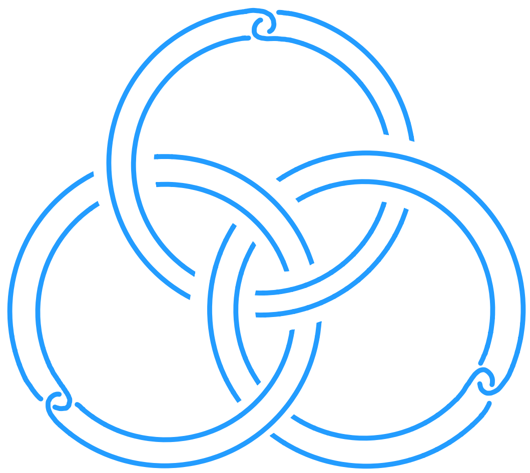
February 20, 2024

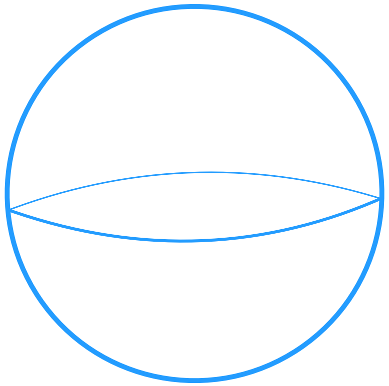
University of Melbourne

Knots, links, and 4-manifolds



Manifolds

- Manifolds are spaces which are locally homeomorphic to Euclidean space



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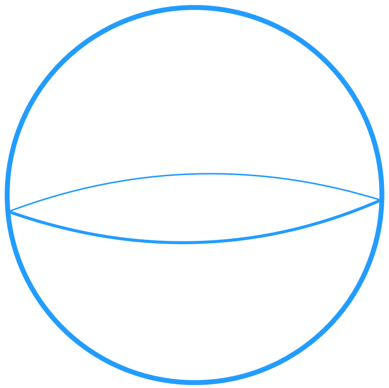

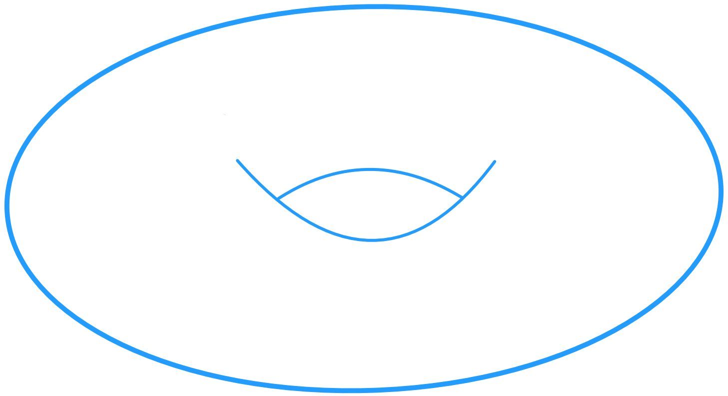

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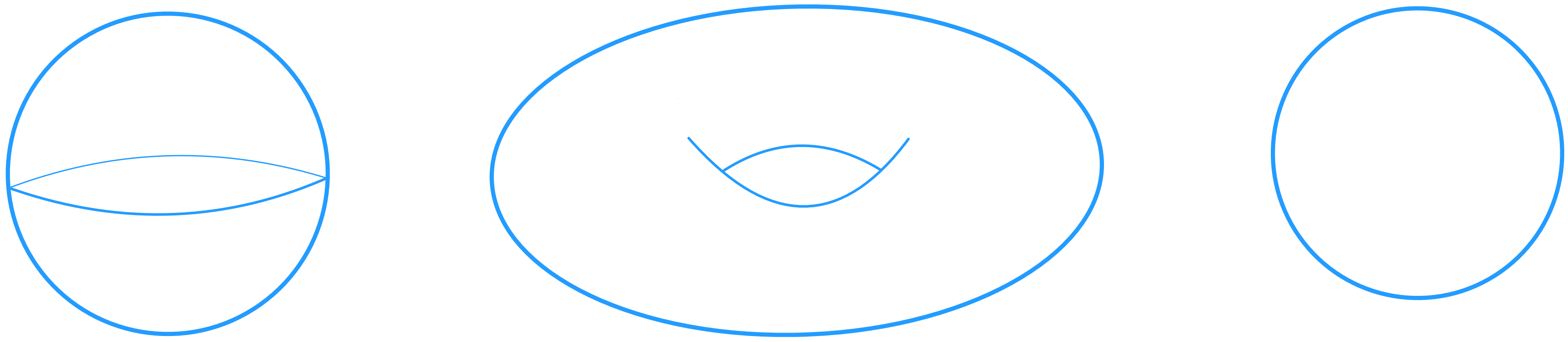

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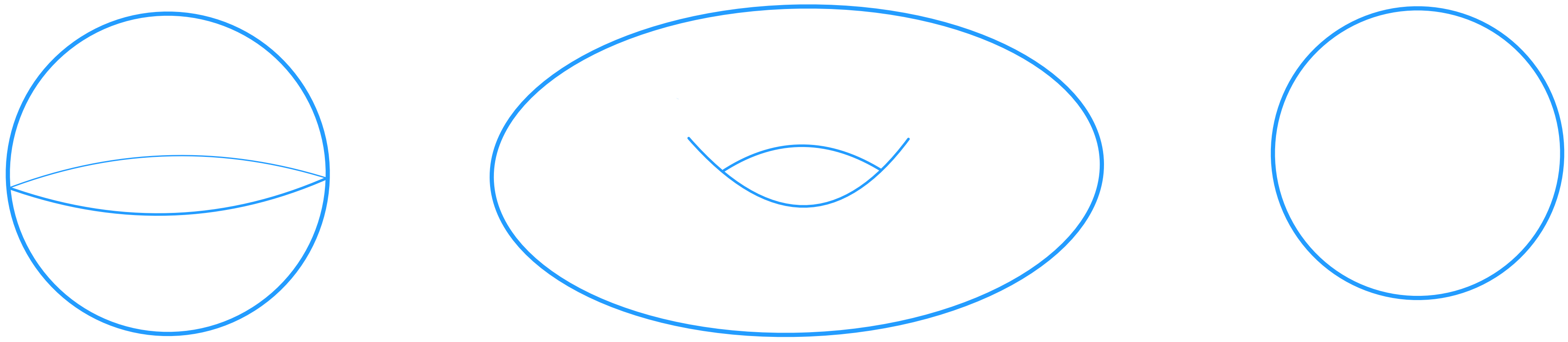

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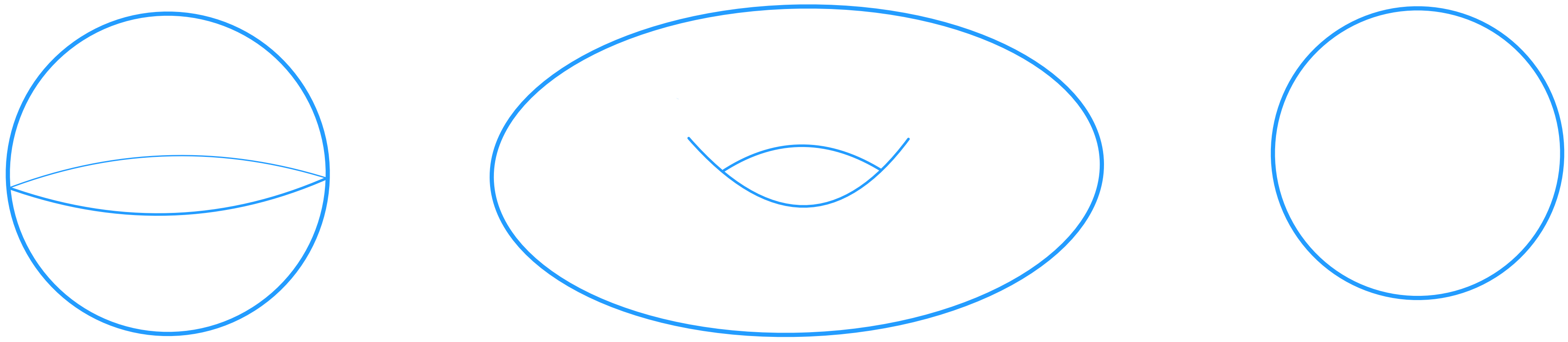

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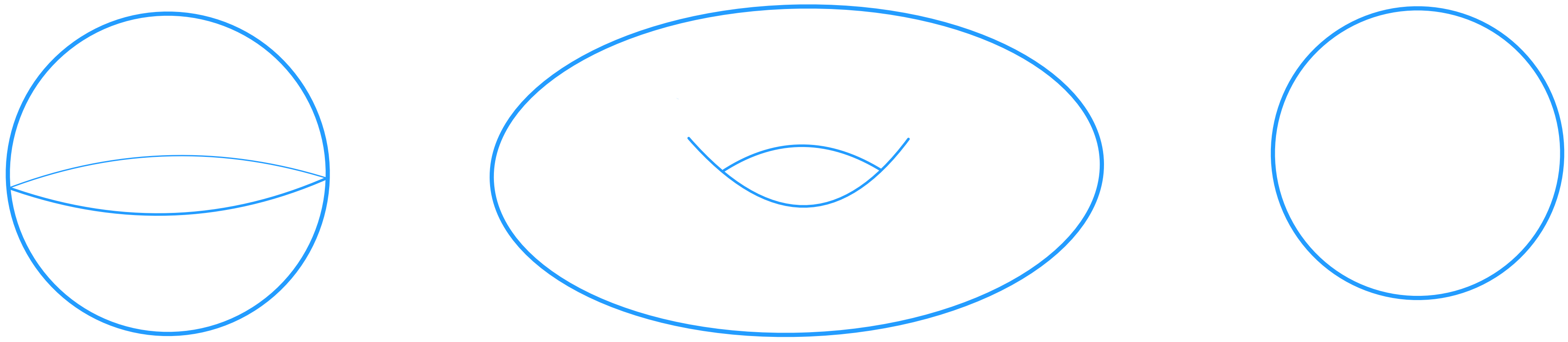

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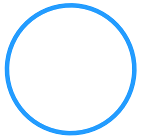
- Space-time appears to be a 4-dimensional manifold
- The n-dimensional sphere is given by $\{(x_1, x_2, \dots, x_{n+1}) \mid x_1^2 + x_2^2 + \dots + x_{n+1}^2 = 1\}$

Classifying manifolds (compact, connected, orientable, empty boundary)

- Dimension 1: the circle



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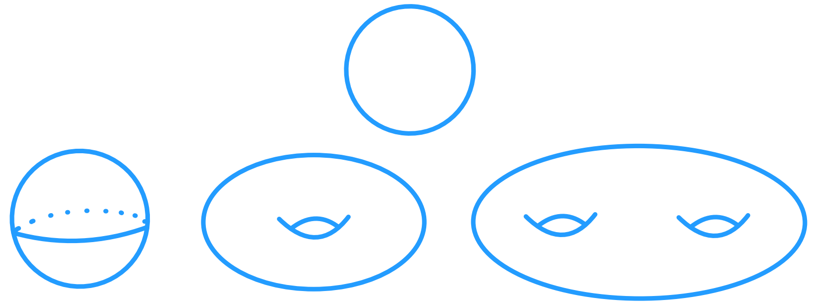
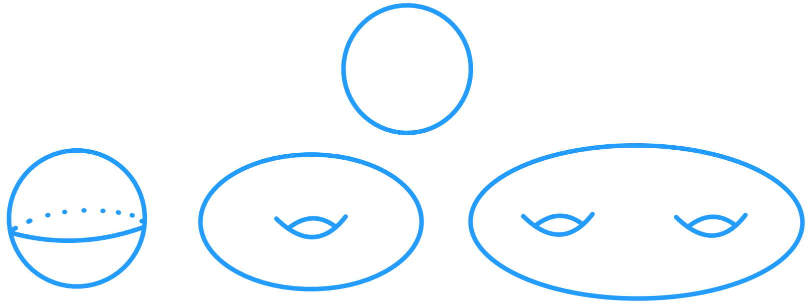
- Dimension 2: classified by genus



Classifying manifolds (compact, connected, orientable, empty boundary)

- Dimension 1: the circle
- Dimension 2: classified by genus
- Dimension 3: decompose into geometric pieces



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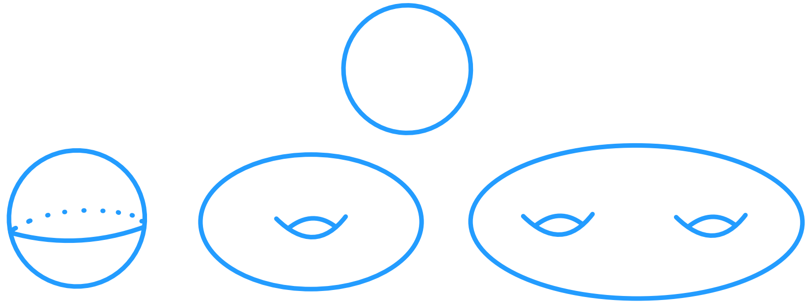

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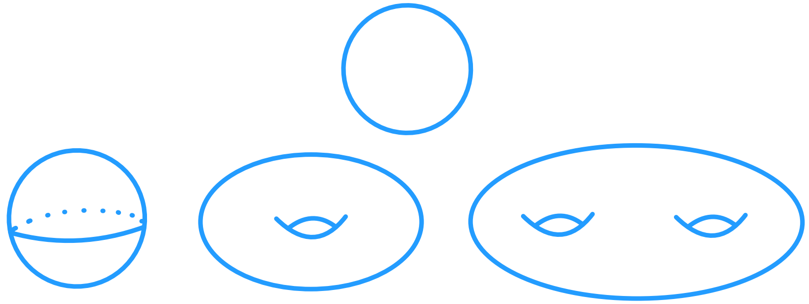

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controlled by fundamental group, not all groups realised

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4-dimensional manifolds display unique behaviour

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- Closed, simply connected 4-manifolds are classified up to homeomorphism
- Topological surgery theory and the s-cobordism theorem hold for some fundamental groups

Constructing a non smoothable 4-manifold

- Given closed, oriented M^4, we have the symmetric, bilinear, unimodular intersection form $Q_X : H_2(M;\mathbb{Z})/_{\text{torsion}} \times H_2(M;\mathbb{Z})/_{\text{torsion}} \to \mathbb{Z}$

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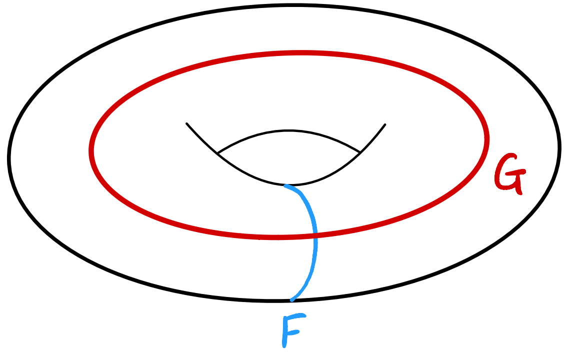

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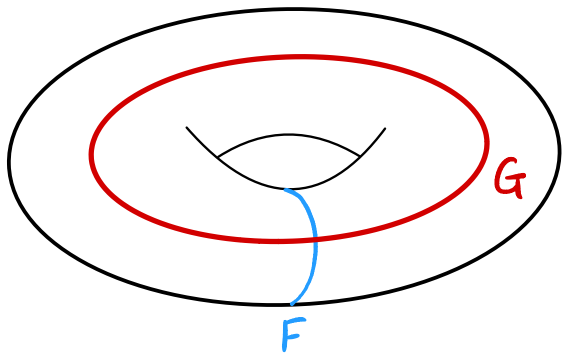
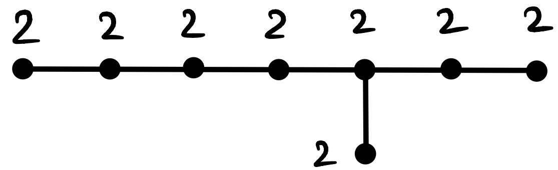


- Let $E8 \cong \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & 0 & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & 0 & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}$



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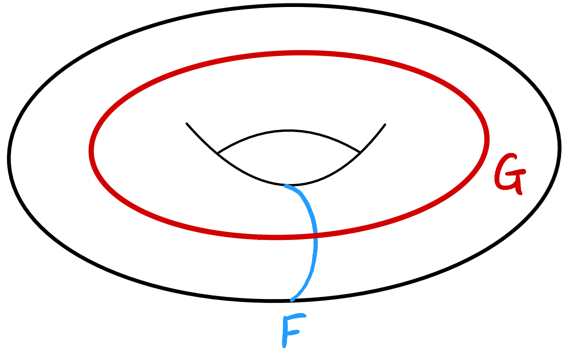

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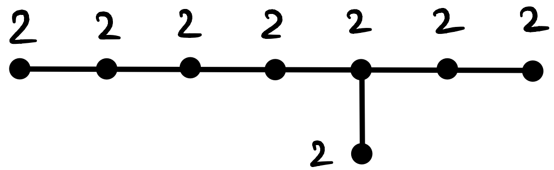

- E8 and $E8 \oplus E8$ are realised as intersection forms of nonsmoothable 4-manifolds

Realising algebra by topology

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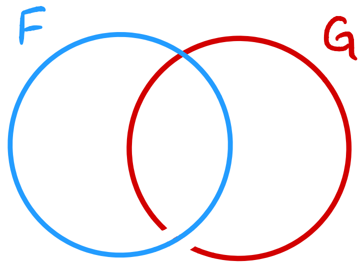
Dream



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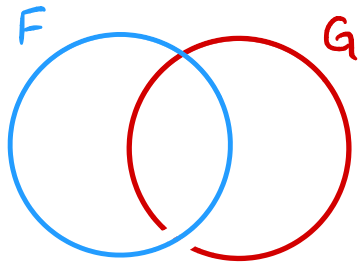

$$\Big\downarrow \text{Surgery}$$

X

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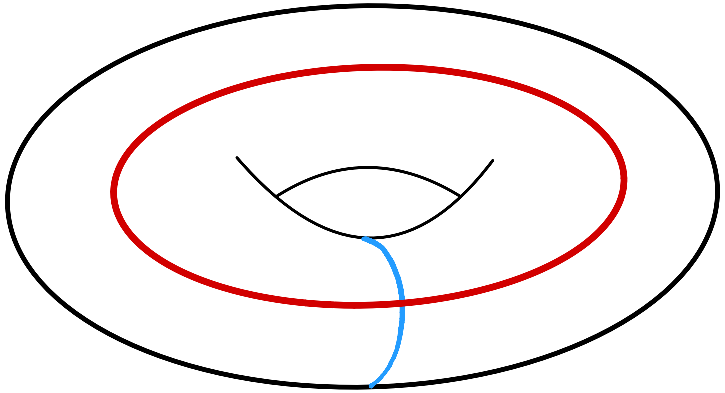

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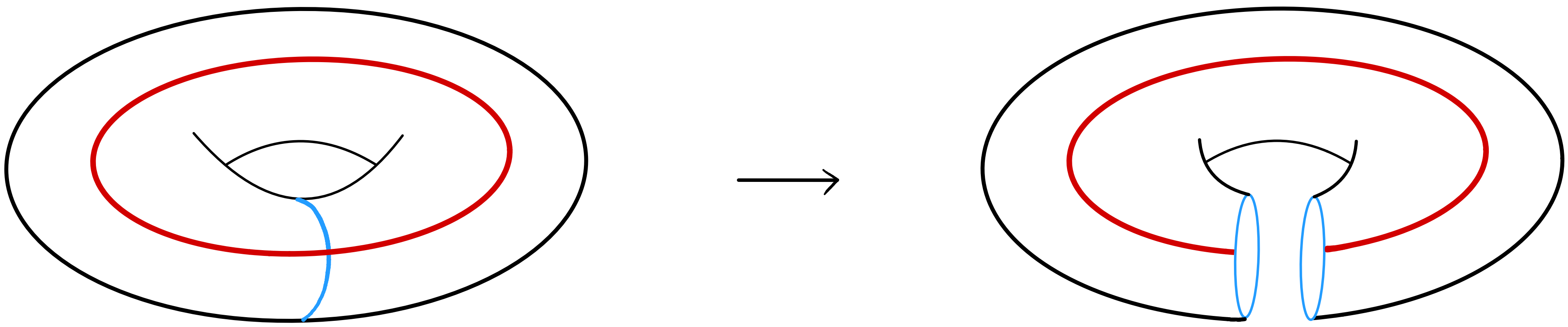

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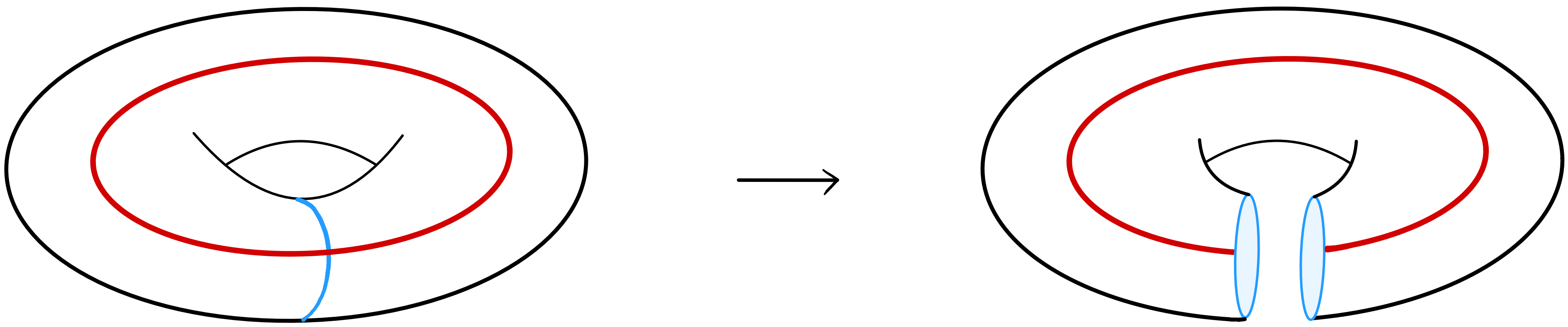

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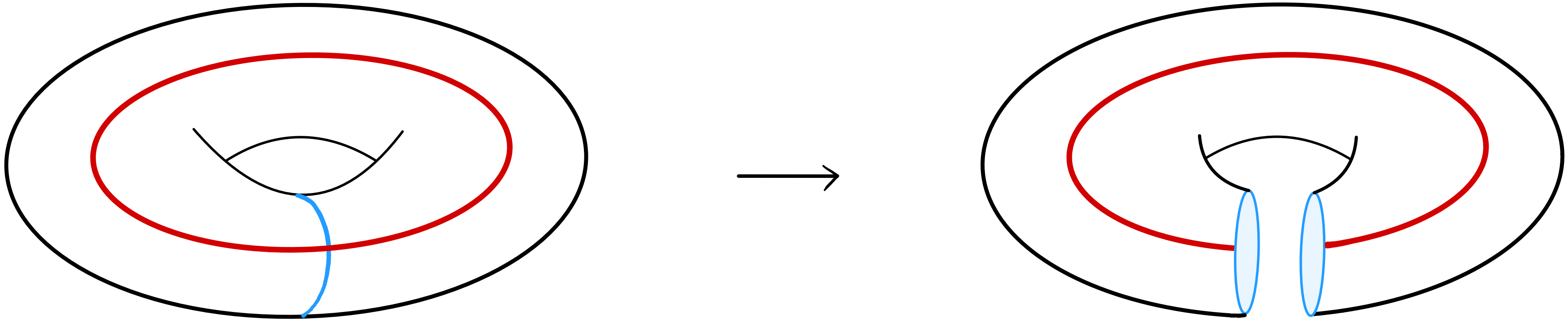
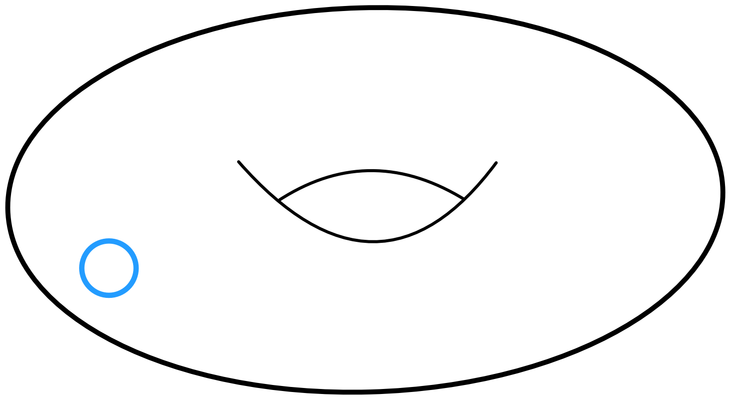


- The geometrically dual circle ensures the result is less complex



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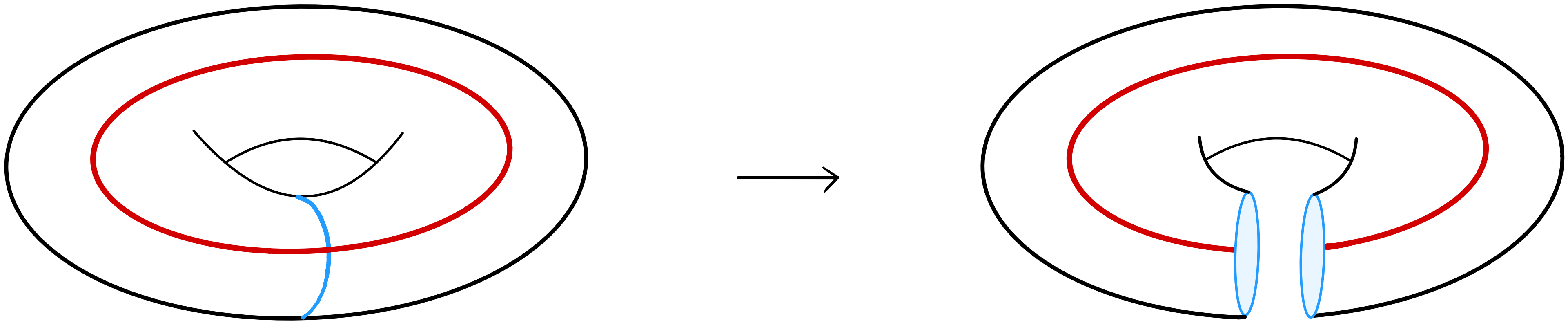

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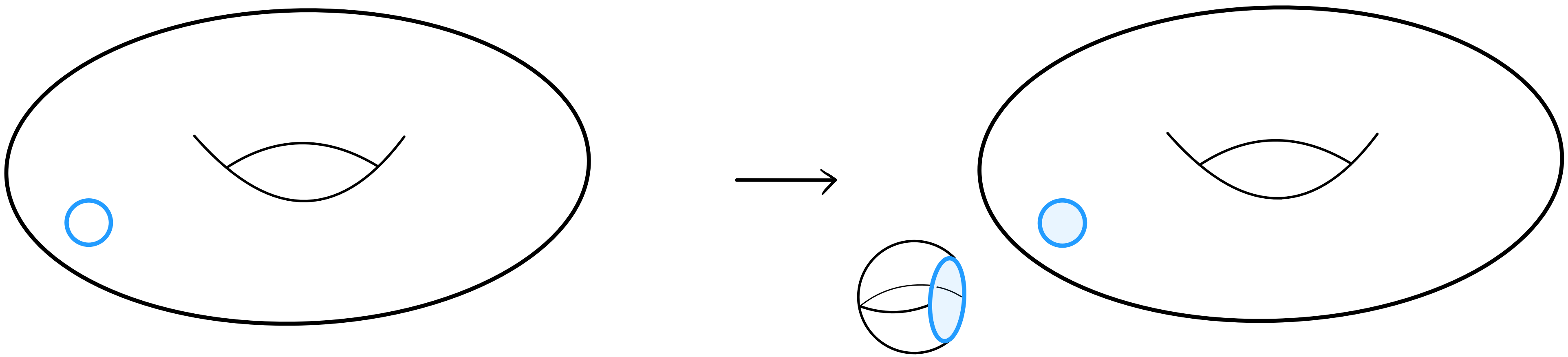

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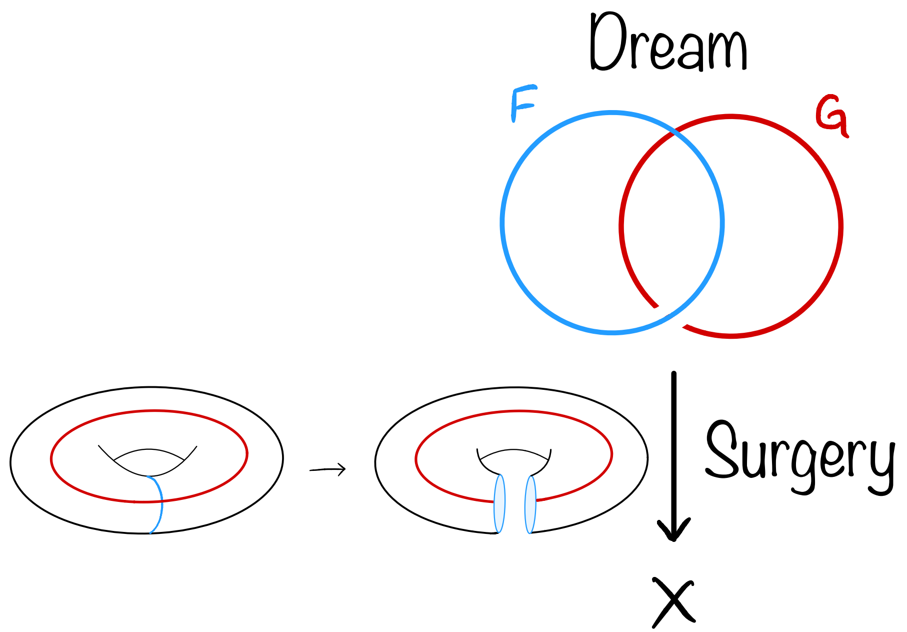

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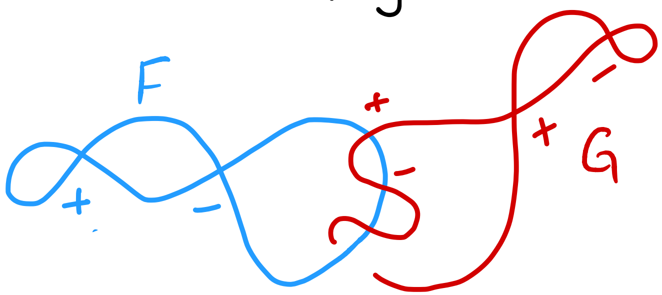
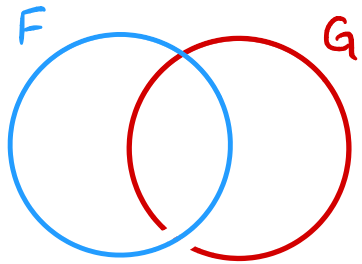
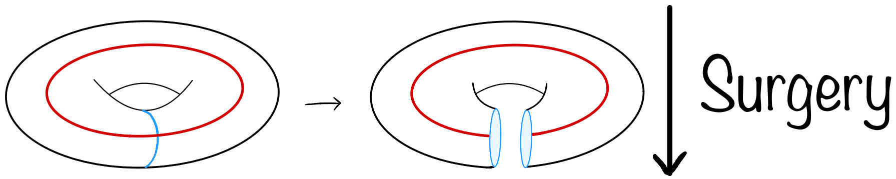
Reality



Dream





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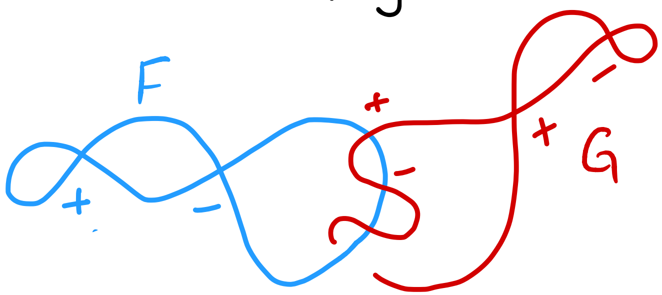

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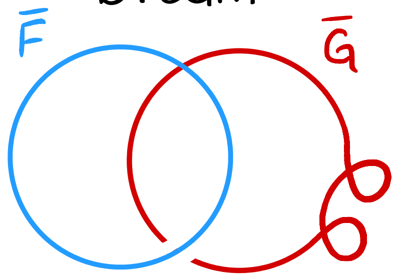

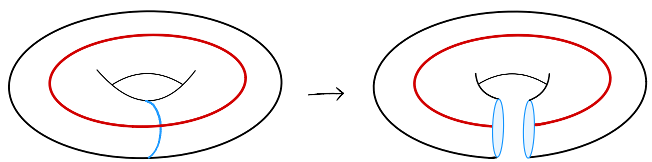

$\big\downarrow$ Surgery

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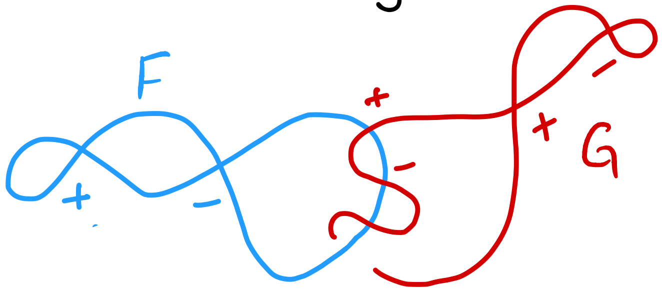
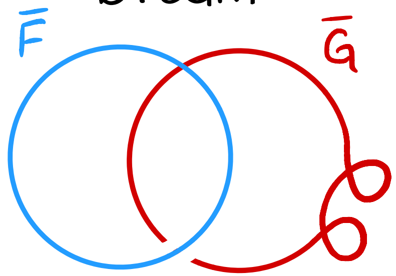
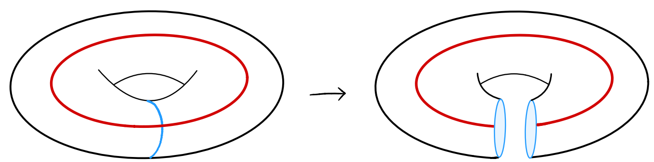


$\xrightarrow{\exists?\ \text{homotopy}}$

Dream





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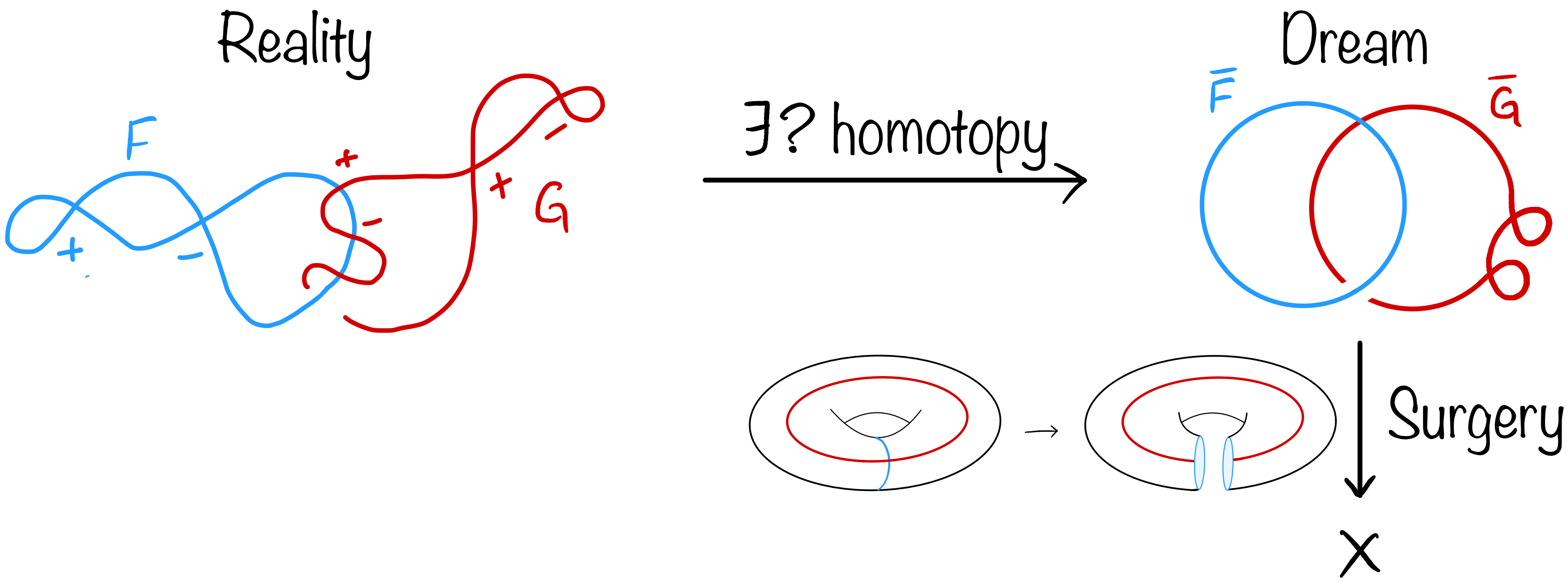

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- Enriques surface: $K3/_{\text{involution}}$. $Q_{Enr} = E8 \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Realising algebra by topology

Theorem [Casson 1970s, Freedman 1982, Freedman-Quinn 1990]

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- Topological h- and s-cobordism theorems (for good fundamental groups)
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  - If the form is even, there is precisely one such 4-manifold
  - If the form is odd, there are precisely two such 4-manifolds, distinguished by the Kirby-Siebenmann invariant

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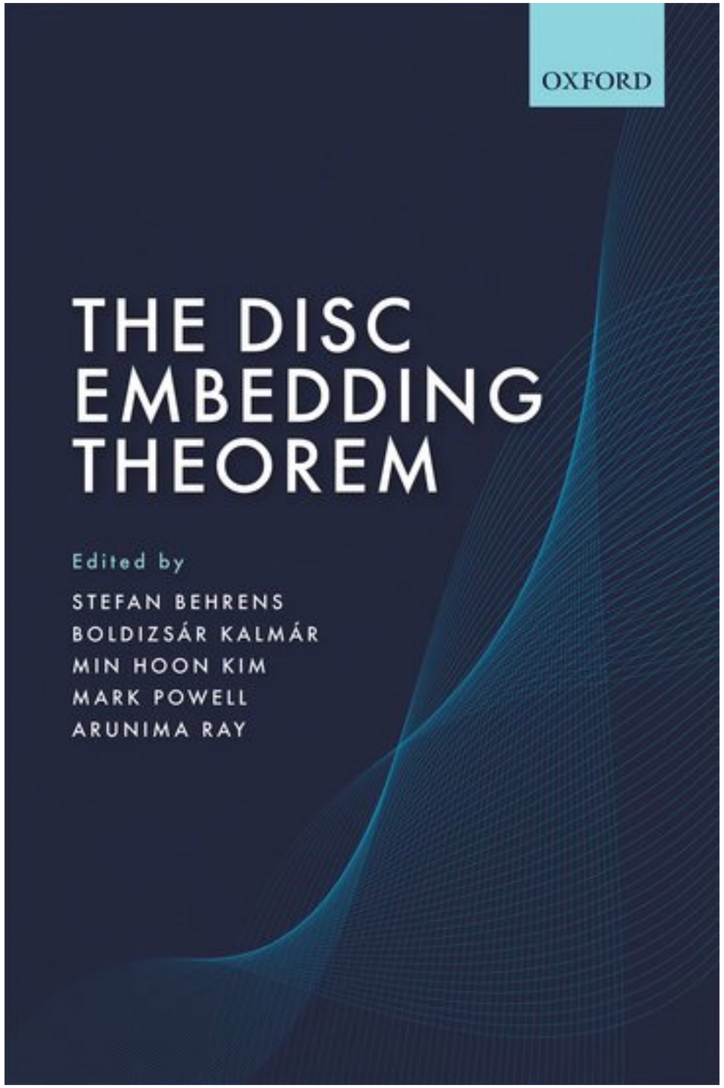
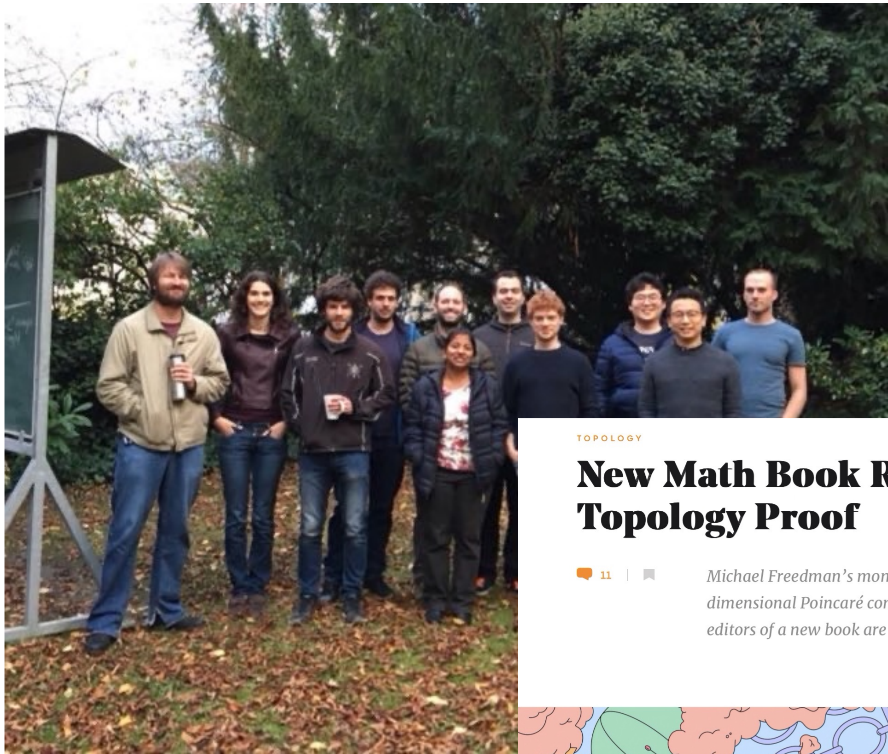
The disc embedding theorem



- Published 2021, Oxford University Press



TOPOLOGY

New Math Book Rescues Landmark Topology Proof

11

Michael Freedman's momentous 1981 proof of the four-dimensional Poincaré conjecture was on the verge of being lost. The editors of a new book are trying to save it.

Grace Park for Quanta Magazine

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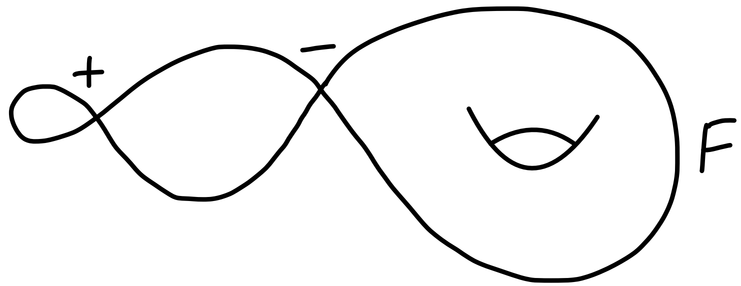
Then F is homotopic to a locally flat embedding $\bar{F}$, with a geometrically dual sphere $\bar{G}$ if and only if the Kervaire-Milnor invariant $km(F) \in \mathbb{Z}/2$ is trivial.

The Kervaire-Milnor invariant

- Assume that F has trivial algebraic self-intersection number



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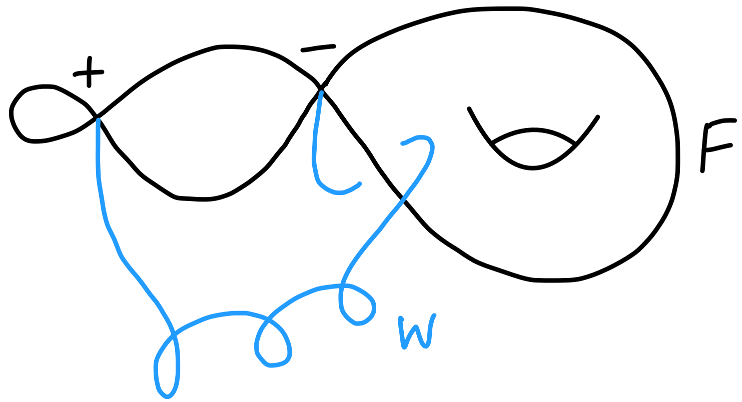

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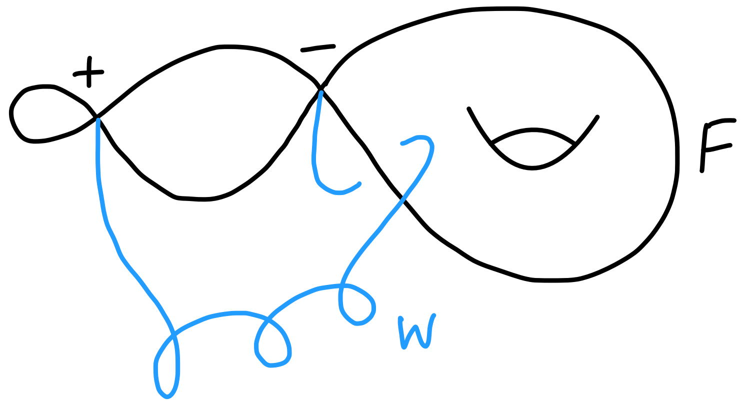

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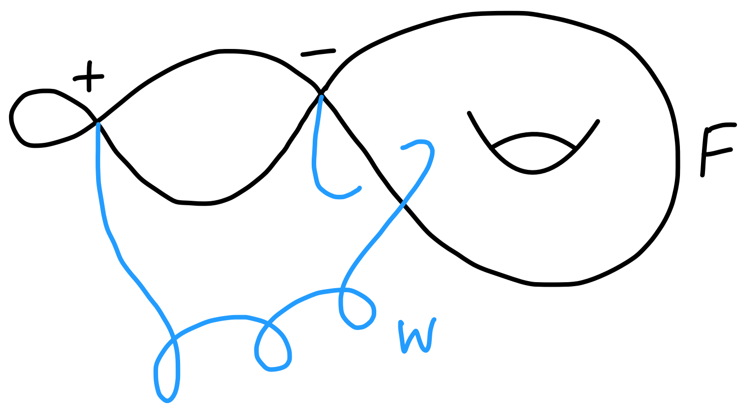

- Then the self-intersections are in isolated double points, paired by Whitney discs
- $km(F, \{W\}) = \sum |\mathrm{Int}(W) \pitchfork F| \mod 2$
- When is this independent of $\{W\}$?

Primitive homology classes have genus one

Corollary [Kasprowski-Powell-R.-Teichner 2022, Lee-Wilczyński 1997]

Every primitive class in $H_2(M;\mathbb{Z})$ for M^4 simply connected is represented by a locally flat embedded torus

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- The analogue does not hold for smooth embeddings
- cf. the minimal genus problem

Embedding spheres in knot traces

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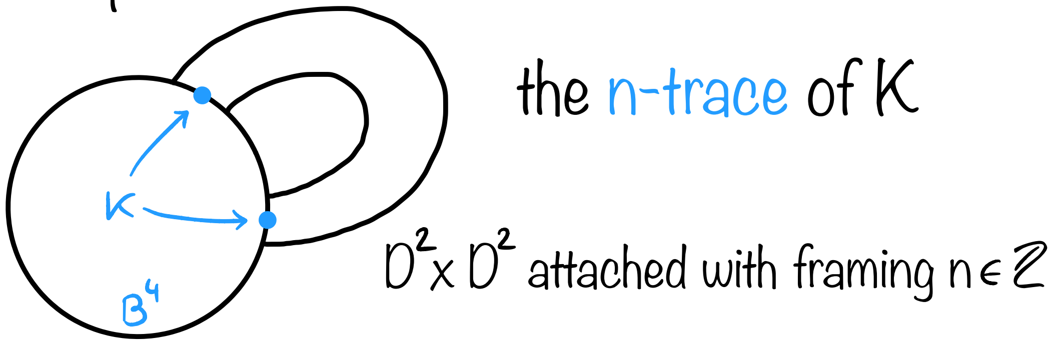
E.g. let $K \subset S^3$; then $X_n(K) :=$



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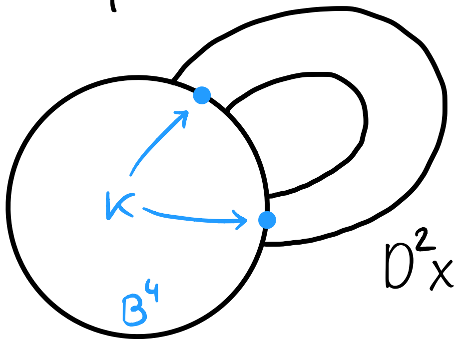


the n-trace of K

$D^2 \times D^2$ attached with framing $n \in \mathbb{Z}$

Theorem [Feller-Miller-Nagel-Orson-Powell-R. 2021]

Fix $n \in \mathbb{Z}$, and $K \subset S^3$ a knot. A generator of $\pi_2(X_n(K); \mathbb{Z}) = \mathbb{Z}$ is represented by a locally flat embedded sphere S such that $\pi_1(X_n(K) \setminus S)$ is abelian

if and only if

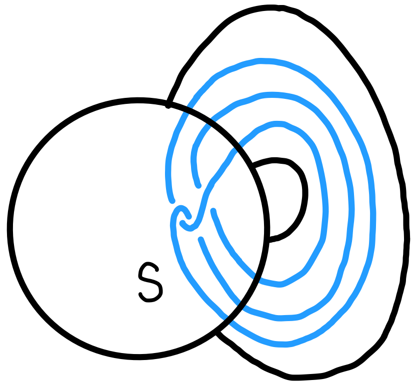
(i) $\mathrm{Arf}(K) = 0$

(ii) $\Delta_K(t) \doteq 1$, if $n = 0$
$\prod_{\xi^n = 1} \Delta_K(\xi) = 1$, if $n \neq 0$

(iii) $\sigma_K(\xi) = 0$, for all $\xi \in S^1$ such that $\xi^n = 1$

Embedding spheres in knot traces

- Step 1: topological surgery to build a homology cobordism between $S^3_n(K)$ and $L(n,1)$



- Step 2: apply a classification result [Freedman, Boyer]

- Step 3: final check using a Kervaire-Milnor invariant

Classifying manifolds up to homotopy equivalence

Theorem [Hillman-Kasprowski-Powell-R. 2024+]

Let M and N be closed, oriented 4-manifolds with fundamental group

(i) a finite free product of cyclic groups;

(ii) $\mathbb{Z} \times \mathbb{Z}/2$; or

(iii) a torsion-free 3-manifold group.

Then M and N are homotopy equivalent if and only if they have isomorphic quadratic 2-types, i.e. the same π_2, k-invariant, and equivariant intersection form.

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- [Baues-Bleile] The homotopy type is determined by the image of the fundamental class in $H_4(B;\mathbb{Z})$, where B is the Postnikov 2-type
- We generalise a criterion of Hambleton-Kreck from finite to infinite groups, showing when this image is determined by the intersection form

Further results in dimension four

- Conditions allowing surgery without ambient good fundamental group
  E.g. "π_1-good surgery theorem" [R. 2024+]

- The effect of Gluck twisting on general 4-manifolds [Kasprowski-Powell-R. 2022]

- Constructing arbitrarily large families of topological 4-manifolds which are simple homotopy equivalent and h-cobordant, but are pairwise non-s-cobordant [Kasprowski-Powell-R. 2022]

- Constructing pairs of smooth 4-manifolds that are homotopy equivalent but not homeomorphic [Kasprowski-Powell-R. 2024+]

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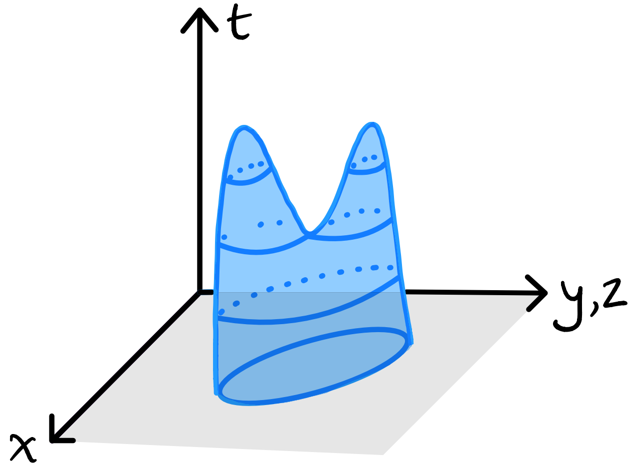

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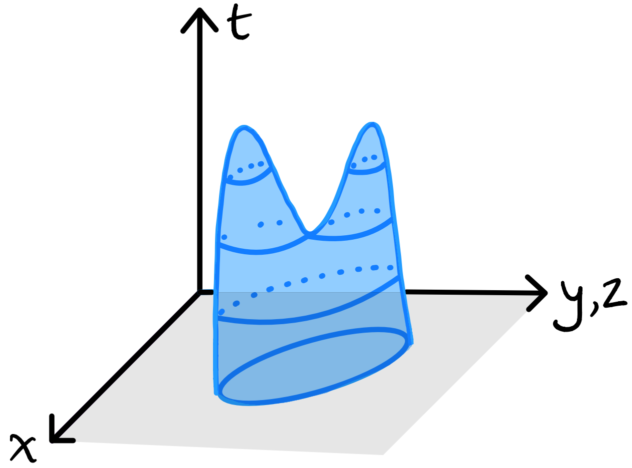

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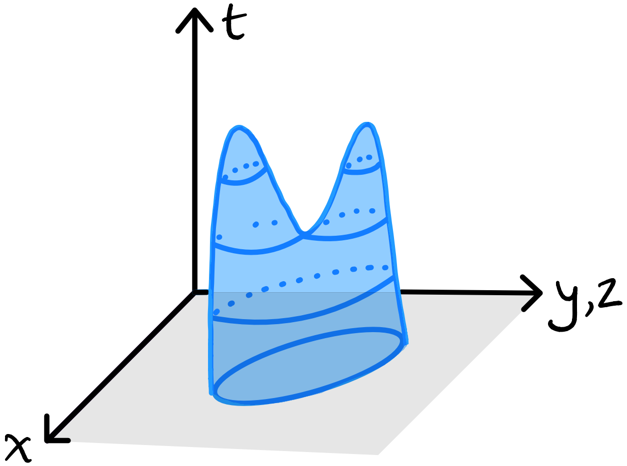

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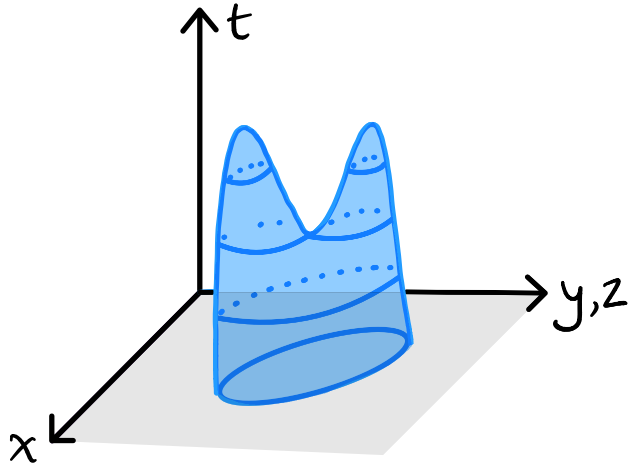

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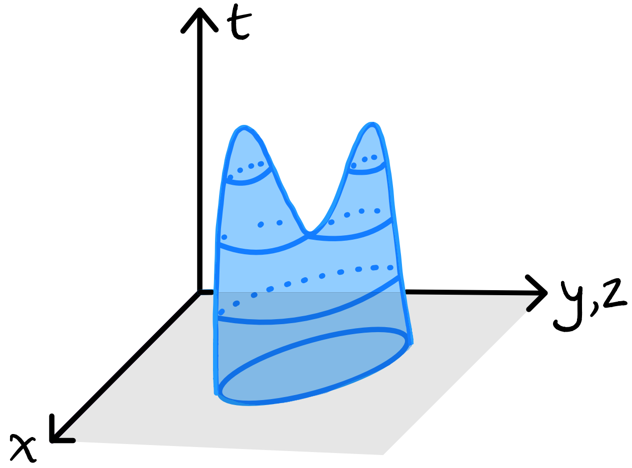

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- cf. earlier work on slice knots, esp. satellite knots, e.g. [R. 2013, Cochran-Davis-R. 2014, R. 2015, Cochran-R. 2016, Feller-Park-R. 2019, Davis-Park-R. 2021]

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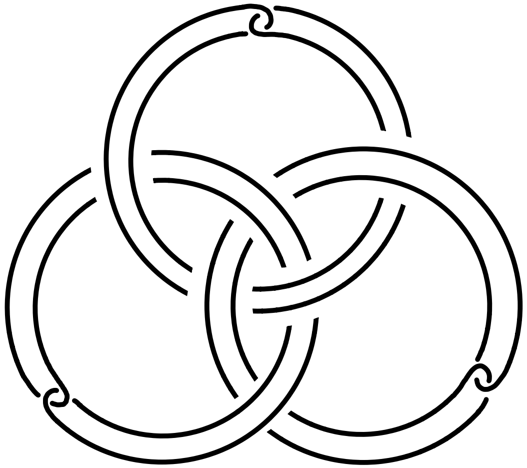

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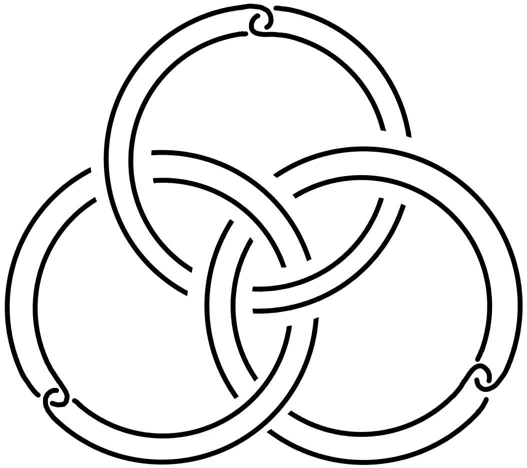

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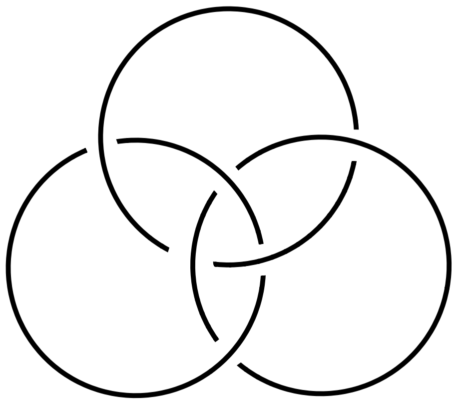

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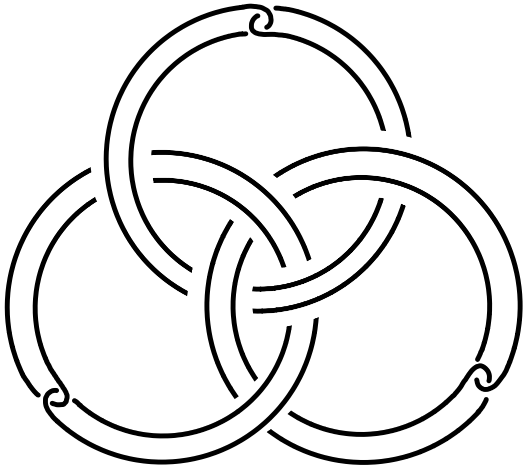

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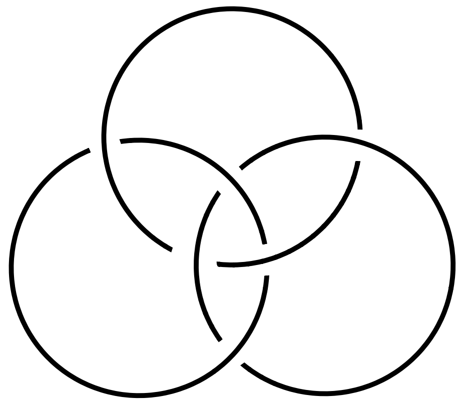

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Key open question for classifying 4-manifolds: decide if these links are slice

Slicing knots in general 4-manifolds

- Potential strategy to disprove smooth 4D Poincaré conjecture: find a homotopy 4-sphere Σ and a knot K such that K is smoothly slice in Σ but not in S^4

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- [Feller-Miller-R. ongoing] Is there a universal topological genus bound for knots in (most) topological 4-manifolds?

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- What can we say about this relationship? [Lidman-Miller-R. ongoing]

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- [Gompf-Orson-R. ongoing]
  - Use high-dimensional techniques to study the pseudo-mapping class groups of $\mathcal{R}$-homeomorphs
  - Use Freedman-Quinn-Taylor machinery to upgrade to mapping class groups and mapping class groups at infinity

Questions?