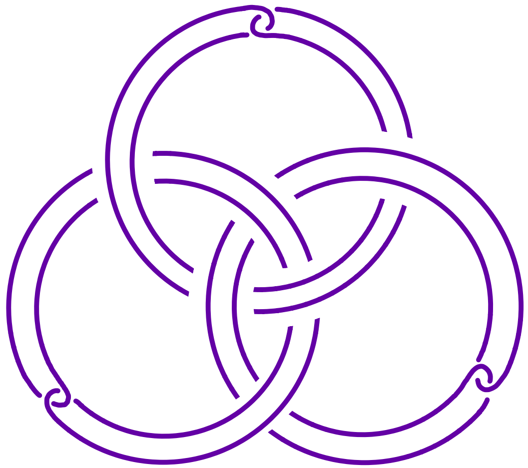
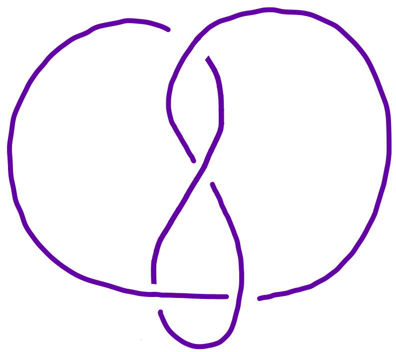
December 5, 2022

Knotted loops
& 4-dimensional spaces



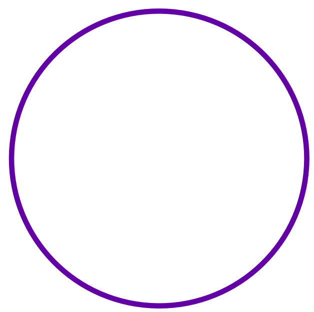
Knots and links

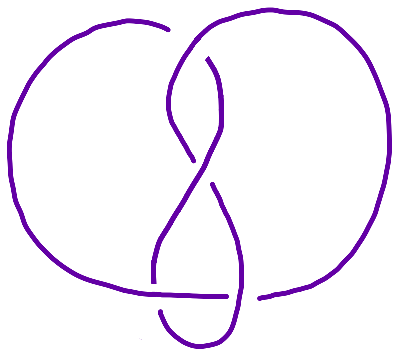
- A knot is a embedded circle in 3-dimensional space

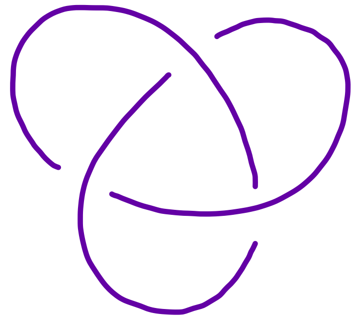


Knots and links

- A knot is a embedded circle in 3-dimensional space

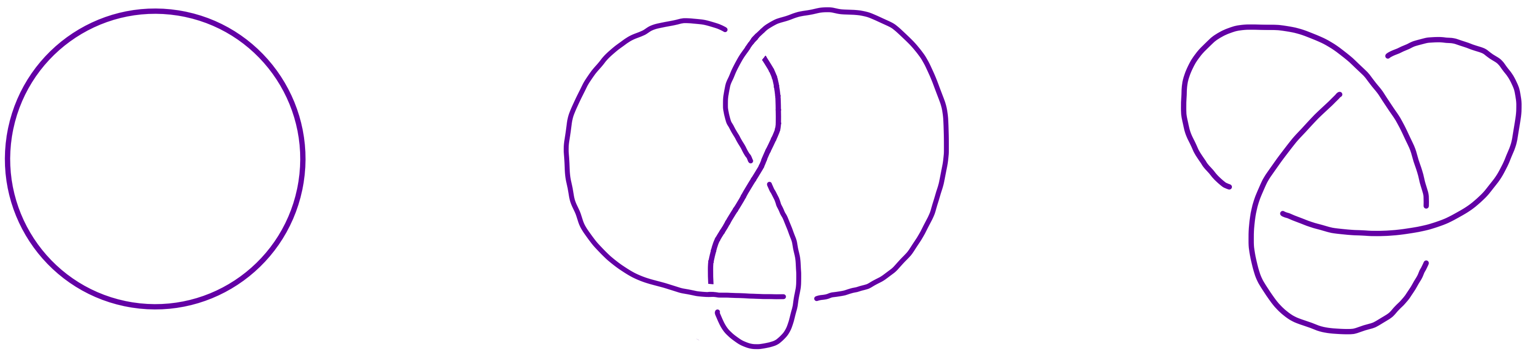




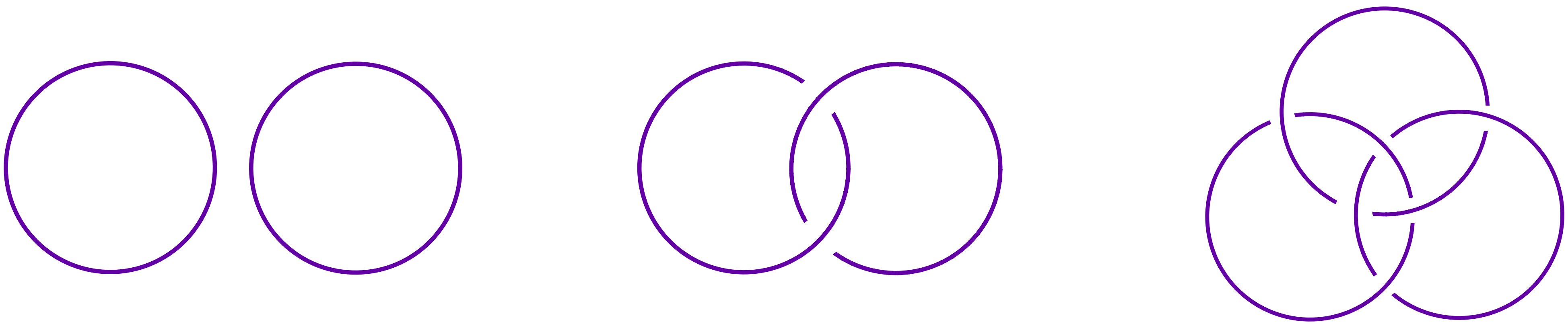


Knots and links

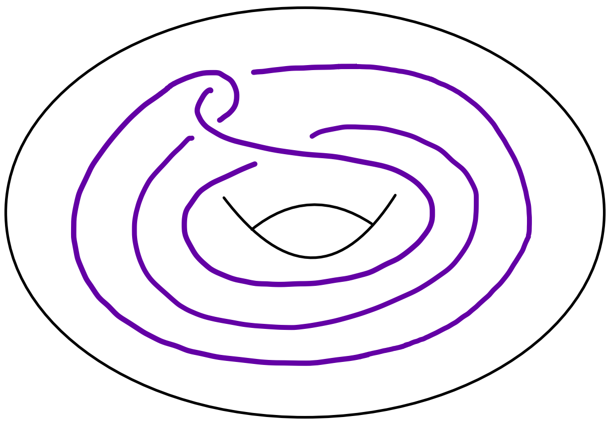
- A knot is a embedded circle in 3-dimensional space



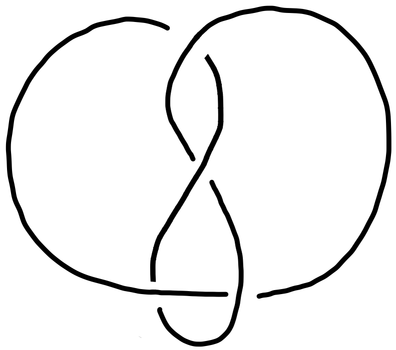
- A link is a collection of knots



Satellite operations

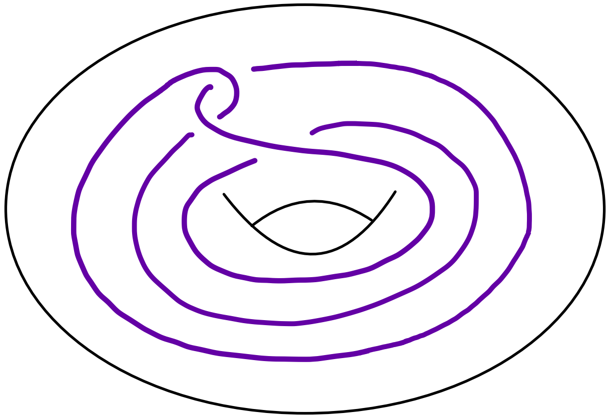


A pattern P

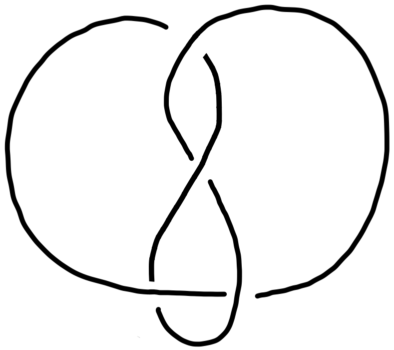


A knot K

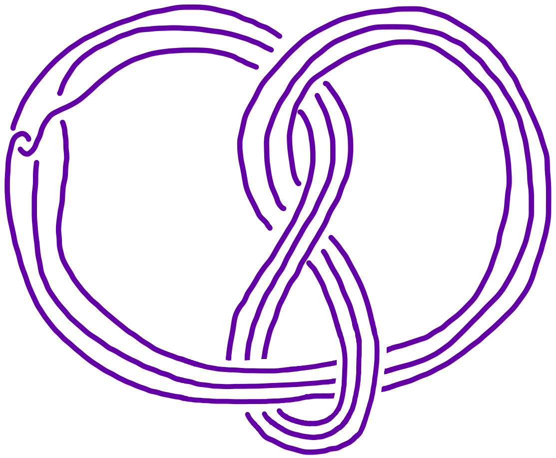
Satellite operations



A pattern P

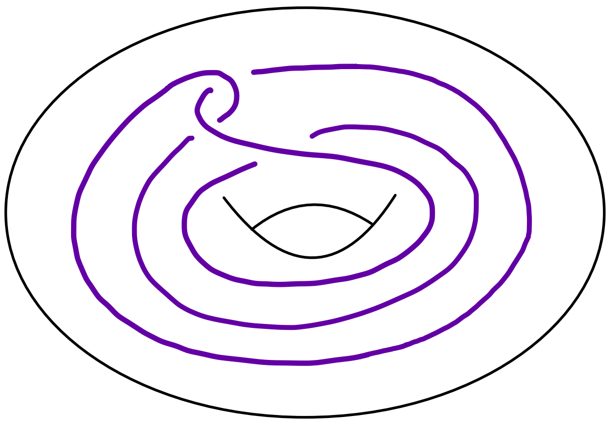


A knot K

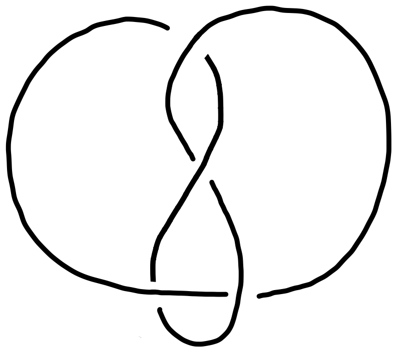


The satellite $P(K)$

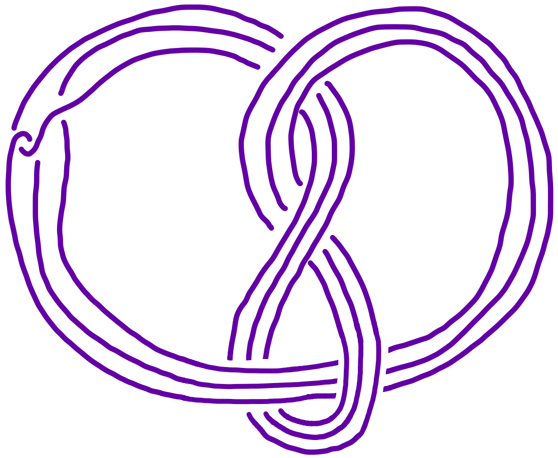
Satellite operations



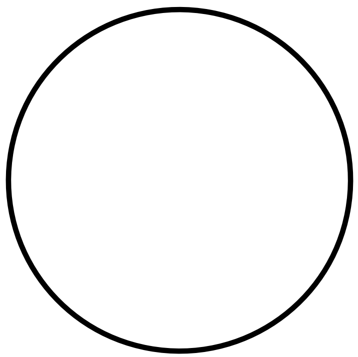
A pattern P

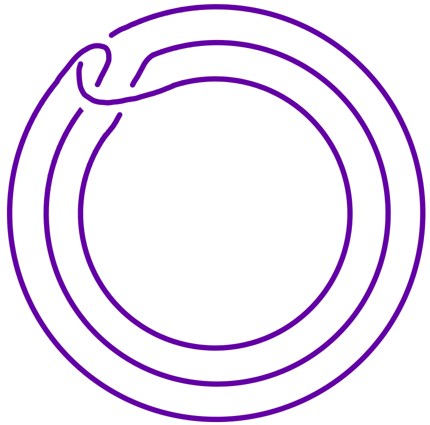


A knot K

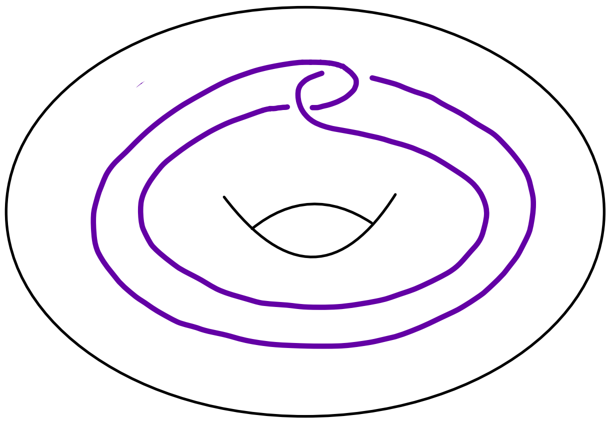


The satellite $P(K)$

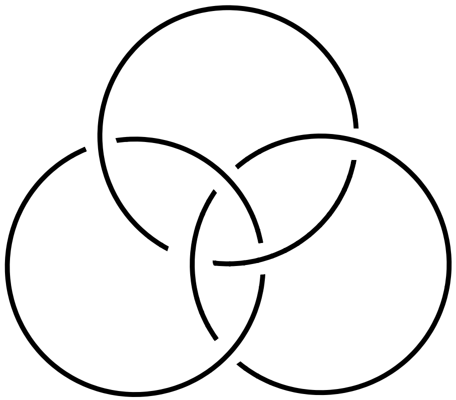




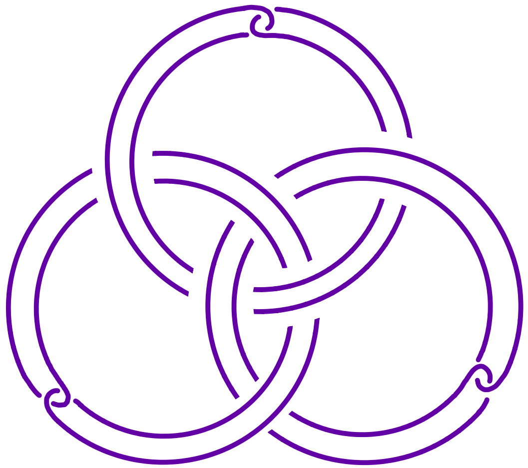
Satellite operations



A pattern Q



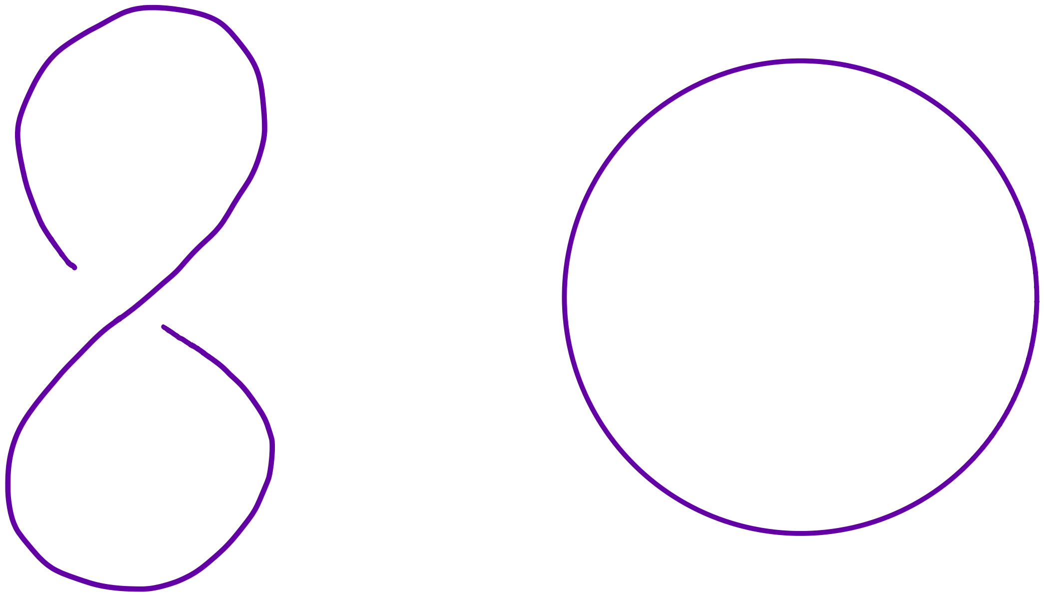
A link L



The satellite Q(L)

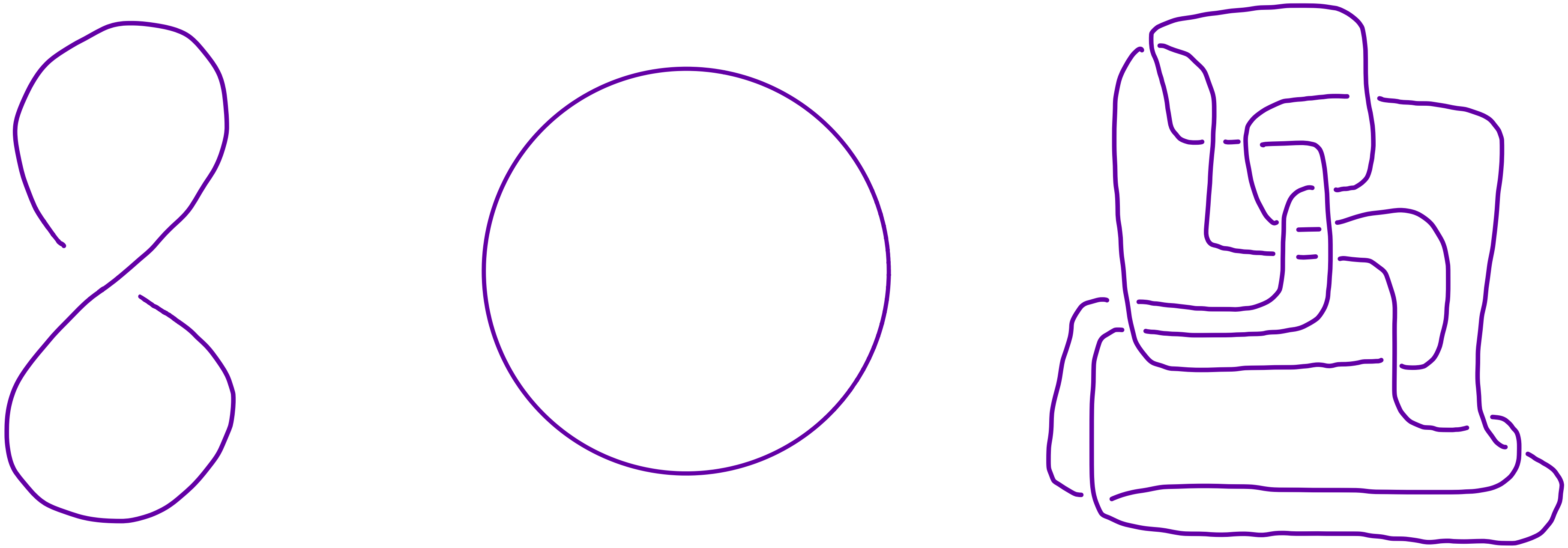
Isotopy of knots

- Two knots/links are isotopic if one can be deformed to the other



Isotopy of knots

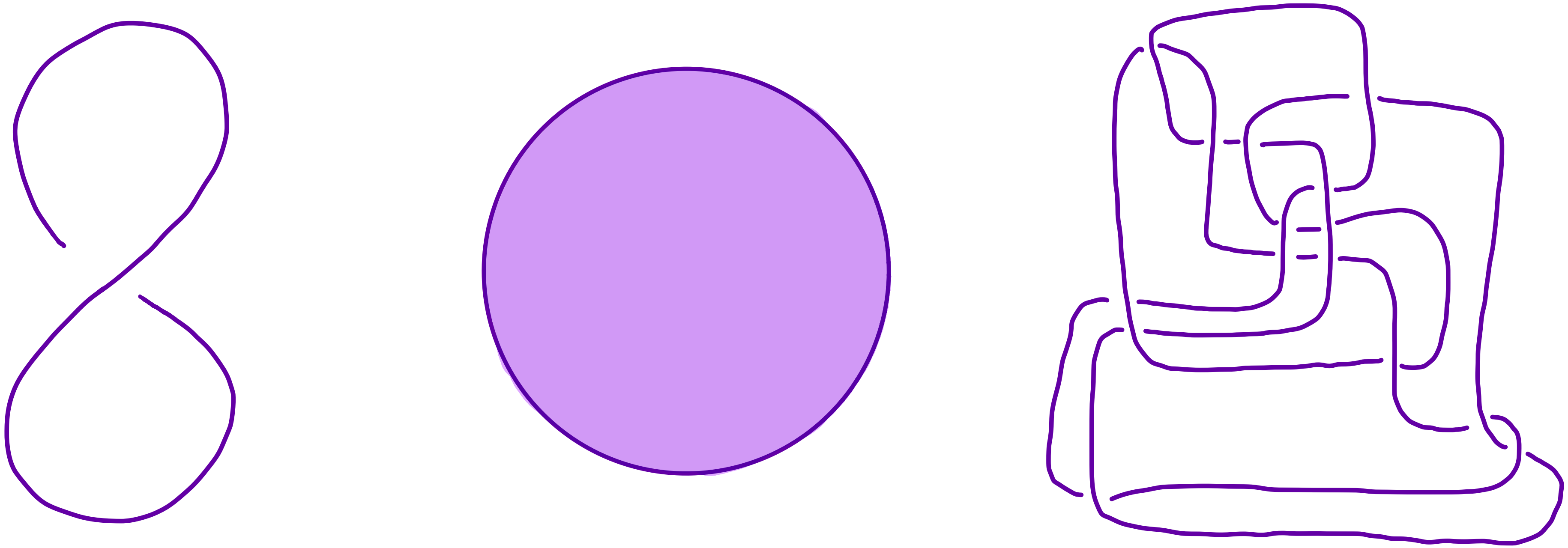
- Two knots/links are isotopic if one can be deformed to the other



Three pictures of the trivial knot, or unknot

Isotopy of knots

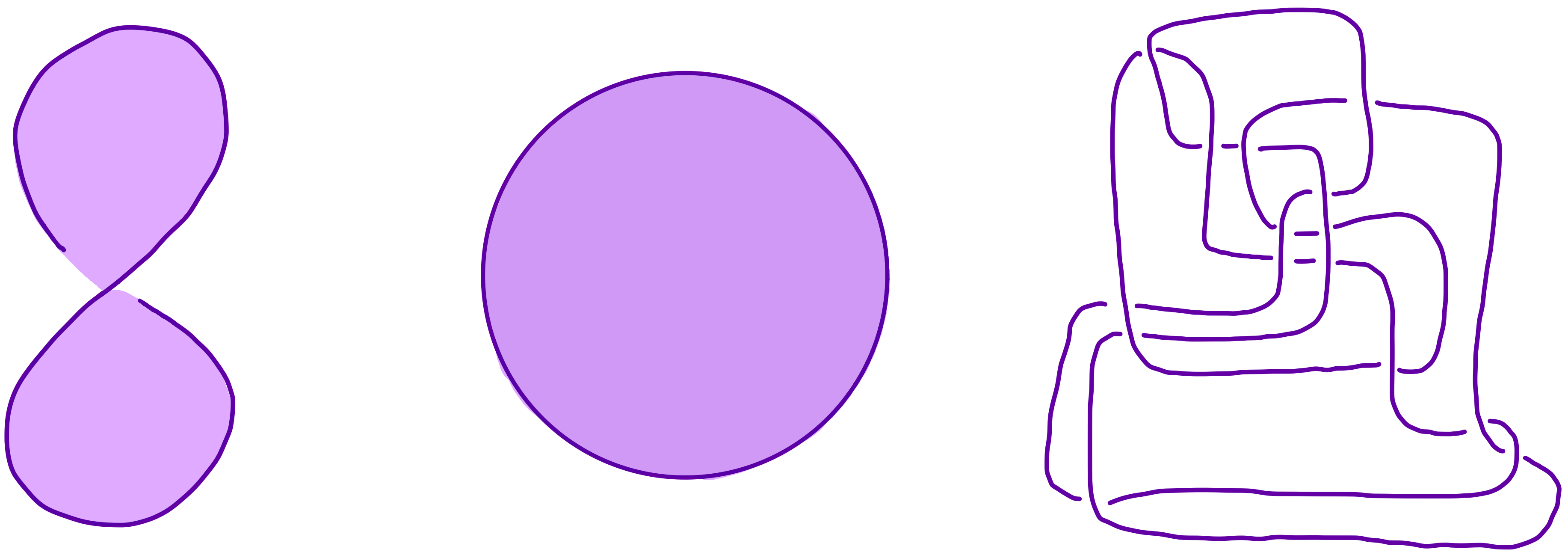
- Two knots/links are isotopic if one can be deformed to the other



Three pictures of the trivial knot, or unknot

Isotopy of knots

- Two knots/links are isotopic if one can be deformed to the other



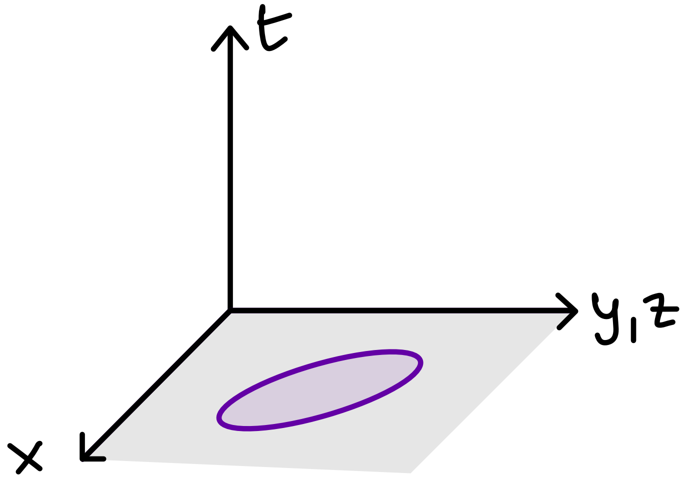
Three pictures of the trivial knot, or unknot

Slice knots and links

- A knot is trivial precisely if it bounds a disc in 3-dimensional space

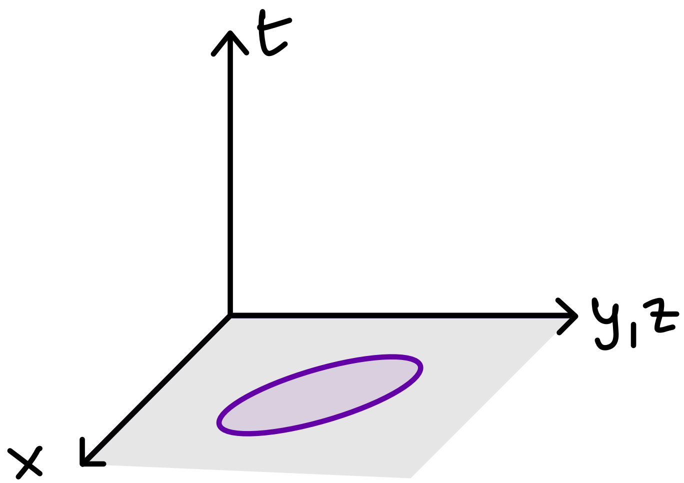
Slice knots and links

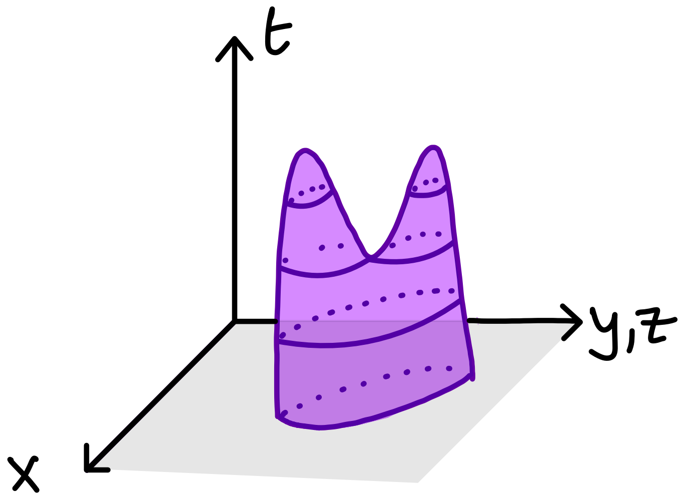
- A knot is trivial precisely if it bounds a disc in 3-dimensional space



Slice knots and links

- A knot is trivial precisely if it bounds a disc in 3-dimensional space

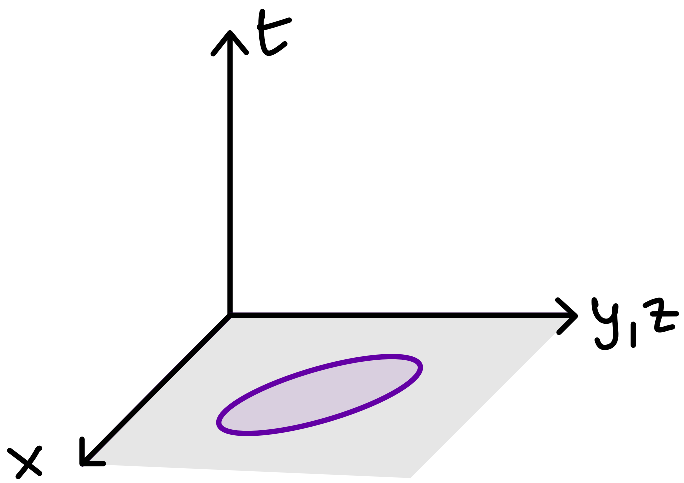


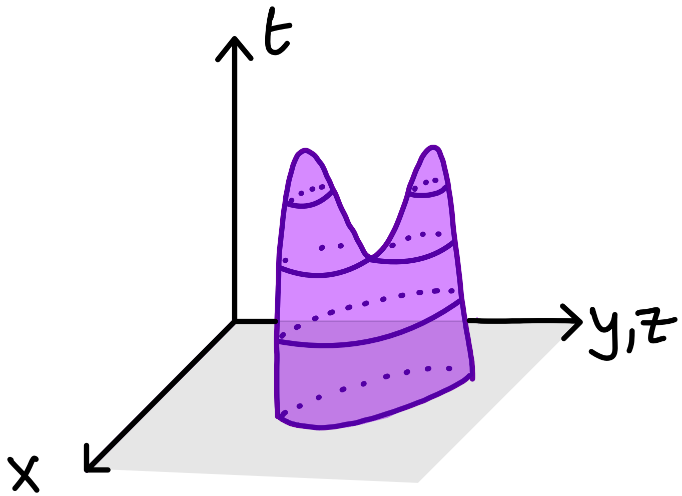


- A knot is slice if it bounds a disc in $\mathbb{R}^4_+$, the upper half-space in 4-dimensional space

Slice knots and links

- A knot is trivial precisely if it bounds a disc in 3-dimensional space

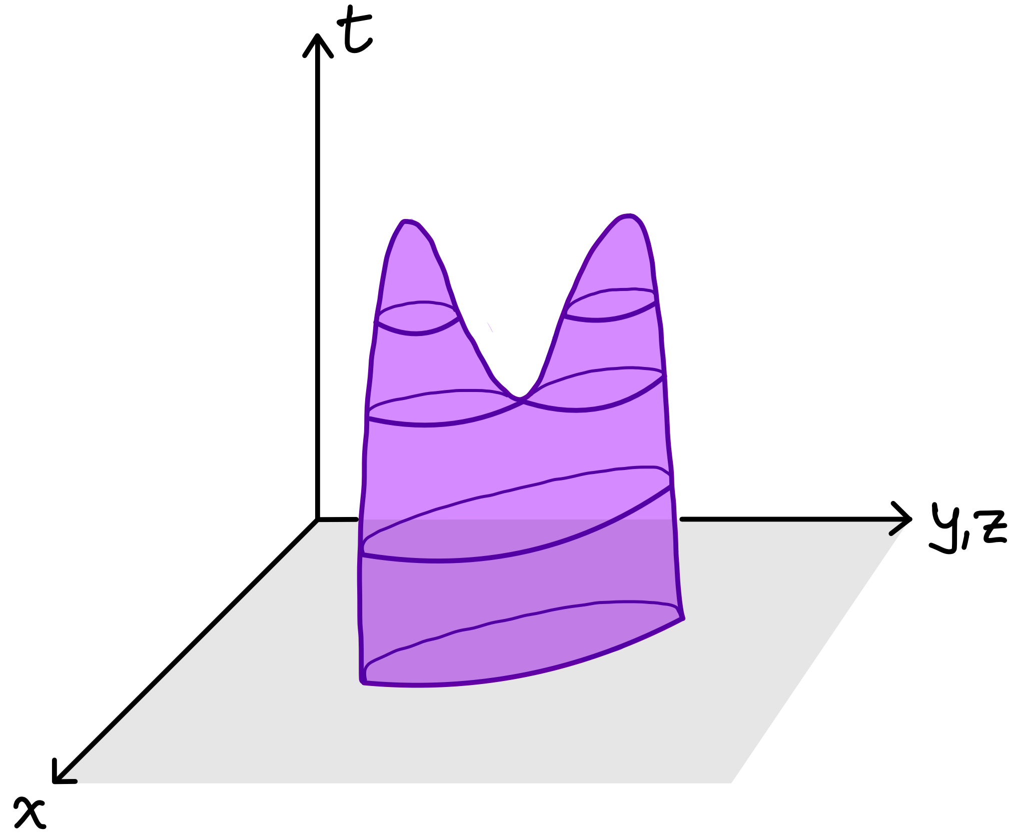


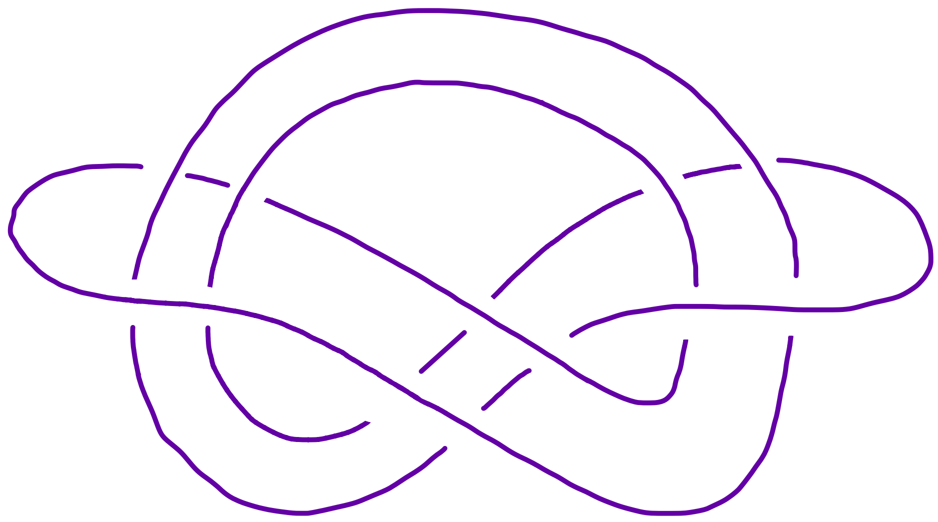


- A knot is slice if it bounds a disc in $\mathbb{R}^4_+$, the upper half-space in 4-dimensional space

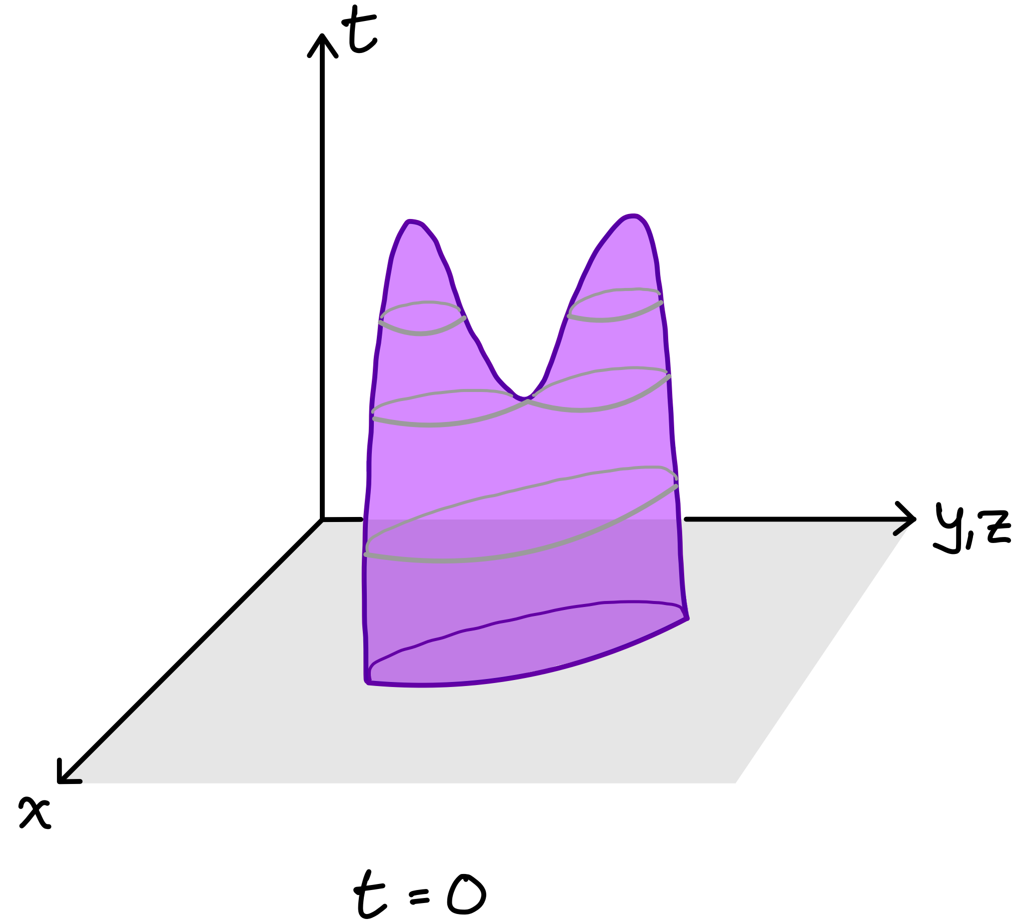
- A link is slice if it bounds a collection of discs in $\mathbb{R}^4_+$

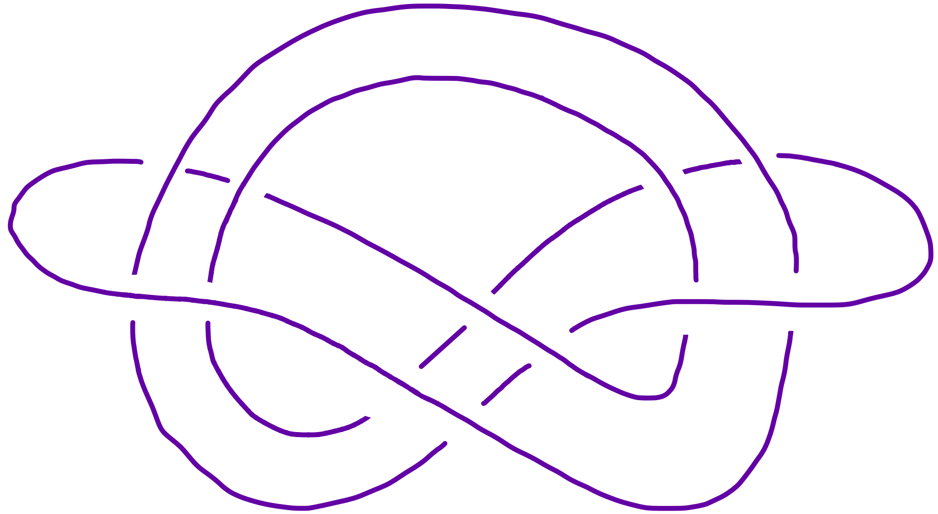
A slice knot



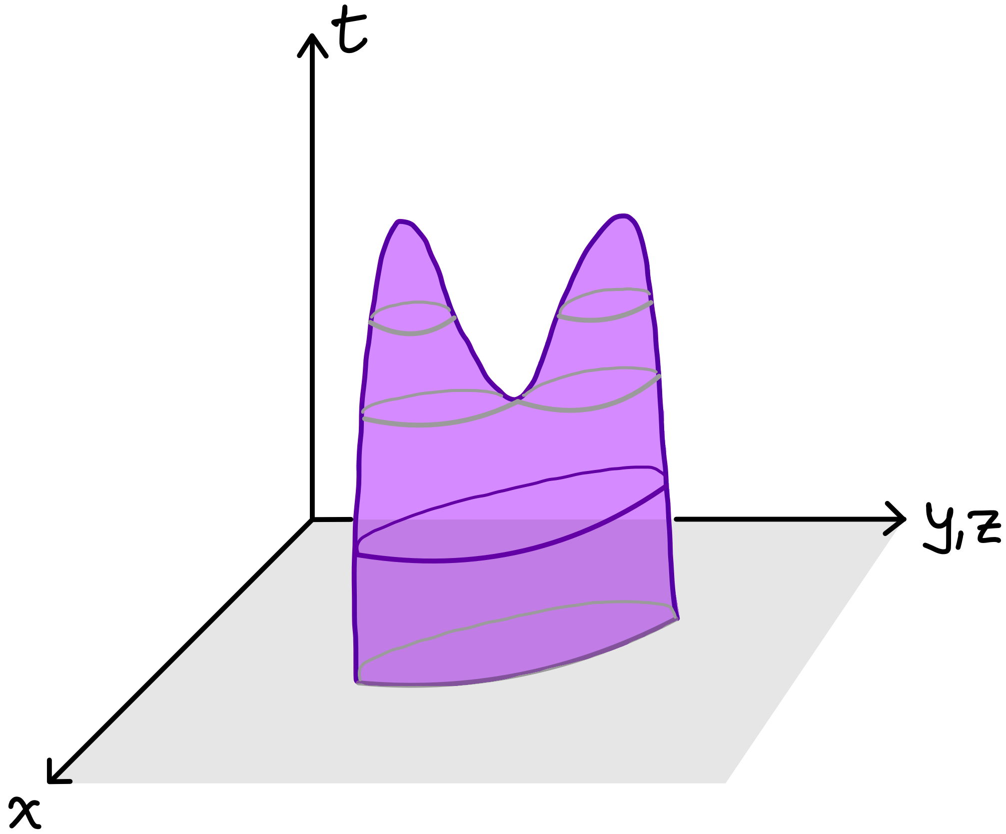


A slice knot

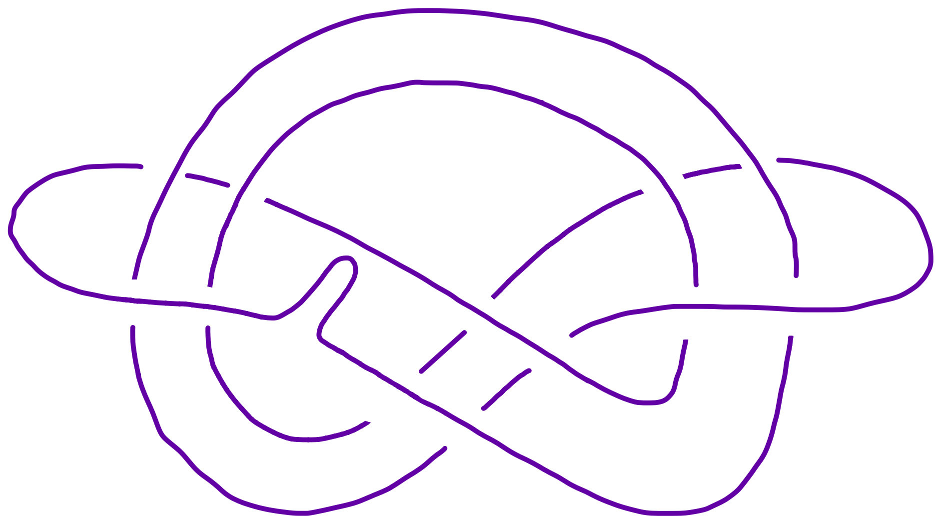




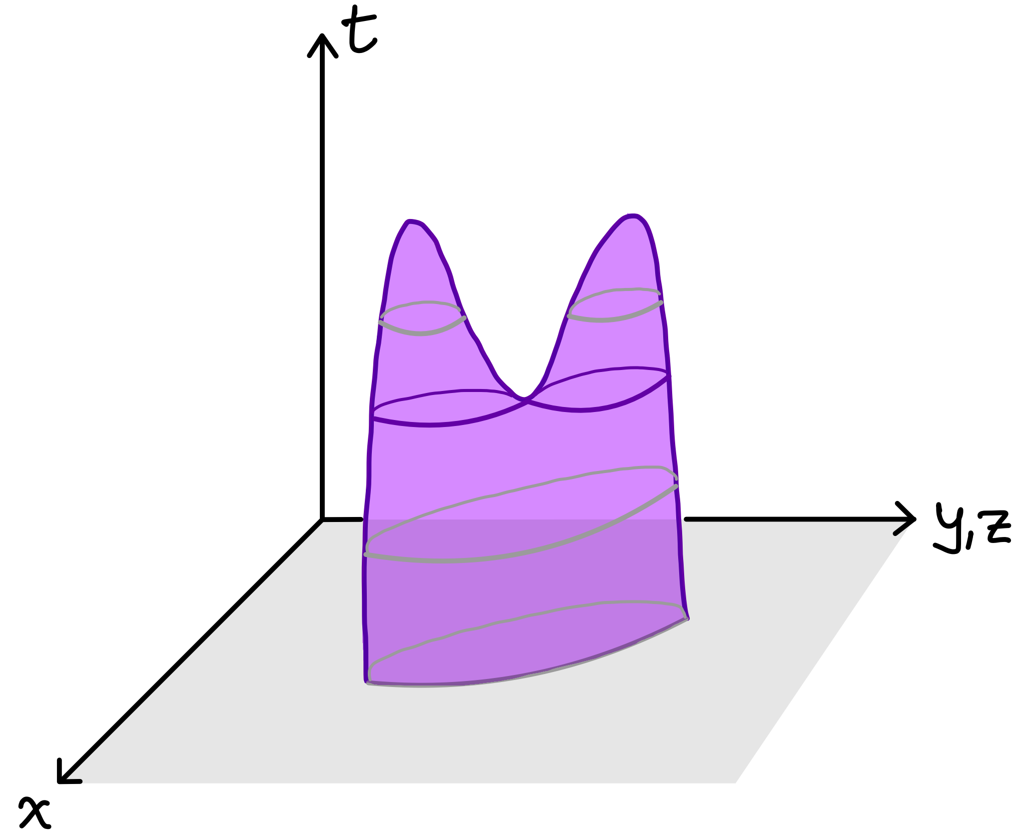
A slice knot



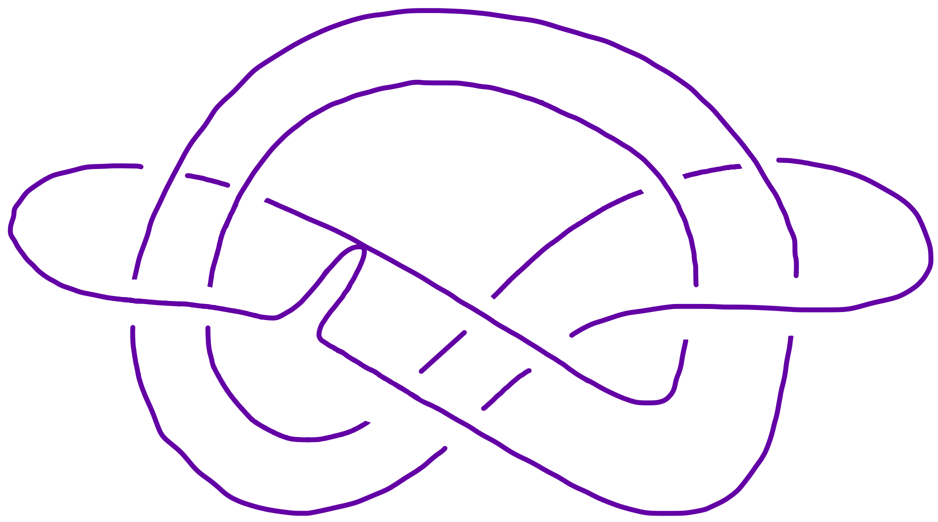
t = 1



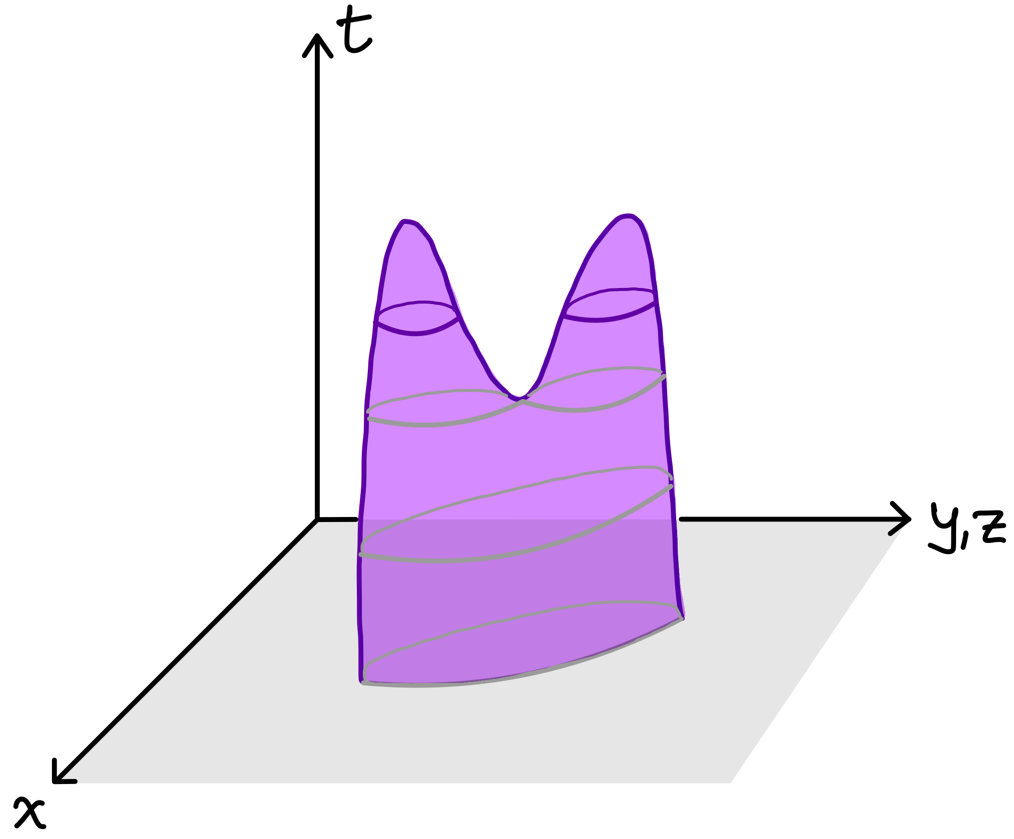
A slice knot



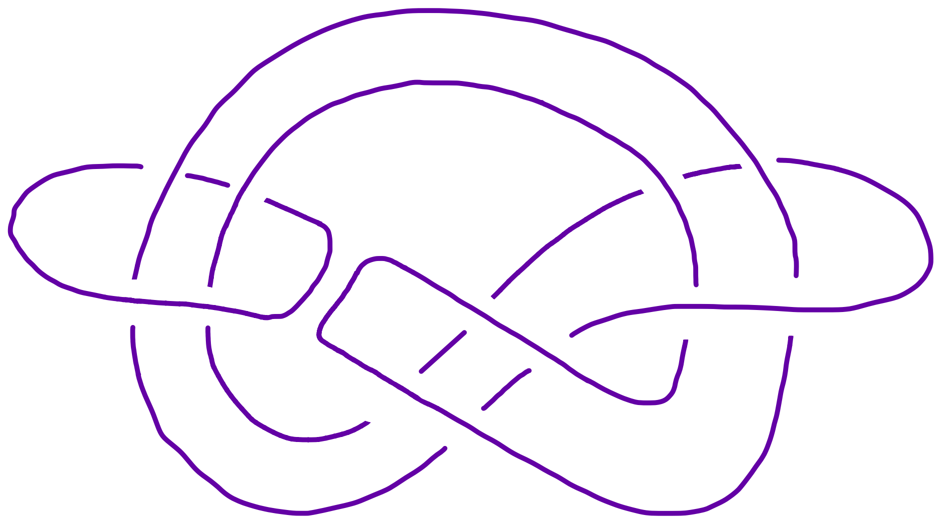
$t = 2$



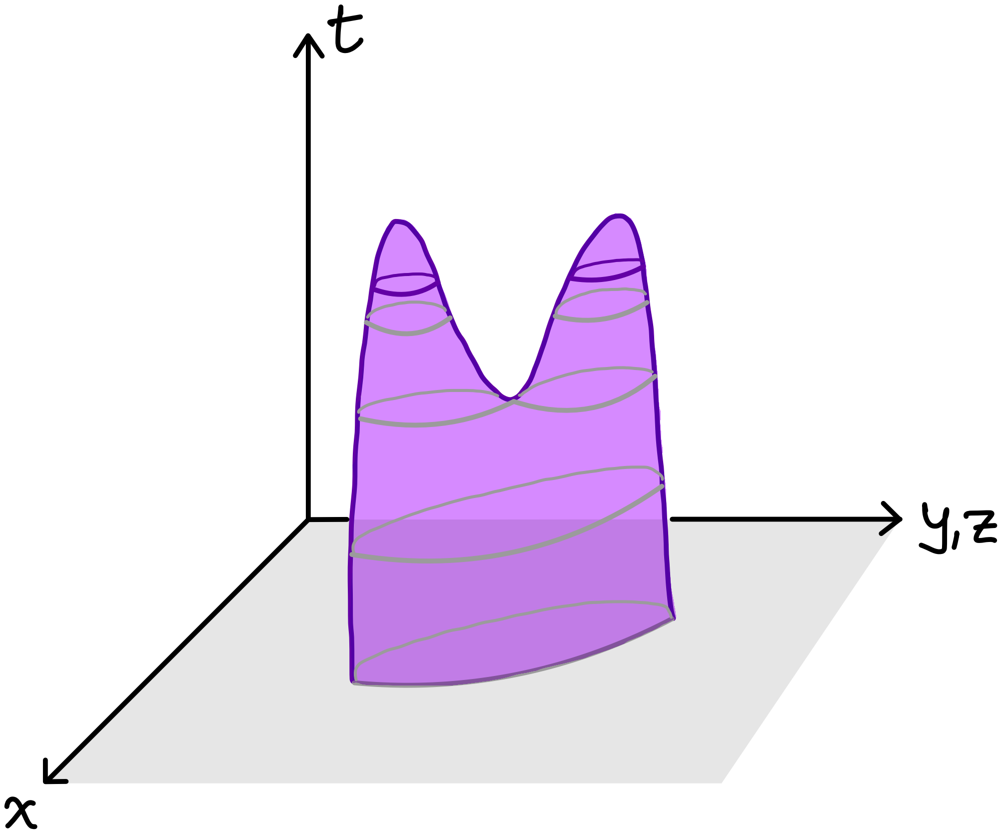
A slice knot



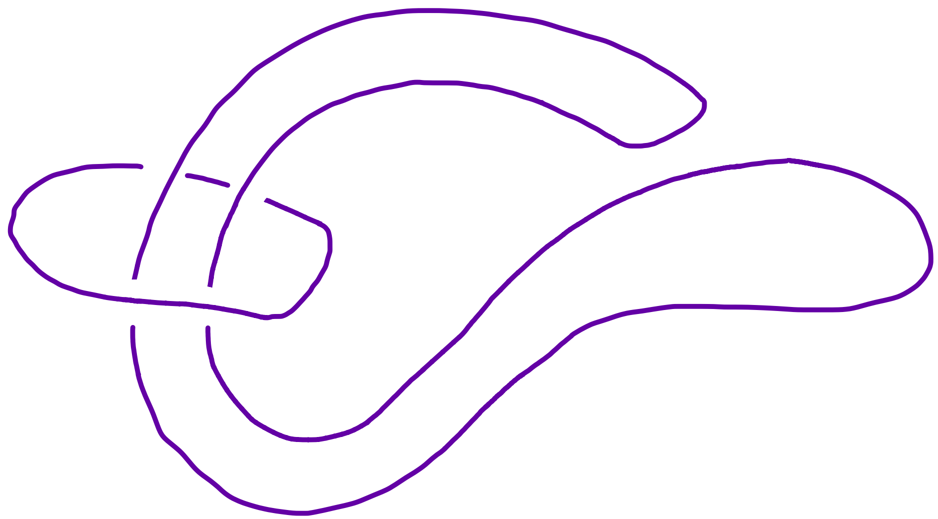
t = 3



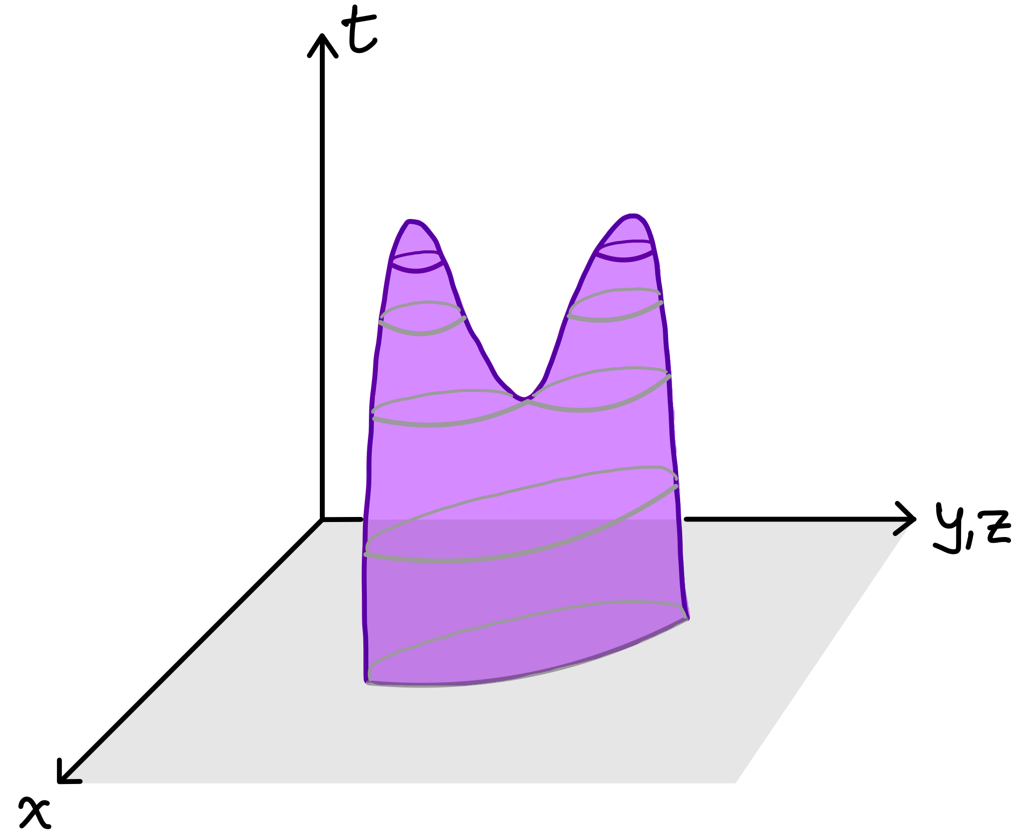
A slice knot



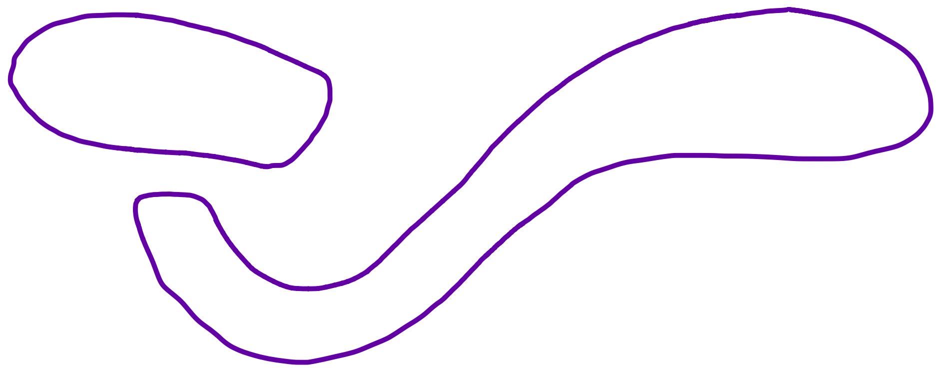
$t = 3\cdot 2$



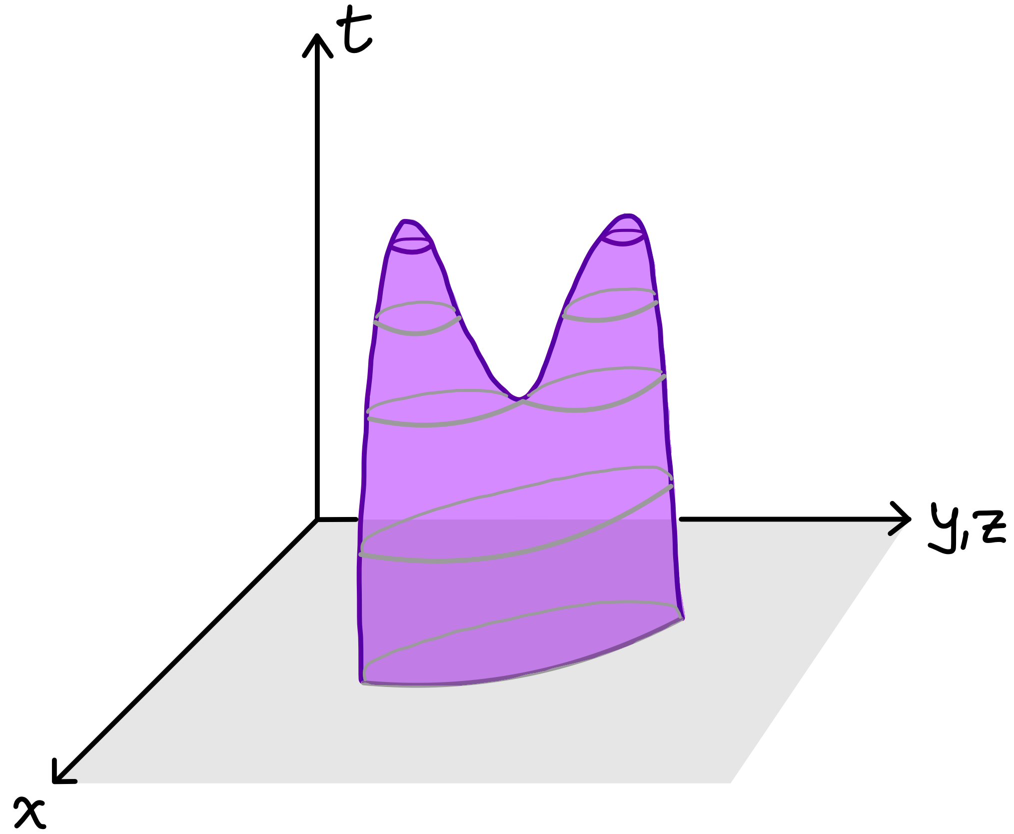
A slice knot



$t = 3.5$

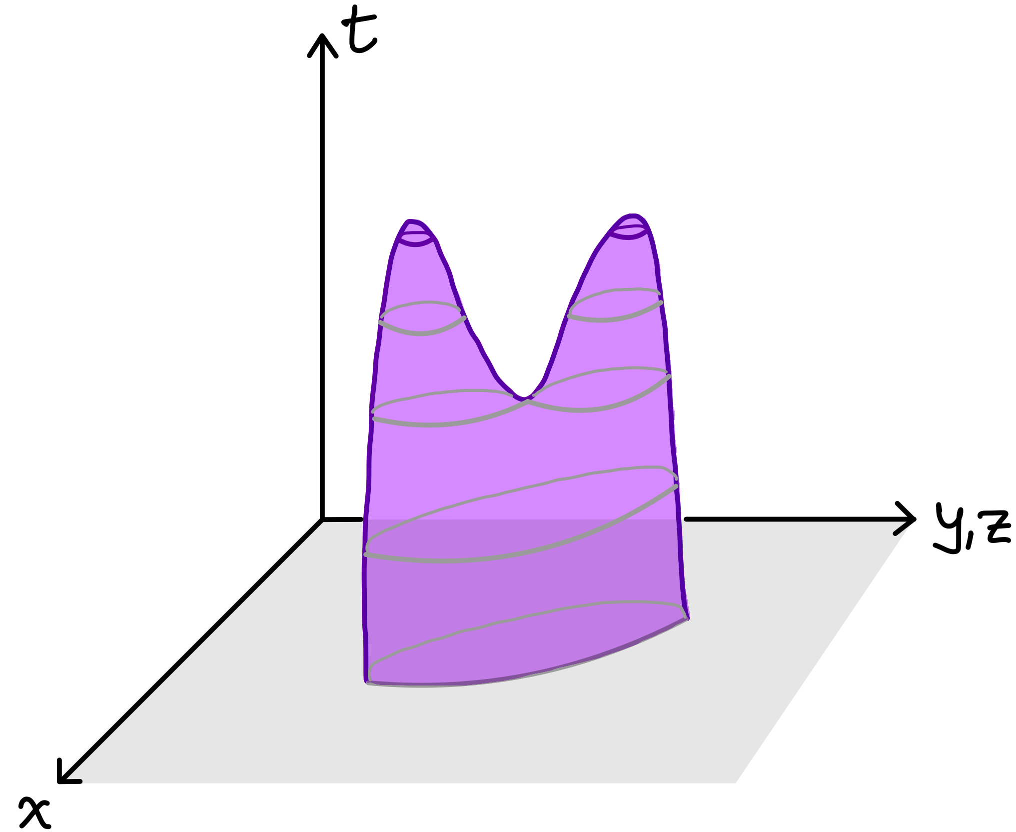


A slice knot



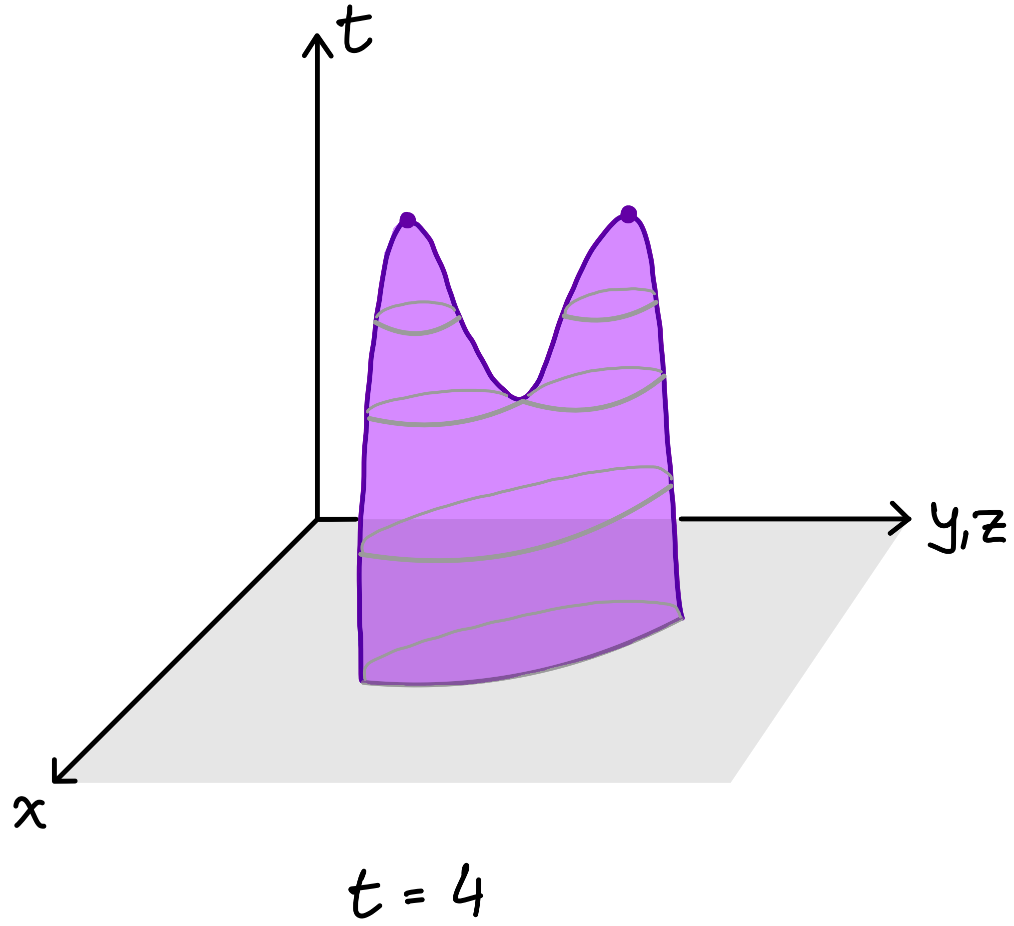
$t = 3 \cdot 7$

A slice knot

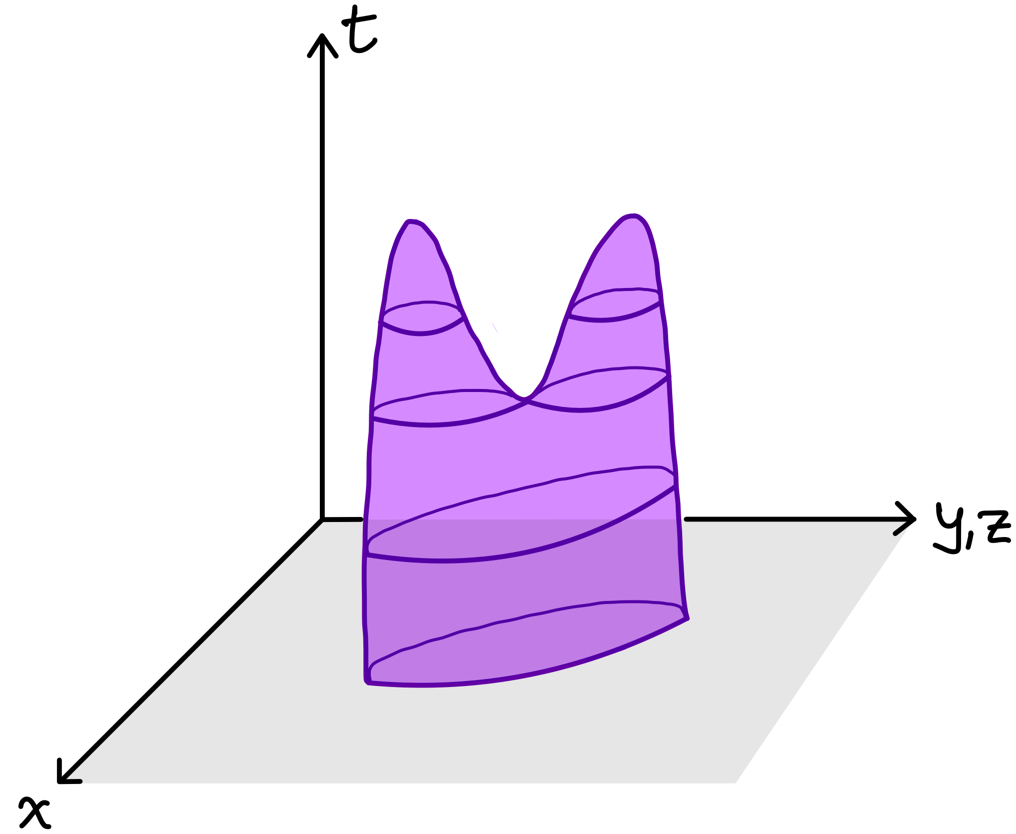


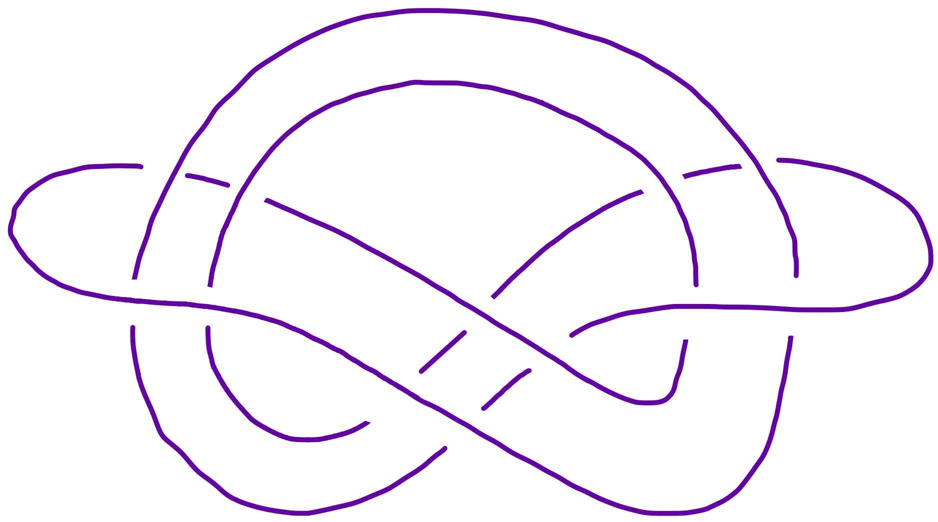
$t = 3.8$

A slice knot



A slice knot



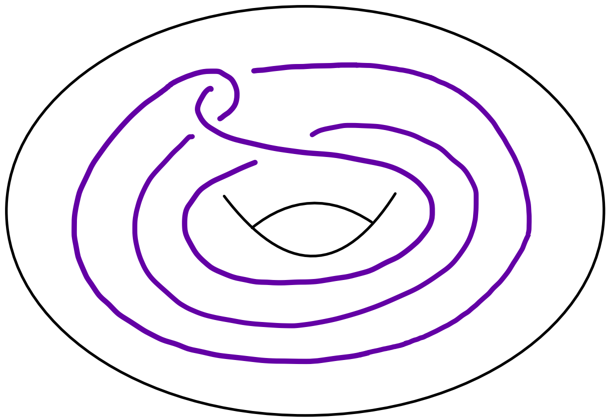


Satellite knots and sliceness

Theorem [Cochran-Davis-R. 2014]

For infinitely many patterns P, and all knots K, if $P(K)$ is slice, then K is slice.

E.g.

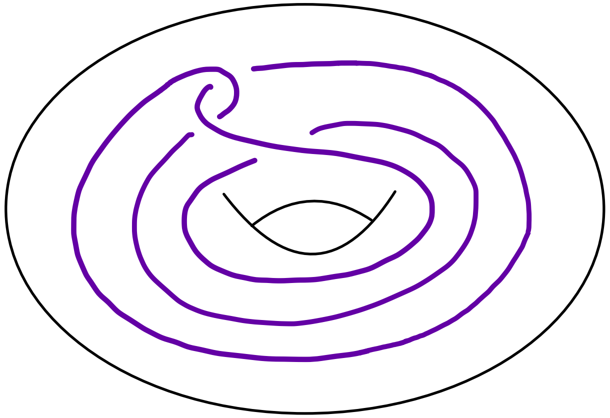


Satellite knots and sliceness

Theorem [Cochran-Davis-R. 2014]

For infinitely many patterns P, and all knots K, if $P(K)$ is slice, then K is slice.

E.g.



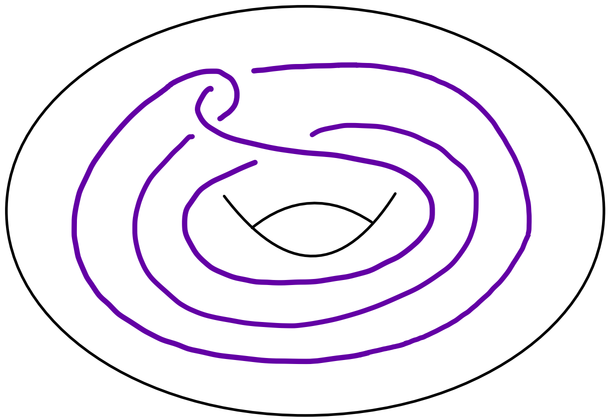
- More generally, with respect to the notion of concordance, these patterns are injective functions

Satellite knots and sliceness

Theorem [Cochran-Davis-R. 2014]

For infinitely many patterns P, and all knots K, if $P(K)$ is slice, then K is slice.

E.g.

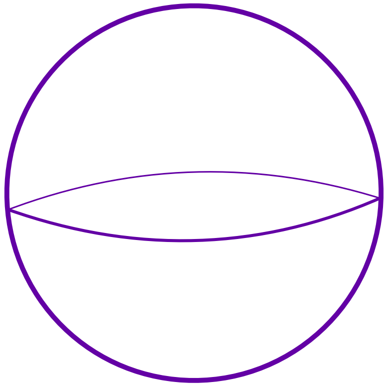


- More generally, with respect to the notion of concordance, these patterns are injective functions
- Further work on iterated satellites, bijectivity of patterns, ...
  [R. 2013, R. 2015, Cochran-R. 2016, Feller-Park-R. 2019, Davis-Park-R. 2021]

4-dimensional spaces

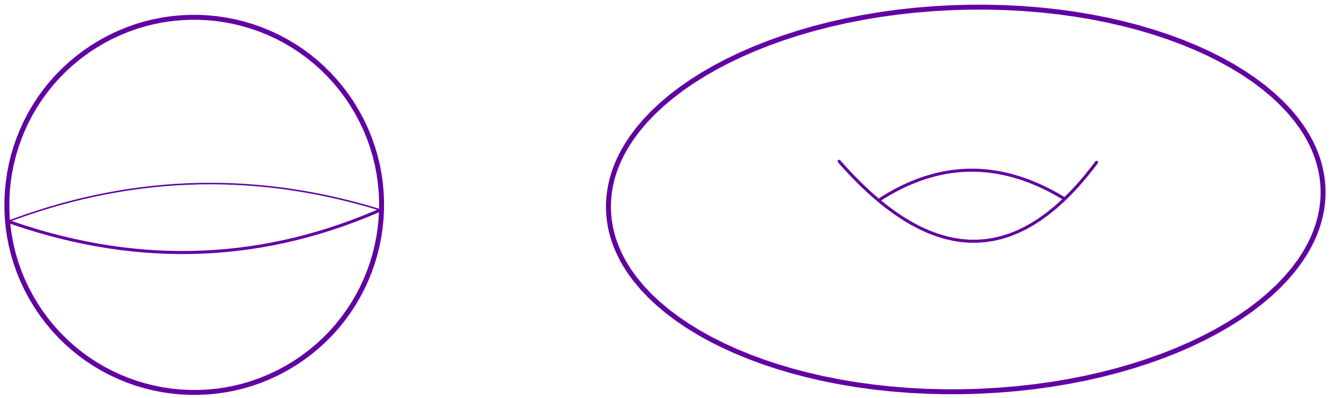
Manifolds

- Manifolds are spaces which are locally like Euclidean space



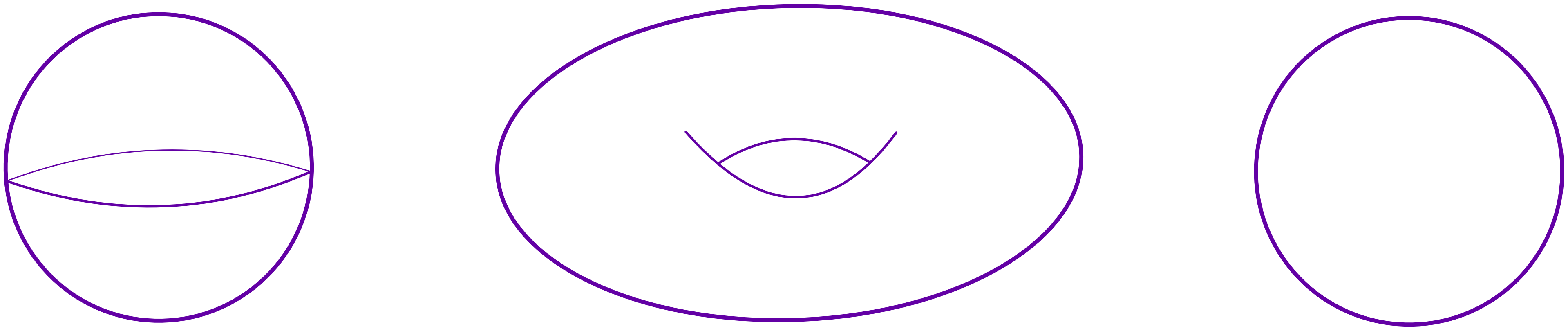
Manifolds

- Manifolds are spaces which are locally like Euclidean space



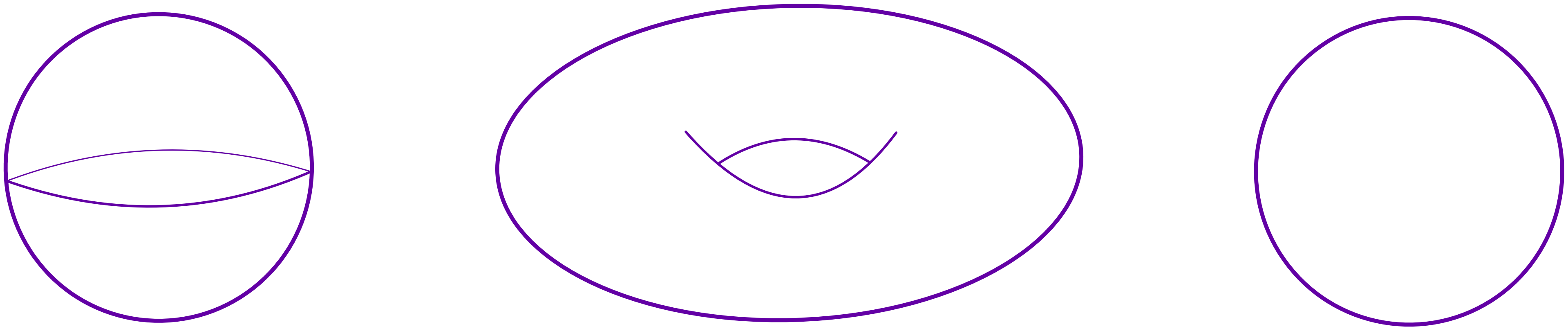
Manifolds

- Manifolds are spaces which are locally like Euclidean space



Manifolds

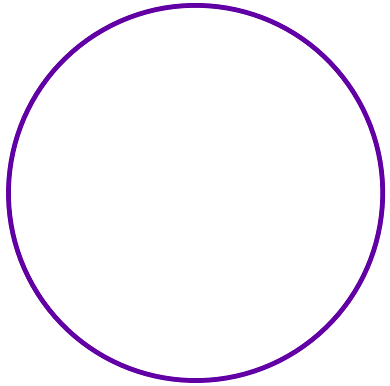
- Manifolds are spaces which are locally like Euclidean space



- The universe appears to be a 3-dimensional manifold
- Space-time forms a 4-dimensional manifold
- The n-dimensional sphere is given by $\{(x_1, x_2, \ldots, x_{n+1}) \mid x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\}$

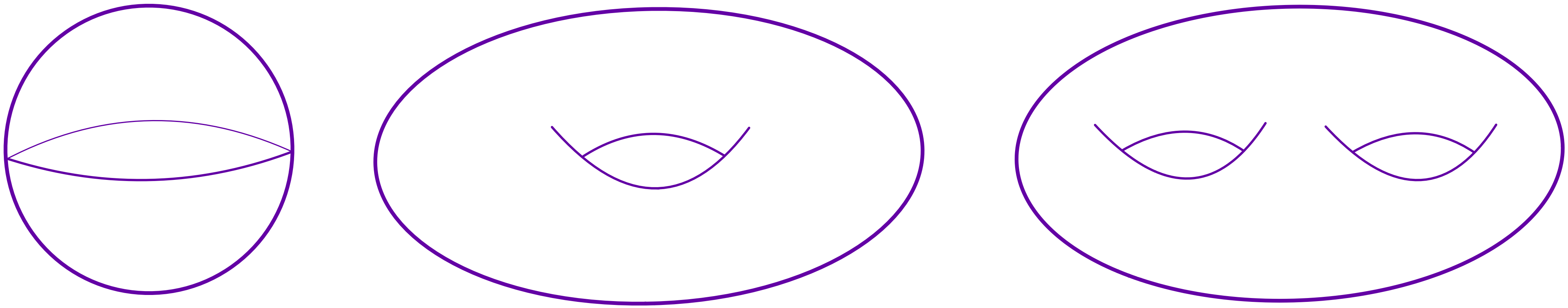
Classifying manifolds

- Dimension 1: the circle



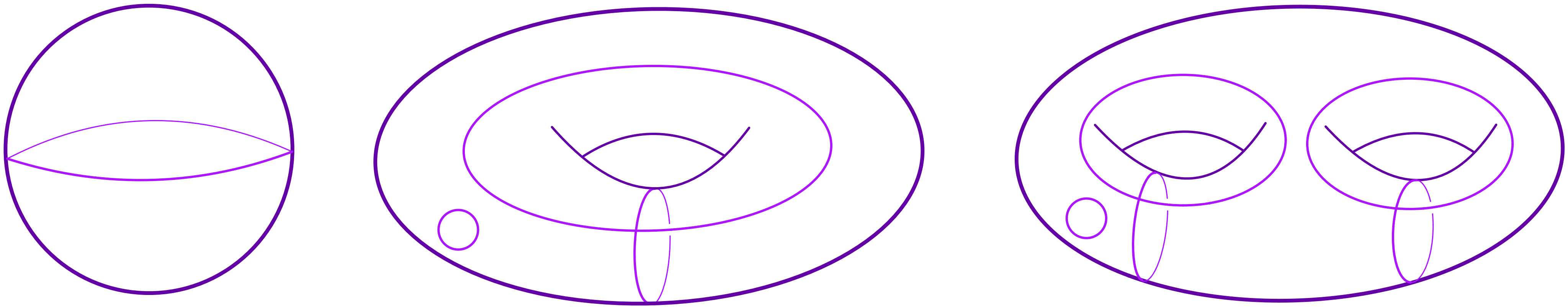
Classifying manifolds

- Dimension 1: the circle
- Dimension 2: classified by genus



Classifying manifolds

- Dimension 1: the circle
- Dimension 2: classified by genus



The distinct circles in a manifold M are recorded in the fundamental group $\pi_1(M)$

Classifying manifolds

- Dimension 1: the circle
- Dimension 2: classified by genus
- Dimension 3: classified, decompose into geometric pieces [Perelman 2003]

Classifying manifolds

- Dimension 1: the circle
- Dimension 2: classified by genus
- Dimension 3: classified, decompose into geometric pieces [Perelman 2003]

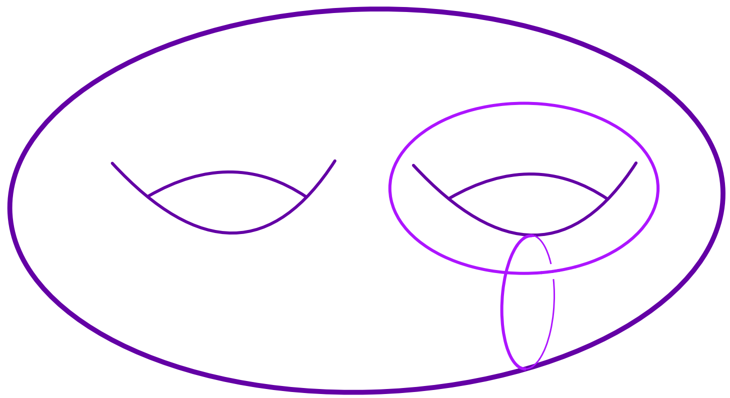
- Dimensions 5 & higher: surgery theory

Classifying manifolds

- Dimension 1: the circle
- Dimension 2: classified by genus
- Dimension 3: classified, decompose into geometric pieces [Perelman 2003]
- What about dimension four?
- Dimensions 5 & higher: surgery theory

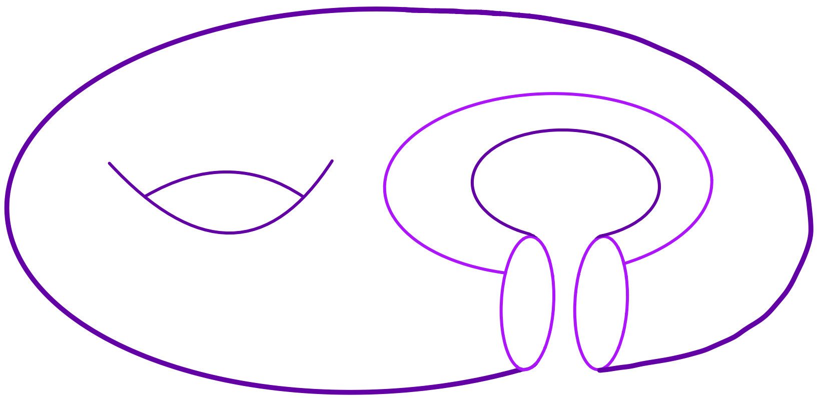
Surgery to classify surfaces

- Idea: modify the surface in a controlled manner to reduce complexity
  - find geometrically dual pairs of circles, detecting genus



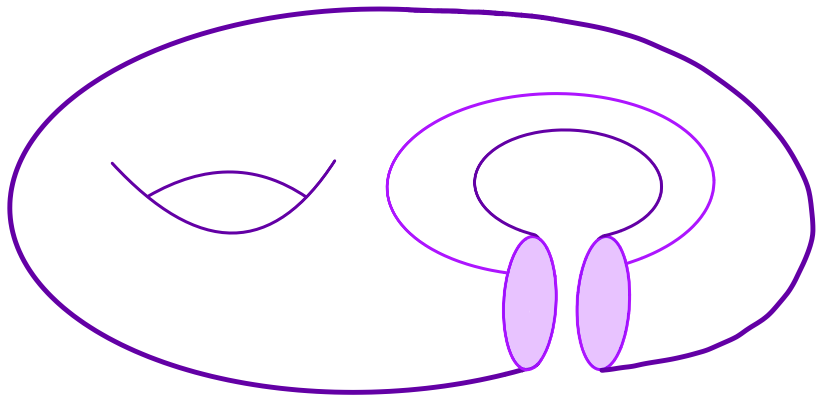
Surgery to classify surfaces

- Idea: modify the surface in a controlled manner to reduce complexity
  - find geometrically dual pairs of circles, detecting genus



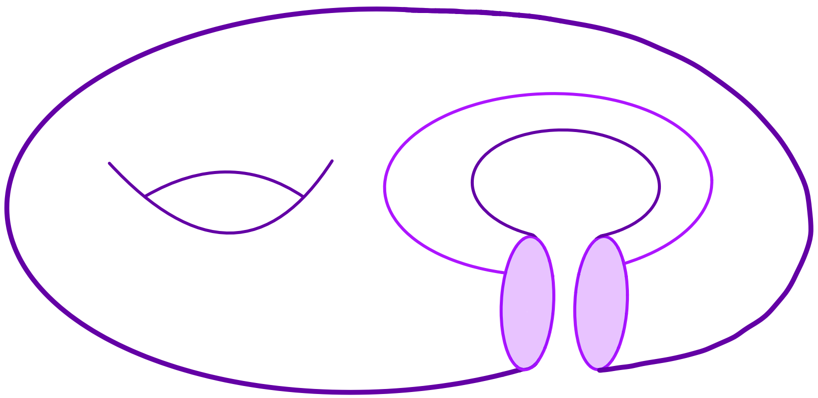
Surgery to classify surfaces

- Idea: modify the surface in a controlled manner to reduce complexity
  - find geometrically dual pairs of circles, detecting genus

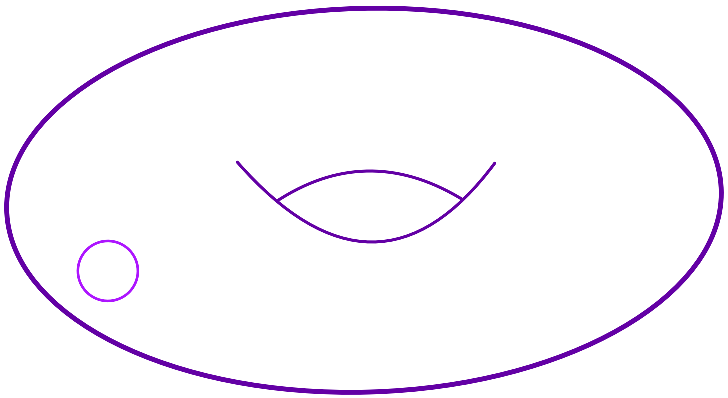


Surgery to classify surfaces

- Idea: modify the surface in a controlled manner to reduce complexity
  - find geometrically dual pairs of circles, detecting genus

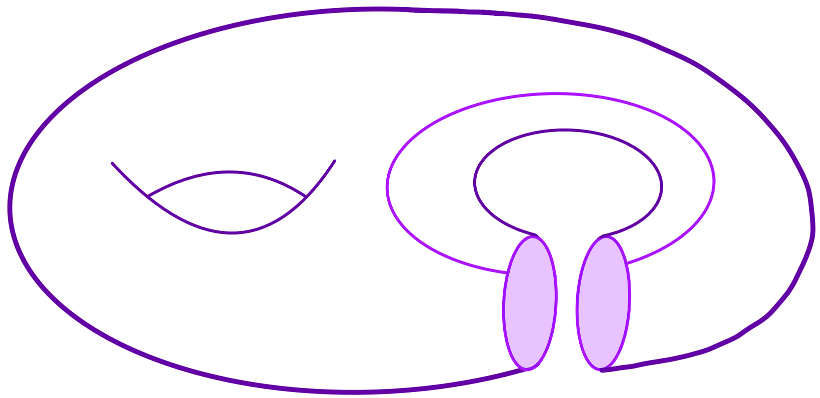


  - the dual circle ensures that the result is less complex

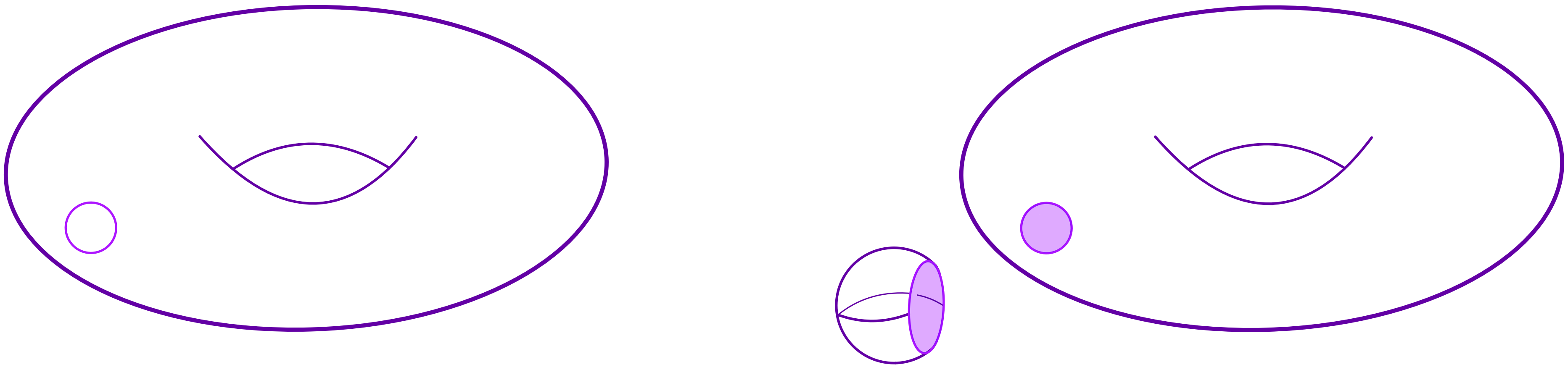


Surgery to classify surfaces

- Idea: modify the surface in a controlled manner to reduce complexity
  - find geometrically dual pairs of circles, detecting genus



  - the dual circle ensures that the result is less complex



Surgery for 4-manifolds

- For 4-manifolds, we need to find and control circles as well as spheres
    - In general, we must control spheres up to the middle dimension

- The group $\pi_2(M)$ records the types of spheres in a manifold M

Surgery for 4-manifolds

- For 4-manifolds, we need to find and control circles as well as spheres
  - In general, we must control spheres up to the middle dimension

- The group $\pi_2(M)$ records the types of spheres in a manifold M

Goal for surgery in 4-manifolds

Let M be a 4-manifold. Try to represent every algebraically dual pair in $\pi_2(M)$ by a geometrically dual pair of spheres in M.

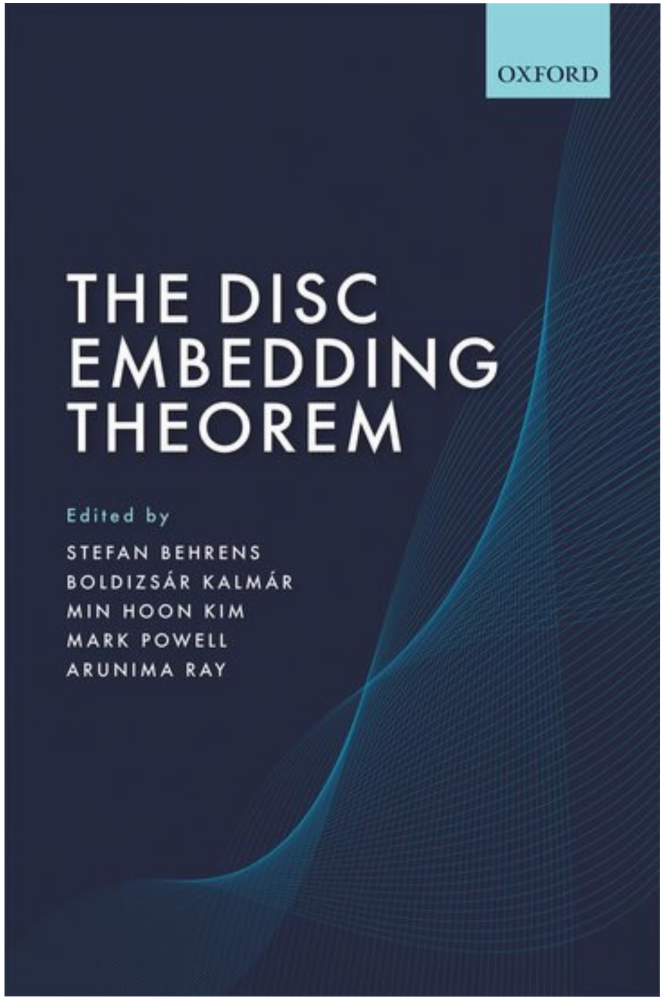
Sphere embedding theorem

Let M be a 4-manifold. Every algebraically dual pair in $\pi_2(M)$ can be represented by a geometrically dual pair of spheres in M.

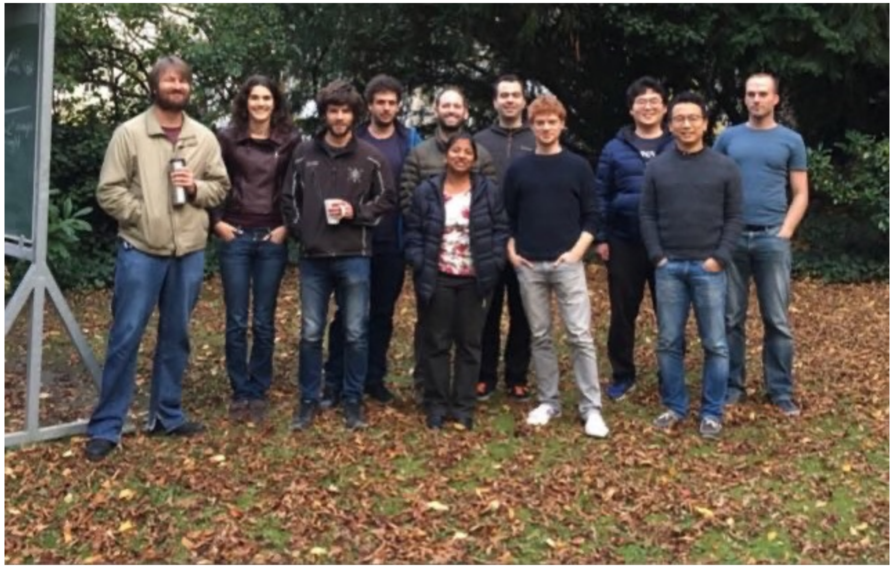
For $\pi_1(M)$ trivial: Freedman 1982
For $\pi_1(M)$ good: Freedman-Quinn 1990 + Powell-R.-Teichner 2018
For general $\pi_1(M)$: open

- Good groups include all finite groups, abelian groups, solvable groups, elementary abelian groups, groups of subexponential growth, ...
- Consequences include: 4-dimensional Poincaré conjecture, classification of simply connected 4-manifolds, topological transversality, ...

Explaining 4-manifolds



- 2013+2016:
  - semester programmes in Bonn, Germany
  - lectures by Freedman and others
  - notes developed into accessible textbook
  - published 2021, Oxford University Press



Sphere embedding theorem

Theorem [Casson 1970s, Freedman 1982, Freedman-Quinn 1990]

M a connected 4-manifold, $\pi_1(M)$ good, Σ a simply connected 2-manifold.

$$\begin{array}{ccc} F\colon \Sigma & \longrightarrow & M \\ \uparrow & & \uparrow \\ \partial\Sigma & \hookrightarrow & \partial M \end{array}$$

such that
- the algebraic self-intersection number of F vanishes

&
- F has a framed algebraically dual sphere.

Then F is homotopic, relative to $\partial\Sigma$, to an embedding, with a geometrically dual sphere.

Sphere embedding theorem

Theorem [Casson 1970s, Freedman 1982, Freedman-Quinn 1990]

M a connected 4-manifold, $\pi_1(M)$ good, Σ a simply connected 2-manifold.

$$\begin{array}{ccc} F\colon \Sigma & \longrightarrow & M \\ \uparrow & & \uparrow \\ \partial\Sigma & \hookrightarrow & \partial M \end{array}$$

such that • the algebraic self-intersection number of F vanishes

& • F has a framed algebraically dual sphere.

Then F is homotopic, relative to $\partial\Sigma$, to an embedding, $\underbrace{\text{with a geometrically dual sphere.}}_{\pi_1(M)\neq 1,\ \text{Powell-R.-Teichner 2018}}$

~~Sphere~~ Surface embedding theorem

Theorem [Casson 1970s, Freedman 1982, Freedman-Quinn 1990, Powell-R.-Teichner 2018, Stong 1994, Kasprowski-Powell-R.-Teichner 2022]

M a connected 4-manifold, $\pi_1(M)$ good, Σ a simply connected 2-manifold.

$$\begin{tikzcd} F\colon \Sigma \arrow[r] & M \\ \partial\Sigma \arrow[u, hook] \arrow[r, hook] & \partial M \arrow[u, hook] \end{tikzcd}$$

such that
- the algebraic self-intersection number of F vanishes

&
- F has a framed algebraically dual sphere.

Then F is homotopic, relative to $\partial\Sigma$, to an embedding, with a geometrically dual sphere

~~Sphere~~ Surface embedding theorem

Theorem [Casson 1970s, Freedman 1982, Freedman-Quinn 1990, Powell-R.-Teichner 2018, Stong 1994, Kasprowski-Powell-R.-Teichner 2022]

M a connected 4-manifold, $\pi_1(M)$ good, Σ a ~~simply~~ connected 2-manifold.

$$\begin{tikzcd} F\colon \Sigma \arrow[r] & M \\ \partial\Sigma \arrow[u, hook] \arrow[r, hook] & \partial M \arrow[u, hook] \end{tikzcd}$$

such that

- the algebraic self-intersection number of F vanishes

&

- F has a ~~framed~~ algebraically dual sphere.

Then F is homotopic, relative to $\partial\Sigma$, to an embedding, with a geometrically dual sphere

~~Sphere~~ Surface embedding theorem

Theorem [Casson 1970s, Freedman 1982, Freedman-Quinn 1990, Powell-R.-Teichner 2018, Stong 1994, Kasprowski-Powell-R.-Teichner 2022]

M a connected 4-manifold, $\pi_1(M)$ good, Σ a ~~simply~~ connected 2-manifold.

$$\begin{array}{ccc} F\colon \Sigma & \longrightarrow & M \\ \uparrow & & \uparrow \\ \partial\Sigma & \hookrightarrow & \partial M \end{array}$$

such that

- the algebraic self-intersection number of F vanishes

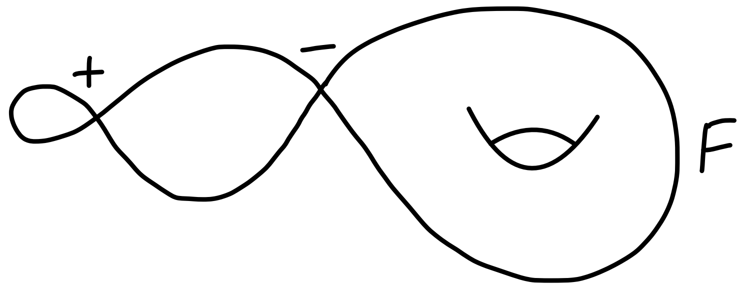
&

- F has a ~~framed~~ algebraically dual sphere.

Then F is homotopic, relative to $\partial\Sigma$, to an embedding, with a geometrically dual sphere if and only if the Kervaire-Milnor invariant $km(F) \in \mathbb{Z}/2$ vanishes.

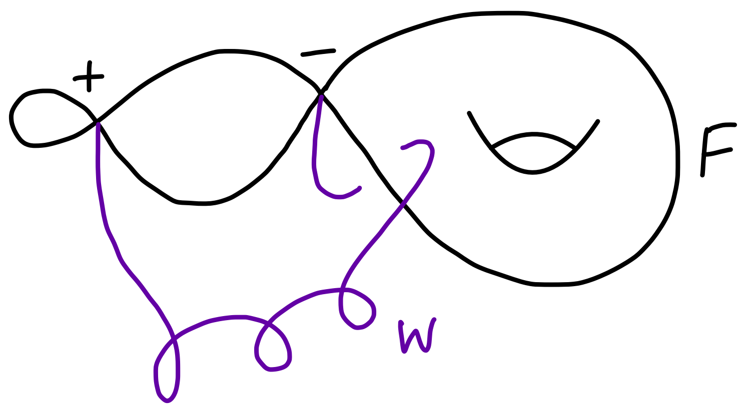
The Kervaire-Milnor invariant

- Assume that F has trivial algebraic self-intersection number



The Kervaire-Milnor invariant

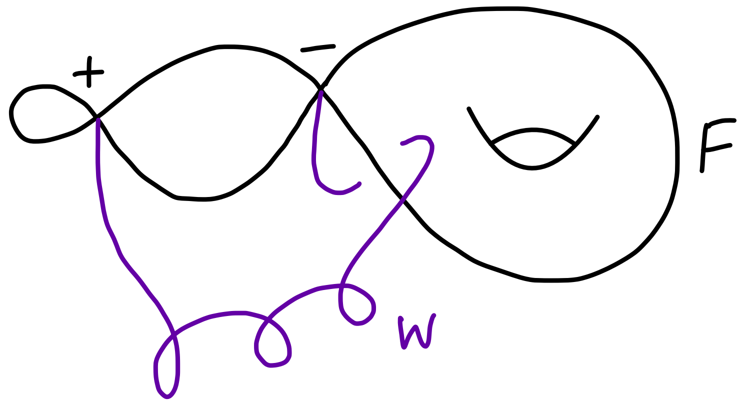
- Assume that F has trivial algebraic self-intersection number



- Then the self-intersections are in isolated double points, paired by Whitney discs

The Kervaire-Milnor invariant

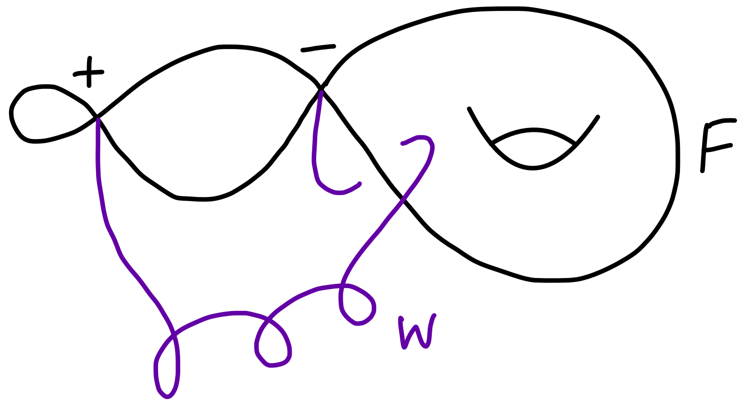
- Assume that F has trivial algebraic self-intersection number



- Then the self-intersections are in isolated double points, paired by Whitney discs
- $km(F, \{W\}) = \sum |\operatorname{Int}(W) \pitchfork F| \mod 2$

The Kervaire-Milnor invariant

- Assume that F has trivial algebraic self-intersection number



- Then the self-intersections are in isolated double points, paired by Whitney discs
- $km(F, \{W\}) = \sum |\mathrm{Int}(W) \pitchfork F| \mod 2$
- When is this independent of $\{W\}$?

Other directions

- More classification results for 4-manifolds [Kasprowski-Powell-R. 2022+]
- The effect of Gluck twisting on general 4-manifolds [Kasprowski-Powell-R. 2022]
- Construction of pairs of smooth 4-manifolds that are homotopy equivalent but not homeomorphic [Kasprowski-Powell-R. 2022+]
- Study of the mapping class group of the universal exotic smooth structure on $\mathbb{R}^4$ [Gompf-Orson-R. 2022+]
- Finding embedded spheres in knot traces (no algebraically dual sphere) [Feller-Miller-Nagel-Orson-Powell-R. 2021]
- Slicing knots in general 4-manifolds [Kjuchukova-Miller-R.-Sakalli 2021, Kasprowski-Powell-R.-Teichner 2022, Marengon-Miller-R.-Stipsicz 2022]

Links & 4-manifolds

Links and 4-manifolds

Theorem [Casson-Freedman 1984]

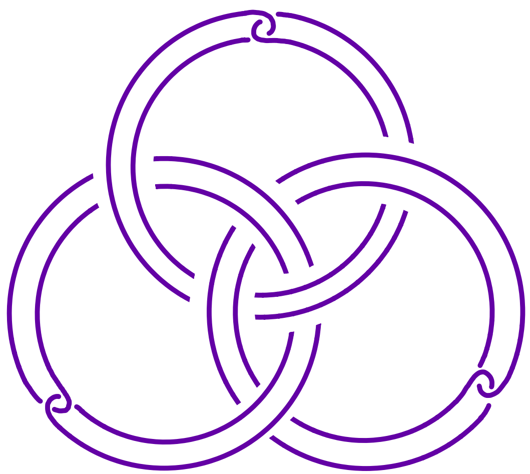
Surgery works for all 4-manifolds precisely if every member of a certain family of links is slice (with free fundamental group).

Links and 4-manifolds

Theorem [Casson-Freedman 1984]

Surgery works for all 4-manifolds precisely if every member of a certain family of links is slice (with free fundamental group).

E.g.



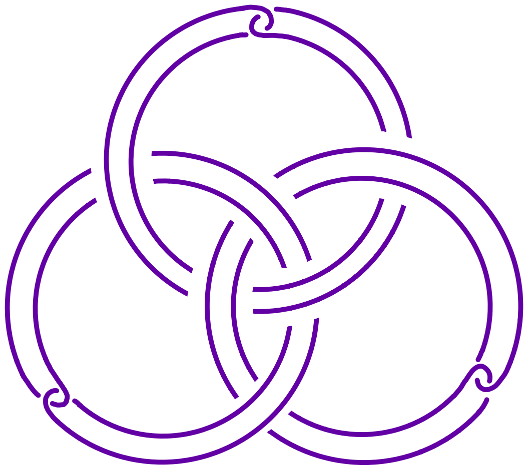
Slice?

Links and 4-manifolds

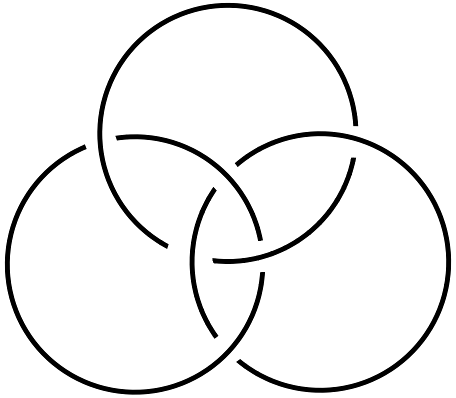
Theorem [Casson-Freedman 1984]

Surgery works for all 4-manifolds precisely if every member of a certain family of links is slice (with free fundamental group).

E.g.



Slice?



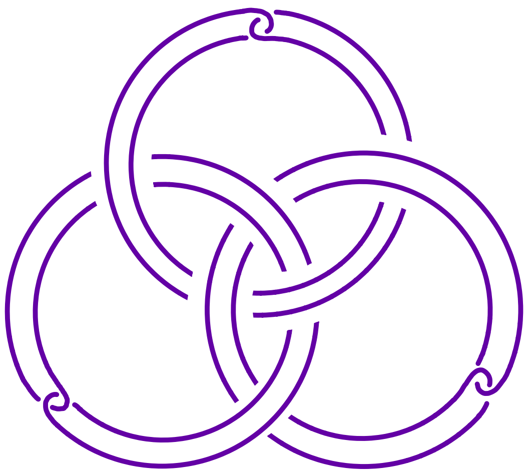
Not slice!

Links and 4-manifolds

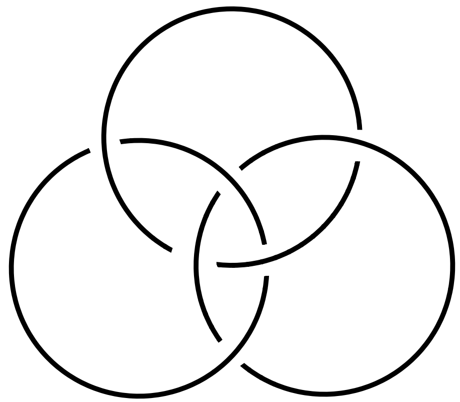
Theorem [Casson-Freedman 1984]

Surgery works for all 4-manifolds precisely if every member of a certain family of links is slice (with free fundamental group).

E.g.



Slice?



Not slice!

Key open question for classifying 4-manifolds: decide if these links are slice.

Questions?