**Proceedings of the 18th World Congress**
**The International Federation of Automatic Control**
**Milano (Italy) August 28 - September 2, 2011**

# On Registration of Uncertain Three-dimensional Vectors with Application to Robotics ⋆

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**Abstract:** In this paper we investigate the problem of finding a suitable roto-translation achieving the optimal matching between two uncertain three-dimensional vector sets. State-of-the-art approaches for vector registration are based on strict assumptions on the covariance matrices describing the uncertainty on the vectors, hence they can be too conservative or inaccurate when the actual uncertainty differs from the employed model. After discussing the problem we propose two iterative solutions for matching the 3D vectors, showing that a suitable uncertainty model allows reducing the estimation error while preserving the real-time nature of the computation. We further derive the covariance matrices for such estimates, evaluating their consistency through extensive numerical experiments. The results appear particularly suitable for robotic applications, since the vector sets constitute a natural representation of three-dimensional perception of a robot, interacting with complex non-planar environments.

Keywords: Registration; Optimization; 3D Vectors; Robotics; Robot Navigation.

## 1. INTRODUCTION

The problem of three-dimensional vectors registration occurs in several engineering applications, including photogrammetry, Oswal and Balasubramanian [1968], attitude determination, Shuster and Oh [1981], manufacturing tolerancing, Calafiore and Bona [1998] and robotics, Borrmann et al. [2008]. The registration (or *matching*) problem consists in estimating a suitable roto-translation that minimize a distance between two vector sets. A formal statement of the problem is provided in Section 2. Literature on the topic has quite a long history, dating back to the sixties, Schonemann [1966], Wahba [1965], although it includes some recent contributions, see Shuster [2006] and Zhang et al. [2005]. The renewed interest towards this problem lies in its application to 3D navigation, since vectors are particularly suitable for modeling three-dimensional perception, as measurements from stereo-cameras, Garcia and Solanas [2004], 3D laser scanners, Cole and Newman [2006], Nüchter et al. [2004], Weingarten and Siegwart [2005], or bearing only sensors (camera measurements related to far landmarks, star-trackers, sun sensors, etc.), Lemaire et al. [2007], Shuster [2006]. In the context of mobile robotics, 3D vector sets constitute a compact representation for several types of information, i.e., (i) the position of salient points (or objects) in the environment, (ii) directions towards landmarks, (iii) orientation of objects or topological places (e.g., orientation of edges of an object, normal to planes, or, for instance, the orientation of a corridor, which can be easily extracted from laser or camera measurements). As a consequence the roto-translation to be estimated often corresponds to the robot displacement between two consecutive sensor acquisitions and the vector registration acquires the meaning of robot self-localization (Zhang et al. [2005]). All the mentioned types of sensor measurements are unavoidably affected by noise, therefore a proper model of the underlying uncertainty is needed for retrieving accurate estimates; moreover, in the framework of *probabilistic robotics* (Thrun et al. [2005]) it is crucial to associate the estimate with a measurement of its uncertainty, in the form of a covariance matrix. The latter is often incorporated in wider frameworks as Extended Kalman Filter (EKF), Cole and Newman [2006], or maximum likelihood estimation (ML), Thrun and Montemerlo [2006], and for this reason it is fundamental that the covariance matrix is consistent with the actual estimation error. Well-known registration techniques consider simplified instances of the problem, in order to attain effective closed-form solutions, see Arun et al. [1987], Horn [1987]. On the other hand, recent approaches propose complex solutions to the general case, which neglect relevant sources of uncertainty or are unsuitable for being incorporated in EKF or ML frameworks. For instance in Zhang et al. [2005] the authors derive a 4 by 4 covariance matrix for the estimated rotation (using unit quaternion representation) and a 3 by 3 covariance matrix for the translation estimate. In this case, however, the correlation between the two quantities is neglected; moreover a 4 by 4 representation of the covariance matrix of a unit quaternion in known to be singular, Lefferts et al. [1982], due to the unit norm constraint, making prohibitive its use for filtering. In this work we discuss extensively the problem of vector registration and we propose two iterative solutions to the general problem, which are demonstrated to improve the accuracy of the estimates, while preserving the on-line nature of the estimation process. Although the formulation is tailored on 3D navigation with application to robotics, the results can be easily extended to different technological fields.

⋆ This work was partially funded by Ministero dell'Istruzione, dell'Università e della Ricerca (MIUR) under MEMONET National Research Project, and by Regione Piemonte under STEPS project (Project co-financed by EC Platform: POR FESR - 2007/2013).

The rest of this article is organized as follows. The problem formulation is presented in Section 2 and some concepts and related results are briefly recalled at the beginning of Section 3. An iterative solution based on variable decoupling is presented in Section 3.1, whereas an approach based on the Gauss-Newton method is reported in Section 3.2. Numerical simulation and experimental results are reported in Section 4 and conclusions are drawn in Section 5.

## 2. PROBLEM STATEMENT

An agent (robot, vehicle, sensor) is deployed in a three-dimensional environment and is able to observe 3D features of the scenario, being them points, line directions or normals to planes. The point features are described by $n_p$ 3D vectors, whereas normals and directions correspond to $n_u$ three-dimensional unit vectors. Hence we can define the *observation set* as $B = \{b_i \in \mathbb{R}^3, i = 1, \ldots, n_p, \ldots, n\}$, being $n = n_p + n_u$ the total amount of observed features, expressed in the reference frame of the agent $\mathcal{R}_B$. A *reference set* $\Omega$ is also supposed to be available from a previous observation or from a known features map. This second set contains a collection of 3D features, expressed in a reference frame $\mathcal{R}_A$, which can correspond to a previous agent pose (if the set is obtained from a previous observation) or to an absolute reference frame (if the set corresponds to a given features map). The agent is capable of selecting a sub-set $A \subseteq \Omega$, such that each element in $A$ corresponds to an observed feature in $B$ (in the same order). In robotics such selection is often referred to as *data association*, Thrun et al. [2005]. Therefore the *reference set* is now restricted to $A = \{a_i \in \mathbb{R}^3, i = 1, \ldots, n_p, \ldots, n\}$. The *vector registration problem* consists in finding the roto-translation that relates the reference frames $\mathcal{R}_A$ and $\mathcal{R}_B$, given the two vector sets $A$ and $B$.

Let $R$ and $t$ be a rotation matrix in $SO(3)$ and a translation vector in $\mathbb{R}^3$, respectively. It is clear that, in a noiseless situation, the desired roto-translation has to satisfy the following equalities:

$$\begin{array}{ll} a_i = Rb_i + t, & i = 1, \ldots, n_p \\ a_i = Rb_i, & i = n_p + 1, \ldots, n. \end{array}$$

In such an ideal case the problem can be formulated as an over-determined linear system in the unknown variables $(R, t)$ and solved by discarding the superfluous constrains Horn [1987]. In realistic situations, however, the exact match cannot be achieved in general, since the observed features are acquired through sensor measurements, and they are affected by noise. As a consequence the problem needs be reformulated as a minimization of the squares of the *registration errors*:

$$E(R, t) = \sum_{i=1}^{n_p} \|a_i - Rb_i - t\|^2 + \sum_{i=n_p+1}^{n} \|a_i - Rb_i\|^2. \tag{1}$$

This formulation (also studied without the translation term) is often referred to as Procrustes, Schonemann [1966], or Wahba problem, Shuster [2006], and several approaches for efficiently solving it do exist, ranging from singular value decomposition, Arun et al. [1987], to quaternion-based closed-form solutions, Horn [1987]. In the literature on the topic several authors proposed to include some weights on the summands of the cost function, with the meaning of *confidence indices* on the corresponding vector match:

$$E(R, t) = \sum_{i=1}^{n_p} \omega_i \|a_i - Rb_i - t\|^2 + \sum_{i=n_p+1}^{n} \omega_i \|a_i - Rb_i\|^2. \tag{2}$$

In the previous equation it is assumed $\omega_i > 0, i = 1, \ldots, n$ and $\sum_{i=1}^{n} \omega_i = 1$. The solution to the weighted problem (2) easily follows from the solution of problem (1), see Horn [1987].

As we will see in the rest of this section the Wahba problem (1) and its weighted form (2) are particular cases of a more general registration problem. For exploiting such observation we rewrite the measured vectors as $a_i = \alpha_i - \delta_{a_i}$ and $b_i = \beta_i - \delta_{b_i}, i = 1, \ldots, n$, where $\alpha_i$ and $\beta_i$ are the true vectors, whereas the terms $\delta_{a_i}$ and $\delta_{b_i}$ are zero mean independent gaussian noises, i.e., $\delta_{a_i} \sim N(0, P_i^a)$ and $\delta_{b_i} \sim N(0, P_i^b)$, being $P_i^a$ and $P_i^b$ the 3 by 3 covariance matrices associated to the vectors $a_i$ and $b_i$, respectively. Under this setup the vector registration problem becomes a minimization of a Mahalanobis distance, and the cost function can be formulated as follows:

$$\begin{aligned} E(R, t) = \sum_{i=1}^{n_p} &(a_i - Rb_i - t)^\top P_i^{-1} (a_i - Rb_i - t) + \\ \sum_{i=n_p+1}^{n} &(a_i - Rb_i)^\top P_i^{-1} (a_i - Rb_i), \end{aligned} \tag{3}$$

where $P_i = P_i^a + RP_i^b R^\top$ is the *registration covariance* expressed in the reference frame $\mathcal{R}_A$. It is quite easy to verify that equation (3) can be formulated according to (1) when $P_i^a = P_i^b = I_3$, with $I_n$ denoting the $n \times n$ identity matrix. It is also easy to show that (3) reduces to (2) when $P_i^a = (\sigma_i^a)^2 I_3$ and $P_i^b = (\sigma_i^b)^2 I_3$, with a suitable choice of the weights:

$$\omega_i = \frac{[(\sigma_i^a)^2 + (\sigma_i^b)^2]^{-1}}{\sum_{k=1}^{n} [(\sigma_k^a)^2 + (\sigma_k^b)^2]^{-1}}.$$

As shown in the next section the *generalized cost function* (3) cannot be solved in closed-form as the two particular cases (1) and (2) but requires an iterative procedure to attain *a minimum* of the objective function.

## 3. REGISTRATION OF UNCERTAIN 3D VECTORS

Before presenting our contribution we briefly recall the solution to the minimization of the weighted cost function (2). Such review will be useful later to gain a deeper insight on the problem at hand. Let us define the weighted centroids of the two vector sets as:

$$\bar{a} = \frac{\sum_{i=1}^{n_p} \omega_i a_i}{\sum_{i=1}^{n_p} \omega_i}, \quad \bar{b} = \frac{\sum_{i=1}^{n_p} \omega_i b_i}{\sum_{i=1}^{n_p} \omega_i}, \tag{4}$$

and let the *normalized vectors* be:

$$\begin{array}{ll} \tilde{a}_i = a_i - \bar{a}, & i = 1, \ldots, n_p, \\ \tilde{a}_i = a_i & i = n_p + 1, \ldots, n, \\ \tilde{b}_i = b_i - \bar{b}, & i = 1, \ldots, n_p, \\ \tilde{b}_i = b_i & i = n_p + 1, \ldots, n. \end{array} \tag{5}$$

It is worth noticing that the $n_u$ unit vectors (line directions and normal to planes) do not enter in the definition of the centroid since they do not provide position information, see Calafiore and Bona [1998]. Then the following result holds.

*Theorem 1.* (Minimum of the weighted cost function). Let $E(R, t)$ be given by (2), and let the centroids and the normalized coordinates be defined as in (4) and (5), respectively. Then, the minimum of the objective function is $E^* = E(R^*, t^*)$, where:

$$t^* = \bar{a} - R^*\bar{b}, \tag{6}$$

and the optimal rotation $R^*$ is:

$$R^* = \arg\max_R \left( \sum_{i=i}^{n} \omega_i \tilde{a}_i^\top R \tilde{b}_i \right). \tag{7}$$

**Proof.** Let us express the vectors in (2) as a function of the normalized coordinates and the corresponding centroids:

$$E(R,t) = \sum_{i=1}^{n_p} \omega_i \|\bar{a}_i + \tilde{a}_i - R\bar{b}_i - R\tilde{b}_i - t\|^2 + \sum_{i=n_p+1}^{n} \omega_i \|\tilde{a}_i - R\tilde{b}_i\|^2.$$

Developing the square in the first sum we obtain:

$$\begin{aligned} E(R,t) = &\sum_{i=1}^{n_p} \omega_i \|\bar{a}_i - R\bar{b}_i - t\|^2 + 2(\bar{a}_i - R\bar{b}_i - t)^\top \\ &\sum_{i=1}^{n_p} \omega_i (\tilde{a}_i - R\tilde{b}_i) + \sum_{i=1}^{n} \omega_i \|\tilde{a}_i - R\tilde{b}_i\|^2, \end{aligned} \tag{8}$$

but:

$$\sum_{i=1}^{n_p} \omega_i (\tilde{a}_i) = \sum_{i=1}^{n_p} \omega_i (a_i - \bar{a}_i) = \sum_{i=1}^{n_p} \omega_i (a_i) - \bar{a}_i \sum_{i=1}^{n_p} \omega_i = 0_3,$$

and:

$$\sum_{i=1}^{n_p} \omega_i (R\tilde{b}_i) = R \sum_{i=1}^{n_p} \omega_i (b_i - \bar{b}_i) = 0_3,$$

hence the second term in (8) is identically zero. Moreover the first summand is non negative and it is the only term depending on the translation $t$. Therefore the translation minimizing the cost function is clearly:

$$t^* = \bar{a} - R\bar{b},$$

for any rotation $R$. The demonstration can be then completed by rewriting the only non-zero term in (8) and developing the square of the summands:

$$E(R,t) = \sum_{i=1}^{n} \omega_i \|\tilde{a}_i\|^2 - 2 \sum_{i=1}^{n} \omega_i \tilde{a}_i^\top R \tilde{b}_i + \sum_{i=1}^{n} \omega_i \|R\tilde{b}_i\|^2. \tag{9}$$

We notice that rotation preserves vector length, hence $\|R\tilde{b}_i\| = \|\tilde{b}_i\|$. Hence, in the previous expression the first and the last terms do not depend on the optimization variable $R$. It is now clear that the rotation that minimizes the objective function (2) also attains the maximum of the expression (7), thus proving the thesis. □

For sake of completeness we mention that the solution to the optimization problem (7) can be computed in closed-form and different solutions for computing it exists. In Horn [1987] a solution based on unit quaternions is reported, whereas Arun et al. [1987] first proposed the use of singular value decomposition for estimating the optimal rotation.

It is evident that Theorem 1 encompasses the solution to the Wahba problem (1), if the weights are chosen to be equal. Our purpose is now to further generalize the previous formulation in order to perform a proper registration of two uncertain vector sets, i.e., to compute a solution for the generalized problem (3).

### *3.1 Registration with variable decoupling*

It is possible to observe that a key property that was exploited along the previous proof regards the rotational symmetry of the covariance matrices involved in the optimization problem. As mentioned, the particular cases (1) and (2) correspond to the choice of spherical acceptance regions, in which the uncertainty has the same behavior in all the directions (*invariance under rotation*).

As we will see in this section this peculiarity does not hold for the general case, hence requiring the adoption of an iterative solution, which assumes that the invariance property only holds locally for small rotations.

For the purpose of solving the optimization problem over the cost function (3) we define the *generalized centroids* as:

$$\bar{a} = \sum_{i=1}^{n_p} \Gamma_i a_i, \quad \bar{b} = \sum_{i=1}^{n_p} \left(R^\top \Gamma_i R\right) b_i, \tag{10}$$

with

$$\Gamma_i = \left( \sum_{k=1}^{n_p} P_k^{-1} \right)^{-1} P_i^{-1}.$$

This choice of weights for the centroids resembles somehow the definition of the weighted centroids (4) and it has a nice geometric interpretation as discussed later. The corresponding normalized coordinates can be now defined as in (5). According to these definitions we can state the following theorem.

*Theorem 2.* (Minimization of the generalized cost function). Let us consider the variables as defined in (5) and (10). The optimal rotation, minimizing of the objective function (3) can be attained by minimizing the following cost:

$$F(R) = \sum_{i=1}^{n} (\tilde{a}_i - R\tilde{b}_i)^\top P_i^{-1} (\tilde{a}_i - R\tilde{b}_i). \tag{11}$$

Moreover, if we call $R^*$ the optimal solution obtained from the previous minimization, the optimal translation vector, can be expressed in the form:

$$t^* = \bar{a} - R^*\bar{b}. \tag{12}$$

**Proof.** Let us express the objective function (3) in terms of the centroids and the normalized coordinates:

$$\begin{aligned} E(R,t) = &\sum_{i=1}^{n_p} (\bar{a}_i + \tilde{a}_i - R\bar{b}_i - R\tilde{b}_i - t)^\top P_i^{-1} \\ &(\bar{a}_i + \tilde{a}_i - R\bar{b}_i - R\tilde{b}_i - t) + \sum_{i=n_p+1}^{n} (\tilde{a}_i - R\tilde{b}_i)^\top P_i^{-1} (\tilde{a}_i - R\tilde{b}_i). \end{aligned}$$

Developing the first sum in the objective function we can rewrite $E(R,t)$ as:

$$\begin{aligned} E(R,t) = &\sum_{i=1}^{n_p} (\bar{a}_i - R\bar{b}_i - t)^\top P_i^{-1} (\bar{a}_i - R\bar{b}_i - t) + \\ &+ 2(\bar{a}_i - R\bar{b}_i - t)^\top \sum_{i=1}^{n_p} P_i^{-1} (\tilde{a}_i - R\tilde{b}_i) + \sum_{i=1}^{n} (\tilde{a}_i - R\tilde{b}_i)^\top P_i^{-1} (\tilde{a}_i - R\tilde{b}_i). \end{aligned} \tag{13}$$

But, recalling our choice of the centroids, we can see that:

$$\sum_{i=1}^{n_p} P_i^{-1}(\tilde{a}_i) = \sum_{i=1}^{n_p} P_i^{-1}(a_i - \bar{a}) =$$
$$= \left(\sum_{k=1}^{n_p} P_k^{-1}\right)\left(\sum_{k=1}^{n_p} P_k^{-1}\right)^{-1}\sum_{i=1}^{n_p} P_i^{-1}a_i - \left(\sum_{i=1}^{n_p} P_i^{-1}\right)\bar{a}$$
$$= \left(\sum_{k=1}^{n_p} P_k^{-1}\right)\left(\sum_{i=1}^{n_p} \Gamma_i^{-1}a_i\right) - \left(\sum_{i=1}^{n_p} P_i^{-1}\right)\bar{a} = 0_3,$$

and analogously:

$$\sum_{i=1}^{n_p} P_i^{-1}(R\tilde{b}_i) = \sum_{i=1}^{n_p} P_i^{-1}(Rb_i - R\bar{b}) =$$
$$= \left(\sum_{k=1}^{n_p} P_k^{-1}\right) RR^\top \left(\sum_{k=1}^{n_p} P_k^{-1}\right)^{-1}\sum_{i=1}^{n_p} P_i^{-1}Rb_i - \left(\sum_{i=1}^{n_p} P_i^{-1}\right)R\bar{b} =$$
$$= \left(\sum_{k=1}^{n_p} P_k^{-1}\right) R\left(\sum_{i=1}^{n_p}\left(R^\top\Gamma_i^{-1}R\right)b_i\right) - \left(\sum_{i=1}^{n_p} P_i^{-1}\right)R\bar{b} = 0_3.$$

It is now evident that the second term in the objective function is zero; the first term in (13) is non-negative, and it is annihilated by the following choice of the translation vector:

$$t^* = \bar{a} - R\bar{b}.$$

Moreover the optimal rotation matrix $R^*$, minimizing the registration error, attains the minimum of the following cost:

$$F(R) = \sum_{i=1}^{n}(\tilde{a}_i - R\tilde{b}_i)^\top P_i^{-1}(\tilde{a}_i - R\tilde{b}_i),$$

and this fact concludes the proof. □

It can be seen that $\Gamma_i$ contains the information matrix of the $i$-th vector match, and the summation in parenthesis only provides a normalization term. On the other hand the weights for the points in $B$ are $R^\top\Gamma_i R = (\sum_{k=1}^{n_p} R^\top P_k^{-1}R)^{-1}R^\top P_i^{-1}R$. But if we further develop the term $P_{i,b}^{-1} = R^\top P_i^{-1}R$ we obtain:

$$P_{i,b}^{-1} = R^\top\left(P_i^a + RP_i^bR^\top\right)^{-1}R = \left(R^\top P_i^a R + P_i^b\right)^{-1}.$$

Hence the weight for the point features in $B$ still depends on the registration uncertainty, but now the corresponding information matrix is expressed in the reference frame $\mathcal{R}_b$.

The solution to the optimization problem (11) is all but straightforward. Both the normalized variables and the covariance matrices $P_i$ are function of the optimization variables $R$. We attack this problem by assuming that for small angular variations the covariance matrices can be considered constant (*local invariance to rotations*), i.e., for small rotations $P_i = \bar{P}_i$ and $P_{i,b} = \bar{P}_{i,b}$. Under such hypothesis we can solve iteratively the problem by considering a convex approximation of the objective function around an initial guess for the rotation matrix, that we shall call $\bar{R}$. In particular we rewrite the optimization variable as $R = \bar{R}R(\tilde{\theta})$, where $R(\tilde{\theta})$ is the rotation corresponding to a correction angle $\tilde{\theta}$ with respect to $\bar{R}$. If the correction angle is small we can adopt the approximation $R(\tilde{\theta}) \approx I + S(\tilde{\theta})$, being $S([\tilde{\theta}_1\ \tilde{\theta}_2\ \tilde{\theta}_3])$ the skew symmetric matrix:

$$S(\tilde{\theta}) = \begin{bmatrix} 0 & -\tilde{\theta}_3 & \tilde{\theta}_2 \\ \tilde{\theta}_3 & 0 & -\tilde{\theta}_1 \\ -\tilde{\theta}_2 & \tilde{\theta}_1 & 0 \end{bmatrix}. \quad (14)$$

Hence a convex approximation of the objective function around the initial guess $\bar{R}$ can be written as:

$$F(R) \approx \sum_{i=1}^{n}(\tilde{a}_i - \bar{R}\tilde{b}_i - J_i\tilde{\theta})^\top \bar{P}_i^{-1}(\tilde{a}_i - \bar{R}\tilde{b}_i - J_i\tilde{\theta}), \quad (15)$$

where $J_i = -\bar{R}S(\tilde{b}_i)$, and the normalized coordinates are computed with respect to the centroids:

$$\bar{a} = \sum_{i=1}^{n_p}\bar{\Gamma}_i a_i, \quad \bar{b} = \sum_{i=1}^{n_p}\left(R^\top\bar{\Gamma}_i R\right)b_i,$$

with

$$\bar{\Gamma}_i = \left(\sum_{k=1}^{n_p}\bar{P}_k^{-1}\right)^{-1}\bar{P}_i^{-1}.$$

We can solve the linearized problem (15) by taking the derivative with respect to $\tilde{\theta}$ and setting it to zero, obtaining:

$$\tilde{\theta}^* = \left(\sum_{k=1}^{n} J_k^\top \bar{P}_k^{-1}J_k\right)^{-1}\sum_{i=1}^{n} J_i^\top \bar{P}_i^{-1}(\tilde{a}_i - \bar{R}\tilde{b}_i). \quad (16)$$

Hence the initial guess for the rotation matrix can be improved by including the correction angle $\tilde{\theta}^*$, i.e., $\bar{R} \longleftarrow \bar{R}R(\tilde{\theta}^*)$, and the procedure is iterated until the local correction becomes negligible.

As we will see in Section 4, a natural and cheap initialization for $\bar{R}$ is the solution to the Wahba problem (1), that enables the previous iterative procedure to quickly converge to the minimum of the cost function. After the rotation $R^*$ is retrieved using such iterative technique, the translation vector can be easily computed according to (12).

In several application it is often required to compute a covariance matrix of the estimation error, which provides probabilistic assessments on the computed roto-translation. According to the *variable decoupling* (VD) technique presented so far we can state the following result.

*Theorem 3.* (Covariance of the estimation error using VD). Let us consider the variables as defined in (5) and (10). Moreover let $\delta_t \in \mathbb{R}^3$ and $\delta_\theta \in \mathbb{R}^3$ be the translation and angular errors, respectively. If a solution $(R^*, t^*)$ of the generalized registration problem is obtained according to Theorem 2, then a first-order approximation of the covariance matrix of the estimation error $\delta_x = [\delta_t^\top\ \delta_\theta^\top]^\top$ can be computed as:

$$P_x = \sum_{i=1}^{n}\begin{bmatrix} A_iP_iA_i^\top & A_iP_iB_i^\top \\ B_iP_iA_i^\top & B_iP_iB_i^\top \end{bmatrix}, \quad (17)$$

where:

$$A_i = \left(\sum_{k=1}^{n_p}P_k^{-1}\right)^{-1}c_iP_i^{-1} + R^*S(\bar{b})\left(\sum_{k=1}^{n}J_kP_k^{-1}J_k^\top\right)^{-1}$$
$$\left(J_i^\top P_i^{-1} - \left(\sum_{k=1}^{n}J_kP_k^{-1}\right)\left(\sum_{k=1}^{n_p}P_k^{-1}\right)^{-1}c_iP_i^{-1}\right),$$

and

$$B_i = \left(\sum_{k=1}^{n} J_k P_k^{-1} J_k^\top\right)^{-1}$$
$$\left(J_i^\top P_i^{-1} - \left(\sum_{k=1}^{n} J_k P_k^{-1}\right)\left(\sum_{k=1}^{n_p} P_k^{-1}\right)^{-1} c_i P_i^{-1}\right),$$

with $J_i = -R^* S(\tilde{b}_i)$, $S(\bar{b})$ and $S(\tilde{b}_i)$ are matrices built as in (14), and $c_i$ is a binary support variable, so that $c_i = 1,\ i = 1, \ldots, n_p$ and $c_i = 0,\ i = n_p + 1, \ldots, n$.

**Proof.** See Appendix A.

*3.2 A Gauss-Newton approach to registration*

In this section we will present a second approach for computing the roto-translation achieving the optimal matching between the given vector sets. This technique is based on a Gauss-Newton methods for solving non linear optimization problems. Also in this case it turns out to be useful to assume *local invariance to rotations* for the involved covariance matrices. Let us start from the original problem (3) and consider the expression of a single residual in the summand:

$$r_i = a_i - R b_i - c_i t, \tag{18}$$

where the support variable $c_i$ is equal to 1 for $i = 1, \ldots, n_p$ and equal to zero for the other vectors. It is possible to see that the non linearities in the residual are simply connected with the rotation matrix. Let us assume that an initial guess $\bar{R}$ for optimization is available and let us express the rotation $R$ in the residual (18) as a function of such known rotation and an unknown correction $R(\tilde{\theta})$:

$$r_i = a_i - \bar{R} R(\tilde{\theta}) b_i - c_i t. \tag{19}$$

Assuming the correction angle $\tilde{\theta}$ to be small we can approximate $R(\tilde{\theta}) = I_3 + S(\tilde{\theta})$, hence (19) becomes:

$$r_i \approx a_i - \bar{R} b_i - \bar{R} S(\tilde{\theta}) b_i - c_i t =$$
$$= a_i - \bar{R} b_i - c_i t + \bar{R} S(b_i) \tilde{\theta} = a_i - \bar{R} b_i - C_i x,$$

with $C_i = [c_i \ -\bar{R} S(b_i)], i = 1, \ldots, n$ and $x = [t^\top \ \tilde{\theta}^\top]^\top$. Recalling the *local invariance to rotation*, we can now rewrite a quadratic approximation of the objective function (3) around the initial guess $\bar{R}$:

$$E(R, t) = \sum_{i=1}^{n} r_i^\top P_i^{-1} r_i \approx$$
$$\approx \sum_{i=1}^{n} \left(a_i - \bar{R} b_i - C_i x\right)^\top \bar{P}_i^{-1} \left(a_i - \bar{R} b_i - C_i x\right).$$

The solution to the previous convex optimization problem can be computed as Boyd and Vandenberghe [2004]:

$$x^* = \left(\sum_{k=1}^{n} C_k^\top \bar{P}_k^{-1} C_k\right)^{-1} \sum_{i=1}^{n} C_i^\top \bar{P}_i^{-1} (a_i - \bar{R} b_i).$$

The previous solution provides an estimate of the translation $t^*$ and a local angular correction $\tilde{\theta}^*$. The algorithm can be iterated until convergence using the refined rotation matrix estimate $\bar{R} R(\tilde{\theta}^*)$ as initial guess for a new iteration.

According to the framework presented so far it is also easy to devise a covariance matrix of the estimated roto-translation:

$$P_x = \left(\sum_{k=1}^{n} C_k^\top \bar{P}_k^{-1} C_k\right)^{-1}. \tag{20}$$

For sake of readability we report the derivation of $P_x$ in Appendix B.

## 4. NUMERICAL EXPERIMENTS

In this section we present an experimental analysis aimed at testing the accuracy, the computational effort and the consistency of the proposed algorithms. The two techniques are compared with the state-of-the-art approaches to registration, in order to evaluate trade-offs and limitations.

The numerical tests are tailored on mobile robotics applications, hence the vector sets are supposed to be a representation of 3D perception of a rover, equipped with some exteroceptive sensors. Since the uncertainty model of the perception plays a key role in our derivation we will first present the sensor models, being used in the tests, and then we will report the outcome of the analysis.

*4.1 Uncertainty and sensor models*

For our numerical tests we consider two types of sensors: a 3D laser scanner and a stereo camera. This choice is motivated by the fundamental role of the mentioned devices in mobile robotics applications, see Lemaire et al. [2007] and Weingarten and Siegwart [2005], and by the different underlying uncertainty models, which turn out to be a useful discriminant for the applicability of the proposed approaches to registration. A 3D laser scanner is usually a pan-tilt structure, equipped with a laser range finder, Weingarten and Siegwart [2005]. In order to obtain the uncertainty of a 3D point, namely $a = [a_x \ a_y \ a_z]^\top$, measured with laser, we express the generic three-dimensional vector in polar coordinates:

$$\begin{cases} a_x = \rho \cos(\psi) \cos(\gamma) \\ a_y = \rho \cos(\psi) \sin(\gamma) \\ a_z = \rho \sin(\psi) \end{cases} \tag{21}$$

where $\rho$ is the distance of the range finder from the measured point, whereas $\psi$ and $\gamma$ are the angles describing the elevation and the azimuth to the point, respectively. It can be seen that the right-hand side of the equations contains parameters whose uncertainty is usually known, since the accuracy of the angles depends on the materials and on the mechanical assembly of the pan-tilt structure, whereas the uncertainty of $\rho$ is connected with the employed range finder. Therefore the uncertainty of the vector $[\rho \ \psi \ \gamma]$ can be described by the covariance matrix $P_s = \mathrm{diag}(\sigma_\rho^2, \sigma_\psi^2, \sigma_\gamma^2)$, being $\sigma_\rho$, $\sigma_\psi$, and $\sigma_\gamma$ the standard deviations of the sensor parameters. Hence it is now simple to compute a first-order uncertainty propagation to obtain the covariance matrix $P_a$ for point $a$.

For modeling the perception of a stereo camera we considered the following model:

$$\begin{cases} a_x = (1/d) \cos(\psi) \cos(\gamma) \\ a_y = (1/d) \cos(\psi) \sin(\gamma) \\ a_z = (1/d) \sin(\psi) \end{cases} \tag{22}$$

where the point is expressed in function of the *inverse depth d*. The inverse depth is related to the *disparity* that can be computed from a pair of stereo images, see

Hartley and Zisserman [2000]. As in the previous case a straightforward propagation of the covariance from the sensor parameters $[d\ \psi\ \gamma]$ provides the uncertainty of the generic point $a$. Although appearing similar to the previous representation, such model leads to completely different covariance matrices, since in the camera model the accuracy in the distance measurements decreased with the the actual distance from the measured point. An example of the covariance matrices obtained with the laser model and the camera model are reported in Figure 1(a) and 1(b), respectively. The reader is referred to Hartley and Zisserman [2000] for further details on the camera uncertainty.

We also considered a third setup in which a random covariance matrix is associated to each vector in the two sets. This last case is only reported to evaluate the numerical stability of the algorithms and it is of scarce practical interest. In order to implement such a setup we computed a generic positive definite matrix for each point, with the meaning of covariance matrix. A simple way to compute a positive definite matrix $P_a$ it to extract a random full-rank 3 by 3 matrix $M$, and to compute $P_a = M^\top M$. The latter is a positive definite matrix, see Horn and Johnson [1985].

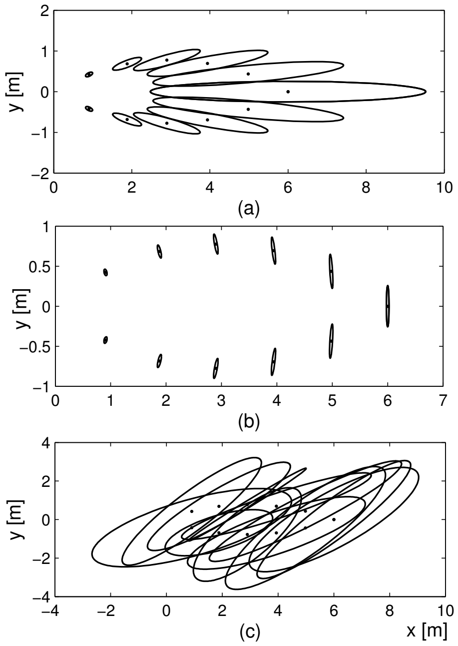


Fig. 1. Examples of 95% confidence regions (projected on the ground plane) for the different sensor models: (a) stereo camera model, (b) laser model, (c) random covariance.

We conclude this section by noticing that the statement of the registration problem (see Section 2) assumes that the vectors of the two sets can be correctly associated (*data association*). If this is a mild assumption when using salient features from the stereo camera Lemaire et al. [2007], it constraints the vector sets obtained from a 3D laser scanner to contain only distinguishable points or directions (corners, reflective tags, landmarks, etc.).

### *4.2 Results and discussion*

In order to evaluate the proposed techniques we considered a point cloud $B$, with points randomly extracted in the cube $[-5,\ 5]^3$. The reference set $A$ is then obtained by applying a random roto-translation to the set $B$. Since we are interested in the realistic case of noisy measurements, the vectors of the two sets are perturbed with additive noise, according to the models introduced in the previous section. The numerical analysis encompasses the two proposed techniques, namely registration with variable decoupling (VD) and with Gauss-Newton approach (GN), and the solution to the simplified problems (1) and (2). For sake of readability we will refer to the solution of (1) as *noiseless registration* (NR) and to the solution of (2) as *weighted registration* (WR). For the latter the weights were chosen as:

$$\omega_i = k\frac{1}{\text{Tr }(P_i)} = k\frac{1}{\text{Tr }(P_i^a + RP_i^b R^\top)} = k\frac{1}{\text{Tr }(P_i^a) + \text{Tr }(P_i^b)}$$

where $k$ is a normalization factor that makes the weights sum up to 1, Tr $(\cdot)$ is the trace operator, and $P_i$ is the registration uncertainty, as defined in Section 2. Since the trace of the covariance matrix is function of the sum of the length of the semiaxes of the confidence region defined by the covariance matrix $P_i$, Bar-Shalom et al. [2001], this metric appears suitable to quantify the reliability of the $i$-th pair of points. Moreover the weights do not depend on the rotation matrix, and this is a desirable property for an effective implementation of the technique. The two proposed iterative techniques uses the solution to the problem (1) as initial guess for non-linear optimization. In Table 1 we report the statistics on the accuracy of the four techniques, considering sets of 100 vectors, for the following selection of sensor parameters uncertainty: $\sigma_\rho = 0.01$ m, $\sigma_\psi = \sigma_\gamma = 1^o$ and $\sigma_d = 0.05$ m$^{-1}$. The translation error for a single test is computed as $\bar{\delta_t} = \|\delta_t\| = \|\tau - t^*\|$, being $\tau$ and $t^*$ the actual translation and the corresponding estimate, respectively; the rotation error is computed as $\bar{\delta_\theta} = \|\delta_\theta\|$, where the rotation vector $\delta_\theta$ satisfies $\mathcal{R} = R^* R(\delta_\theta)$, being $\mathcal{R}$ and $R^*$ the true rotation and the estimate, respectively. The results are averaged over 1000 Monte Carlo runs.

| | | | Camera | Laser | Random |
|---|---|---|---|---|---|
| NR | $\bar{\delta_t}$ | [m] | 0.188 (0.08) | 0.014 (0.006) | 0.214 (0.121) |
| | $\bar{\delta_\theta}$ | [$^o$] | 0.385 (0.185) | 0.285 (0.121) | 3.131 (1.502) |
| WR | $\bar{\delta_t}$ | [m] | 0.058 (0.037) | 0.011 (0.005) | 0.208 (0.116) |
| | $\bar{\delta_\theta}$ | [$^o$] | 0.517 (0.246) | 0.248 (0.104) | 3.031 (1.456) |
| VD | $\bar{\delta_t}$ | [m] | 0.014 (0.007) | 0.004 (0.001) | 0.146 (0.084) |
| | $\bar{\delta_\theta}$ | [$^o$] | 0.219 (0.093) | 0.211 (0.095) | 1.364 (0.626) |
| GN | $\bar{\delta_t}$ | [m] | 0.014 (0.007) | 0.004 (0.001) | 0.146 (0.084) |
| | $\bar{\delta_\theta}$ | [$^o$] | 0.219 (0.093) | 0.211 (0.095) | 1.364 (0.626) |

Table 1. Registration accuracy for different uncertainty models: mean translation ($\bar{\delta_t}$) and angular ($\bar{\delta_\theta}$) errors for the considered techniques. The standard deviation, evaluated over 1000 Monte Carlo runs, is reported in parenthesis.

In almost all the tests the two proposed techniques produce a remarkable improvement in the registration accuracy. In particular for the camera model, the translation errors are reduced of an order of magnitude and the angular errors are reduced by a factor of 2; in no test the iterative approaches converged to local minima. It can be seen that the statistics on the errors of the two proposed approaches are exactly the same and the techniques produce the same estimate for the roto-translation; this is quite an intuitive result, since they simply attain the same minimum of the objective function. It is worth noticing that the approach with weighted objective function turns out to be less accurate when considering the camera model. This may depend on the fact that the scalar weights cannot disambiguate the direction of the uncertainty: for far points the distance measurements is inaccurate and the dimension of the confidence region increases, see Figure 1(a). As a consequence such points have smaller weights, although they carry on significant information for the rotation estimation.

The VD and GN approaches are iterated until a stopping condition is met, e.g., the correction in the optimization variable does not exceed a given threshold. In Table 2 we show the average number of iterations and the average time required for the registration of the point clouds. The tests are performed in Matlab, using a standard laptop. Both techniques are able to converge in few iterations,

| | | Camera | Laser | Random |
|---|---|---|---|---|
| NR | CPU time [s] | 0.013 | 0.013 | 0.013 |
| | Iterations | - | - | - |
| WR | CPU time [s] | 0.013 | 0.013 | 0.013 |
| | Iterations | - | - | - |
| VD | CPU time [s] | 0.172 | 0.338 | 0.276 |
| | Iterations | 5.2 | 11.7 | 9.7 |
| GN | CPU time [s] | 0.072 | 0.064 | 0.124 |
| | Iterations | 3.4 | 2.9 | 6.6 |

Table 2. Computational effort for different uncertainty models: average number of iteration and CPU time for performing registration with the compared techniques.

but the computational effort for registration with variable decoupling is considerably higher. Although in the VD approach the optimization variable does not include the translation term, the computational cost of computing the centroids and the slightly higher number of required iterations constitute a limitation to its computational effectiveness.

The consistency of the proposed approaches was evaluated by computing the *Normalized Estimation Error Squared* (NEES) Bar-Shalom et al. [2001]:

$$\mathrm{NEES} = \delta_x^\top P_x^{-1} \delta_x \leq \chi^2_{r,1-\alpha}$$

where $\delta_x = [\delta_t^\top \ \delta_\theta^\top]^\top \in \mathbb{R}^6$ and $P_x$ are the estimation error and the corresponding covariance matrix, and $\chi^2_{r,1-\alpha}$ is the quantile of the $\chi^2$ distribution with $r = \dim(\delta_x)$ degrees of freedom and upper tail probability equal to $\alpha$. In Table 3 we report the *consistency ratio* ($\mathrm{NEES}/\chi^2_{r,1-\alpha}$) for the different sensor models, averaged over 1000 Monte Carlo runs Bar-Shalom et al. [2001]. It is possible to see that, for the

| | | Camera | Laser | Random |
|---|---|---|---|---|
| VD | Average consistency ratio | 0.65 | 3.96 | 0.68 |
| | % of inconsistent estimates | 6 | 43 | 15 |
| GN | Average consistency ratio | 0.46 | 0.37 | 0.46 |
| | % of inconsistent estimates | 4 | 1 | 5 |

Table 3. Consistency of the proposed approaches: consistency ratio ($\mathrm{NEES}/\chi^2_{r,1-\alpha}$) and percentage of inconsistent estimates for $\alpha = 0.01$.

Gauss-Newton approach, the percentage of inconsistent estimates is comparable with the probability tail $\alpha = 0.01$, for which the estimation errors of a consistent estimator need be bounded by the ellipsoid defined by the inequality (23) with probability 0.99, see Bar-Shalom et al. [2001]. Moreover the average consistency ratio shows that the violations of such ellipsoidal constraints are always mild for the Gauss-Newton approach, confirming the consistency of the registration technique. The results obtained with the variable decoupling technique, however, suggest that this approach cannot be used as a general purpose algorithm for registration, because in some cases the covariance estimation turns out to be optimistic. It is not immediate to devise the reason for the inconsistency. According to the numerical results the reasons are essentially twofold: (i) the optimization problem (11) is heavily non linear, including non-linear terms in the expression of the centroids; heavy non-linearity in the objective function makes less effective the first-order approximation of the covariance matrix, see Appendix A; for the same reason the approach requires a larger number of iterations than the GN-approach, confirming that the linear approximation of the residual is not as accurate as in the Gauss-Newton method; (ii) in the derivation of the covariance matrix it is necessary to assume local invariance to rotation also for the expression of the centroids, whereas the Gauss-Newton approach requires this assumption only for the information matrix in (3), and, the uncertainty carried on by this simplification cannot be easily evaluated.

To complete our analysis we report the percentage of inconsistent estimates for different values of the sensor parameters, limiting the analysis to the stereo camera model. In Figure 2 we show the results regarding registration with variable decoupling. It is worth noticing that the consistency of the estimator worsens when the uncertainty of the distance measurement is sensibly lower than the uncertainty of the angular measures, and this conclusion is compatible with the results obtained with the laser model, see Table 3. Repeating the same experiment with the Gauss-Newton approach no significative variation in the consistency of the estimator was observed (results are omitted for space reasons).

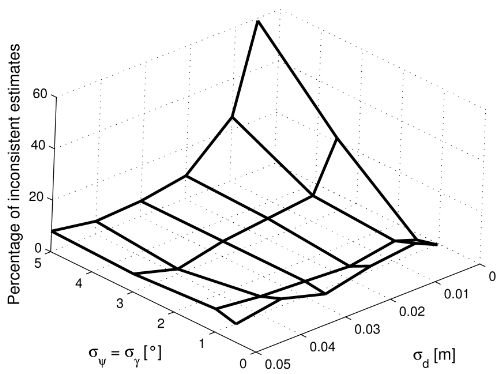


Fig. 2. Percentage of inconsistent estimates for different sensor parameters $[\sigma_d, \sigma_\psi, \sigma_\gamma]$ using camera model. The results, evaluated over 100 Monte Carlo runs, are obtained with the variable decoupling registration approach.

We conclude this experimental section with some statistics on the translation and angular error for different sizes of the point clouds, see Figure 3(a) and 3(b).

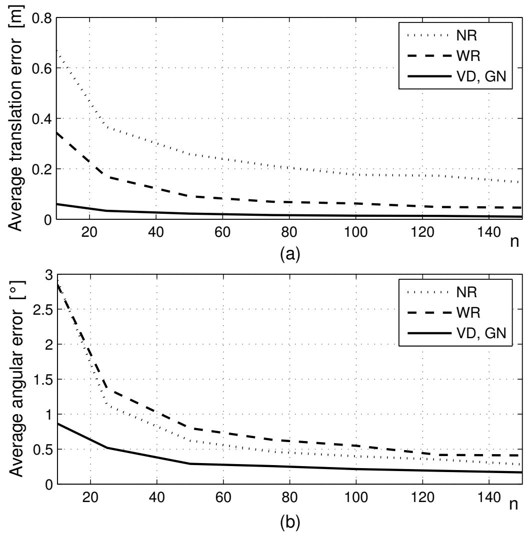


Fig. 3. Average translation error (a) and average angular error (b) for registration of $n$ points using camera model. The reported curves correspond to: NR (dotted line), WR (dashed line), VD and GN (solid line).

## 5. CONCLUSION

In this work we discuss the problem of probabilistic registration of uncertain 3D vectors, which arises in several technological fields, ranging from manufacturing tolerancing to mobile robotics. The problem is first formulated in terms of a minimization of the weighted residual errors on vector matching and the limitations of the state-of-the-art approaches for vector registration are discussed. Then we propose the use of two iterative algorithms for solving the problem, namely a registration technique based on *variable decoupling* and an algorithm based on a Gauss-Newton approach to non-linear optimization. Both algorithms are verified to remarkably improve the accuracy in the estimation of the roto-translation between the vector sets. The techniques are extensively evaluated through numerical tests, both in terms of computational effort and consistency of the registration errors with the covariance matrices of the estimates.

## 6. APPENDIX

### *6.1 Appendix A*

In this section we derive the expression of the covariance matrix, as reported in Theorem 3. Assuming that the *true* vectors sets $(\alpha_i, \beta_i), i = 1, \ldots, n$ and the actual rotation matrix $\mathcal{R}$ are exactly known, it is possible to compute the *true* translation vector $\tau$, according to Theorem 2:

$$\tau = \bar{\alpha} - \mathcal{R}\bar{\beta}, \tag{23}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are the centroids, computed by means of equation (10) from the corresponding true quantities. We can now rewrite the true quantities (unknown in practice) in terms of estimated quantities and residual errors:

$$\tau = (\bar{a} + \delta_{\bar{a}}) - R^* R(\delta_\theta)\left(\bar{b} + \delta_{\bar{b}}\right), \tag{24}$$

where $\bar{a}$ and $\bar{b}$ are the estimated centroids of the vector sets, $\delta_{\bar{a}}$ and $\delta_{\bar{b}}$ are the corresponding estimation errors, $R^*$ is the rotation matrix computed according to Theorem 2, and $R(\delta_\theta)$ is the rotation matrix corresponding to the angular error vector $\delta_\theta$. Assuming small angular errors we can approximate $R(\delta_\theta) = I_3 + S(\delta_\theta)$; Hence, neglecting higher order terms in the residual errors, we can rewrite equation (24) as:

$$\tau = \bar{a} - R^*\bar{b} + \delta_{\bar{a}} - R^*\delta_{\bar{b}} - R^* S(\delta_\theta)\bar{b}. \tag{25}$$

Now it is easy to recognize the first two terms to be the estimated translation $t^*$, see (12), and, recalling that $S(\delta_\theta)\bar{b} = -S(\bar{b})\delta_\theta$, we can recover the expression of the translation error $\delta_t$:

$$\delta_t = \tau - t^* = \delta_{\bar{a}} - R^*\delta_{\bar{b}} + R^* S(\bar{b})\delta_\theta. \tag{26}$$

Finally we can express the translation error in function of the residual error of each given vector, namely $\delta_{a_i}, \delta_{b_i}, i = 1, \ldots, n$, by recalling the definition of the generalized centroids:

$$\delta_t = \left(\sum_{k=1}^{n_p} P_k^{-1}\right)^{-1} \sum_{i=1}^{n_p} P_i^{-1}\left(\delta_{a_i} - R^*\delta_{b_i}\right) + R^* S(\bar{b})\delta_\theta. \tag{27}$$

The latter provides a first-order approximation of the translation error, in function of the error on each vector and on the rotation error. In order to complete the derivation we need to devise an expression of the residual error $\delta_\theta$. The latter can be obtained by writing the optimization problem (11) in terms of the unknown true vectors $(\alpha_i, \beta_i), i = 1, \ldots, n$ and the corresponding centroids (5):

$$\begin{aligned} F(R) &= \sum_{i=1}^{n} \phi_i^\top P_i^{-1} \phi_i = \\ &= \sum_{i=1}^{n} \left(\alpha_i - \bar{\alpha} - R\left(\beta_i - \bar{\beta}\right)\right)^\top P_i^{-1}\left(\alpha_i - \bar{\alpha} - R\left(\beta_i - \bar{\beta}\right)\right). \end{aligned} \tag{28}$$

It is now clear that the true rotation attains the minimum of the previous objective function, i.e., $\mathcal{R} = \arg\min F(R)$. For such optimal solution we can now rewrite the residuals in (28) by substituting the true quantities with the corresponding estimates and the corresponding errors:

$$\begin{aligned} \phi_i &= \alpha_i - \bar{\alpha} - \mathcal{R}\left(\beta_i - \bar{\beta}\right) = \alpha_i - \mathcal{R}\beta_i - \bar{\alpha} + \mathcal{R}\bar{\beta} = \\ &= \alpha_i - \mathcal{R}\beta_i - \tau = a_i + \delta_{a_i} - R^* R(\delta_\theta)\left(b_i + \delta_{b_i}\right) - t^* - \delta_t. \end{aligned} \tag{29}$$

Taking the small angle approximation for the rotation matrix $R(\delta_\theta)$, and neglecting higher order terms, we can express (29) as:

$$\phi_i = a_i - R^* b_i - t^* + \delta_{a_i} - R^*\delta_{b_i} - R^* S(\delta_\theta) b_i - \delta_t. \tag{30}$$

Substituting the expression (27) in (30) we obtain:

$$\begin{aligned} \phi_i = a_i - R^* b_i - t^* + \delta_{a_i} - R^*\delta_{b_i} + R^* S(\tilde{b}_i)\delta_\theta + \\ - \left(\sum_{k=1}^{n_p} P_k^{-1}\right)^{-1} \sum_{i=1}^{n_p} P_i^{-1}\left(\delta_{a_i} - R^*\delta_{b_i}\right). \end{aligned} \tag{31}$$

We can now solve the optimization problem (28), using the expression (31) for the residual errors, hence obtaining the rotation error in function of the errors in the input vectors:

$$\delta_\theta = \left(\sum_{k=1}^{n} J_k^\top \bar{P}_k^{-1} J_k\right)^{-1} \sum_{i=1}^{n} J_i^\top \bar{P}_i^{-1}$$
$$\left(a_i - R^* b_i - t^* + \left(\delta_{a_i} - R^*\delta_{b_i}\right) - \left(\sum_{k=1}^{n_p} P_k^{-1}\right)^{-1} \sum_{i=1}^{n_p} P_i^{-1}\left(\delta_{a_i} - R^*\delta_{b_i}\right)\right) \quad (32)$$

where $J_i = -R^* S(\tilde{b}_i)$. In the previous expression $a_i$, $b_i$, $R^*$ and $t^*$ are known quantities, whereas $\delta_{a_i}$ and $\delta_{b_i}$ are normally distributed random variables. Recalling that, by definition, the uncertainty of the term $\delta_{a_i} - R^*\delta_{b_i}$ is $\bar{P}_i$, and using simple rules of uncertainty propagations in linear systems, it is now straightforward to obtain the expression (17) from (27) and (32).

### *6.2 Appendix B*

In this appendix we prove the correctness of the covariance matrix for the estimate obtained with the Gauss-Newton approach. Let us formulate the optimization problem (3) in terms of the corresponding (unknown) true quantities $(\alpha_i, \beta_i), i = 1, \ldots, n$:

$$E(R,t) = \sum_{i=1}^{n} \left(\alpha_i - R\beta_i - c_i t\right)^\top P_i^{-1} \left(\alpha_i - R\beta_i - c_i t\right),$$

with $c_i = 1, i = 1, \ldots, n_p$ and $c_i = 0, i = n_p + 1, \ldots, n$. It is now clear that the solution of the previous optimization problem provides the true roto-translation between the vector sets, namely the pair $(\mathcal{R}, \tau)$. Let us write the true quantities in terms of the estimated (measured) quantities and the corresponding errors:

$$E = \sum_{i=1}^{n} \left(a_i + \delta_{a_i} - R^* R(\delta_\theta)(b_i + \delta_{b_i}) - c_i(t^* + \delta_t)\right)^\top$$
$$P_i^{-1}\left(a_i + \delta_{a_i} - R^* R(\delta_\theta)(b_i + \delta_{b_i}) - c_i(t^* + \delta_t)\right).$$

Assuming that the solution computed from the Gauss-Newton approach, i.e., $(R^*, t^*)$, is sufficiently close to the true roto-translation, we can take a small angle approximation for the rotation matrix $R(\delta_\theta)$, and, neglecting higher order terms, we obtain:

$$E = \sum_{i=1}^{n} \left(a_i - R^* b_i - c_i t^* + \delta_{a_i} - R^*\delta_{b_i} + R^* S(b_i)\delta_\theta - c_i\delta_t\right)^\top$$
$$\bar{P}_i^{-1}\left(a_i - R^* b_i - c_i t^* + \delta_{a_i} - R^*\delta_{b_i} + R^* S(b_i)\delta_\theta - c_i\delta_t\right).$$

From the previous convex optimization problem we can derive the expression of the roto-translation errors in function of the errors $(\delta_{a_i}, \delta_{b_i}), i = 1, \ldots, n$ in the input vectors:

$$\delta_x = \left(\sum_{k=1}^{n} C_k^\top \bar{P}_k^{-1} C_k\right)^{-1} \sum_{i=1}^{n} C_i^\top \bar{P}_i^{-1} \quad (33)$$
$$\left(a_i + R^* b_i - c_i t^* + \delta_{a_i} - R^*\delta_{b_i}\right).$$

where $C_i = [c_i \;\; -\bar{R}^* S(b_i)]$ and $\delta_x = [\delta_t^\top \; \delta_\theta^\top]^\top$. From the previous expression it is now easy to recognize that the only uncertain term in the right-hand side of the equation is $\delta_{a_i} - R^*\delta_{b_i}$, whose covariance matrix is $\bar{P}_i$. Therefore it is possible to recover the covariance matrix (20) of the random roto-translation error $\delta_x$, by propagating the uncertainty through the linear transformation (33).

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