Homework 1

1 Problem

(a) Write down a state equation for a dynamical system in state-space form that you obtain from a research paper of your choosing (e.g., search Google Scholar or IEEE Xplore for a paper that is related to your research or interests). Your system may or may not include a control input. If needed, reduce the equation to a system of first order ODEs and show your work. Explain what are the components of the state vector and what process/system the equation represents. Define any symbols introduced.

(b) State the measurement equation for the system. If a measurement equation is not included in the paper hypothesize what sensor could be used to observe the behavior of the system and what the corresponding measurement equation would look like.

(c) Provide the citation for the paper you referred to.

2 Problem

Use the Peano-Baker series to prove that the state transition matrix $\boldsymbol{\Phi}(t,0)$ for the system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ with

$$A = \begin{bmatrix} t & t \\ 0 & t \end{bmatrix} \qquad \text{is} \qquad \boldsymbol{\Phi}(t,0) = \begin{bmatrix} e^{\frac{1}{2}t^2} & \frac{1}{2}t^2 e^{\frac{1}{2}t^2} \\ 0 & e^{\frac{1}{2}t^2} \end{bmatrix}$$

Hint: Use the series definition of the exponential function.

Solution.

Solution: The transition matrix is computed using the following

$$\begin{aligned}\Phi(t,0) = I &+ \int_0^t A(\sigma)d\sigma + \int_0^t A(\sigma_1)\int_0^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 \\ &+ \int_0^t A(\sigma_1)\int_0^{\sigma_1} A(\sigma_2)\int_0^{\sigma_2} A(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 + \cdots\end{aligned}$$

and we compute each integral term iteratively.

(a)

$$\int_0^t A(\sigma)d\sigma = \begin{bmatrix} \frac{1}{2}t^2 & \frac{1}{2}t^2 \\ 0 & \frac{1}{2}t^2 \end{bmatrix}$$

(b)

$$\int_0^t A(\sigma_1)\left(\int_0^{\sigma_1} A(\sigma_2)d\sigma_2\right)d\sigma_1 = \int_0^t A(\sigma_1)\begin{bmatrix} \frac{1}{2}\sigma_1^2 & \frac{1}{2}\sigma_1^2 \\ 0 & \frac{1}{2}\sigma_1^2 \end{bmatrix} d\sigma_1 = \begin{bmatrix} \frac{1}{8}t^4 & \frac{1}{4}t^4 \\ 0 & \frac{1}{8}t^4 \end{bmatrix}$$

(c)

$$\begin{aligned}\int_0^t A(\sigma_1)\left(\int_0^{\sigma_1} A(\sigma_2)\int_0^{\sigma_2} A(\sigma_3)d\sigma_3 d\sigma_2\right)d\sigma_1 &= \int_0^t A(\sigma_1)\begin{bmatrix} \frac{1}{8}\sigma_1^4 & \frac{1}{4}\sigma_1^4 \\ 0 & \frac{1}{8}\sigma_1^4 \end{bmatrix} d\sigma_1 \\ &= \begin{bmatrix} \frac{1}{48}t^6 & \frac{1}{16}t_1^6 \\ 0 & \frac{1}{48}t^6 \end{bmatrix}\end{aligned}$$

Thus

$$\Phi(t,0) = \begin{bmatrix} 1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 + \frac{1}{48}t^6 + \cdots & \frac{1}{2}t^2 + \frac{1}{4}t^4 + \frac{1}{16}t_1^6 + \cdots \\ 0 & 1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 + \frac{1}{48}t^6 + \cdots \end{bmatrix} \quad (2)$$

Using the identity,

$$\frac{1}{2n(2n-2)(2n-4)\cdots(2)} = \frac{1}{n!2^n}$$

the infinite series in (2) are written

$$\begin{aligned} 1 + \frac{1}{2}t^2 + \frac{1}{8}t^4 + \frac{1}{48}t^6 + \cdots &= \sum_{k=0}^{\infty} \frac{t^{2k}}{k!2^k} \\ &= \sum_{k=0}^{\infty} \frac{(\frac{1}{2}t^2)^k}{k!} \\ &= e^{\frac{1}{2}t^2} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}t^2 + \frac{1}{4}t^4 + \frac{1}{16}t_1^6 + \cdots &= \sum_{k=1}^{\infty} \frac{kt^{2k}}{2^k k!} \\ &= \frac{1}{2}t^2 \sum_{k=1}^{\infty} \frac{t^{2(k-1)}}{2^{k-1}(k-1)!} \\ &= \frac{1}{2}t^2 \sum_{k=0}^{\infty} \frac{t^{2(k)}}{2^k(k}!) \\ &= \frac{1}{2}t^2 e^{\frac{1}{2}t^2} \end{aligned}$$

Thus,

$$\Phi(t,0) = \begin{bmatrix} e^{\frac{1}{2}t^2} & \frac{1}{2}t^2 e^{\frac{1}{2}t^2} \\ 0 & e^{\frac{1}{2}t^2} \end{bmatrix}$$

3 Problem

Consider the system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$ with initial condition $\boldsymbol{x}_0 = [1, 0]^{\mathrm{T}}$. Find the solution $\boldsymbol{x}(t)$ using the Laplace inverse approach with partial fractions.

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

Solution.

Case 1: Laplace inverse approach

Step 1: Find the matrix sI

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

Step 2: Find the matrix $[sI–A]$

$$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s-3 & -4 \\ -2 & s-1 \end{bmatrix}$$

Step 3: Find the inverse of the matrix $[sI–A]$

$$[sI - A]^{-1} = \frac{Adj(sI - A)}{|sI - A|}$$

$$|\, sI - A \,| = (s-3)\,(s-1) - 8$$

$$= s^2 - s - 3s + 3 - 8 \;=\; s^2 - 4s - 5$$

$$= (s-5)(s+1)$$

$$Adj(sI-A)=\begin{bmatrix} s-1 & 4 \\ 2 & s-3 \end{bmatrix}$$

$$[sI-A]^{-1}=\frac{Adj(sI-A)}{|sI-A|}=\frac{\begin{bmatrix} s-1 & 4 \\ 2 & s-3 \end{bmatrix}}{(s-5)(s+1)}$$

$$=\begin{bmatrix} \frac{s-1}{(s-5)(s+1)} & \frac{4}{(s-5)(s+1)} \\ \frac{2}{(s-5)(s+1)} & \frac{s-3}{(s-5)(s+1)} \end{bmatrix}$$

Step 4: Find the laplace inverse of $[sI–A]^{-1}$

$$e^{At}=L^{-1}[sI-A]^{-1}$$

$$=L^{-1}\begin{bmatrix} \frac{s-1}{(s-5)(s+1)} & \frac{4}{(s-5)(s+1)} \\ \frac{2}{(s-5)(s+1)} & \frac{s-3}{(s-5)(s+1)} \end{bmatrix}$$

Finding the partial fraction

$$\Rightarrow \frac{s-1}{(s-5)(s+1)}=\frac{A}{(s-5)}+\frac{B}{(s+1)}$$

$$s-1=A(s+1)+B(s-5)$$

When $s=5$	When $s=-1$
$5–1 = A(5+1) + 0$	$–1–1 = A(0) + B(–1–5)$
$4 = 6A$	$–2 = –6B$
$A = 4/6$	$B = 2/6$

$$\Rightarrow \frac{4}{(s-5)(s+1)}=\frac{A}{(s-5)}+\frac{B}{(s+1)}$$

$$4=A(s+1)+B(s-5)$$

When $s=5$	When s = –1
$4 = A(5+1) + 0$	$4 = A(0) + B(–1–5)$
$4 = 6A$	$4 = –6B$
$A = 4/6$	$B = –4/6$

$$\Rightarrow \frac{2}{(s-5)(s+1)} = \frac{A}{(s-5)} + \frac{B}{(s+1)}$$

$$2 = A(s+1) + B(s-5)$$

When $s=5$	When $s=-1$
$2 = A(5+1) + 0$	$2 = A(0) + B(-1-5)$
$2 = 6A$	$2 = -6B$
$A = 2/6$	$B = -2/6$

$$\Rightarrow \frac{s-3}{(s-5)(s+1)} = \frac{A}{(s-5)} + \frac{B}{(s+1)}$$

$$s-3 = A(s+1) + B(s-5)$$

When $s=5$	When $s=-1$
$5-3 = A(5+1) + 0$	$-1-3 = A(0) + B(-1-5)$
$2 = 6A$	$-4 = -6B$
$A = 2/6$	$B = 4/6$

$$\therefore e^{At} = L^{-1}\begin{bmatrix} \frac{4/6}{(s-5)} + \frac{2/6}{(s+1)} & \frac{4/6}{(s-5)} - \frac{4/6}{(s+1)} \\ \frac{2/6}{(s-5)} - \frac{2/6}{(s+1)} & \frac{2/6}{(s-5)} + \frac{4/6}{(s+1)} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} \frac{4}{6}e^{5t} + \frac{2}{6}e^{-t} & \frac{4}{6}e^{5t} - \frac{4}{6}e^{-t} \\ \frac{2}{6}e^{5t} - \frac{2}{6}e^{-t} & \frac{2}{6}e^{5t} + \frac{4}{6}e^{-t} \end{bmatrix}$$

$$\ldots \therefore e^{At} = \begin{bmatrix} \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t} & \frac{2}{3}e^{5t} - \frac{2}{3}e^{-t} \\ \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} \end{bmatrix}$$

Step 5: Find the free response $x(t)$

$$x(t) = e^{At}x(0)$$

$$x(t) = e^{At} = \begin{bmatrix} \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t} & \frac{2}{3}e^{5t} - \frac{2}{3}e^{-t} \\ \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{2}{3}e^{5t} + \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} \end{bmatrix}$$

4 Problem

For the same system as given in the problem above, find the solution $\boldsymbol{x}(t)$ using the eigenvalue approach (with Cayley-Hamilton Theorem).

Solution.

Case 3: Cayley Hamilton approach

Given

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}; \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Step 1: Find the eigen values

$$[\lambda I - A] = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \lambda - 3 & -4 \\ -2 & \lambda - 1 \end{bmatrix}$$

$$[\lambda I - A]^{-1} = \frac{Adj(\lambda I - A)}{|\lambda I - A|}$$

$$|\lambda I - A| = (\lambda - 3)(\lambda - 1) - 8$$
$$= \lambda^2 - \lambda - 3\lambda + 3 - 8 = \lambda^2 - 4\lambda - 5$$
$$= (\lambda - 5)(\lambda + 1)$$

The eigen values are $\lambda_1 = 5$, $\lambda_2 = -1$

Step 2: Find $g(\lambda_1) = \beta_0 + \beta_1 \lambda_1$, $g(\lambda_2) = \beta_0 + \beta_1 \lambda_2$

$\because$ The matrix A is second order, and hence, the polynomial $g(\lambda)$ will be of the form, $g(\lambda) = \beta_0 + \beta_1 \lambda$

For $\lambda_1 = 5$, we have

$$g(\lambda_1) = \beta_0 + \beta_1 \lambda_1$$
$$g(5) = \beta_0 + 5\beta_1$$

For $\lambda_2 = -1$, we have

$$g(\lambda_2) = \beta_0 + \beta_1 \lambda_2$$
$$g(-1) = \beta_0 + \beta_1(-1)$$
$$g(-1) = \beta_0 - \beta_1$$

Step 3: Find $f(\lambda_1) = e^{\lambda_1 t}$ and $f(\lambda_2) = e^{\lambda_2 t}$

The coefficients $g(\lambda) = \beta_0 + \beta_1 \lambda$ are evaluated using the following relations:

$$f(\lambda_1) = e^{\lambda_1 t} = g(\lambda_1) = \beta_0 + \beta_1 \lambda_1$$

$$f(\lambda_2) = e^{\lambda_2 t} = g(\lambda_2) = \beta_0 + \beta_1 \lambda_2$$

For $\lambda_1 = 5$, we have

$$f(\lambda_1) = e^{5t}$$

$$g(\lambda_1) = \beta_0 + 5\beta_1$$

$$f(\lambda_1) = g(\lambda_1)$$

$$e^{5t} = \beta_0 + 5\beta_1 \tag{i}$$

For $\lambda_1 = -1$, we have

$$f(\lambda_2) = e^{-t}$$

$$g(\lambda_2) = \beta_0 + \beta_1$$

$$f(\lambda_2) = g(\lambda_2)$$

$$e^{-t} = \beta_0 - \beta_1 \tag{ii}$$

By solving equations (i) and (ii), we get

$$6\beta_1 = e^{5t} - e^{-t} \tag{iii}$$

$$\beta_1 = (e^{5t} - e^{-t})/6 \tag{iv}$$

Substituting (iv) in (i), we get

$$\beta_0 = \left(e^{5t} + 5e^{-t}\right)/6 \tag{v}$$

Step 5: Find $e^{At} = \beta_0 I + \beta_1 A$

$$e^{At} = \left[\left(e^{5t} + 5e^{-t}\right)/6\right]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left[\left(e^{5t} - e^{-t}\right)/6\right]\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{6}e^{-5t} + \frac{5}{6}e^{-t} & 0 \\ 0 & \frac{1}{6}e^{-5t} + \frac{5}{6}e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{3}{6}e^{-5t} - \frac{3}{6}e^{-t} & \frac{4}{6}e^{-5t} - \frac{4}{6}e^{-t} \\ \frac{2}{6}e^{-5t} - \frac{2}{6}e^{-t} & \frac{1}{6}e^{-5t} - \frac{1}{6}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{6}e^{-5t} + \frac{2}{6}e^{-t} & \frac{4}{6}e^{-5t} - \frac{4}{6}e^{-t} \\ \frac{2}{6}e^{-5t} - \frac{2}{6}e^{-t} & \frac{2}{6}e^{-5t} + \frac{4}{6}e^{-t} \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} \frac{2}{3}e^{-5t} + \frac{1}{3}e^{-t} & \frac{2}{3}e^{-5t} - \frac{2}{3}e^{-t} \\ \frac{1}{3}e^{-5t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{-5t} + \frac{2}{3}e^{-t} \end{bmatrix}$$

Step 6: Find the free response $x(t)$

$$x(t) = e^{At}x(0)$$

$$x(t) = e^{At} = \begin{bmatrix} \frac{2}{3}e^{-5t} + \frac{1}{3}e^{-t} & \frac{2}{3}e^{-5t} - \frac{2}{3}e^{-t} \\ \frac{1}{3}e^{-5t} - \frac{1}{3}e^{-t} & \frac{1}{3}e^{-5t} + \frac{2}{3}e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow x(t) = \begin{bmatrix} \frac{2}{3}e^{-5t} + \frac{1}{3}e^{-t} \\ \frac{1}{3}e^{-5t} - \frac{1}{3}e^{-t} \end{bmatrix}$$

Note: The solution above has a typo (replace e^{-5t} with e^{5t})

5 Problem

Consider the following system of coupled second-order equations

$$\ddot{y} - a(\dot{z} - \dot{y}) - b(z - y) = cu_1 + du_2$$
$$\ddot{z} - e\dot{z} - f(y + z) = gu_1$$

where (a, b, c, d, e, f, g) are all constants and $z(t)$ and $y(t)$ are states and $u_1(t)$ and $u_2(t)$ are control inputs. Define an appropriate state vector $\boldsymbol{x} \in \mathbb{R}^n$ and control vector $\boldsymbol{u} \in \mathbb{R}^m$ and then re-write this system as n first-order differential equations in the form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$.

Solution. Define the state as $\boldsymbol{x} = [y, \dot{y}, z, \dot{z}]^{\mathrm{T}}$ and the control input as $\boldsymbol{u} = [u_1, u_2]^{\mathrm{T}}$. The system can be written in first order form as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= a(x_4 - x_2) + b(x_3 - x_1) + cu_1 + du_2 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= gu_1 + ex_4 + f(x_1 + x_3)
\end{aligned}$$

This is equivalent to the matrix form $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}$ with

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b & -a & b & a \\ 0 & 0 & 0 & 1 \\ f & 0 & f & e \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ c & d \\ 0 & 0 \\ g & 0 \end{bmatrix}$$

Answers may vary depending on defnition of $\boldsymbol{x}$ and $\boldsymbol{u}$.

6 Problem

An equilibrium point for a nonlinear system $\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ is a state $\boldsymbol{x}^*$ for which the state-rate is zero (i.e., $\dot{\boldsymbol{x}} = f(\boldsymbol{x}^*) = 0$). For the following nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_1x_2 \\ -x_1 + x_1^2 + x_2^2 \end{bmatrix}$$

determine the four equilibrium points $\boldsymbol{x}_a^*$, $\boldsymbol{x}_b^*$, $\boldsymbol{x}_c^*$ and $\boldsymbol{x}_d^*$. (Hint: one of them is $[1/2, -1/2]^{\mathrm{T}}$). Then state the linearized dynamics $\Delta\dot{\boldsymbol{x}} = \boldsymbol{A}\Delta\boldsymbol{x}$ around each nominal equilibrium. Define exactly what you mean by $\Delta\boldsymbol{x}$ for each linearized system.

Solution. To find the equilibirum points set the state-rate to zero:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_1x_2 \\ -x_1 + x_1^2 + x_2^2 \end{bmatrix}$$

which gives two nonlinear equations in two unknowns. From the first equation a condition for equilibrium is that

$$x_2 = 2x_1x_2$$

Which is satisfied if either $x_2 = 0$ or if $x_1 = 1/2$. First consider $x_2 = 0$. From the second equation

$$0 = -x_1 + x_1^2$$

which is satisfied if either $x_1 = 0$ or if $x_1 = 1$. Hence we've found two equilibrium points $\boldsymbol{x}_a^* = [1, 0]^{\mathrm{T}}$ and $\boldsymbol{x}_b^* = [0, 0]^{\mathrm{T}}$. Now consider the second case that $x_1 = 1/2$. Substituting this requirement into the second equation

$$\begin{aligned}
0 &= -(1/2) + (1/4) + x_2^2 \\
x_2^2 &= 1/4
\end{aligned}$$

which is satisfied if either $x_2 = 1/2$ or $x_2 = -1/2$. Hence the remaining two equilibrium points are $\boldsymbol{x}_c^* = [1/2, 1/2]^\mathrm{T}$ and $\boldsymbol{x}_d^* = [1/2, -1/2]^\mathrm{T}$. Now, to compute the linearized system first find the Jacobian

$$\boldsymbol{J_x f} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_n 2} \end{bmatrix} = \begin{bmatrix} -2x_2 & 1-2x_1 \\ -1+2x_1 & 2x_2 \end{bmatrix}$$

We will have a total of four linearized systems of the form

$$\Delta \boldsymbol{x} = \boldsymbol{A} \Delta \boldsymbol{x}$$

with one for each equilibrium point.

For $\boldsymbol{x}_a^* = [1, 0]^\mathrm{T}$

$$\boldsymbol{A} = \boldsymbol{J_x f}\Big|_{\boldsymbol{x}_a^*} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and the deviation states are defined as $\Delta \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_a^*$.

For $\boldsymbol{x}_b^* = [0, 0]^\mathrm{T}$

$$\boldsymbol{A} = \boldsymbol{J_x f}\Big|_{\boldsymbol{x}_b^*} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and the deviation states are defined as $\Delta \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_b^*$.

For $\boldsymbol{x}_c^* = [1/2, 1/2]^\mathrm{T}$

$$\boldsymbol{A} = \boldsymbol{J_x f}\Big|_{\boldsymbol{x}_c^*} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the deviation states are defined as $\Delta \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_c^*$.

For $\boldsymbol{x}_d^* = [1/2, -1/2]^\mathrm{T}$

$$\boldsymbol{A} = \boldsymbol{J_x f}\Big|_{\boldsymbol{x}_d^*} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and the deviation states are defined as $\Delta \boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x}_d^*$.