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UWM seminar

Embedding surfaces in 4-manifolds

joint with
Daniel Kasprowski
Mark Powell
Peter Teichner

Embedding surfaces in 4-manifolds

(joint w. Kasprowski, Powell, Teichner)

Q: Given a map of a surface in a 4-mfld, when is it homotopic to a (loc. flat or smooth) embedding?

- an embedding $\Sigma \subset M$ is loc. flat if each pt in Σ has a nbd $\mathcal{U}$ s.t. $(\mathcal{U}, \mathcal{U} \cap \Sigma) \approx (\mathbb{R}^4, \mathbb{R}^2)$

- generically the image of $\Sigma^2 \longrightarrow M^4$ has isolated double point singularities

Why is this an interesting question?

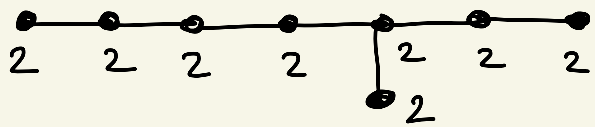
Example:

- By Poincaré duality, every closed 4-mfld has an bilinear, unimodular intersection form

$$Q_M : H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

- e.g. $Q_{S^2 \times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

- $$E8 := \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & 0 & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & 0 & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}$$



Q: Is $E8 \oplus E8$ the intersection form of a closed, simply connected 4-mfld?

Idea:

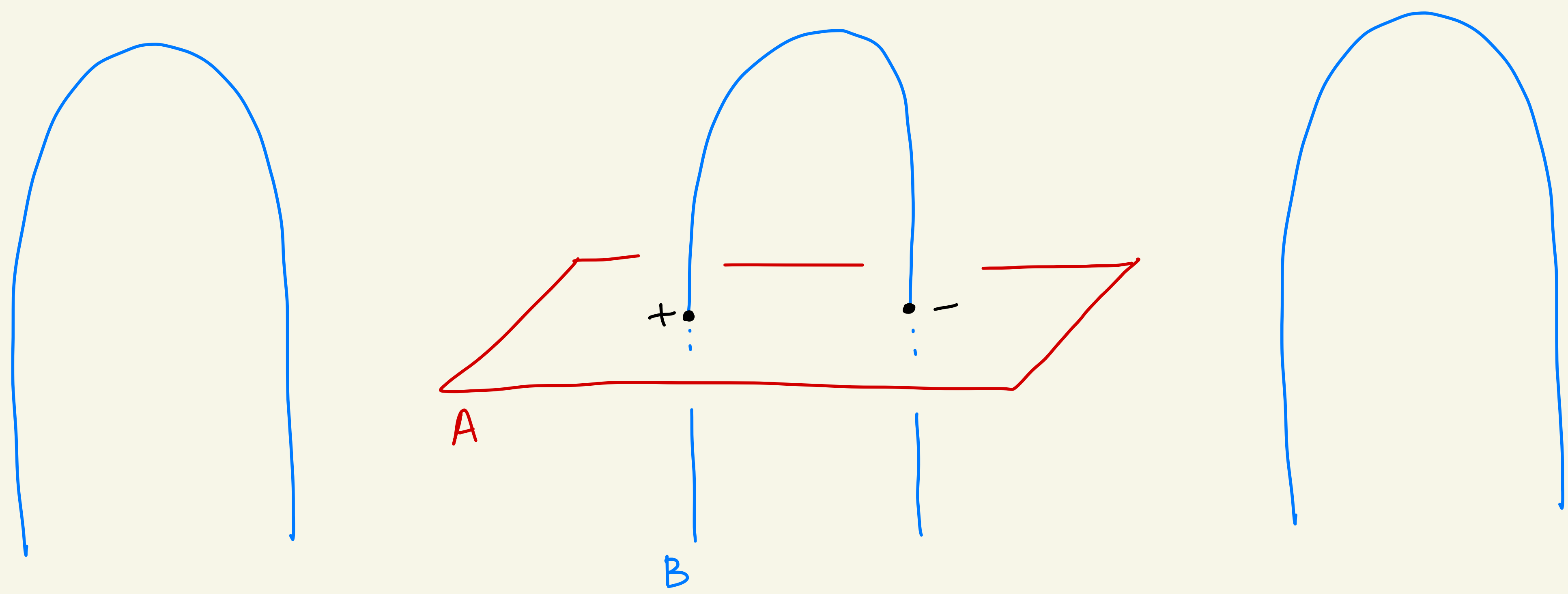
The K3 surface $:= \{[x,y,z,w] \in \mathbb{CP}^3 \mid x^4+y^4+z^4+w^4=0\}$

$$\pi_1(K3) = 1$$

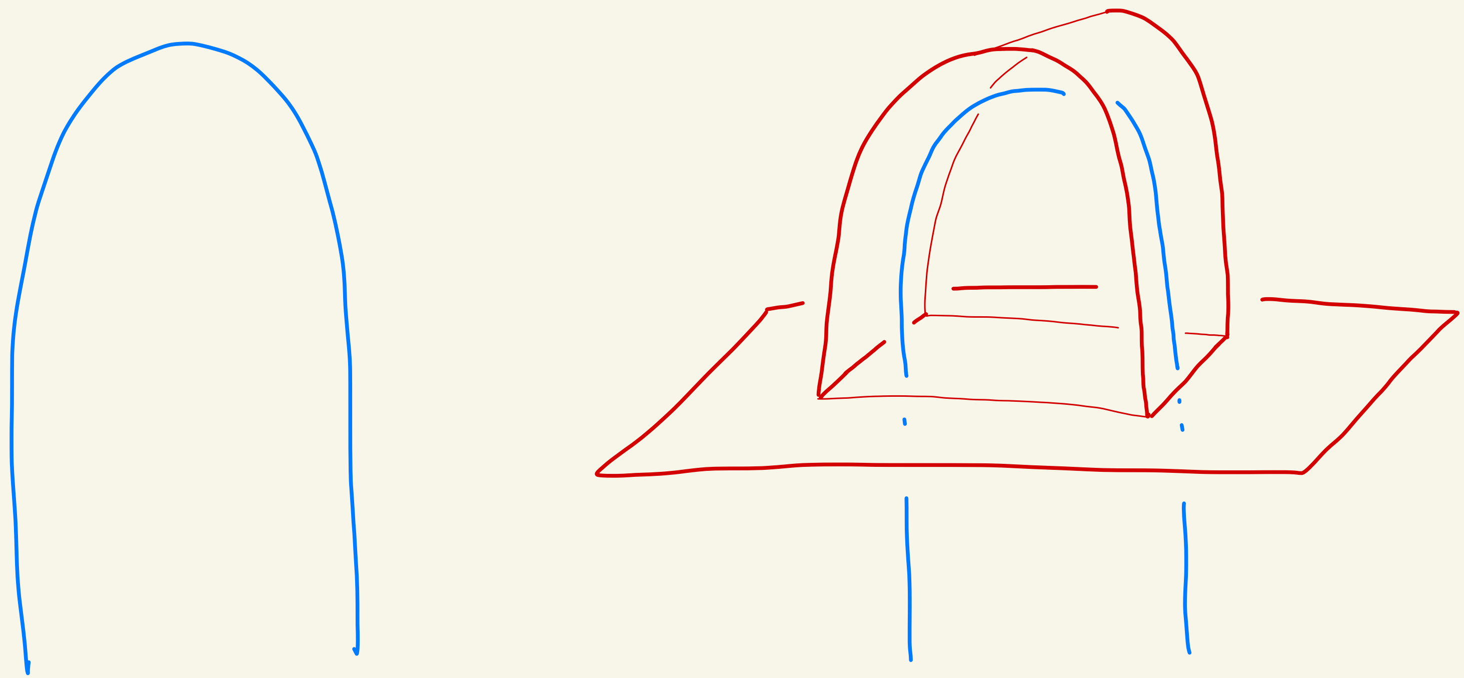
$$Q_{K3} \cong E8 \oplus E8 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Goal: realise algebra by geometry.

The Whitney trick

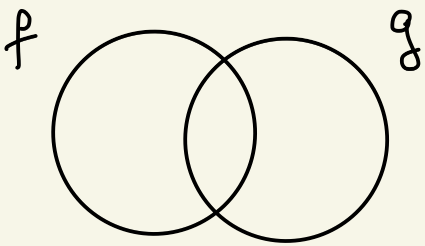


The Whitney trick



- if $\exists$ (framed) embedded Whitney disc, can remove the pair of intersections
- using the Whitney trick, Smale proved the smooth h-cob theorem in dim ≥ 5
- what about dimension 4?

Intersection numbers



$\lambda(f,g) := \qquad \in \mathbb{Z}[\pi_1 M]$

$*$
basept.

$\lambda(f,g) = 0 \iff$

Self-intersection number $\mu(f) = 0 \iff$

f, g are alg. dual if $\lambda(f,g) = 1 \iff$

f, g are geom. dual if $f \pitchfork g = \{pt\}$

Breakthrough result : Disc embedding theorem (Casson, Freedman '82, Freedman - Quinn '90)

M^4 connected, topological manifold. $\pi_1 M$ good

$\Sigma = \sqcup \Sigma_i$ compact surface, each Σ_i simply connected

$$\begin{tikzcd} F : \Sigma \arrow[r, looseness=1, "\circ"] & M \\ \partial\Sigma \arrow[u, hook] \arrow[r, hook] & \partial M \arrow[u, hook] \end{tikzcd}$$ generic immersion

such that • algebraic intersection numbers of F vanish

• $\exists\, G : \sqcup S^2 \looparrowright M$ framed alg. dual to F

Then F is (reg.) htpic rel ∂ to a loc. flat emb $\bar{F}$

with geom dual spheres $\bar{G}$ with $G \simeq \bar{G}$.

Consequences of the disc embedding theorem

Good groups

- abelian gps, finite gps, solvable groups, ...
- gps of subexp growth [Krushkal-Quinn, Freedman-Teichner]
- closed under subgps, quotients, direct limits, extensions.
-

Disc embedding theorem (Casson, Freedman '82, Freedman–Quinn '90)

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such that
- algebraic intersection numbers of F vanish
- $\exists G: \sqcup S^2 \looparrowright M$ framed alg. dual to F

Then F is (reg.) htpic rel ∂ to a loc. flat emb $\overline{F}$ with geom dual spheres $\overline{G}$ with $G \simeq \overline{G}$

Corollary 1: $F : \Sigma^2 \looparrowright M^4$ with

- Σ connected
- alg int numbers vanish
- $\exists G$ alg dual sphere

$F' :=$

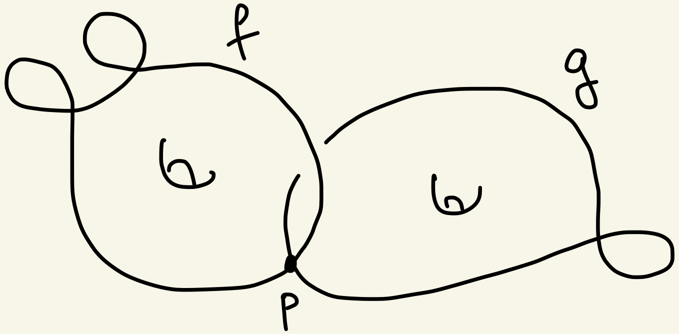
Then f' is (reg) htpic to an embedding

Corollary 2: $F : \Sigma^2 \looparrowright M^4$ with

- Σ connected, $g(\Sigma) > 0$
- alg int numbers vanish
- $\exists G$ alg dual sphere
- $\pi_1 M = 1$

Then f is (reg) htpic to an embedding

Intersection numbers



$\lambda(f,g)$ not well defined in $\mathbb{Z}[\pi_1 M]$!

$*$

$\lambda(f,g)=0 \iff$ all pts in $f \pitchfork g$ paired by gen imm. coll of wh discs

$\mu(f)=0 \iff$ all ints in $f \pitchfork f$ paired by gen imm coll of wh discs

Definition of the Kervaire-Milnor invariant

- for discs/spheres due to FQ90 §10 + Stong

$\Sigma = \sqcup \Sigma_i$

$F: \Sigma \looparrowright M$, alg int numbers of F vanish, $\exists G: \sqcup S^2 \looparrowright M$ alg dual

Let $F^{\infty} \subseteq F$ subset with twisted dual spheres

Let $\{W_\ell^{\infty}\} \subseteq \{W_\ell\}$ subset pairing ints of F^{∞}.

$$km(F; \{W_\ell\}) := \sum_\ell |\operatorname{Int} W_\ell^{\infty} \pitchfork F^{\infty}| \mod 2$$

When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

Proposition (KPRT): $km(F; \{W_e\})$ is well defined

iff

F is b-characteristic.

Definition: $$km(F) = \begin{cases} 0 & \text{if } F \text{ not b-char} \\ km(F; \{W_e\}) & \text{for any choice of } \{W_e\} \text{ if } F \text{ b-char} \end{cases}$$

Proof outline: Suppose $\exists \{W_e\}$ s.t. $\mathrm{km}(F; \{W_e\}) = 0 \in \mathbb{Z}/2$

Step 1: By neg htpy, make F and G geom dual (still immersed)

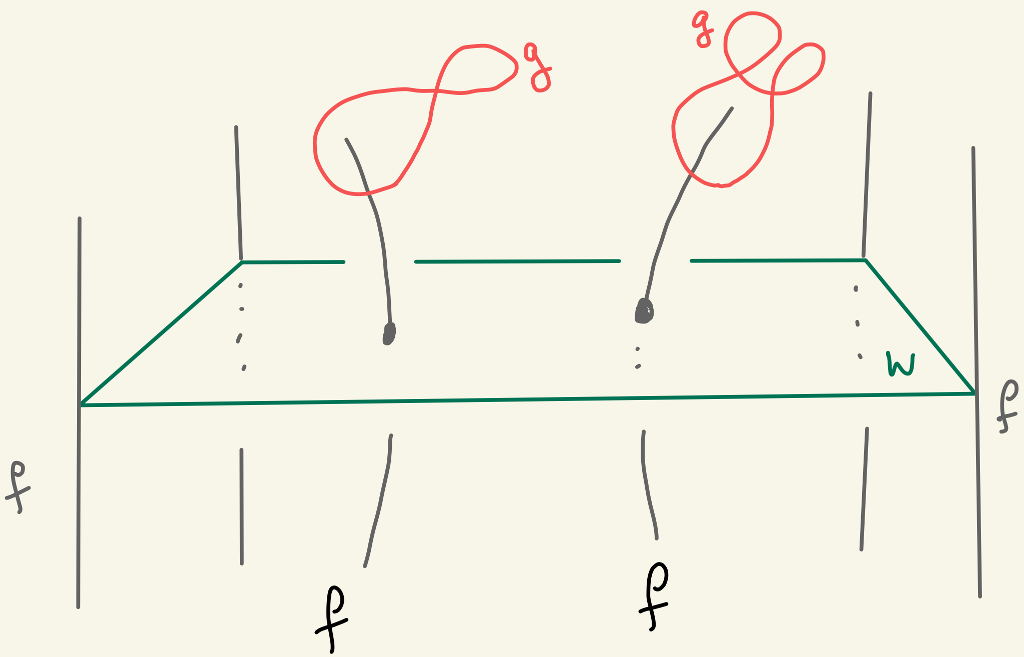
Step 2: Upgrade $\{W_e\}$ and F by neg htpy s.t. $\{\mathrm{Int}\, W_e\} \cap F = \emptyset$

Step 3: Use (whitney) disc embedding theorem to replace $\{W_e\}$ by $\{V_e\}$ with
- $\{\mathrm{Int}\, V_e\} \cap F = \emptyset$
- $\{V_e\}$ flat, embedded, disjoint
- geom dual spheres $\{V_e^T\}$ in $M \setminus F$

Step 4: Tube G into $\{V_e^T\}$ to get $\bar{G}$ geom dual to F, disjoint from $\{V_e\}$
(ignore if only care about $\bar{F}$)

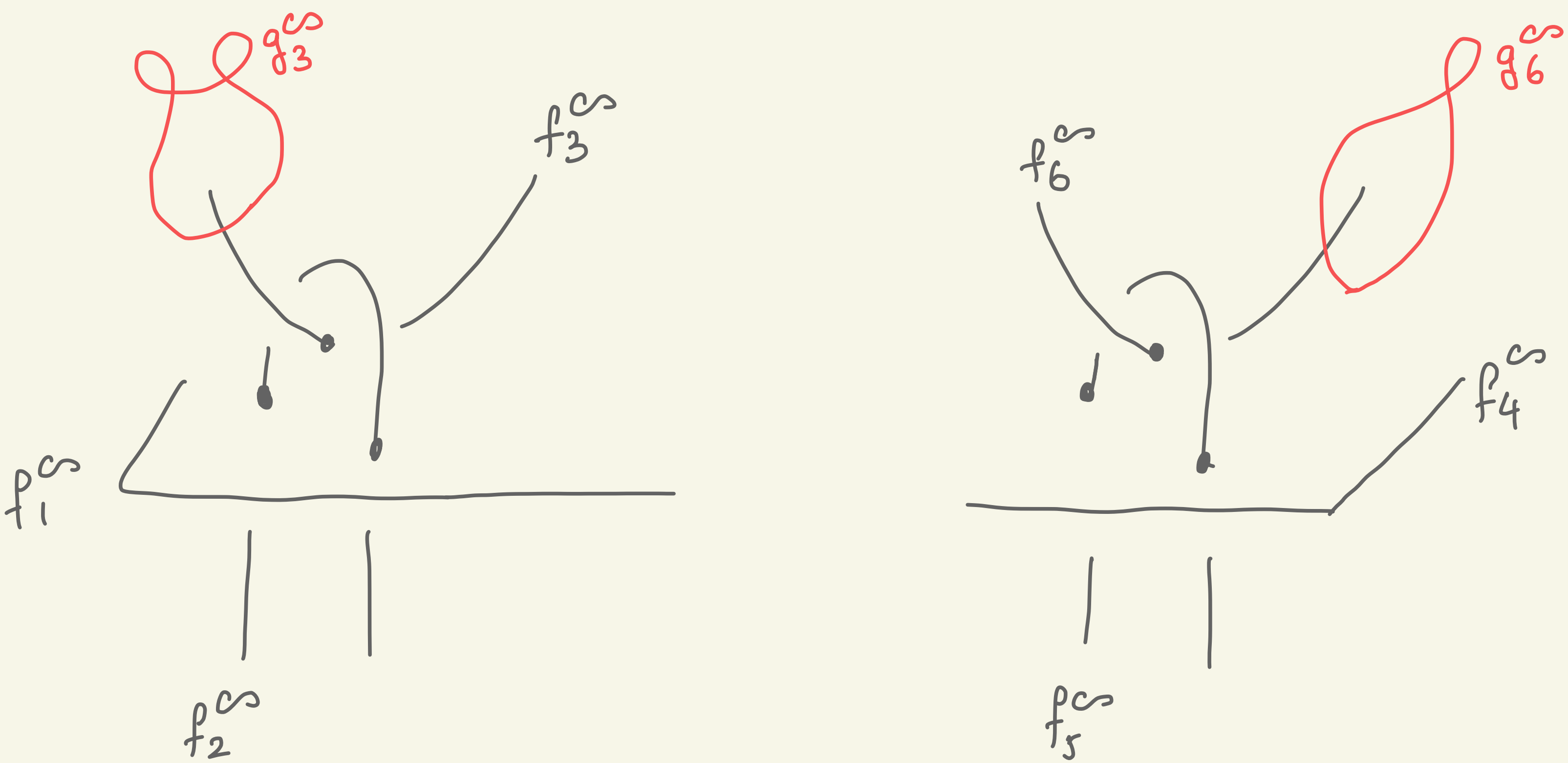
Step 5: Wh move F over $\{V_e\}$ to produce desired $\bar{F}$.

Step 2: Upgrade $\{W_e\}$ and F by neg htpy s.t. $\{\mathrm{Int}\, W_e\} \cap F = \emptyset$



Step 2: Upgrade $\{W_e\}$ and F by neg htpy s.t. $\{\mathrm{Int}\, W_e\} \cap F = \emptyset$

Remaining problem:



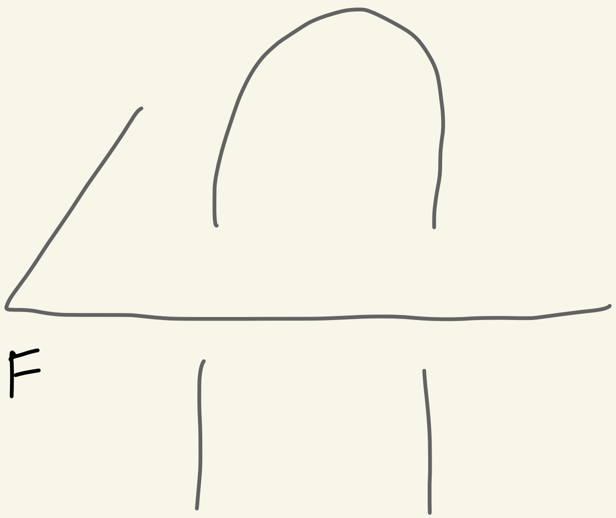
$\square$

When is $\mathrm{km}(F; \{W_e\})$ independent of $\{W_e\}$?

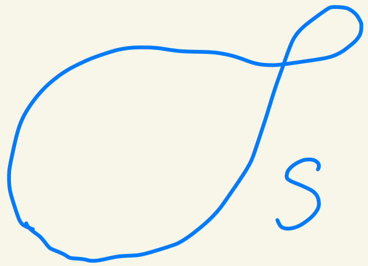
- For convenience let Σ connected, M, Σ oriented.

Suppose $\exists$ immersed sphere $S \looparrowright M$

s.t. $F \cdot S \not\equiv S \cdot S \mod 2$.



F



S

(i) $F \cdot S$ odd
$S \cdot S$ even

(ii) $F \cdot S$ even
$S \cdot S$ odd

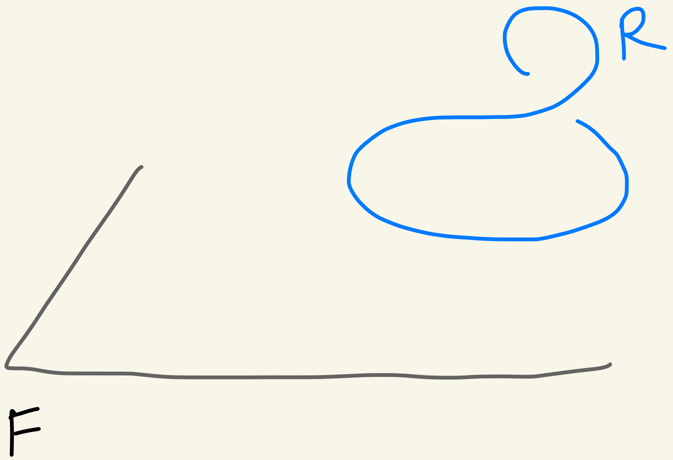
Otherwise, F is called s-characteristic.

When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

- For convenience let Σ connected, M, Σ oriented

Suppose $\exists$ immersed $\mathbb{RP}^2$ $R \looparrowright M$

s.t. $F \cdot R \not\equiv R \cdot R \mod 2$



Otherwise, F is called r-characteristic.

When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

- For convenience let Σ connected, M, Σ oriented

Let $\mathcal{B} \subseteq H_2(M, \Sigma^{cs}; \mathbb{Z}/2)$ the subset rep by maps of annuli or Möbius bands

Suppose the $\mathbb{Z}/2$ int form $\lambda_{\Sigma^{cs}}$ on $H_1(\Sigma^{cs}; \mathbb{Z}/2)$ is nontrivial on $\partial\mathcal{B}$

When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

- for convenience let Σ connected, M, Σ oriented

If $\lambda_{\Sigma^{cs}}|_{\partial B}$ trivial, define for a band B and A a collection of wh arcs for F^{cs}

$$\Theta_A(B) := \mu_{\Sigma^{cs}}(\partial B) + \partial B \pitchfork A + B \pitchfork F^{cs} + e(B) \mod 2$$

Suppose $\exists B$ s.t. $\Theta_A(B) \neq 0$

When is $km(F; \{W_e\})$ independent of $\{W_e\}$?

- for convenience let Σ connected, M, Σ oriented

Lemma: $\Theta_A(B)$ depends only on the homology class of B

If $\lambda_{\Sigma^{cs}}|_{\partial B} = 0$, Θ_A does not depend on A.

so there is a well defined map $\Theta : \mathcal{B} \longrightarrow \mathbb{Z}/2$.

Definition: F is b-characteristic if $\lambda_{\Sigma^{cs}}|_{\partial B} = 0$ & $\Theta = 0$.

Corollary 1: $F : \Sigma^2 \looparrowright M^4$ with

- Σ connected
- alg int numbers vanish
- $\exists G$ alg dual sphere

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Then f' is (reg) htpic to an embedding

Corollary 2: $F : \Sigma^2 \looparrowright M^4$ with

- Σ connected, $g(\Sigma) > 0$
- alg int numbers vanish
- $\exists G$ alg dual sphere
- $\pi_1 M = 1$

Then f is (reg) htpic to an embedding

Thanks for your attention!