Topics in topology || Topological manifolds

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Plan for the course:

A manifold is a Hausdorff, second countable topological space that is locally homeomorphic to $\mathbb{R}^n$. One frequently encounters manifolds with additional structure, such as smooth, piecewise-linear (PL), Riemannian, or symplectic. The aim of this course is to learn about unadulterated topological manifolds. For such manifolds, many essential tools from the start of differential topology are deep theorems. Examples include the existence and uniqueness of collar neighbourhoods for boundaries of manifolds, the existence of tubular neighbourhoods and transversality for submanifolds, and the well-definedness of the connected sum operation on closed, oriented manifolds. We will learn about these results, and what goes into their proofs, for topological manifolds.

Although we will primarily study topological manifolds, smooth and PL manifolds are never far away. We shall need facts from these theories, that we shall recall or prove as needed. Moreover, a natural and fascinating question arises as to whether a given topological manifold admits a smooth/PL structure, and if so how many up to equivalence. An interesting aspect is the strong dependence on the dimension of the manifold in question. The character of the theory is markedly different for each of the dimensions $1, 2, 3, 4,$ and 5, whereas for dimensions ≥ 5, there is a more coherent theory with a similar flavour. We shall try to give a feeling for all dimensions, and the connections between them.

Specific topics:

(1) Introduction to topological manifolds. Invariance of domain.
(2) Definition of PL and smooth structures. Combinatorial manifolds. Review of basic differential topology and PL topology.
(3) Decomposition spaces.
(4) Collars of boundaries, existence and uniqueness.
(5) The Schoenflies problem.
(6) Stallings' and Zeeman's unknotting theorems.
(7) The isotopy extension theorem.
(8) Local contractibility of homeomorphisms.
(9) A review of surgery and the s-cobordism theorem (3 categories), discuss unknotting in codimension 2.
(10) The torus trick, the stable homeomorphism theorem, and the annulus theorem.

(11) Connected sums, homeomorphisms of S^n.
(12) Microbundles, tangent and normal. The Kister-Mazur theorem.
(13) The product structure theorem, concordance implies isotopy.
(14) Transversality, existence of handle structures.
(15) Handle structures.
(16) Smoothing theory.
(17) Normal bundles in codimension one and two.
(18) Homotopy groups of classifying spaces and the Hauptvermutung.
(19) Low dimensions.
(20) The manifold recognition problem. Disjoint discs, ANR homology manifolds and the Quinn invariant.
(21) The double suspension theorem.
(22) Triangulation conjecture, modulo Floer theory.
(23) $\mathbb{R}^n$ has a unique smooth structure except in dim 4. Discuss exotic $\mathbb{R}^4$s.