SURFACES IN 4-MANIFOLDS
PROBLEMS

ARUNIMA RAY

Within each section, the exercises are in roughly increasing order of difficulty and amount of outside information required.

Link to these problems: tinyurl.com/aruray-problems
Link to lecture notes: tinyurl.com/aruray-notes
Link to feedback form: tinyurl.com/aruray-feedback

1. Exercises for Lecture 1

Exercise 1.1. Let $K\colon S^1 \to S^3$ be a 1-knot. Consider the disc given by $\operatorname{cone}(K) \subseteq \operatorname{cone}(S^3) = B^4$. Show that this disc is locally flat if and only if K is unknotted. Feel free to use, without proof, that a 1-knot K is unknotted if and only if $\pi_1(S^3 \setminus K) \cong \mathbb{Z}$.

Exercise 1.2. Let $K\colon S^1 \to S^3$ be a 1-knot, and let M be a smooth 4-manifold. Let $\mathring{M}$ denote $M \setminus \mathring{B}^4$. We say that K is smoothly slice in M if $K \subseteq \partial\mathring{M} = S^3$ bounds a smoothly embedded disc in $\mathring{M}$.

Suppose Σ is a 4-manifold homotopy equivalent to S^4. Suppose there is a knot K which is smoothly slice in Σ but not in S^4. Under these hypotheses prove that there is an exotic smooth structure on S^4.

Feel free to use, without proof (!), the 4-dimensional topological Poincaré conjecture: every 4-manifold homotopy equivalent to S^4 is homeomorphic to S^4.

Exercise 1.3. Show that every 1-knot in S^4 bounds a smoothly embedded disc in S^4.

Consider an arbitrary 1-knot $K\colon S^1 \to M^4$, where M^4 is some smooth 4-manifold. Give a complete characterisation of when K bounds a smoothly embedded disc in M^4.

Exercise 1.4. Let $J\colon S^1 \to S^3$ be an (oriented) 1-knot. Let $-J$ denote the knot obtained from J by changing the orientations on both S^1 and S^3. From the point of view of a diagram of J, the knot $-J$ is obtained by changing all the crossings of J and then changing the direction of the arrow.

Show that the knot $J \# -J$ is slice. One approach to this question would be to consider the spinning construction from the lecture.

Exercise 1.5. Let K be the 2-knot obtained by spinning a 1-knot k. Show that $\pi_1(S^4 \setminus K) \cong \pi_1(S^3 \setminus k)$. Conclude that there are infinitely many non-isotopic 2-knots in S^4.

2. Exercises for Lecture 2

Exercise 2.1. Prove that every smooth 2-knot $K\colon S^2 \to S^4$ bounds some compact, connected, oriented 3-manifold smoothly embedded in S^4. This is called a *Seifert solid* for K. What about arbitrary knotted surfaces $\Sigma \to S^4$?

One way to approach this problem is to construct a map to S^1 and pull back a regular value. Constructing this map carefully might require some obstruction theory.

Exercise 2.2. Prove that every 1-knot in S^3 is the boundary of some smoothly embeddded, compact, oriented surface in B^4.

Last updated: 11:47am Grenoble time on 21st June, 2024.

Let k be a 1-knot in S^3 which can be changed to the unknot by c crossing changes. Show that k bounds a genus c compact, oriented, smoothly embedded surface in B^4.

Exercise 2.3. Prove that every 1-knot in S^3 is smoothly slice in $S^2 \times S^2$. Prove that every 1-knot in S^3 is smoothly slice in $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Can you say anything about the homology class of the discs that you constructed?

Note: this is a hard problem. Look up something called the *Norman trick*. For the second question, it might be helpful to know that $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is diffeomorphic to $S^2 \widetilde{\times} S^2$, the twisted S^2-bundle over S^2.

Exercise 2.4. Let D denote the unit 4-ball in $\mathbb{R}^4$. Let A denote the xy-plane, and B denote the zw-plane, both in $\mathbb{R}^4$. Show that $\partial D \cap (A \cup B)$ is a Hopf link. Consider how orientations on A and B induce orientations on the Hopf link.

It will be helpful to come up with a nice description of D. A useful idea is to think of $\mathbb{R}^4$ as $\mathbb{R}^3 \times \mathbb{R}$, thinking of the last coordinate as time, and then drawing 'stills' from movies. How will D show up in these stills? Can you identify the boundary of D with S^3, specifically your usual picture of S^3?

3. Exercises for Lecture 3

Exercise 3.1. Confirm that the Alexander polynomial of an arbitrary Whitehead double is 1. Recall that the Alexander polynomial is only well defined up to multiplication by $\pm t^k$, for $k \in \mathbb{Z}$.

Exercise 3.2. Let K be a smoothly slice 1-knot in S^3. Show that the Whitehead double, with either clasp, is smoothly slice.

It might help to consider a helpful Seifert surface for the Whitehead double.

Exercise 3.3. Let Σ be a topologically unknotted, closed, orientable surface in S^4. Show that $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}$.

Exercise 3.4. In the lecture we used the disc embedding theorem to construct a locally flat slice disc Δ for the Whitehead double of an arbitrary 1-knot. Show that $\pi_1(B^4 \setminus \Delta) \cong \mathbb{Z}$.

This is a challenging problem. Recall there is a geometrically dual sphere in the outcome of the disc embedding theorem.

4. Exercises for Lecture 4

Exercise 4.1. Let $\Sigma \subseteq S^4$ be a smoothly embedded closed orientable surface. Show that after some number of weak internal stabilisations Σ becomes unknotted.

It might be helpful to consider a Seifert solid for Σ and a Heegaard decomposition for it.

Exercise 4.2. Let $\Sigma \subseteq S^4$ be a surface knot. What is the effect of weak internal stabilisation on $\pi_1(S^4 \setminus \Sigma)$?

Hint: A single relation is added.

Exercise 4.3. Use the s-cobordism theorem to prove the topological 4-dimensional Poincaré conjecture: any 4-manifold homotopy equivalent to S^4 is homeomorphic to S^4.

I don't know how to do this without the following consequence of a theorem of Bryant and Lacher: if Σ is a manifold homotopy equivalent to S^4 then the cone on Σ is a 5-manifold.

Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
Email address: aruray@mpim-bonn.mpg.de
URL: http://people.mpim-bonn.mpg.de/aruray/