Homework 3

1 Problem

Consider the following system that we examined in Lecture 3:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x} \tag{2}$$

with initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ at initial time $t_0 = 0$ and the corresponding discrete-time system

$$\boldsymbol{x}_k = \boldsymbol{F}\boldsymbol{x}_{k-1} + \boldsymbol{G}\boldsymbol{u}_{k-1} \tag{3}$$

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x}_k \tag{4}$$

where

$$\boldsymbol{F} = e^{\boldsymbol{A}\Delta t} = \begin{bmatrix} \cos\Delta t & \sin\Delta t \\ -\sin\Delta t & \cos\Delta t \end{bmatrix} \tag{5}$$

(with a constant sampling interval Δt). Determine the observability of both systems. Is the discrete-time system observable for all $\Delta t > 0$? Explain your conclusion.

2 Problem

For the ROV system in the Appendix: Suppose that you wish to observe the full state $\boldsymbol{x} = [x, z, \theta, \dot{x}, \dot{z}, \dot{\theta}]^{\mathrm{T}}$ using only three sensors. That is, your output is constrained to be $\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x}$ with $\boldsymbol{y} \in \mathbb{R}^3$. Provide one set of measurements (i.e., a matrix $\boldsymbol{C}$) that is sufficient for observability and another set that is not. Justify your choices. You may use the MATLAB `obsv` and `rank` commands.

3 Problem

Using your choice of an observable output matrix $\boldsymbol{C}$ from the previous problem, design a Luenberg observer to estimate the full system state. That is, provide the expression for the observer and an appropriate matrix $\boldsymbol{H}$ and show that it has stable error dynamics. For this question you may use the MATLAB `eig` and/or `place` commands.

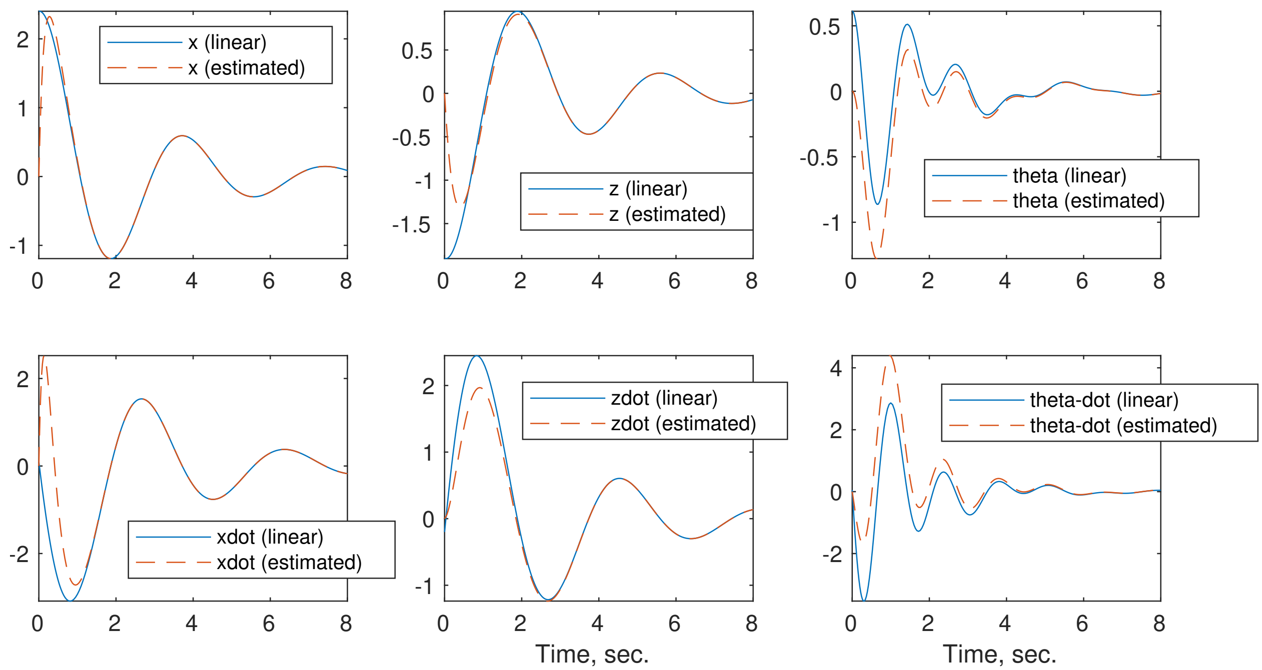
4 Problem

For the ROV system in the Appendix: Create an augmented system that includes the Luenberg observer you designed in the previous problem. Simulate the linearized system and observer using `ode45` for an interval of 8 seconds and assuming an initial condition

$$\boldsymbol{x}_0 = [2.4, -1.9, 35\pi/180, 0.1, -0.2, 3\pi/180]^{\mathrm{T}} \tag{6}$$

with an initial guess for the observer of $\hat{\boldsymbol{x}}_0 = \boldsymbol{0}$. Create a subplot array in MATLAB (2 rows and 3 columns) that shows the actual state of the linearized system and the observer's estimated state.

Answers will vary depending on your choice of $\boldsymbol{C}$ and $\boldsymbol{H}$ but your solution should look similar to the image below. Choose poles that will clearly show the estimate converging in the first few seconds of your simulation (i.e., select poles in Problem 3 that are not too fast or too slow).

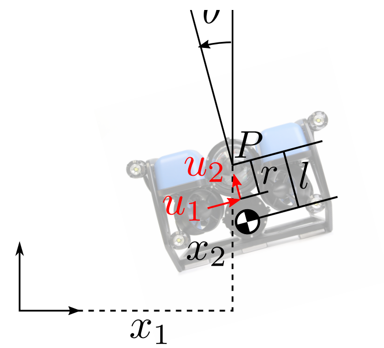


5 Problem

Write a MATLAB routine that runs at the end code of your simulation of the linearized ROV system to determine the initial condition $\boldsymbol{x}_0$ using only the data provided by the output of the linearized system $\boldsymbol{y}(t)$ and knowledge of the linearized system matrices $\boldsymbol{A}$ and $\boldsymbol{C}$.

Appendix: ROV System Model

An underwater remotely operater vessel (ROV) shown below moves in the vertical plane and the position of a reference point P on the ROV is given by the coordinates x_1 (horizontal position) and x_2 (vertical position) as measured from a reference frame placed at a desired ROV location. Several thrusters that produce a net lateral thrust u_1 and a net upward thrust u_2 (in the body's frame of reference).



A simplified model of the vehicle's dynamics derived is:

$$m\ddot{x} = -c_d\dot{x} + u_1\cos\theta - u_2\sin\theta \tag{7}$$
$$m\ddot{z} = -c_d\dot{z} + u_1\sin\theta + u_2\cos\theta \tag{8}$$
$$J\ddot{\theta} = -mgL\sin\theta - c_\theta\dot{\theta} + ru_1 \tag{9}$$

where the mass of the ROV is $m = 6.621$ kg and its inertia is $J = 0.61$ kg·m^2. The center of mass is located a distance $L = 0.2$ m below the reference point P and the net thruster force are applied at a distance $r = 0.1$ m below P. The vehicle experiences linear damping (drag) with coefficients $c_d = 5$ and $c_\theta = 0.88$. Assume $g = 9.81$ m/s^2. To reach the goal location the thrusters use the following proportional control laws:

$$u_1 = -k_p x_1 \tag{10}$$
$$u_2 = -k_p x_2 \tag{11}$$

with $k_p = 20$. We can linearize the closed-loop system around the reference point where the vehicle is stationary, level, and hovering at the origin to give a linearized system of the form

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x} \tag{12}$$

where the state vector is defined as $\boldsymbol{x} = [x, z, \theta, \dot{x}, \dot{z}, \dot{\theta}]^{\mathrm{T}}$. Re-write the system in first-order form with feedback

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ (-c_d/m)x_4 - (k_p/m)x_1\cos x_3 + (k_p/m)x_2\sin x_3 \\ (-c_d/m)x_5 - (k_p/m)x_1\sin x_3 - (k_p/m)x_2\cos x_3 \\ \dot{-}(mgL/J)\sin x_3 - (c_\theta/J)x_6 - (rk_p/J)x_1 \end{bmatrix} \tag{13}$$

The Jacobian of the system willl have entries

$$\boldsymbol{J}_x \boldsymbol{f}(1,4) = 1 \tag{14}$$
$$\boldsymbol{J}_x \boldsymbol{f}(2,5) = 1 \tag{15}$$
$$\boldsymbol{J}_x \boldsymbol{f}(3,6) = 1 \tag{16}$$
$$\boldsymbol{J}_x \boldsymbol{f}(4,1) = -k_p \cos x_3/m \tag{17}$$
$$\boldsymbol{J}_x \boldsymbol{f}(4,2) = k_p \sin x_3/m \tag{18}$$
$$\boldsymbol{J}_x \boldsymbol{f}(4,3) = ((-c_d/m)x_4 - u_1 \sin x_3/m - u_2 \cos x_3/m) \tag{19}$$
$$\boldsymbol{J}_x \boldsymbol{f}(4,4) = (-c_d/m)x_3 \tag{20}$$
$$\boldsymbol{J}_x \boldsymbol{f}(5,1) = -k_p \sin x_3/m \tag{21}$$
$$\boldsymbol{J}_x \boldsymbol{f}(5,2) = -k_p \cos x_3/m \tag{22}$$
$$\boldsymbol{J}_x \boldsymbol{f}(5,3) = ((-c_d/m)x_5 + u_1 \cos x_3/m - u_2 \sin x_3/m) \tag{23}$$
$$\boldsymbol{J}_x \boldsymbol{f}(5,5) = (-c_d/m) \tag{24}$$
$$\boldsymbol{J}_x \boldsymbol{f}(6,1) = -(rk_p/J) \tag{25}$$
$$\boldsymbol{J}_x \boldsymbol{f}(6,3) = -(mgL/J)\cos x_3 \tag{26}$$
$$\boldsymbol{J}_x \boldsymbol{f}(6,6) = (-c_\theta/J) \tag{27}$$

Evaluating for $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 0$

$$\boldsymbol{A} = \boldsymbol{J}_x \boldsymbol{f} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ (-k_p/m) & 0 & 0 & (-c_d/m) & 0 & 0 \\ 0 & (-k_p/m) & 0 & 0 & (-c_d/m) & 0 \\ (-rk_p/J) & 0 & -(mgL/J) & 0 & 0 & (-c_\theta/J) \end{bmatrix} \tag{28}$$

You can confirm that you've obtained the correct system by evaluating your $\boldsymbol{A}$ matrix numerically:

```
A =
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
   -3.0207         0         0   -0.7552         0         0
         0   -3.0207         0         0   -0.7552         0
   -3.2787         0  -21.2957         0         0   -1.4426
```