Satellite operators as group actions on knot concordance

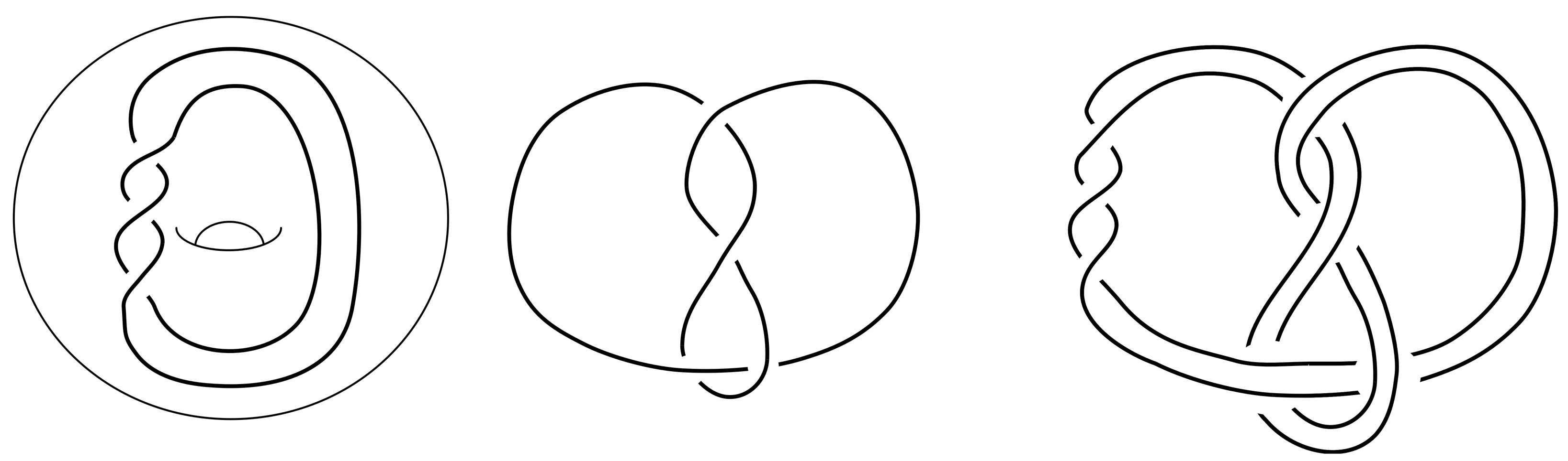
Arunima Ray (Brandeis University) – Joint work with Christopher W. Davis (University of Wisconsin–Eau Claire)

Any knot in a solid torus, i.e. a *satellite operator*, acts on knots in S^3 via the satellite construction. We introduce a generalization of satellite operators which form a group (unlike classical satellite operators) modulo a generalization of concordance. This group has an action on knots in homology spheres which is compatible with the classical satellite construction. Using this action, we obtain several results about the classical satellite construction on knot concordance classes.

Satellite operators act on knots in S^3

A *satellite operator* is a knot in the solid torus $S^1 \times D^2$ considered up to isotopy.

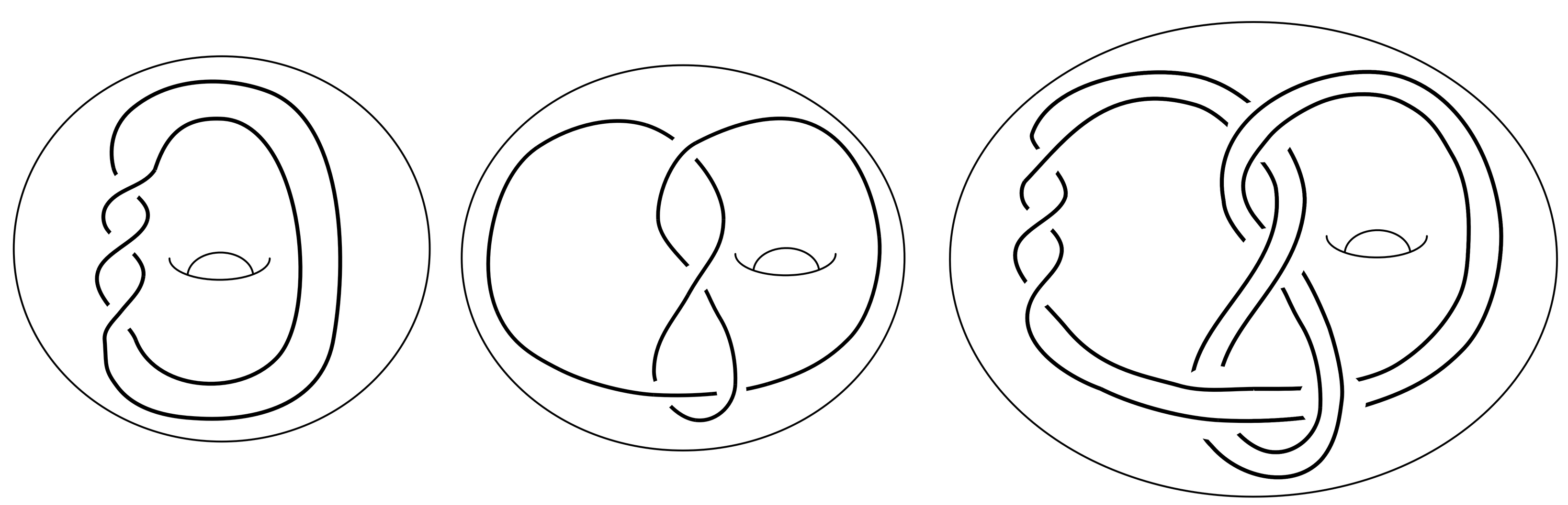
Satellite operators act on knots in S^3 via the satellite construction:



P $\quad$ K $\quad$ $P(K)$

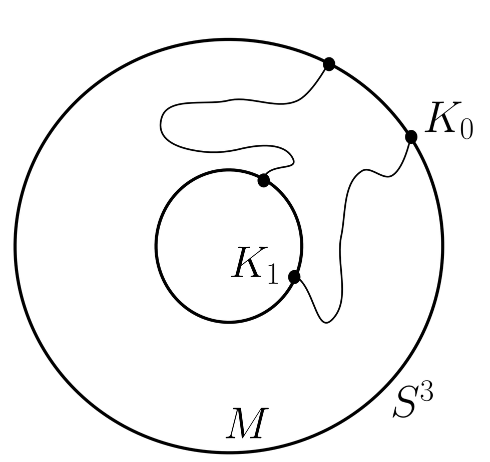
Under the operation below, satellite operators form a monoid $\mathcal{S}$ with a monoid action on knots, i.e. for any knot K,

$$(P \star Q)(K) = P(Q(K))$$



P $\quad$ Q $\quad$ $P \star Q$

However, satellite operators do not have inverses, i.e. $\mathcal{S}$ is *not* a group.



Knots K_0, K_1 are *concordant* if they cobound a smoothly embedded annulus in $S^3 \times [0,1]$. Knots modulo concordance form the *knot concordance group* $\mathcal{C}$.

Knots K_0, K_1 are *topologically concordant* if they cobound a locally flat, topologically embedded annulus in $S^3 \times [0,1]$. Knots modulo topological concordance form the *topological knot concordance group* $\mathcal{C}_{\text{top}}$.

Knots K_0, K_1 are *exotically concordant* if they cobound a smoothly embedded annulus in a possibly exotic $S^3 \times [0,1]$. Knots modulo exotic concordance form the *exotic knot concordance group* $\mathcal{C}_{\text{ex}}$.

For each $* \in \{\emptyset, \text{ex}, \text{top}\}$, any satellite operator P gives a map

$$\begin{aligned} P : \mathcal{C}_* &\to \mathcal{C}_* \\ K &\mapsto P(K) \end{aligned}$$

We can also consider knots up to (smooth, topological or exotic) concordance in homology 3–spheres. These yield the groups $\widehat{\mathcal{C}}_*$ for each $* \in \{\emptyset, \text{ex}, \text{top}\}$.

Let T be the torus $S^1 \times S^1$. A *homology cylinder* on T is a triple (V, i_+, i_-) where

- V is a compact, connected, oriented 3–manifold
- For $\epsilon = \pm 1$, $i_\epsilon : T \to \partial V$ is an embedding
- i_+ is orientation-preserving
- i_- is orientation-reversing
- $\partial V = i_+(T) \sqcup i_-(T)$
- $(i_\epsilon)_* : H_*(T) \to H_*(V)$ is an isomorphism

Two homology cylinders (U, i_+, i_-) and (V, j_+, j_-) are said to be *homology cobordant* if there is a smooth 4–manifold W with

$$\partial W = U \sqcup -V \Big/ i_+(x) = j_+(x), i_-(x) = j_-(x), \forall x \in T,$$

such that

$$H_*(U; R) \to H_*(W; R)$$
$$H_*(V; R) \to H_*(W; R)$$

are isomorphisms.

Homology cylinders form a monoid under the operation of 'stacking'. Under homology cobordism, homology cylinders form a group (Levine'01), denoted $\widehat{\mathcal{S}}$. In fact, we get such groups $\widehat{\mathcal{S}}_*$ for each $* \in \{\text{ex}, \text{top}\}$.

Each $\widehat{\mathcal{S}}_*$ has a group action on $\widehat{\mathcal{C}}_*$, the group of knots in homology 3–spheres up to concordance in the category $*$.

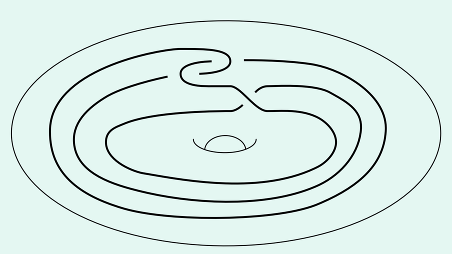
Main theorem

Let $* \in \{\text{ex}, \text{top}\}$. For each $\mathcal{C}_*$, there is a submonoid $\mathcal{S}_*$ of $\mathcal{S}$, an enlargement $\Psi : \mathcal{C}_* \hookrightarrow \widehat{\mathcal{C}}_*$, and a monoid morphism, $E : \mathcal{S}_* \to \widehat{\mathcal{S}}_*$, such that the following diagram commutes for all $P \in \mathcal{S}_*$:

$$\begin{tikzcd} \mathcal{C}_* \arrow[r, "P"] \arrow[d, "\Psi"'] & \mathcal{C}_* \arrow[d, "\Psi"] \\ \widehat{\mathcal{C}}_* \arrow[r, "E(P)"] & \widehat{\mathcal{C}}_* \end{tikzcd}$$

Here $\widehat{\mathcal{C}}_*$ is the group of knots in homology 3–spheres up to concordance in the category $*$. Ψ is the natural inclusion.

The map E is given as follows: for a satellite operator P in a solid torus V, carve out a neighborhood of P inside V. The resulting 3–manifold has two toral boundary components, with canonical maps from the torus $T = S^1 \times S^1$.



The submonoids $\mathcal{S}_*$ correspond to certain 'strong winding number $\pm$ 1' satellite operators, such as the one shown above. Any winding number $\pm$ 1 satellite operator which is unknotted as a knot in S^3 is strong winding number $\pm$ 1.

Consequences

Suppose that P is a satellite operator with strong winding number ± 1. Then

- $P : \mathcal{C}_{ex} \to \mathcal{C}_{ex}$ is injective,
- $P : \mathcal{C}_{top} \to \mathcal{C}_{top}$ is injective, and
- if the smooth 4-dimensional Poincaré Conjecture holds, $P : \mathcal{C} \to \mathcal{C}$ is injective.

This was known previously by Cochran–Davis–R.'14

Let $P \subseteq V = S^1 \times D^2$ be a satellite operator with winding number ± 1. If the meridian of P is in the normal subgroup of $\pi_1(E(P))$ generated by the meridian of V, then P is strong winding number ± 1 and there exists another strong winding number ± 1 satellite operator $\overline{P}$ such that $E(P)^{-1} = E(\overline{P})$ as homology cylinders. Therefore, $\overline{P}(P(K))$ is concordant to K for any knot K. Moreover, the function $P : \mathcal{C}_* \to \mathcal{C}_*$ is bijective for each $* \in \{\text{ex}, \text{top}, \}$.

As a result, for each $m \geq 0$, the satellite operator P_m shown below has an inverse satellite operator $\overline{P_m}$ which can be explicitly drawn, i.e. $\overline{P_m}(P_m(K))$ is concordant to K for any knot K. Moreover, each P_m gives a bijective map $P_m : \mathcal{C}_* \to \mathcal{C}_*$ for $* \in \{\text{ex}, \text{top}\}$ and is distinct from all connected sum operators.

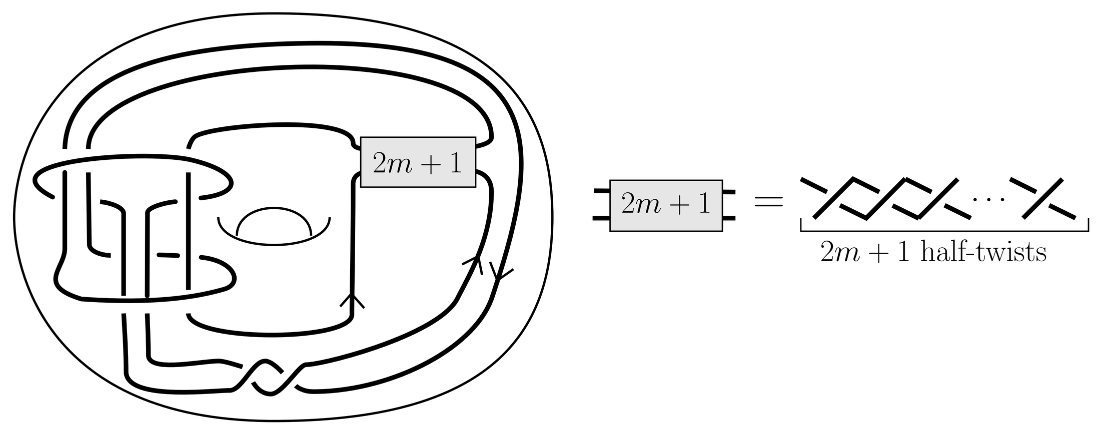


Fig. 5: A class of bijective satellite operators $\{P_m\}_{m \geq 0}$, with inverses $\overline{P_m}$

The inverse satellite operators $\overline{P_m}$ can be drawn by using a handlebody diagram of $E(P_m)$ as the complement of a 2–component link in S^3 with certain marked curves.

A. R. was partially supported by NSF–DMS–1309081.