law must satisfy the following *probability axioms*. Let Ω be a sample space with events $\{x_1, x_2, \cdots, x_n\}$. To every sample element $x \in \Omega$ (or event $x \subseteq \Omega$) the probability law must assign a number called the probability $P(x)$, that satisfies:

- (Nonegativity) $P(x) \geq 0$ for all $x \in \Omega$

- (Additivity) If x_i and x_j are disjoint events, then $P(\{x_i \cup x_j\}) = P(x_i) + P(x_j)$. Note this is the union of two elements so it gives the probability of either x_i or x_j. This is also denoted $P(x_i, x_j)$.

- (Normalization) The probability of the entire sample space is equal to 1.

$$P(\{x_1 \cup \cdots \cup x_n\}) = 1$$

Also, the set $A = \Omega$ is called a *certain event*. Conversely, the complement of event $A = \Omega$ is the emptyset, i.e., $\Omega^c = \emptyset$ and is called the *impossible event*. From these axioms we can deduce that, for some events in the event space $A, B \in \mathcal{F}$:

- If $A \subset B$, the $P(A) \leq P(B)$

- $P(\{A \cup B\}) = P(A) + P(B) - P(A \cap B)$
- $P(\{A \cup B\}) \leq P(A) + P(B)$

When there is partial information available about the outcome of the random experiment we can use a *conditional probability law*. Suppose we know the outcome of the experiment is contained in event B then we wish to determine the likelihood it also belongs to event A. Introduce the conditional probability law as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

where the " | " symbol in this context means "given". That is, the probability of event A given the outcome is in event B. One can confirm that this new probability law also satisfies the required probability axioms and thus inherits all of the properties introduced previously. For example, recall that $P(A \cup C) \leq P(A) + P(C)$ and it is also true that

$$P(A \cup C|B) \leq P(A|B) + P(C|B) \ . \tag{4}$$

The conditional probability formula can be rearranged as $P(A \cap B) = P(A|B)P(B)$ and applying this iteratively gives a multiplication rule for several related events $A_1, A_2, \ldots, A_n$:

$$\begin{aligned}
P(A_1 \cap A_2 \cap \cdots \cap A_n) &= P(A_n|A_{n-1} \cap \cdots \cap A_1)P(A_{n-1} \cap \cdots \cap A_1) \\
&= P(A_n|A_{n-1} \cap \cdots \cap A_1)P(A_{n-1}|A_{n-2} \cap \cdots \cap A_1)P(A_{n-2} \cap \cdots \cap A_1) \\
&= P(A_n|A_{n-1} \cap \cdots \cap A_1)P(A_{n-1}|A_{n-2} \cap \cdots \cap A_1)P(A_{n-2}|A_{n-3} \cap \cdots \cap A_1) \\
&\quad \cdots \ P(A_2|A_1)P(A_1) \\
&= \prod_{i=2}^{n} P(A_i|A_{i-1} \cap \cdots \cap A_1)P(A_1)
\end{aligned}$$

Earlier in this lecture we alluded that the event space $\mathcal{F}$ can be the powerset of Ω. However, more preciesely, any $\mathcal{F}$ that is a *σ-algebra* suffices. A σ-algebra $\mathcal{F}$ is a collection of sets in Ω (not necessarily the powerset) that is "consistent" in the following sense:

1. If $A \in \mathcal{F}$ then its complement is also in $\mathcal{F}$, that is $A^c \in \mathcal{F}$
2. If $A_1, A_2 \in \mathcal{F}$ then their union is also in $\mathcal{F}$, that is $A_1 \cup A_2 \in \mathcal{F}$
3. The sample space itself is in $\mathcal{F}$, that is $\Omega \in \mathcal{F}$

Example: Consider a two-coin toss with possible outcomes of heads H or tails T for each coin. The sample space is $\Omega = \{HH, HT, TH, TT\}$. A valid σ-algebra (event space) is

$$\mathcal{F} = \{\emptyset, \Omega, \{TT\}, \{HT, TH, HH\}\} \ .$$

Indeed, other valid σ-algebras may be constructed with the sample space and note that the choice of $\mathcal{F}$ is not the same as the powerset of Ω.

For our purposes, we will not delve further into the measure-theoretic aspects of probability theory. The axiomatic definition of probability described earlier assumes a probability space defined by the triplet $(P, \Omega, \mathcal{F})$ which consists of a probability function, a sample space, and the event space sigma-algebra.

Random Variables

In many cases the probabilistic model introduced above has outcomes $\omega \in \Omega$ in the sample space that can be associated with numeric values (e.g., taking the temperature of an object with a noisy thermometer gives a sample space that consists of temperature values). A *random variable* (or r.v. for short) is a function that maps the outcomes of an experiment to the real number line $\mathbb{R} = (-\infty, \infty)$. For example, suppose we toss a coin 5 times, then the number of heads is an appropriate random variable. In this example, the random variable can only take on values of $\{1, 2, 3, 4, 5\}$. Random variables that only take on finite values are called *discrete random variables* whereas r.v.s that take on a continuum of values are called *continuous random variables*. Formally, a random variable can be considered a function $X(\omega) : \Omega \to T \subseteq \mathbb{R}$ that maps sample points ω to the real line $\mathbb{R} = (-\infty, \infty)$ (or a subset thereof). Random variables are typically denoted by capital letters, e.g., X. The range of X is the subset $T \subseteq \mathbb{R}$ of the real line that the random variable maps to: $T = X(\Omega) = \{X(\omega) : \omega \in \Omega\} \subseteq \mathbb{R}$. A r.v. is similar to a typical function, except that the argument (ω in our notation) lives in an abstract sample space. (For example, the sample space can be the outcome of a coin toss.) We are often interested in the probability that a r.v. takes on some particular value i.e., that $Z(\omega)$ is equal to some $z \in T$, where T is range of the r.v. This probability is denoted in shorthand as $P(Z = z)$ or $P(z)$. However, to be more precise, we could write

$$P(Z = z) = P(\omega \in \Omega : Z(\omega) = z) \tag{5}$$

to emphasize that in fact we are seeking the probability of the outcome in the sample space that maps through the random variable to $Z(\omega) = z$.

Discrete Random Variables

Consider a discrete r.v. $Z : \Omega \to T$, where $T = \{z_1, z_2, \ldots, z_n\}$ is a finite set of values. A *probability mass function* (p.m.f) is a function $p_Z(z) = P(Z = z)$ that assigns to each value $z \in T$ a probability. (Note that p_Z assigns probabilities to random values whereas our earlier capital P assigns probabilities to sample points or events.) Summing the p.m.f. over the range of possible values adds to one

$$\sum_{z \in T} p_Z(z) = 1 \; .$$

The expectation of Z (also called the mean or average) is the sum of the values in T weighted by their probabilities

$$\begin{aligned} \mu = E[Z] &= z_1 p_Z(z_1) + \cdots + z_n p_Z(z_n) \\ &= \sum_{z \in T} z p_Z(z) \; . \end{aligned} \tag{6}$$

The second statistical moment of the r.v. is the expected value of Z^2,

$$\begin{aligned} E[Z^2] &= z_1^2 p_Z(z_1) + \cdots + z_n^2 p_Z(z_n) \\ &= \sum_{z \in T} z^2 p_Z(z) \; , \end{aligned} \tag{7}$$

and the Nth statistical moment is the expected value of Z^N

$$\begin{aligned} E[Z^N] &= z_1^N p_Z(z_1) + \cdots + z_n^N p_Z(z_n) \\ &= \sum_{z \in T} z^N p_Z(z) \ . \end{aligned} \tag{8}$$

When the moment is taken around the mean, i.e., when μ is subtracted from the r.v. in the expectation, we call this a *central moment*. The second central moment is the variance,

$$\begin{aligned} \sigma^2 = \mathrm{Var}(Z) &= E[(Z - E[Z])^2)] \\ &= E[(Z^2 - 2ZE[Z] + (E[Z])^2)] \\ &= E[Z^2] - 2ZE[Z] + (E[Z])^2] \\ &= E[Z^2] - 2E[Z]E[Z] + (E[Z])^2 \\ &= E[Z^2] - (E[Z])^2 \\ &= E[Z^2] - \mu^2 \end{aligned} \tag{9}$$

where σ is called the standard deviation. The *skew* of a random variable is a measured of the asymmetry of the pdf around its mean

$$\text{skew} = E[(X - \mu)^3]$$

In general, we also can define the following *moments* (sometimes called *point estimates*):

$$i\text{th moment of } X = E[(X)^i]$$
$$i\text{th central moment of } X = E[(X - \mu_x)^i]$$

It is also convenient to normalize the central moment by the corresponding power of the standard deviation. This corresponds to computing the statistical moments for a new *standardized r.v.*

$$\bar{Z} = \left(\frac{Z - \mu}{\sigma} \right) \ . \tag{10}$$

The standardized central moments are then:

$$E[\bar{Z}] = E\left[\left(\frac{Z-\mu}{\sigma} \right) \right] = \left[\left(\frac{E[Z]-\mu}{\sigma} \right) \right] = \left[\left(\frac{\mu-\mu}{\sigma} \right) \right] = 0 \quad \text{(standardized mean)} \tag{11}$$

$$E[\bar{Z}^2] = E\left[\left(\frac{Z-\mu}{\sigma} \right)^2 \right] = \left(\frac{E[(Z-\mu)^2]}{\sigma^2} \right) = \left(\frac{\sigma^2}{\sigma^2} \right) = 1 \quad \text{(standardized variance)} \tag{12}$$

$$E[\bar{Z}^3] = E\left[\left(\frac{Z-\mu}{\sigma} \right)^3 \right] \quad \text{(skew)} \tag{13}$$

$$E[\bar{Z}^4] = E\left[\left(\frac{Z-\mu}{\sigma} \right)^4 \right] \quad \text{(kurtosis)} \ . \tag{14}$$

Continuous Random Variables

We can extend the concept of p.m.f. for discrete r.v.s to continuous r.v.s by introducing the *probability density function* (p.d.f.). For a continuous random variable, $Z : \Omega \to T$, where $T \subseteq \mathbb{R}$, the p.d.f. $f_Z(z)$ satisfies

$$P(Z \in B) = \int_B f_Z(s) ds \ , \tag{15}$$

where $B \subseteq T$. In other words, the probability that the continuous r.v. Z takes on a value in the interval B is equal to the integral (15). The function $f_Z(\cdot)$ is a curve over the real line that (to satisfy the probability axioms) must integrate to one

$$\int_{\mathbb{R}} f_Z(s)ds = 1 \; . \tag{16}$$

Note that a function may integrate to one even if it is greater than one at some points, i.e., the above requirement does not restrict $f_Z(\cdot) \leq 1$.

For the particular choice of the set $B_c(z) = \{s \in \mathbb{R} \; : \; s \leq z\}$, equivalently $B_c(z) = (-\infty, z]$ the integral (15) is

$$F_Z(z) = P(Z \in B(z)) = \int_{B_c(z)} f_Z(s)ds \tag{17}$$

$$= P(Z \leq z) = \int_{-\infty}^{z} f_Z(s)ds \tag{18}$$

and the function $F_Z(z)$ is called the *cumulative distribution function* (c.d.f.). Some properties obtained from this definition include:

$$\begin{aligned} F_Z(z) &\in [0,1] \\ F_Z(-\infty) &= 0 \\ F_Z(\infty) &= 1 \\ F_Z(a) &\leq F_Z(b) \quad \text{if} \quad a \leq b \\ P(a \leq Z \leq b) &= F_Z(b) - F_Z(a) \end{aligned}$$

The relationship of the c.d.f. with the p.d.f. is that

$$f_Z(s) = \left[\frac{dF_Z(\tau)}{d\tau} \right]_{\tau = s} \; .$$

Expected Value Operator for Continuous R.V.s and Functions

The expected value of any function $g(X)$ that depends on a random variable can be computed as follows

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)dx$$

where X is a continuous random variable and $p(x)$ is the probability density function of X (i.e., $p(x) = f_X(x)$). For $g(x) = X$ we obtain an expression for the mean/expected value:

$$E[X] = \int_{-\infty}^{\infty} xp(x)dx \tag{19}$$

For $g(X) = (X - \mu_x)^2$ we obtain the variance

$$\begin{aligned}
E[(X-\mu_x)^2] &= \int_{-\infty}^{\infty} (X-\mu_x)^2 p(x)dx \\
&= \int_{-\infty}^{\infty} X^2 p(x)dx - \int_{-\infty}^{\infty} 2X\mu_x p(x)dx + \int_{-\infty}^{\infty} \mu_x^2 p(x)dx \\
&= \int_{-\infty}^{\infty} X^2 p(x)dx - 2\mu_x \int_{-\infty}^{\infty} Xp(x)dx + \mu_x^2 \underbrace{\int_{-\infty}^{\infty} p(x)dx}_{=1} \\
&= E[X^2] - 2\mu_x E[X] + \mu_x^2 \\
&= E[X^2] - \mu_x^2
\end{aligned}$$

The definitions for central and standardized moments given above can also be obtained by replacing the summations with integrals, as needed.

Linearity of the Expected Value Operator

An operator L is said to be linear if

1. $L(f+g) = Lf + Lg$
2. $L(\alpha f) = \alpha L f$

From the definition of the expected value (19)

$$E[f(X)+g(X)] = \int_{-\infty}^{\infty} (f(x)+g(x))p(x)dx \tag{20}$$

$$= \int_{-\infty}^{\infty} f(x)p(x)dx + \int_{-\infty}^{\infty} g(x)p(x)dx \tag{21}$$

$$= E[f(X)] + E[g(X)] \tag{22}$$

and

$$E[\alpha g(X)] = \int_{-\infty}^{\infty} \alpha g(x)p(x)dx \tag{23}$$

$$= \alpha \int_{-\infty}^{\infty} g(x)p(x)dx \tag{24}$$

$$= \alpha E[g(X)] \tag{25}$$

hence the expected value is linear. Values that are not random e.g., a constant k, remain unchaged by the expected value ($E[k] = k$).

Gaussian Random Variables

The p.d.f. of a Gaussian continuous r.v. has a *normal distribution* given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-\mu)^2/2\sigma^2}\ , \tag{26}$$

where $\mu \in \mathbb{R}$ is the mean and $\sigma \in \mathbb{R}$ is the standard deviation. The mean μ determines where

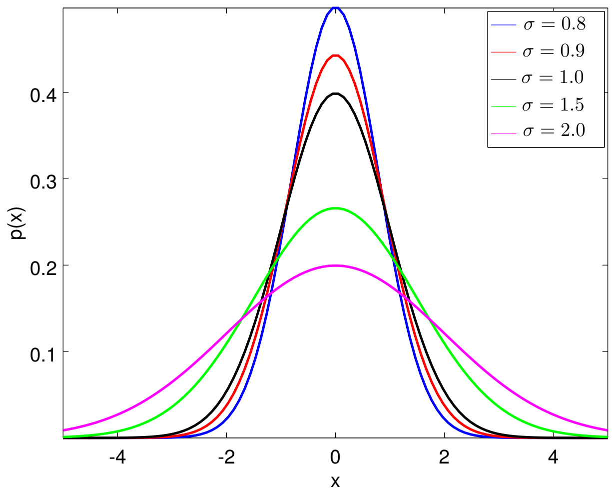


Figure 3: Gaussian probability distributions with a zero mean $\mu = 0$ and varying standard deviation σ

the Gaussian is centered and the variance σ^2 (or standard deviation σ) determines how spread out the distribution is i.e., the larger the standard deviation σ the wider the bell curve. A random variable Z that is normally distributed with mean μ and variance σ^2 is denoted by

$$Z \sim \mathcal{N}(\mu, \sigma^2) .$$

For example, the blue curve in Fig. 3 represents the p.d.f (26) of $Z \sim \mathcal{N}(0, 0.8^2)$.

As evident from (26), the Gaussian p.d.f mean and variance fully define the shape of the p.d.f curve. For a Gaussian r.v. Z, the standardized moments are

$$E[\bar{Z}] = 0 \quad \text{(standardized mean for Gaussian r.v.)} \tag{27}$$

$$E[\bar{Z}^2] = 1 \quad \text{(standardized variance for Gaussian r.v.)} \tag{28}$$

$$E[\bar{Z}^3] = 0 \quad \text{(skew for Gaussian r.v.)} \tag{29}$$

$$E[\bar{Z}^4] = 3 \quad \text{(kurtosis for Gaussian r.v.)} \tag{30}$$

As discussed previously, the probability that a random variable Z takes on a value in some set (e.g., $B_l(a,b) = \{s \in \mathbb{R} \ : \ a \leq s \leq b\}$) is found by integrating the p.d.f over the set:

$$P[B_l(a,b)] = \int_a^b f_Z(s) ds . \tag{31}$$

If Z is Gaussian, then (31) requires integrating (26) which has no closed-form expression. However, some values of the integral (31) can be found in a *cumulative probability table* that was computed numerically. For example, the probability of Z taking on a value within one standard deviation of the mean is about 68%:

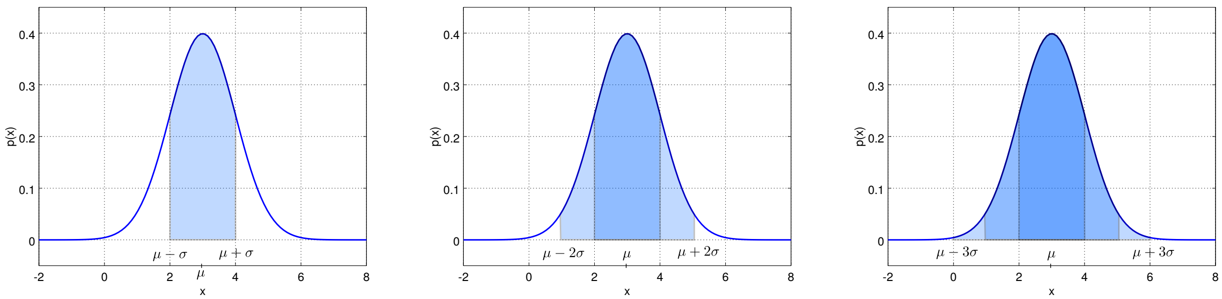
$$P[B_l(\mu - \sigma, \mu + \sigma)] = 0.68 . \tag{32}$$

Similarly, the probability of X taking on a value within two or three standard deviations of the mean is:

$$P[B_l(\mu - 2\sigma, \mu + 2\sigma)] = 0.95 \tag{33}$$

$$P[B_l(\mu - 3\sigma, \mu + 3\sigma)] = 0.997 \tag{34}$$

Refer to Fig. 4 for a graphical representation.



(a) 68% of the time the random number is within $\mu \pm \sigma$

(b) 95% of the time the random number within $\mu \pm 2\sigma$

(c) 99.7% of the time the random number is within $\mu \pm 3\sigma$

Figure 4: Probability distribution for the random number $X \sim \mathcal{N}(3, 1^2)$. The mean is $\mu = 3$ and the standard deviation is $\sigma = 1$.

References

[Bertsekas, 2008] Bertsekas, D. (2008). *Introduction to Probability*. Athena Scientific.