DIRECT AND INDIRECT CONSTRUCTIONS OF LOCALLY FLAT SURFACES IN 4-MANIFOLDS

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Abstract. There are two main approaches to building locally flat surfaces in 4-manifolds: direct methods applying Freedman-Quinn's disc embedding theorem, and indirect methods using surgery theory. Notably the second method also requires the disc embedding theorem, but only indirectly. In this minicourse we will give an introduction to both methods, by sketching the proofs of the following results: every primitive second homology class in a closed, simply connected 4-manifold is represented by a locally flat torus [Lee-Wilczyński [LW97]]; Alexander polynomial one 1-knots are topologically slice [Freedman-Quinn [FQ90]]; and, if time permits, 2-knots in the 4-sphere with infinite cyclic fundamental group of the complement are topologically unknotted [Freedman-Quinn [FQ90]].

Contents

1. Introduction

1.1. **Plan for the course and logistics.** The other three courses in the summer school are focussed on smoothly embedded surfaces in smooth 4-manifolds. By contrast, we will work entirely in the topological setting. Four is the lowest dimension where there are manifolds that do not admit any smooth structure. Locally flat surfaces are therefore the most we can hope to find most generally in 4-manifolds.

In addition, there is a remarkable disparity between the smooth and topological settings in dimension four, in particular related to the behaviour of embedded surfaces. Hopefully this difference will become clear during this summer school. Even in a smooth 4-manifold it is interesting to consider locally flat surfaces, e.g. in order to detect when an invariant or phenomenon is 'purely smooth' vs 'purely topological'.

The main goal of the lectures will be to give the audience an overview of the tools and techniques available in the purely topological setting, with the hope of emboldening them to use them to attack some of the many interesting open problems about locally flat surfaces in topological 4-manifolds.

Broadly speaking there are two flavours of proofs and techniques in this setting. The first is very direct and hands-on. We draw explicit pictures and modify them, keeping careful track of how intersection points are created or removed. For example, this includes the manoeuvres in the constructive part of the proof of the disc embedding theorem

Last updated: 11:59:39 GA time on 2024-05-31.

(Theorem 1.12). These manoeuvres will be the focus of the first 2-3 lectures. Specifically we will see how they can be used to prove the following theorem.

Theorem 1.1 ([LW97, $d = 1$ case of Theorem 1.1])**.** *Let M be a closed, simply connected 4-manifold. Then every primitive class in $H_2(M;\mathbb{Z})$ is represented by a locally flat torus.*

Here a class is said to be primitive if it is not a nonzero multiple of another class. The original proof of Lee and Wilczyński is not especially direct. We will give a more geometric proof from [KPRT22]. The above statement is a (shared) special case of two distinct general results, from [LW97] and [KPRT22], which we state in Section 2.

In the second half of the minicourse we will use more abstract techniques, specifically surgery theory and the h-cobordism theorem. We will see how these techniques can be used to show the following results.

Theorem 1.2 ([FQ90])**.** *Every 1-knot $K\colon S^1 \hookrightarrow S^3$ with Alexander polynomial one is (topologically) slice.*

Theorem 1.3 ([FQ90])**.** *Every 2-knot $K\colon S^2 \hookrightarrow S^4$ with $\pi_1(S^2 \setminus K) \cong \mathbb{Z}$ is (topologically) unknotted.*

Logistics. The teaching assistant for the course is Daniel Hartman. Please bring questions to the TA sessions, about the lectures but also about the exercises, described below. These notes can be found at `www.tinyurl.com/GTSSC-TOPnotes`. Feedback is also most welcome. Possible avenues for feedback are: via questions and comments in lectures, by talking to Daniel, by talking to me, or by anonymously filling out this form `www.tinyurl.com/GTSSC-TOPfeedback`.

Exercises. There are exercises at the end of these notes. The problems are separated into three levels. **Green** △ exercises should be attempted if you are seeing all of this material for the first time. Prerequisites are courses in introductory geometric and algebraic topology. **Orange** □ exercises are for readers who are already comfortable with some of the terminology; they may require nontrivial input from outside these lectures, which we have tried to indicate as hints. Finally, **red** ◯ exercises are challenge problems. Of course this separation into levels is based on what I personally find difficult, which might be different from what you find difficult. Open problems will be marked as such, and do not intersect with the exercises. Please send me (TEXed) solutions to the exercises, and I will include them (with credit) in these notes.

Relationship between these notes and the lectures. Many details and references in these notes were not mentioned in the accompanying lectures. The order of topics has also been slightly modified.

1.2. **Locally flat embeddings.** We will be considering locally flat embeddings of surfaces in 4-manifolds, so we begin by defining these. For $m \geq 0$ let

$$\mathbb{R}^m_+ := \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1 \geq 0\}$$

Definition 1.4. An embedding $f\colon (X, \partial X) \hookrightarrow (M, \partial M)$, i.e. a continuous map which is a homeomorphism onto its image, of a k-manifold X in a 4-manifold M is said to be *locally flat* if for all $x \in \Sigma$ there is a neighbourhood $U \subseteq M$ of $f(x)$ such that $(U, U \cap f(\Sigma))$ is homeomorphic to either $(\mathbb{R}^4, \mathbb{R}^k)$, in the case that $x \in \operatorname{Int}\Sigma$, or to $(\mathbb{R}^4_+, \mathbb{R}^k_+)$, in the case that $x \in \partial\Sigma$. See the schematic in Figure 1.

For smooth 4-manifolds one usually considers smoothly embedded surfaces. In the case of 4-manifolds which might not admit smooth structures, locally flat embeddings are the correct analogue. In particular, submanifolds of a topological manifold are locally flat by definition. There do exist embeddings which are not locally flat (see e.g. Exercise △ 5.1.1). However, these lack some very useful properties enjoyed by locally flat embeddings. Let us quickly review these.

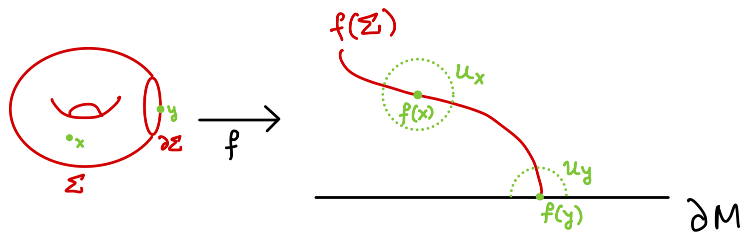

FIGURE 1. A locally flat embedding of a surface Σ. Here we would have homeomorphisms $(U_x, U_x \cap f(\Sigma) \approx (\mathbb{R}^4, \mathbb{R}^2)$ and $(U_y, U_y \cap f(\Sigma) \approx (\mathbb{R}^4_+, \mathbb{R}^2_+)$.

Theorem 1.5 ([Qui82; FQ90, Theorems 9.3A and 9.5A]). *Let M be a 4-manifold.*

(1) *(Existence of normal vector bundles) Every (locally flat) submanifold of M has a normal vector bundle, which is unique up to bundle isomorphism and ambient isotopy.*
(2) *(Topological transversality) Let Σ_1, Σ_2 be (locally flat) submanifolds of M. There is an ambient isotopy of M taking Σ_1 to some Σ_1' such that Σ_1' and Σ_2 intersect transversely.*

Without going into too many details, we give the definition of a normal vector bundle.

Definition 1.6. Let M be a 4-manifold and let $(X, \partial X) \subseteq (M, \partial M)$ be a k-dimensional submanifold. A *normal vector bundle* of X in M is a pair $(E, p\colon E \to X)$ with the following properties.

(1) E is a neighbourhood of X in M;
(2) E is a codimension zero submanifold of M;
(3) the map $p\colon E \to X$ is an $(n-k)$-dimensional vector bundle such that $p(x) = x$ for all $x \in X$;
(4) $\partial E = p^{-1}(\partial X)$; and
(5) the data above is *extendable*, i.e. given any $(n-k)$-dimensional vector bundle $(F, q\colon F \to X)$, any radial homeomorphism from an open convex disc bundle of F to E can be extended to a homeomorphism from all of F to a neighbourhood of E in M.

The purpose of the first four properties is for the normal vector bundle to mimic the notion of an open tubular neighbourhood in the smooth setting. There is a technical problem that the closure of such an open neighbourhood might have undesirable self-intersections. The fifth property of extendability is designed to avoid this.

Similarly, we give a very quick definition of transversality.

Definition 1.7. Let $(X_1, \partial X_1), (X_2, \partial X_2) \subseteq (M, \partial M)$ be locally flat subamnifolds of a 4-manifold M, of dimension k_1 and k_2 respectively. We say that X_1 and X_2 *intersect transversely* if for any point $x \in X_1 \cap X_2$, there exists a chart $U \subseteq M$ such that $(U, U \cap X_1, U \cap X_2)$ is homeomorphic to either $(\mathbb{R}^4, \mathbb{R}^{k_1} \times \{0\}, \{0\} \times \mathbb{R}^{k_2})$ if $x \in \operatorname{Int} U$, or to $(\mathbb{R}^4_+, \mathbb{R}^1_+ \times \mathbb{R}^{k_1-1} \times \{0\}, \mathbb{R}^1_+ \times \{0\} \times \mathbb{R}^{k_2-1})$ if $x \in \partial M$.

The formulation in the boundary of M is unnecessarily complicated. Intuitively, you should think of the statement of topological transversality as saying that, after an isotopy of one of the submanifolds, we can assume that the intersections are of the smallest possible dimension. In case $k_1 + k_2 < 4$, the definition of transversality does not immediately imply that X_1 and X_2 can be made disjoint. But one sees immediately that a small further isotopy around all of the remaining intersections produces disjoint submanifolds.

Remark 1.8. An often repeated slogan is that topological 4-manifolds behave like high-dimensional manifolds (whereas smooth 4-manifolds do not). However there are situations where topological 4-manifolds are even better behaved than high-dimensional manifolds. As an example of this, we note that (locally flat) submanifolds of high-dimensional manifolds do not necessarily have normal vector bundles. For more on this, see [FQ90, Section 9.4; FNOP19, Section 5.3].

On the other hand, topological transversality holds in all dimensions and codimensions, but the definition is much more complicated to parse, in particular due to the unavailability of normal bundles. The results are due to Marin [Mar77] and Quinn [Qui82, Qui88] (see also [FQ90, Section 9.5; FNOP19, Chapter 10]).

1.3. **Topological generic immersions.** In addition to locally flat embeddings, there is also a useful notion of a *topological generic immersion*, and a result saying that continuous maps can be approximated by these. To state this precisely, we first give the definition of an immersion of manifolds in the topological setting. We limit ourselves to surfaces. We have the standard inclusions:

$$\begin{aligned}\iota\colon \mathbb{R}^2 &= \mathbb{R}^2 \times \{0\} \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4,\\ \iota_+\colon \mathbb{R}^2_+ &= \mathbb{R}^2_+ \times \{0\} \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4, \text{ and}\\ \iota_{++}\colon \mathbb{R}^2_+ &= \mathbb{R}^k_+ \times \{0\} \hookrightarrow \mathbb{R}^k_+ \times \mathbb{R}^2 = \mathbb{R}^2_+.\end{aligned}$$

Definition 1.9. Let Σ be a surface and M be a 4-manifold. A continuous map $f\colon \Sigma \to M$ is an *immersion* if for each point $p \in \Sigma$ there is a chart φ around p and a chart Ψ around $f(p)$ fitting into one of the following commutative diagrams. The first diagram is for $p \in \operatorname{Int}\Sigma$ and $f(p) \in \operatorname{Int} M$, the second diagram is for $p \in \partial\Sigma$ and $f(p) \in \operatorname{Int} M$, and the third is for $p \in \partial\Sigma$ and $f(p) \in \partial M$. In particular f is required to map interior points of Σ to interior points of M, but it is possible that $\partial\Sigma$ is mapped to $\operatorname{Int} M$.

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "\iota"] \arrow[d, "\varphi"] & \mathbb{R}^4 \arrow[d, "\Psi"] \\ \Sigma \arrow[r, "f"] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^2_+ \arrow[r, "\iota_+"] \arrow[d, "\varphi"] & \mathbb{R}^4 \arrow[d, "\Psi"] \\ \Sigma \arrow[r, "f"] & M \end{tikzcd} \qquad \begin{tikzcd} \mathbb{R}^2_+ \arrow[r, "\iota_{++}"] \arrow[d, "\varphi"] & \mathbb{R}^4_+ \arrow[d, "\Psi"] \\ \Sigma \arrow[r, "f"] & M \end{tikzcd} \tag{1.1}$$

Some authors prefer to call this notion a *locally flat immersion*.

The *singular set* of an immersion $f\colon \Sigma \to M$ is the set

$$\mathcal{S}(f) := \{m \in M \mid |f^{-1}(m)| \geq 2\}.$$

[1]

Definition 1.10. Let Σ be a surface and M be a 4-manifold. A continuous map $f\colon \Sigma \to M$ is said to be a *generic immersion*, denoted $f\colon \Sigma \looparrowright M$, if it is an immersion and the singular set is a closed, discrete subset of M consisting only of transverse double points, each of whose preimages lies in the interior of Σ. In particular whenever $m \in \mathcal{S}(f)$, there are exactly two points $p_1, p_2 \in \Sigma$ with $f(p_i) = m$, and there are disjoint charts φ_i around p_i, for $i = 1, 2$, where φ_1 is as in the left-most diagram of (1.1), and φ_2 is the same, with respect to the same chart Ψ around m, but with ι replaced by

$$\iota'\colon \mathbb{R}^2 = \{0\} \times \mathbb{R}^2 \hookrightarrow \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4.$$

Theorem 1.11 (Immersion lemma [FQ90, Corollary 9.5C])**.** *Every continuous map $\Sigma \to M$ of a surface to a 4-manifold is homotopic to a generic immersion.*

We allow generic immersions to to map the boundary of a surface to the interior of a 4-manifold, since we will often apply the immersion lemma to find generically immersed Whitney discs.

These fundamental results for surfaces in 4-manifolds (Theorems 1.5 and 1.11) were proven by Quinn [Qui82], using Freedman's disc embedding theorem [Fre82]. We will not go into these proofs, which are quite intricate, but rather we will be glad that these

[1] insert figures here

tools exist and use them freely in the rest of these lectures. We observe that analogous properties hold for smooth maps of surfaces in smooth 4-manifolds, and perhaps you use these without thinking. The takeaway is that, at least with respect to these properties, we can also be that casual about locally flat surfaces in topological 4-manifolds.

1.4. **Visualising surfaces in 4-manifolds.** In Section 2, we will primarily modify generically immersed surfaces directly by hand. Therefore it will be crucial for us to visualise them. We will generally draw schematic pictures, but to begin with let us draw some concrete pictures.

By definition locally flat surfaces (and generically immersed surfaces) are standard in small coordinate charts, which we can draw precisely. Since each chart in a 4-manifold is a copy of $\mathbb{R}^4 = \mathbb{R}^3 \times (-\varepsilon, \varepsilon)$, we can draw a sequence of copies of $\mathbb{R}^3$, and see how our surfaces show up within them.

It will be especially useful to consider transverse intersections of surfaces, so we also depict these.

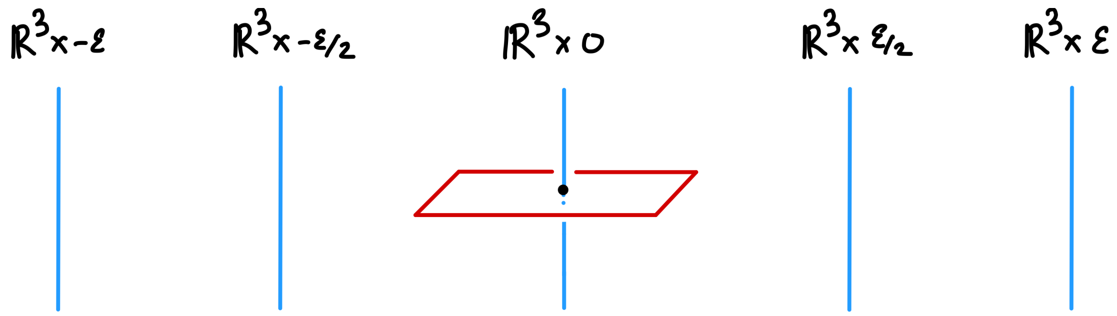

FIGURE 2. The xy-plane is shown in red, and the zw-plane is shown in blue.

2

Let's take a moment to find the *Clifford torus* in this picture. By definition, the Clifford torus is the product (in $\mathbb{R}^4$ of the unit circle in the xy-plane with the unit circle in the zw-plane.

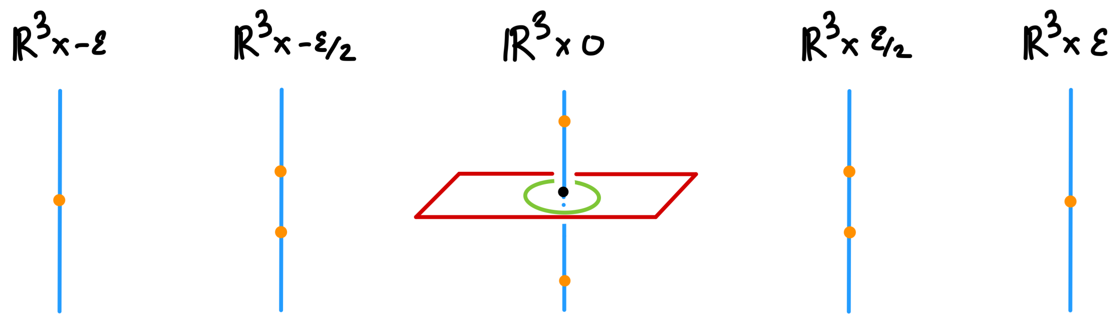

FIGURE 3. The unit circle in the xy-plane is shown in green and the unit circle in the zw-plane is shown in orange.

[2]insert second figure here

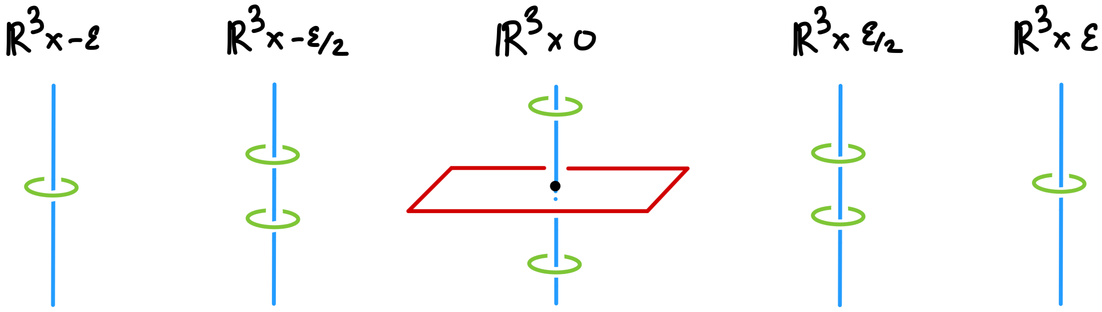

FIGURE 4. The Clifford torus is shown in green.

1.5. **Statement of the disc embedding theorem.** The fundamental breakthrough in the study of topological 4-manifolds (and surfaces within them) is the disc embedding theorem. We begin by stating the simplest version of the theorem, and address more general versions in subsequent remarks.

Theorem 1.12 ([Fre82; FQ90, Theorem 5.1A])**.** *Let M be a topological 4-manifold with $\pi_1(M)$ trivial. Suppose we have*

$$\begin{tikzcd} f\colon D^2 \arrow[r, looparrow] & M^4 \\ \partial D^2 \arrow[u, hook] \arrow[r, hook] & \partial M, \arrow[u, hook] \end{tikzcd}$$

where $f|_{\partial D^2}$ is a locally flat embedding, and we have $g\colon S^2 \looparrowright M^4$, such that

(i) *g has trivial normal bundle;*
(ii) *g has trivial self-intersection, i.e. $g \cdot g = 0$; and*
(iii) *f and g are* algebraically dual *if $f \cdot g = 1$.*

Then f is homotopic to some $\overline{f}$, relative to the boundary, and g is homotopic to some $\overline{g}$ such that

(1) *$\overline{f}$ is a locally flat embedding, and*
(2) *$\overline{f}$ and $\overline{g}$ are* geometrically dual*, i.e. $f \pitchfork g$ is a point.*

[3]

Remark 1.13. There is a version of the theorem for finite collections of discs. The proof is essentially the same. There is a complicated generalisation to infinite collections of discs, called the *disc deployment lemma*, which is significantly harder to prove [Qui82].

Remark 1.14. The theorem also holds for M with more general fundamental group. Specifically, there is a notion of *good* group, whose definition we will not go into (see instead [FT95, KOPR21]). For applications, it suffices to know that the class of good groups is known to contain groups of subexponential growth [FT95, KQ00], and to be closed under subgroups, quotients, extensions, and colimits [FQ90, p. 44]. In particular, all finite groups and all solvable groups are good. It is not known whether non-abelian free groups are good.

In the case of non-trivial fundamental groups, the self-intersection number of g and the intersection between f and g is no longer just the signed count of intersections, but rather an *equivariant* version, with values lying in (a quotient of) $\mathbb{Z}[\pi_1(M)]$.

Remark 1.15. The disc embedding theorem was the key ingredient in the proof of the 4-dimensional h-cobordism theorem. Historically, the first version of the disc embedding theorem proven by Freedman was for discs in an arbitrary smooth, simply connected

[3]insert figure here

4-manifold. This was the ingredient needed by Quinn in [Qui82] to prove the fundamental tools mentioned above (including Theorems 1.5 and 1.11). Using these tools, Freedman's proof could be redone, but now in a topological ambient space. The techniques of the proof were also further developed by Freedman and Quinn to now apply to ambient 4-manifolds with good fundamental group. This was the proof given in [FQ90] and then explained further in [BKK+21].

The disc embedding theorem also implies the *sphere embedding theorem* (see Exercise ◯ 5.3.2), which is the key ingredient in proving the exactness of the topological surgery sequence in dimension four for good fundamental groups.

Conventions. Homeomorphism of manifolds is denoted by the symbol $\approx$, while diffeomorphism is denoted by $\cong$. Manifolds are *not* assumed to be smooth.

Acknowledgements. I would like to thank the organisers of the Georgia Topology Summer School 2024, where these lectures took place, as well as the audience for their lively participation and Daniel Hartman, who was the TA for the accompanying problem sessions. My warm thanks go also to Elise Brod and Megan Fairchild for creating some of the figures.

2. Representing primitive homology classes by locally flat tori

Proof of Theorem 1.1. We split up the proof in a number of steps.

Step 0. *Represent α by a generic immersion $f\colon S^2 \looparrowright M$ and a geometrically dual sphere $g\colon S^2 \looparrowright M$, i.e. $f \pitchfork g$ is a single point.*

First we use that $\pi_1(M) = 1$. This implies that $\pi_2(M) \cong H_2(M;\mathbb{Z})$, so every class in the second homology group can be represented by a map of a sphere. Then by the immersion lemma (Theorem 1.11) we can assume further that this map is a generic immersion.

Next by Poincaré duality, we know that α, being a primitive class, has a dual class. In other words, there is some $\beta \in H_2(M;\mathbb{Z})$ such that $\alpha \cdot \beta = 1$. Like in the previous paragraph, β can be represented by a generic immersion $g\colon S^2 \looparrowright M$, where we know that $f \cdot g = 1$, where this is both the homological intersection number and the signed count of intersections between f and g.

Finally, we have to use the *geometric Casson lemma*. Unfortunately we will not have time to prove this in the lectures, so we leave it as an advanced exercise (Exercise ◯ 5.3.1). By repeated applications of the lemma we arrange and f and g are geometrically dual as desired. We remark that this may greatly increase the number of self-intersections of f and of g.

Step 1. *Arrange that f has trivial self-intersection number, i.e. the signed count of self-intersections of f is zero.*

We will use *interior twisting*. This procedure is best described by a figure. [4] Since this is our first explicit geometric construction, let us take a moment to describe it properly. In the figure on the left, we have a described a small patch on f in a movie picture. In other words, the red vertical lines can be stitched together to give a small patch on f, specifically region with no double points.

The figure describes a modification of this small patch. Notice that the left and right pictures agree on the boundary, so you could imagine taking out the small patch on f shown on the left, and gluing in the surface on the right, like a band-aid. This is allowed and possible since the boundaries agree.

[4] insert figure here

The second key property of the "band-aid" side of the figure is that it has a transverse double point singularity with positive sign. [5] The analogous picture, with the opposite twist has a transverse double point singularity with negative sign. So replacing an embedded patch of f by one of these band-aids adds a new intersection of some desired sign. Do these enough times to arrange that the signed count of intersections of f is zero.

Step 2. *Pair up the points in $f \pitchfork f$ by generically immersed discs.*

Arbitrarily pair up points with opposite sign. For each such pair, the two constituent points can be joined by two arcs – one on each sheet. The union of these arcs is a circle, called a *Whitney circle*. Recalling again that $\pi_1(M) = 1$, we note that each such circle is null homotopic in M. Applying Theorem 1.11 and Theorem 1.5 (2), we can assume that the Whitney circles bound generically immersed discs $\{W_i\}$, which intersect f transversely.

Let us describe what we would like to be true for this collection $\{W_i\}$. In the ideal situation, we would have
[6]

However, in reality, we have a number of problems. We have summarised this in Table 1. These problems may seem independent of one another at first glance, but indeed they are related, and often problems of one sort can be traded for problems of a different sort in a precise way.

Type	Problem	Solution	Cost
1	$\mathring{W}_i \pitchfork \mathring{W}_j$	Disc embedding theorem	free
2	$\mathrm{tw}(\partial W_i)$	Interior twisting	$\mathrm{tw}(\partial W_i) \mapsto \mathrm{tw}(\partial W_i) \pm 2$ $\mathring{W}_i \pitchfork \mathring{W}_j \mapsto \mathring{W}_i \pitchfork \mathring{W}_j + 1$
		Boundary twisting	$\mathrm{tw}(\partial W_i) \mapsto \mathrm{tw}(\partial W_i) \pm 1$ $\mathring{W}_i \pitchfork f \mapsto \mathring{W}_i \pitchfork f + 1$
3	$\partial W_i \pitchfork \partial W_j$	Boundary pushoff	$\partial W_i \pitchfork \partial W_j \mapsto \partial W_i \pitchfork \partial W_j - 1$ $\mathring{W}_i \pitchfork f \mapsto \mathring{W}_i \pitchfork f + 1$
4	$\mathring{W}_i \pitchfork f$	Tubing into g	$\mathring{W}_i \pitchfork f \mapsto \mathring{W}_i \pitchfork f - 1$ $\mathrm{tw}(\partial W_i) \mapsto \mathrm{tw}(\partial W_i) + e(\nu g)$ $\mathring{W}_i \pitchfork \mathring{W}_j$ uncontrolled

TABLE 1. Problems, their solutions, and associated costs

Step 3. *Use geometric manoeuvres to try to minimise the various problems in Table 1.*

In other words, we describe several geometric manoeuvres, which are described in the table as 'solutions'. They are not solutions in a strict sense – while each manoeuvre helps to reduce a certain type of problem, it also has an associated cost. Our goal will be to use these manoeuvres cleverly and in the right order to see whether all problems could be removed.

Before going into the geometric manoeuvres, we justify our statement in the table that problems of type 1 can be solved 'for free' by applying the disc embedding theorem.

[5] it's easy to say the sign precisely when the figure does not yet exist :)

[6] Much more to be added here

Proposition 2.1. *Suppose that* $\operatorname{tw}(\partial W_i) = 0$*,* $\partial W_i \pitchfork \partial W_j = \emptyset$*, and* $\mathring{W}_i \pitchfork f = 0$ *for all* i, j*. Then there exists* $\{\overline{W}_i\}$*, a collection of locally flat embedded and disjoint Whitney discs pairing all the intersection points of* $f \pitchfork f$ *and with interiors in* $M \setminus \nu f$*.*

Proof. The goal is to apply the disc embedding theorem (Theorem 1.12 except for a finite collection of discs) to $\{W_i\}$ in $N := M \setminus \mathring{\nu} f$. We have to check that the hypotheses hold. First we need that $\pi_1(N) = 1$. This follows from Exercise and △ 5.1.3. We also need algebraically dual spheres. For each W_i, let T_i denote the Clifford torus at one of the two double points of f paired by W_i. As we see in Figure 5, we see that T lies in N and is geometrically dual to W_i. Furthermore it satisfies $T_i \pitchfork W_j = \emptyset$ if $i \neq j$. We will modify each T_i to a sphere. Note that a meridional disc for T_i intersects f at a single point. Tube the meridional disc to g, to get a disc bounded by a meridian of T_i lying entirely in N. Now compressing T_i along two copies of this meridional disc produces a sphere S_i. Then we need to check that this collection of spheres satisfies $S_i \cdot W_j = \delta_{ij}$. We also need to check that $S_i \cdot S_j = 0$ for all i, j – this follows from the construction. Moreover, also by construction each S_i has trivial normal bundle. This shows that the hypotheses of the disc embedding theorem are satisfied for $\{W_i\}$ and $\{S_i\}$ in N. Therefore, the theorem provides the desires embedded and disjoint Whitney discs. □

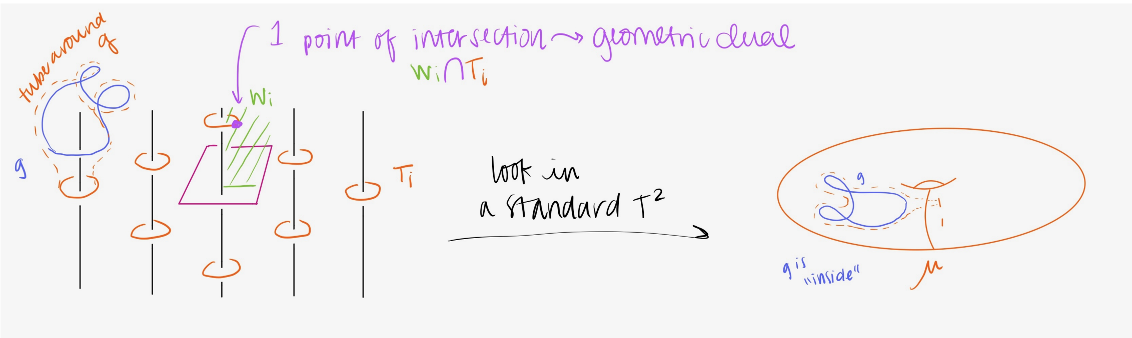

FIGURE 5. Using the Clifford torus to produce algebraically dual spheres

Proposition 2.1 shows that if we can solve all the problems of type 2, 3, and 4, we can find embedded and disjoint Whitney discs for the self-intersections of f, so that if we do the Whitney move along them, we produce a locally flat embedded 2-sphere which is homotopic to f. Note that we can do the Whitney move along locally flat discs, since they have normal bundles, by Theorem 1.5 (1).
[7]

Next we describe the geometric manoeuvres mentioned in Table 1. We already saw interior twisting in a previous step. We also have the operation of *boundary twisting* [8] Determining the effect of interior and boundary twisting on the various problems in Table 1 comprises Exercise □ 5.2.2. The reader might wonder why we need two solutions to problems of type 2. So we remark that interior twisting is *a priori* less effective than boundary twisting, since it can only change the twisting number by even numbers, rather than arbitrary integers. But interior twisting is also 'cheap' – the cost is only problems of type 1, which are 'free' by Proposition 2.1. In contrast, boundary twisting is much more expensive – the cost is problems of type 4, whole solution costs problems of type 2, which is what we were trying to solve in the first place. So one is in danger of getting stuck in a loop of circular reasoning.

[7] More to be added here

[8] Insert boundary twisting figure and explanation

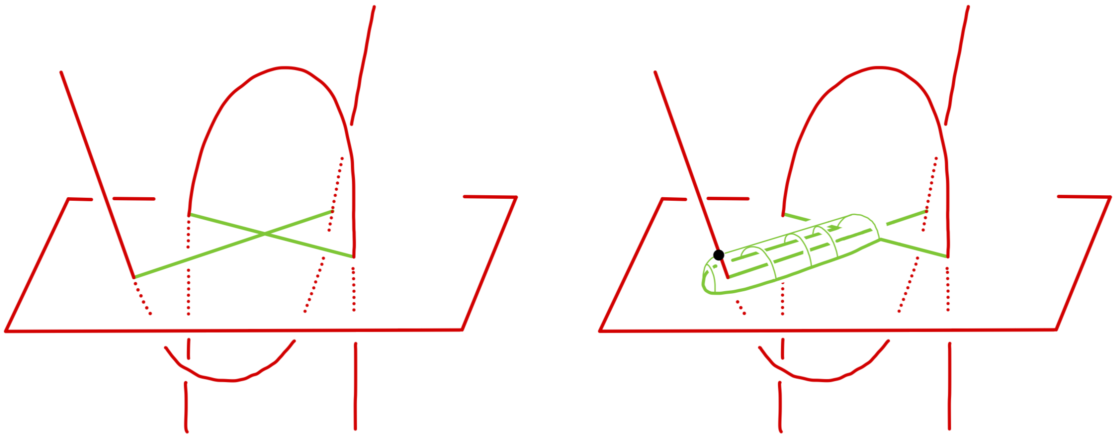

FIGURE 6. Boundary pushoff

We also have the boundary pushoff operation shown in Figure 6. The final operation in the table is to tube intersections of some $\mathring{W}_i$ with f into the geometric dual g. We already saw this operation already in the proof of Proposition 2.1.

Given this toolbox of geometric manoeuvres, we try to solve the problems in the right order to arrange for the hypotheses of Proposition 2.1 to hold. Here is the beginning of a potential strategy.

- Boundary twist, to solve all type 2 problems, creating new type 4 problems.
- Boundary pushoff, to solve all type 3 problems, creating new type 4 problems. (To do this properly one should enumerate the Whitney arc intersections in a reasonable way, and proceed in order. Or, one could choose the Whitney arcs from the start to avoid having intersections. Nonetheless upcoming geometric constructions will create Whitney arc intersections, so we include them as problems in the table.)
- Tube into g, to remove all problems of type 4, creating new type 2 problems.

Now we only have problems of type 1 and 2. By Proposition 2.1, we need only focus on solving these new problems of type 2, but we need to be careful not to create new problems (e.g. of types 3 and 4) in the process. If the current twisting numbers are all even, we could solve them using interior twisting, successfully reducing to the input of Proposition 2.1, which then finishes the argument. What contributed to the current twisting numbers? Since we begun our strategy by arranging the twisting numbers to be zero, the current twisting numbers were only created in the third substep above. Examining that substep more closely, we observe that there are two situations were we can arrange for the twisting numbers to all be even, which we now record.

- Suppose that $e(\nu g)$ is even. [9]
- Suppose that we can arrange for $\{W_i\}$ to have disjoint and embedded boundaries, $\mathrm{tw}(\partial W_i) = 0$ for all i, and each W_i intersects f evenly many times. [10]

Observe that in particular, if the ambient manifold M is spin, then $e(\nu g)$ must be even, and therefore, in such an M, if it is further simply connected, then every primitive class in $H_2(M; \mathbb{Z})$ can be represented by a locally flat, embedded sphere.
[11]

[9]complete this
[10]explain more
[11]Text about transfer move here

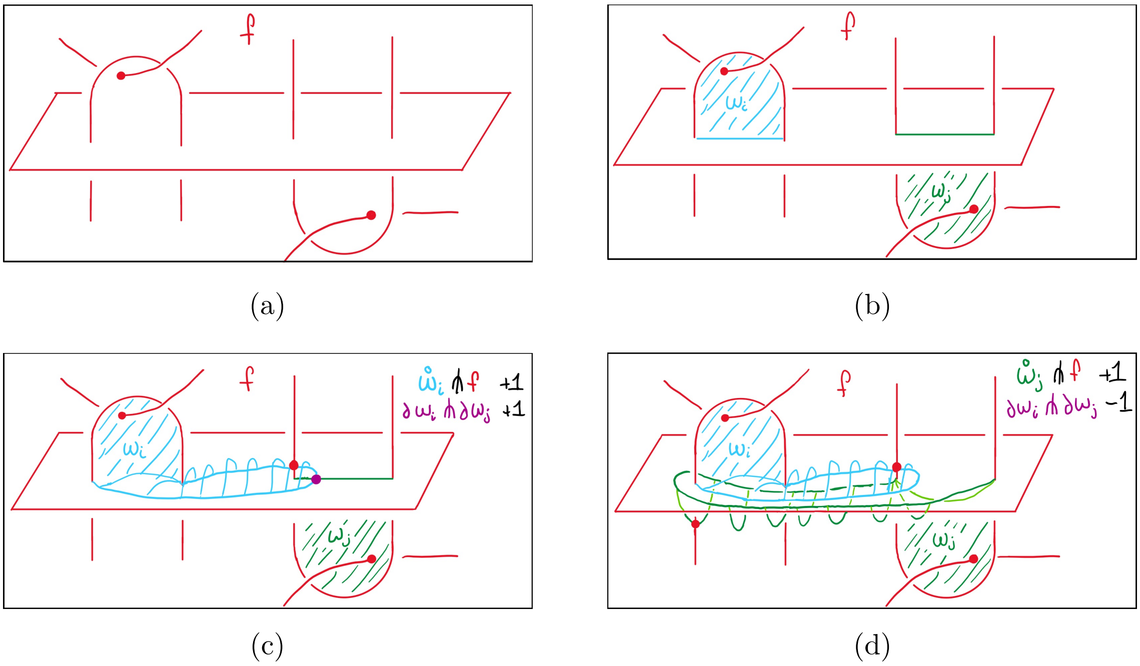

FIGURE 7. Transfer move

By the transfer move we see that we would be able to apply Proposition 2.1, if we could find $\{W_i\}$ with disjoint and embedded boundaries, such that $\mathrm{tw}(\partial W_i) = 0$ for all i, and $\sum_i |\mathring{W}_i \pitchfork f|$ is even.
[12]

St the end of this step of the proof, assume we have arranged for ... [13]

Step 4. *Add genus, do the band-fibre finger move twice to add in two new pairs of self-intersections of f paired by two new Whitney discs.*

[14]

Now we have changed f by a homotopy (but we still call the result f), so that the points of $f \pitchfork f$ are all paired up by Whitney discs $\{W_i\} \cup \{V_1, V_2\}$. By construction, these Whitney discs have trivial twisting number and embedded boundaries. We also know that

$$(\{\mathring{W}_i\} \cup \{\mathring{V}_1, \mathring{V}_2\}) \pitchfork f|$$

is a single point. By relabeling, we can assume that intersection is with W_1. The boundaries are also disjoint, except $\partial V_1 \pitchfork \partial V_2$ is a single point. Then perform the boundary pushoff operation, to trade the Whitney arc intersection for an intersection between V_1 and f. Now the entire set $\{\mathring{W}_i\} \cup \{\mathring{V}_1, \mathring{V}_2\}$ intersects twice with f. The transfer move applied to W_1 and V_1 arranges that each (new) W_1 and V_1 intersects f twice. Then we can tube into g to remove these intersections. Each of $\mathrm{tw}(\partial W_1)$ and $\mathrm{tw}(\partial V_1)$ changes by an even number, which we can fix by interior twisting, paying only the price of type 1 intersections.

Now we have finally arrived at a collection of Whitney discs for $f \pitchfork f$ satisfying the hypotheses of Proposition 2.1.

Step 5. *Apply Proposition 2.1.*

[12]Observe with examples that sometimes this is not possible. If it were, we would be able to always represent primitive classes in simply connected 4-manifolds by embedded spheres, but that is false.

[13]specify what we have at the end, basically there should be a single Whitney f intersection, and all other problems are solved (except type 1).

[14]Figure and explanation for the BFF move

As previously discussed, the Whitney move along the discs produced by Proposition 2.1 produces the desired locally flat embedded torus representing the class α. $\square$

We end this section by stating the more general theorems that were proven by Lee-Wilczyński [LW97, Theorem 1.1] and by myself with Kasprowski, Powell, and Teichner [KPRT22, Theorem 1.2]. For the statement below, we remark that an embedding is said to be *simple* if the fundamental group of the complement is abelian. The divisibility of a class $x \in H_2(N;\mathbb{Z})$ is the least integer d such that $x = dy$ for some $0 \neq y \in H_2(N;\mathbb{Z})$.

Theorem 2.2 ([LW97, Theorem 1.1])**.** *Let M be a compact, oriented, simply connected 4-manifold whose boundary is a disjoint and possibly empty union of integral homology spheres. Suppose $x \in H_2(M;\mathbb{Z})$ is a nonzero class of divisibility d. Then there exists a simple, topologically locally flat embedding $\Sigma \hookrightarrow M$ representing x by an oriented surface of genus $g > 0$ if and only if*

$$b_2(M) + 2g \geq \max_{0\leq j\leq d} \left| \sigma(M) - \frac{2j(d-j)}{d^2} x \cdot x \right|.$$

Note that Theorem 1.1 is the case of $d = 1$. This is a very powerful result, applicable in a variety of situations. There is a companion theorem [LW97, Theorem 1.2] providing one further obstruction in the genus zero case, given by the Kervaire–Milnor condition relating the intersection number $x \cdot x$ to the Kirby–Siebenmann invariant of M and the Rokhlin invariant of the boundary ∂M. However, the condition on the fundamental group of the complement is essential, as is the requirement to work in an ambient 4-manifold that is either closed or has boundary a disjoint union of homology spheres. Roughly speaking, this is required in the surgery-theoretic approach used by Lee–Wilczyński.

Now we state the result of [KPRT22].

Theorem 2.3 ([KPRT22, Theorem])**.** *We assume that M is a connected, topological 4-manifold and that Σ is a nonempty compact surface with connected components $\{\Sigma_i\}_{i=1}^m$. The notation $F = \{f_i\}_{i=1}^m \colon (\Sigma, \partial\Sigma) \looparrowright (M, \partial M)$ represents a generic immersion with components $f_i \colon (\Sigma_i, \partial\Sigma_i) \looparrowright (M, \partial M)$. Suppose that $\pi_1(M)$ is good and that F has algebraically dual spheres $G = \{[g_i]\}_{i=1}^m \subseteq \pi_2(M)$. Then the following statements are equivalent.*

- *(i) The intersection numbers $\lambda(f_i, f_j)$ for all $i < j$, the self-intersection numbers $\mu(f_i)$ for all i, and the Kervaire–Milnor invariant $\mathrm{km}(F) \in \mathbb{Z}/2$, all vanish.*
- *(ii) There is an embedding $\overline{F} = \{\overline{f}_i\}_{i=1}^m \colon (\Sigma, \partial\Sigma) \hookrightarrow (M, \partial M)$, regularly homotopic to F relative to $\partial\Sigma$, with geometrically dual spheres $\overline{G} = \{\overline{g}_i \colon S^2 \looparrowright M\}_{i=1}^m$ such that $[\overline{g}_i] = [g_i] \in \pi_2(M)$ for all i.*

A helpful fact about this theorem is that we can often force the Kervaire–Milnor invariant to be trivial, as we saw in the proof of Theorem 1.1. That is, we have the following corollary.

Corollary 2.4. *Let M is a 4-manifold with $\pi_1(M)$ good and let Σ is a connected, oriented surface with positive genus. Suppose we have a generic immersion $f \colon (\Sigma, \partial\Sigma) \looparrowright (M, \partial M)$ with vanishing self-intersection number and an algebraically dual sphere. Then f is regularly homotopic, relative to $\partial\Sigma$, to an embedding.*

In the case of $\pi_1(M)$ trivial, the self-intersection number of f is an integer, obtained by the signed count of the self-intersections. For more general fundamental groups, we have to use the equivariant self-intersection number mentioned in Remark 1.14.

3. Embedding surfaces using surgery theory

3.1. **Equivariant intersection and self-intersection numbers.** Let M be a connected, oriented, topological manifold, and choose a basepoint $* \in M$. Consider two generic immersions $f, g \colon S^2 \looparrowright M$, intersecting each other transversely. Choose a basepoint $\star \in S^2$ and an orientation for S^2. Choose paths $w_f, w_g \colon [0,1] \to M$ with $w_f(0) = w_g(0) = *$,

$w_f(1) = f(\star)$, and $w_g(1) = g(\star)$. These paths are called *whiskers* for f and g. Define the following sum

$$\lambda(f,g) := \sum_{p \in f \pitchfork g} \varepsilon_p \gamma_p,$$

where

- α_f^p is the image under f in M of a path in S^2 from $\star$ to the preimage of p under f;
- α^p is the image under g in M of a path in S^2 from $\star$ to the preimage of p under g;
- $\varepsilon_p \in \{\pm 1\}$ is the sign of the intersection point p; and
- γ_p is the element of $\pi_1(M, *)$ given by the concatenation

$$w_f \alpha_f^p (\alpha_g^p)^{-1} w_g^{-1}.$$

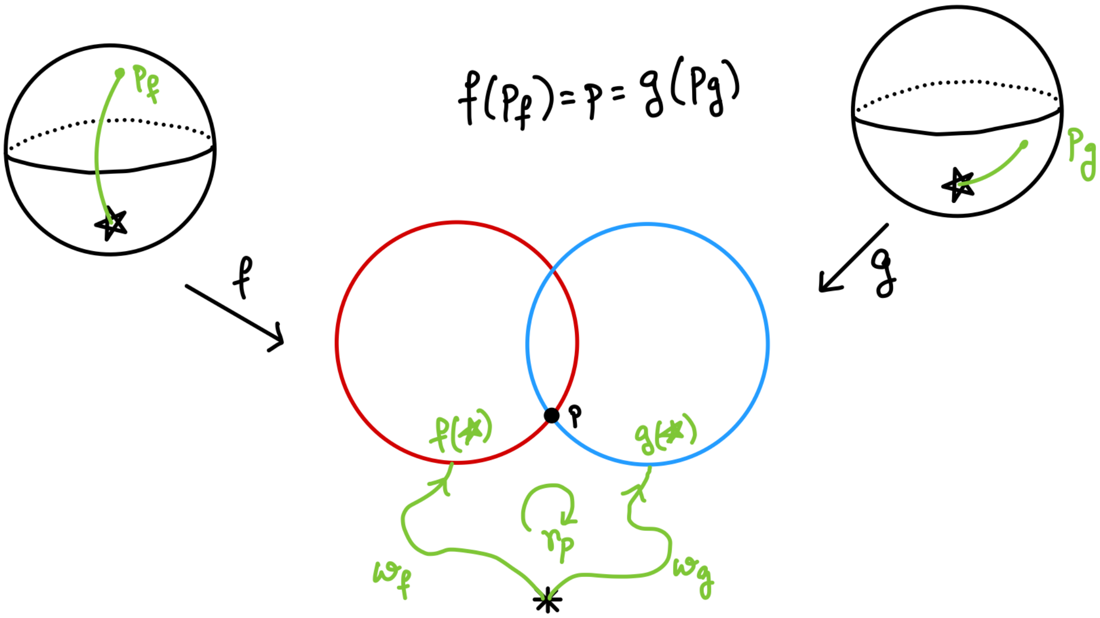

FIGURE 8. Defining the equivariant intersection number

Similarly, define

$$\mu(f) := \sum_{p \in f \pitchfork f} \varepsilon_p \gamma_p,$$

where

- α_1^p and α_2^p are images under f in M of paths in S^2 from $\star$ to the two distinct preimages of p;
- $\varepsilon_p \in \{\pm 1\}$ is the sign of the intersection point p; and
- γ_p is the element of $\pi_1(M, *)$ given by the concatenation

$$w_f \alpha_1^p (\alpha_2^p)^{-1} {w_f}^{-1}.$$

4. Isotopy of surfaces using surgery theory and the h-cobordism theorem

This will be lecture 4 (if time permits). Update: time did not permit :(but thankfully Daniel Hartman stepped in with a mini-talk on this result. Here are the live-typed notes from this.

Our goal is to prove Theorem 1.3. First we will need the following theorem.

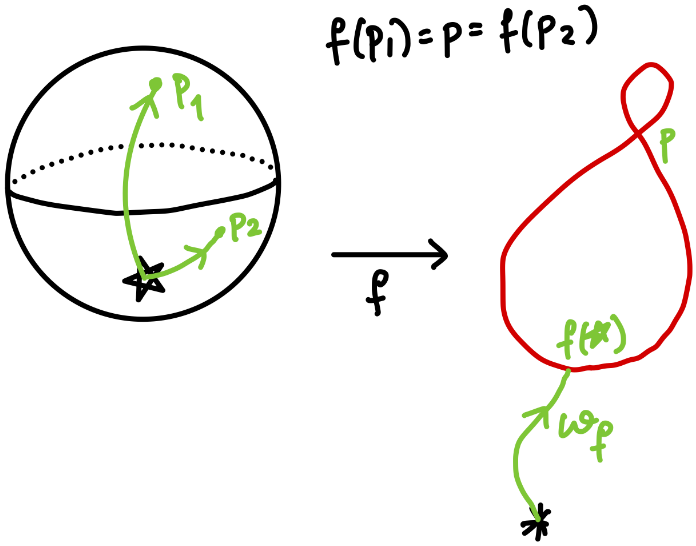

Figure 9. Defining the equivariant self-intersection number.

Theorem 4.1. *Let K be a 2-knot in S^4. If $(S^4 \setminus \mathring{\nu}(K), \partial) \approx (S^1 \times B^3, S^1 \times S^2)$, then K is topologically unknotted.*

We leave this as an exercise, with the following hint: How do you get a homeomorphism of S^4? What does Alexander say about those?

Want to show complements are homeomorphic.

Step 1: $(X, \partial) \xrightarrow{f} (S^1 \times B^3, S^1 \times S^2)$.

Exercise: show (X, ∂) is a homology $(S^1 \times B^3, S^1 \times S^2)$. If $\pi_1(X) \cong \mathbb{Z}$ then homotopy equivalence.

Suppose we have $f\colon (X, \partial) \to (S^1 \times B^3, S^1 \times S^2)$ such that $f|_{\partial X}$ is a homeomorphism.

In Aru's talk, she considered a class in $\Omega_3^{\text{spin}}(S^1)$, which gave us the initial 4-manifold V, which we were able to surgery to a homotopy equivalence. Now we are starting with a homotopy equivalence. In fact, we have two homotopy equivalences - f and the identity. We want to find an *h-cobordism* between these two homotopy equivalences.

Definition 4.2. A triple $(W; M_0, M_1)$ where W is a manifold with $\partial W = M_0 \sqcup M_1$, such that the inclusions $M_i \hookrightarrow W$ are homotopy equivalences.

Theorem 4.3 ([FQ90])**.** *If W is a 5-dimensional h-cobordism with fundamental group $\mathbb{Z}$, then any h-cobordism $(W; M_0, M_1)$ is homeomorphic to the product $M_0 \times [0, 1]$.*

This is an instance of a more general result called the s-cobordism theorem. But since the Whitehead group of $\mathbb{Z}$ is trivial, every h-cobordism with infinite cyclic fundamental group is already an s-cobordism.

Therefore by this theorem, if we are able to find an h-cobordism between the two homotopy equivalences, we will win.

We know that the two maps are cobordant: this is because they have the same signature [15]

In fact this is an bordism of manifolds with boundary. In other words, there exists W such that $\partial W = X \cup S^1 \times S^2 \times [0, 1] \cup S^1 \times B^3$, and there is a map $W \to S^1$. But this is not yet an h-cobordism. We will now modify this cobordism to make it into an h-cobordism as follows.

[15] but this is about bordism over a point. Why is there a bordism over $S^1 \times B^3$?

$$\Omega_5(S^1\times B^3, S^1\times S^2\times I\cup\dots) \to S_{TOP}(S^1\times B^3, S^1\times S^2) \to \Omega_4(S^1\times B^3, S^1\times S^2) \to L_4(\mathbb{Z}[\mathbb{Z}])$$

Here the set $S_{TOP}(S^1 \times B^3, S^1 \times S^2)$ is the *structure set*. The above sequence is called the surgery exact sequence.

The sequence tells us that the obstruction to modifying the given bordism to an h-cobordism lies in $L_5(\mathbb{Z}[\mathbb{Z}])$.

The goal is to find some loop in the current bordism mapping to the generator of $\pi_1(S^1 \times B^3 \times [0,1]$. We cut out a tubular neighbourhood of this loop, and then glue in $S^1 \times E8$, where as before $E8$ refers to the $E8$ manifold. This procedure is called circle summing. (Cut out a tubular neighbourhood of one of the S^1-fibres in $S^1 \times E8$, and glue that in). Here we need to know that $L_5(\mathbb{Z}[\mathbb{Z}]) \cong 8\mathbb{Z}$ via something called Shaneson splitting, so this circle summing process can be used to kill the obstruction.

5. Exercises

5.1. Introductory problems.

Exercise △ 5.1.1. Give an example of a surface in a 4-manifold which is topologically embedded (i.e. there is a continuous map $f\colon \Sigma_g \hookrightarrow M$ where Σ_g is some closed surface, M is some 4-manifold, and f is a homeomorphism onto its image), but not locally flatly embedded.

Hint: Given a knot $K \subseteq S^3$, consider the disc given by $\operatorname{cone}(K) \subseteq \operatorname{cone}(S^3) = D^4$. When is this disc locally flat? It might be useful to remember the fact (without proof) from classical knot theory that a knot K is the unknot if and only if $\pi_1(S^3 \setminus K) \cong \mathbb{Z}$.

Exercise △ 5.1.2. Convince yourself that every smooth embedding of a surface in a smooth 4-manifold is locally flat. Remind yourself of the smooth analogues of Theorems 1.5 and 1.11 and the ideas of their proofs. Without going into the details, consider why those proofs fail in the purely topological setting.

Exercise △ 5.1.3. Let M be a 4-manifold. Suppose we have a generically immersed surface $\Sigma \looparrowright M$ with a geometrically dual sphere, i.e. there is some $g\colon S^2 \looparrowright M$ such that $f \pitchfork g$ is a single transverse point. Show that inclusion induces an isomorphism

$$\pi_1(M \setminus \nu\Sigma) \cong \pi_1(M), \tag{5.1}$$

where νf is the normal bundle of f.

A surface Σ satisfying (5.1) is said to be π_1-*negligible*.

Exercise △ 5.1.4. Let $C\colon S^1 \hookrightarrow M$ be an embedded, orientation preserving loop in a 4-manifold. The procedure of *surgery on M along C* is as follows. Choose a tubular neighbourhood of C, call it $\nu C \cong S^1 \times D^3$. Cut out the interior $\mathring{\nu} C$, and glue in $D^2 \times S^2$, via the identity map along the boundary $S^1 \times S^2$. There are two possible identifications of $\partial\nu C$ with $S^1 \times S^2$, and therefore there are two possible gluing maps.

Suppose we have a map $X \to Y$ of 4-manifolds, such that the induced map on fundamental groups is a surjection. Use surgery on circles in X to change X to some X' with a map to Y inducing an isomorphism on fundamental groups.

5.2. Moderate problems.

Exercise □ 5.2.1. Consider $\mathbb{R}^4$. The xy- and zw-planes intersect transversely at the origin. Let B denote the 4-ball of unit radius at the origin. Show that the intersection of the xy- and zw-planes with $\partial B = S^3$ is the Hopf link.

Suppose that two surfaces f and g in a 4-manifold M intersect transversely at a point $p \in M$. Let $B \subseteq M$ be a small 4-ball at p. Conclude that one can choose B to be small enough so that $\partial B \cap (f \cup g)$ is a Hopf link in ∂B.

Exercise □ 5.2.2. Show the following:

1. Let $f\colon S^2 \looparrowright M$ be a generically immersed sphere in a 4-manifold. Inserting a local trivial double point in f with sign ± 1 changes the euler number of the normal bundle by ∓ 2.
2. Let W be a generically immersed Whitney disc pairing intersections between generically immersed spheres $f, g\colon S^2 \to M$ in a 4-manifold. A single boundary twist changes the twisting number of ∂W by ± 1.

Hint: In both cases, a well-drawn picture could be the answer.

Exercise □ 5.2.3. Consider the equivariant intersection and self-intersection numbers defined in Section 3.1

1. What is the effect on $\lambda(f, g)$ of
   - changing the paths α_f^p and α_g^p?
   - changing the whiskers w_f and w_g?
   - changing the basepoint $*$? (How might you get new whiskers?)

(2) What is the effect on $\mu(f)$ of
   – changing the paths α_1^p and α_2^p?
   – changing the whisker w_f?
   – changing the basepoint $*$? (How might you get a new whisker?)

(3) Conclude there is a well-defined *equivariant intersection number*

$$\begin{aligned}\lambda\colon \pi_2(M) \times \pi_2(M) &\longrightarrow \mathbb{Z}[\pi_1(M)] \\ (f, g) &\longmapsto \lambda(f, g)\end{aligned}$$

(4) As above, try to define the self-intersection number. What should be the domain and codomain? *Hint:* Was there any ambiguity in the definition of $\mu(f)$? Can we change the value of $\mu(f)$ by changing f by a homotopy? (Which homotopies of surfaces in a 4-manifold have we seen in the lectures?) Recall that, generically, a homotopy of surfaces in a 4-manifold is some sequence of isotopies, cusp homotopies, finger moves, and Whitney moves.

Exercise □ 5.2.4. Let f and g be generically immersed spheres in some connected, oriented 4-manifold M. Assume we have chosen a basepoint in M and whiskers for f and g. Show the following.

(1) $\lambda(f, g) = 0$ if and only if all the intersections of f and g can be paired up by immersed Whitney discs in M, with trivial twisting number and disjointly embedded boundaries.

(2) $\mu(f) = 0$ if and only if the self-intersections of f can be paired up by immersed Whitney discs in M, with trivial twisting number and disjointly embedded boundaries.

Exercise □ 5.2.5. Let M be a simply connected 4-manifold, and let $S \subseteq M$ be an embedded sphere with trivial normal bundle. Let M' denote the result of surgery on M along S.

(1) What can you say about the fundamental group of M'?
(2) Can you think of a condition on S to ensure that M' is simply connected?
(3) Find an example of S and M such that M' is simply connected.

Bonus questions: Find an example of S and M such that M' is not simply connected. Find an example of S and M such that $\pi_1(M')$ is nontrivial but $H_1(M;\mathbb{Z})$ is trivial.

Exercise □ 5.2.6. Let Y be an integer homology sphere. Freedman showed that $Y = \partial C$ for some compact, contractible 4-manifold C, which we call the *Freedman filling* of Y. Use the h-cobordism theorem to prove that the Freedman filling of a given integer homology sphere is unique up to homeomorphism.

Use the same idea to show that given any compact, contractible 4-manifold C, every self-homeomorphism $f\colon \partial C \to \partial C$ extends to a homeomorphism of C.

5.3. Challenge problems.

Exercise ○ 5.3.1. Prove the *geometric Casson lemma*: Let f and g be transverse generic immersions of compact surfaces in a connected 4-manifold M. Assume that the intersection points $\{p, q\} \subset f \pitchfork g$ are paired by an immersed Whitney disc W. Then there is a regular homotopy from $f \cup g$ to $\overline{f} \cup \overline{g}$ such that $\overline{f} \pitchfork \overline{g} = (f \pitchfork g) \setminus \{p, q\}$, that is the two paired intersections have been removed.

The regular homotopy may create many new self-intersections of f and g; however, these are algebraically cancelling. Moreover, the regular homotopy is supported in a small neighbourhood of W.

A regular homotopy, by definition, is a sequence of isotopies, finger moves, and Whitney moves.

Exercise ○ 5.3.2. Use the disc embedding theorem to prove the *sphere embedding theorem*: Let M be a topological 4-manifold. Suppose there is $f\colon S^2 \looparrowright M$ with trivial self-intersection number, and an algebraically dual sphere $g\colon S^2 \looparrowright M$ with trivial normal bundle. Show that f is homotopic to $\overline{f}$, a locally flat embedding, with a geometrically dual sphere $\overline{g}$.

Exercise ◯ 5.3.3. Prove the 0-surgery characterisation of sliceness: A knot $K \subseteq S^3$ is topologically slice if and only if the 0-framed Dehn surgery $S^3_0(K) = \partial W^4$ for some compact, connected 4-manifold W such that

(1) the inclusion induced map $\mathbb{Z} \cong H_1(S^3_0(K);\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is an isomorphism;
(2) $\pi_1(W)$ is normally generated by the meridian μ_K of K, considered to lie in $S^3_0(K)$; and
(3) $H_2(W;\mathbb{Z}) = 0$.

Hint: At some point you will probably need the 4-dimensional topological Poincaré conjecture: a homotopy 4-ball with boundary S^3 is homeomorphic to B^4.

6. Solutions

Please send me typed solutions to the exercises and I will incorporate them (crediting you) into these lecture notes. We can draw the movie picture of the xy and zw planes as in Figure 1.

6.1. Introductory problems.

Solution △ 5.1.1. TBD

Solution △ 5.1.2. TBD

Solution △ 5.1.3. TBD

Solution △ 5.1.4. TBD

6.2. Moderate problems.

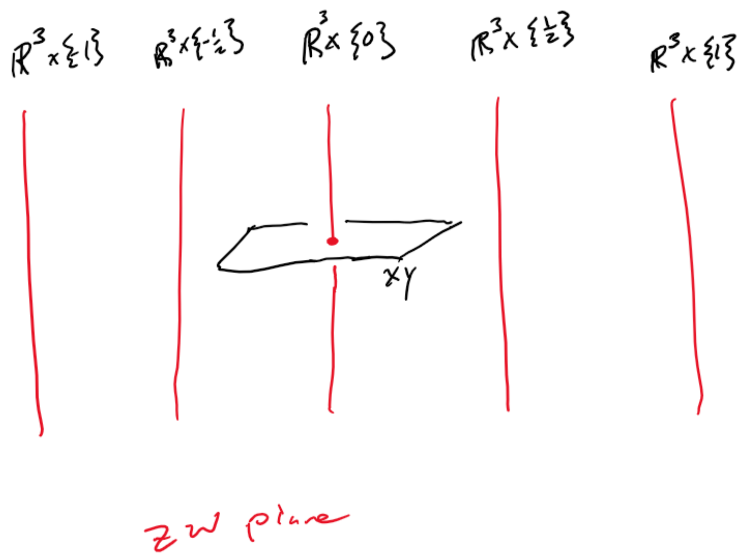

Figure 10. The movie picture of xy and zw planes

Solution □ 5.2.1 (Solution by Malcolm Gabbard and Jesse Osnes)**.** Further, drawing $\partial B^4 = S^3$ in this picture (in green), we have Figure 2.

Note, this is showing S^3 as the suspension of S^2. Also note, the intersection of the xy and zw planes and S^3 are seen as points and a circle in our movie picture. This is drawn in Figure 3. The circle (in dark blue) lies in entirely in the xy plane, and the points (in light blue) are entirely in the zw plane. Further, the light blue points are describing an S^1 as the suspension of S^0 via the movie picture. Denote the circle in the xy plane as S^1_{xy} and the circle in the zw plane as S^1_{zw}. We now show that they are indeed the Hopf Link.

To see that the circles are the Hopf Link, first remove a neighborhood of $S^1_{zw} \subset S^3$. The result is homotopic to $A \times [-1, 1]$, where A is an annulus and where $[-1, 1]$ is in the w direction. We now can deformation retract along the w direction to get just the annulus A at time 0. This can further be contracted to exactly the circle in the xy plane. This is showing that S^1_{xy} is the longitude of the solid torus $S^3 \backslash \nu(S^1_{zw})$. Thus, we have that these circles are indeed the Hopf link.

For the second paragraph, suppose two surfaces f and g in a 4-manifold M intersect transversely at a point $p \in M$. Let $B \subset M$ be a small 4-ball at p. Then, for small enough B, $\partial B \cap (f \cup g)$ is a Hopf Link in ∂B. The claim follows from the above argument because locally f and g behave the same as the xy and zw planes.

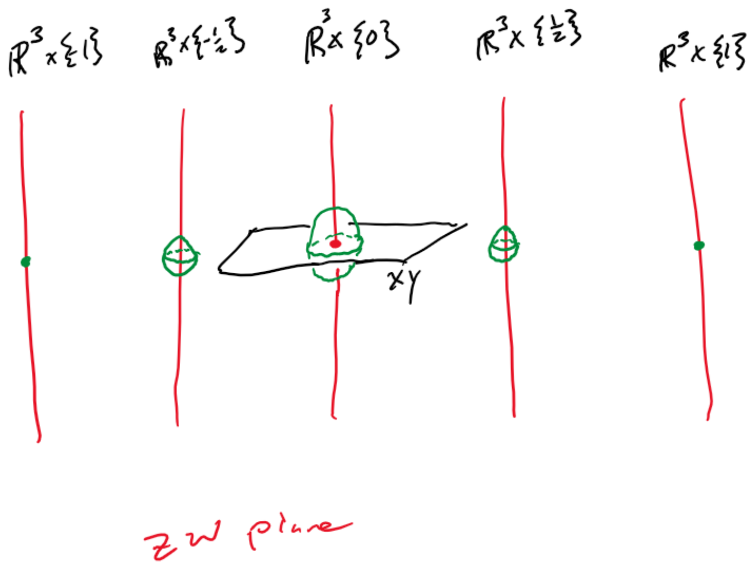

FIGURE 11. The movie picture with S^3 included

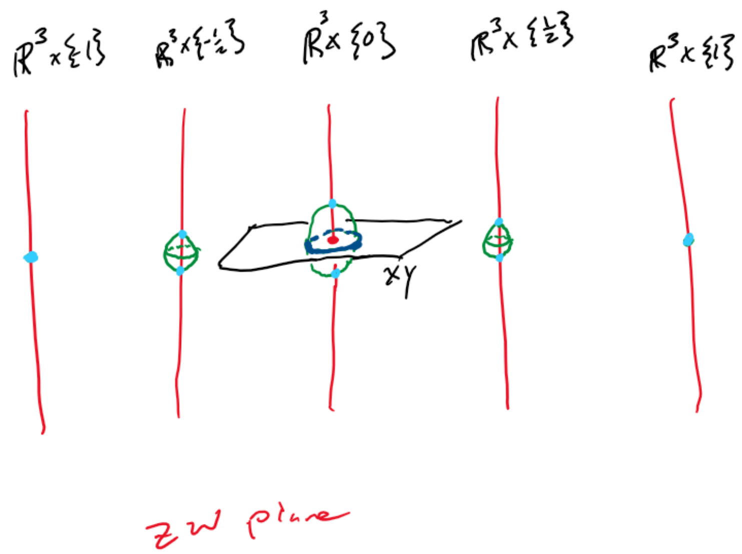

FIGURE 12. Highlighting the intersection of the planes and S^3.

Solution □ **5.2.2.** TBD
Solution □ **5.2.3.** TBD
Solution □ **5.2.4.** TBD
Solution □ **5.2.5.** TBD
Solution □ **5.2.6.** TBD

6.3. **Challenge problems.**
Solution ◯ **5.3.1.** TBD
Solution ◯ **5.3.2.** TBD
Solution ◯ **5.3.3.** TBD

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