Homework 2

1 Problem

Convert the following continuous-time system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$ with

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

into an exact discrete-time system

$$\boldsymbol{x}_k = \boldsymbol{F}\boldsymbol{x}_{k-1} + \boldsymbol{G}\boldsymbol{u}_{k-1}$$

with sampling time Δt. You can check your answer using computer tools but should derive the discrete-time system by hand and show all of your work. Hint: The identity $e^{\boldsymbol{X}} e^{-\boldsymbol{X}} = \boldsymbol{I}$ (for an arbitrary matrix $\boldsymbol{X}$) is useful for computing $\boldsymbol{G}$.

Solution. Since the system is LTI we will use the eigenvalue approach to find the state-transition matrix. (An equally valid approach here is to use Laplace transforms.) The eigenvalues are roots of the characteristic polynomial

$$\det(s\boldsymbol{I} - \boldsymbol{A}) = \det\left(\begin{bmatrix} s+1 & 0 \\ -1 & s \end{bmatrix}\right) = (s+1)s = 0 \tag{1}$$

which implies $\lambda_1 = -1$ and $\lambda_2 = 0$. From the Cayley-Hamilton theorem, the matrix exponential is equal to

$$\boldsymbol{F} = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{A} \tag{2}$$

where the coefficients α_0 and α_1 satisfy

$$e^{-\Delta t} = \alpha_0 - \alpha_1 \tag{3}$$

$$1 = \alpha_0 \tag{4}$$

which implies that $\alpha_0 = 1$ and $\alpha_1 = 1 - e^{-\Delta t}$. Then

$$\boldsymbol{F} = e^{\boldsymbol{A}\Delta t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (1 - e^{-\Delta t}) \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} 1 - 1 + e^{-\Delta t} & 0 \\ 1 - e^{-\Delta t} & 1 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} e^{-\Delta t} & 0 \\ 1 - e^{-\Delta t} & 1 \end{bmatrix} \tag{7}$$

The discrete-time control-input matrix is given by

$$\boldsymbol{G} = e^{\boldsymbol{A}\Delta t} \left(\int_0^{\Delta t} e^{-\boldsymbol{A}\alpha} d\alpha \right) \boldsymbol{B} \tag{8}$$

We already know $e^{\boldsymbol{A}\alpha}$ (can be obtained by substituting Δt for α in (7)) but we do not known $e^{-\boldsymbol{A}\alpha}$. The hint tells us that $e^{\boldsymbol{A}\alpha}e^{-\boldsymbol{A}\alpha} = \boldsymbol{I}$ which implies that $e^{-\boldsymbol{A}\alpha} = \text{inv}(e^{\boldsymbol{A}\alpha})$. Therefore,

$$
\begin{aligned}
e^{-\boldsymbol{A}\alpha} &= \text{inv}(e^{\boldsymbol{A}\alpha}) && (9)\\
&= \frac{1}{e^{-\alpha}}\begin{bmatrix} 1 & 0 \\ -1+e^{-\alpha} & e^{-\alpha} \end{bmatrix} && (10)\\
&= e^{\alpha}\begin{bmatrix} 1 & 0 \\ -1+e^{-\alpha} & e^{-\alpha} \end{bmatrix} && (11)\\
&= \begin{bmatrix} e^{\alpha} & 0 \\ -e^{\alpha}+e^{\alpha}e^{-\alpha} & e^{\alpha}e^{-\alpha} \end{bmatrix} && (12)\\
&= \begin{bmatrix} e^{\alpha} & 0 \\ 1-e^{\alpha} & 1 \end{bmatrix} && (13)
\end{aligned}
$$

Now to compute $\boldsymbol{G}$ using our result above:

$$
\begin{aligned}
\boldsymbol{G} &= \begin{bmatrix} e^{-\Delta t} & 0 \\ 1-e^{-\Delta t} & 1 \end{bmatrix}\left(\int_0^{\Delta t}\begin{bmatrix} e^{\alpha} & 0 \\ 1-e^{\alpha} & 1 \end{bmatrix} d\alpha\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} && (14)\\
&= \begin{bmatrix} e^{-\Delta t} & 0 \\ 1-e^{-\Delta t} & 1 \end{bmatrix}\begin{bmatrix} \int_0^{\Delta t} e^{\alpha} d\alpha \\ \int_0^{\Delta t}(1-e^{\alpha})d\alpha \end{bmatrix} && (15)\\
&= \begin{bmatrix} e^{-\Delta t} & 0 \\ 1-e^{-\Delta t} & 1 \end{bmatrix}\begin{bmatrix} (e^{\Delta t}-1) \\ \Delta t-(e^{\Delta t}-1) \end{bmatrix} && (16)\\
&= \begin{bmatrix} e^{-\Delta t}(e^{\Delta t}-1) \\ (1-e^{-\Delta t})(e^{\Delta t}-1)+\Delta t-(e^{\Delta t}-1) \end{bmatrix} && (17)\\
&= \begin{bmatrix} 1-e^{-\Delta t} \\ (e^{\Delta t}-1)(1-e^{-\Delta t}-1)+\Delta t \end{bmatrix} && (18)\\
&= \begin{bmatrix} 1-e^{-\Delta t} \\ -e^{\Delta t}e^{-\Delta t}+e^{-\Delta t}+\Delta t \end{bmatrix} && (19)\\
&= \begin{bmatrix} 1-e^{-\Delta t} \\ -1+e^{-\Delta t}+\Delta t \end{bmatrix} && (20)
\end{aligned}
$$

2 Problem

Repeat the previous problem using the approximate discretization described in Lecture 4 with $\Delta t = 0.25$. Evaluate both system matrices $\boldsymbol{F}, \boldsymbol{G}$ and $\boldsymbol{F}_{\text{approx}}, \boldsymbol{G}_{\text{approx}}$ (where the former is the exact and the latter is the approximate system). Submit any MATLAB code you use.

Solution. The approximate form from Lecture 4 gives the system matrices

$$
\boldsymbol{F}_{\text{approx}} = \begin{bmatrix} 0.75 & 0 \\ 0.25 & 1 \end{bmatrix} \tag{21}
$$

$$
\boldsymbol{G}_{\text{approx}} = \begin{bmatrix} 0.25 \\ 0 \end{bmatrix} \tag{22}
$$

whereas the exact discrete-time matrices from Problem 1 yield

$$\boldsymbol{F} = \begin{bmatrix} 0.7788 & 0 \\ 0.2211 & 1 \end{bmatrix} \tag{23}$$

$$\boldsymbol{G} = \begin{bmatrix} 0.2212 \\ 0.0288 \end{bmatrix} \tag{24}$$

```
format long;

A = [-1 0;1 0];
B =[1; 0];

syms dt alpha;
F = expm(A*dt)
G = expm(A*dt)*int(expm(-A*alpha),alpha,0,dt)*B

% exact discrete-time system
dtval = 0.25;
Fexact = double(subs(F,dt,dtval))
Gexact = double(subs(G,dt,dtval))

% approximate discrete-time system matrices
Fapprox = eye(2,2) + A*dtval
Gapprox = B*dtval
```

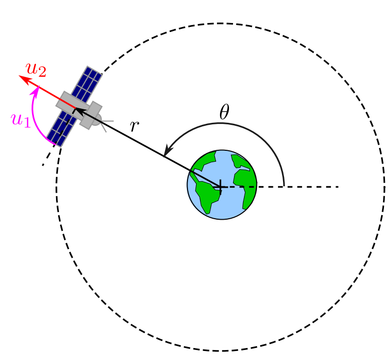
3 Problem

The motion of a satellite in orbit around the earth can be approximately described by the equations:

$$\ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} + 2u_2 \sin u_1 \tag{25}$$

$$\ddot{\theta} = -\frac{2\dot{r}\dot{\theta}}{r} + \frac{u_2}{2r}\cos u_1 \tag{26}$$

where r is the distance from the center of the earth, θ is the angle of the orbit (we're assuming the orbit is in a 2D plane), μ is the standard gravitational parameter, u_1 is the thrust angle and u_2 is a thrust magnitude.



A nominal trajectory for this system is a circular orbit where $r = r_0$ (a constant radius so that $\dot{r} = 0$) and $\dot{\theta} = \omega$ (a constant angular rate) where the radius and angular rate satisfy $\mu = r_0^3\omega^2$. In this nominal trajectory no thrust is needed $u_2 = 0$ and the nominal angle of the thrust can be assumed $u_1 = \pi/2$. Linearize the system around this nominal trajectory to give a system of the form:

$$\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x} + \boldsymbol{B}\Delta\boldsymbol{u} \tag{27}$$

Solution.

The equations of motion of satellite in a planar orbit are:

$$\frac{d\dot{r}}{dt} = \ddot{r} = r\dot{\theta}^2 - \frac{\mu}{r^2} + 2T\sin\phi$$

$$\frac{d\dot{\theta}}{dt} = \ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r} + \frac{T}{2r}\cos\phi$$

State vector $\mathbf{x} = [r \;\; \dot{r} \;\; \theta \;\; \dot{\theta}]^T$

Control input vector $\mathbf{u} = [\phi \;\; T]^T$.

a. Linearized state equation : $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \frac{d\mathbf{x}}{dt} = \frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 - \frac{\mu}{r^2} + 2T\sin\phi \\ \dot{\theta} \\ \frac{-2\dot{r}\dot{\theta}}{r} + \frac{T}{2r}\cos\phi \end{bmatrix} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u} = f(\mathbf{x},\mathbf{u})$$

$$\mathbf{A} = \frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial f}{\partial r} & \frac{\partial f}{\partial \dot{r}} & \frac{\partial f}{\partial \theta} & \frac{\partial f}{\partial \dot{\theta}} \end{bmatrix},$$

Therefore,

$$\mathbf{A} = \frac{\partial f}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial r} & \frac{\partial \dot{x}_1}{\partial \dot{r}} & \frac{\partial \dot{x}_1}{\partial \theta} & \frac{\partial \dot{x}_1}{\partial \dot{\theta}} \\ \frac{\partial \dot{x}_2}{\partial r} & \frac{\partial \dot{x}_2}{\partial \dot{r}} & \frac{\partial \dot{x}_2}{\partial \theta} & \frac{\partial \dot{x}_2}{\partial \dot{\theta}} \\ \frac{\partial \dot{x}_3}{\partial r} & \frac{\partial \dot{x}_3}{\partial \dot{r}} & \frac{\partial \dot{x}_3}{\partial \theta} & \frac{\partial \dot{x}_3}{\partial \dot{\theta}} \\ \frac{\partial \dot{x}_4}{\partial r} & \frac{\partial \dot{x}_4}{\partial \dot{r}} & \frac{\partial \dot{x}_4}{\partial \theta} & \frac{\partial \dot{x}_4}{\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dot{\theta}^2 + \frac{2\mu}{r^3} & 0 & 0 & 2r\dot{\theta} \\ 0 & 0 & 0 & 1 \\ \frac{2\dot{r}\dot{\theta}}{r^2} - \frac{T\cos\phi}{2r^2} & \frac{-2\dot{\theta}}{r} & 0 & \frac{-2\dot{r}}{r} \end{bmatrix}$$

$$\mathbf{B} = \frac{\partial f}{\partial \mathbf{u}} = \begin{bmatrix} \frac{\partial f}{\partial \phi} & \frac{\partial f}{\partial T} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2T\cos\phi & 2\sin\phi \\ 0 & 0 \\ -\frac{T}{2r}\sin\phi & \frac{\cos\phi}{2r} \end{bmatrix}$$

b. At the nominal trajectory $r = r_0$, $\dot{r} = 0$, $\dot{\theta} = \omega$, $T = 0$, $\mu = r_0^3\omega^2$, and $\phi = 90^\circ$.
By substituting these values to above **A** and **B** matrices, we have

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega r_0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2\omega}{r_0} & 0 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

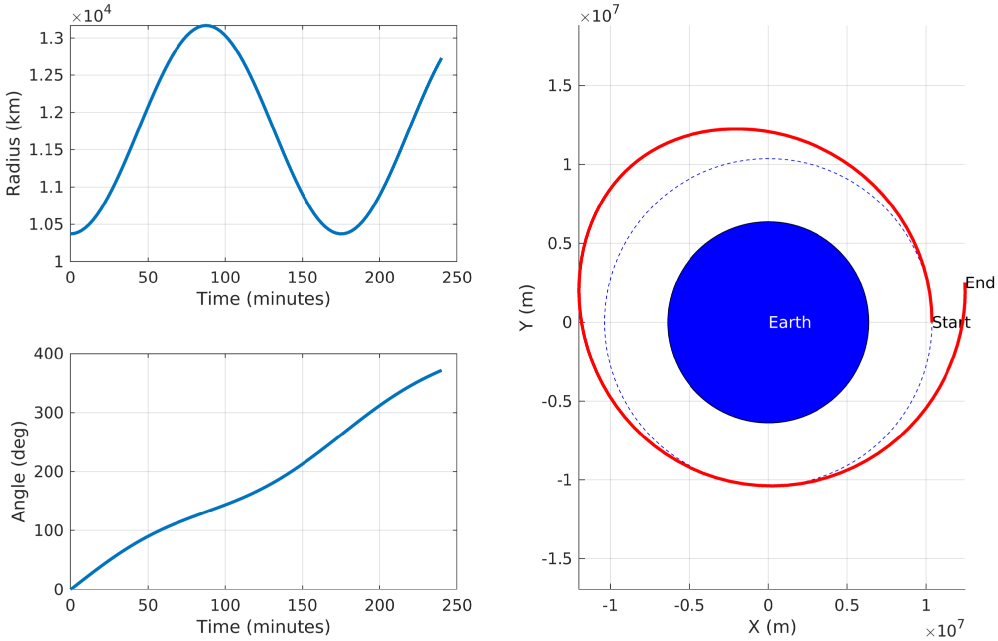
4 Problem

Use MATLAB's `lsim` tool to simulate the system from the previous problem assuming:

- An orbit 4,000 km above the Earth's surface. That is, $r_0 = h + r_{\text{Earth}}$ where $h = 4E6$ meters is the altitude of the orbit and $r_{\text{Earth}} = 6371000$ meters is the radius of the Earth.
- The gravitational parameter for the earth is $\mu = GM$ where $G = 6.674\text{E-}11$ m$^3\cdot$kg$^{-1}\cdot$s^{-2} is

the gravitational constant and $M = 5.97219\text{E}24$ kg is the mass of the Earth.

Simulate the system starting along the reference trajectory while applying a thruster force of 0.25 Newtons for $T = 14400$ seconds (i.e., 4 hours). Plot the actual radius $r(t)$ and angle $\theta(t)$ (not deviations) during the simulation. Your plot should look something like the two left-most panels below (the right panel is optional).



Solution.

```
% Define constants/system matrices here
rEarth = 6371000; % m
h =4E6; % meters (4000 km)
r0 = rEarth + h;
G = 6.674E-11; % gravitational constant
M = 5.97219E24; % mass of earth, kg
mu = G*M;
w = (mu/r0^3)^(1/2);

A = [0 1 0 0;
    3*w^2 0 0 2*w*r0;
    0 0 0 1;
    0 (-2*w/r0) 0 0];
B = [0 0;0 2;0 0;0 0];
C = [];
D = [];
uc = 0.25;
T = 4*60*60;

% Simulate the linear system (deviations)
sys = ss(A,B,C,D);
```

```
t = linspace(0,T); % setup control input u(t)
u = zeros([length(t) 2]);
u(:,2) = uc;
x0 = [0; 0; 0; 0;];
[yhist,thist,xhist] = lsim(sys,u,t,x0);

% convert to original coordinates
r = xhist(:,1)' + r0;
th = xhist(:,3)' + w*t;
x = r.*cos(th);
y = r.*sin(th);

% plot
figure
set(gcf,'Color','w','Units','inches','Position',[1 1 8 6])
subplot(2,2,1)
plot(thist/60,r/1000);
hold on;
xlabel('Time (minutes)')
grid on;
ylabel('Radius (km)')

subplot(2,2,3)
plot(thist/60,th*180/pi);
hold on;
xlabel('Time (minutes)')
ylabel('Angle (deg)')
grid on;

subplot(2,2,[2 4])
xc = cos(linspace(0,2*pi))'; % plot earth
yc = sin(linspace(0,2*pi))';
hold on;
fill(xc*rEarth, yc*rEarth,'b')
text(0,0,'Earth','Color','w')
text(r0,0,'Start')
text(x(end),y(end),'End')
plot(xc*r0, yc*r0,'b--') % plot orbit
plot(x,y,'r-') % plot trajectory
axis equal;
xlabel('X (m)')
ylabel('Y (m)')
grid on;

addpath('../../export_fig')
```

```
export_fig(1,'-dpdf','traj.pdf')
```