Curriculum vitae

Arunima Ray

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Research interests

Low-dimensional topology: knots and links; 3- and 4-manifolds

Employment

Max-Planck-Institut für Mathematik, Bonn, Germany

Lise Meitner research group leader (tenure track W2 position) — Aug 2020 – present
Research group leader/advanced researcher (temporary W2 position) — Aug 2017 – Jul 2020

Brandeis University, Waltham, USA

Postdoctoral instructor — Jul 2014 – Jun 2017

Extended research visits

Hausdorff Institute for Mathematics, Bonn, Germany

Visitor, Junior Hausdorff Trimester Program (Topology) — Sep – Dec 2016

Education

Rice University, Houston, USA

Ph.D. Mathematics — May 2014
Casson towers and filtrations of the smooth knot concordance group
Thesis advisor: Tim D. Cochran

M.A. Mathematics — Dec 2012

State University of New York, College at Geneseo, Geneseo, USA

B.A. Mathematics and Biochemistry — May 2009
summa cum laude
Honours in mathematics

Grants & awards

- **AIM SQuaRE grant**
  American Institute for Mathematics, 2019–present; project titled 'Studying knot concordance via branched covers of S^4', joint with Ryan Blair, Patricia Cahn, Alexandra Kjuchukova, Kent Orr, and Hannah Schwartz.
- **Experiential learning and teaching grant**
  Experiential learning and teaching at Brandeis, Spring 2017; project titled 'An experiential learning approach to MATH 23b', awarded 1,600 USD to support a course assistant in a partially flipped course on mathematical proof-writing.

- **AMS-Simons travel grant**
  American Mathematical Society, Simons Foundation 2015–17; awarded annually to 60 early-career US mathematicians, 2,000 USD per year for two years to be used for research-related travel.
- **AWM-NSF travel grant**
  Association for Women in Mathematics, National Science Foundation, June 2015; awarded 960 USD for travel to the Princeton low-dimensional topology workshop 2015.
- **Nettie S. Autrey fellowship**
  Rice University Office of Graduate & Postdoctoral Studies, 2013–14; awarded annually to a single graduate student in the Schools of Natural Sciences or Engineering.
- **Robert Lowry Patten award**
  Rice University Graduate Student Association, 2013; one of four annual awards recognising service towards improving PhD student life and education.

Past & upcoming research talks

2023

Interactions of low-dimensional topology and quantum field theory, SwissMAP research station • Clay mathematics institute workshop: gauge theory and topology: in celebration of Peter Kronheimer's 60th birthday

2022

Annual conference, Indian women in mathematics • Stanford topology seminar • Colloquium, Georg-August-Universität Göttingen • Montana state university mathematics seminar • Columbia geometric topology seminar • Colloquium in pure mathematics, University of Hamburg • SUNY Geneseo mathematics colloquium • Princeton topology seminar • Durham topology seminar • New developments in four dimensions, conference at University of Victoria • SFB-lecture, Universität Regensburg • Knot online seminar • Universität Bonn topology seminar • Westfälische Wilhelms-Universität Münster topology seminar • MATH+ Colloquium, Berlin mathematical school • MPIM low-dimensional topology seminar (2 talks) • Banff International Research Station Workshop: Topology in dimension 4.5 • MIT topology seminar

2021

AMS special session on invariants of knots and links, joint mathematics meetings, Washington DC • MPIM topology seminar • Building bridges seminar • Regensburg low-dimensional geometry and topology seminar • Nearly carbon neutral geometry and topology conference, plenary talk • Minisymposium on low-dimensional topology, European congress of mathematics • Hausdorff center for mathematics symposium • Special session on recent advances in low-dimensional topology, AMS southeastern sectional meeting • Geometry and topology seminar, Paris 6

2020

Clay mathematics institute workshop: low-dimensional topology • MPIM topology seminar • Geometry and topology seminar, CIRGET (Montreal) • University of Wisconsin-Milwaukee topology seminar • University of North Carolina, Chapel Hill AWM lecture series • Tech topology conference

2019

Duke university geometry/topology seminar • Banff international research station workshop: unifying 4-dimensional knot theory • LMS Durham symposium: pseudoholomorphic curves and gauge theory in low-dimensional topology • Swissknots 2019 • Conference on knot concordance and low-dimensional manifolds, Le Croisic, France • Knots and braids in Norway • Indiana university topology seminar • Rice university topology seminar • Winterbraids IX (contributed talk)

2018
Workshop on twisted and quantum knot invariants, Durham university • Workshop on Topologie, Oberwolfach (contributed talk) • 29. Nordrhein-Westfalen topology meeting, Universität Bonn • MPIM topology seminar

2017
Université de Genève séminaire de topologie et géométrie • Universität Regensburg SFB seminar • Universität Bern mathematics colloquium • Universität Bonn topology seminar • Georgia international topology conference, UGA • AMS special session on invariants of knots, links and 3-manifolds, AMS spring eastern sectional meeting, Hunter College, City university of New York • Rice university topology seminar • UT Austin topology seminar • Conference on Floer homologies and topology of 4-manifolds, UMass Amherst • Notre Dame topology seminar • Geometry and topology seminar, CIRGET (Montreal) • Brandeis university topology seminar • UC Riverside special colloquium, • Bowdoin college colloquium • AMS Session on topology and manifolds, joint mathematics meetings, Atlanta (contributed talk)

2016
UNC Charlotte special colloquium • Hausdorff center for mathematics seminar • Hausdorff institute for mathematics knot concordance and 4-manifolds seminar • Topology in dimension 3.5: a conference in memory of Tim Cochran • Indiana university topology seminar • Rice university topology seminar • San Francisco state university special colloquium • University of Virginia geometry seminar • Lafayette college mathematical adventures and diversions • SUNY Geneseo mathematics department colloquium • AMS Session on topology and knot theory, joint mathematics meetings, Seattle (contributed talk)

2015
Brandeis University IGERT seminar • Session on low-dimensional topology and geometric group theory, 2015 CMS winter meeting, Montreal • AMS special session on geometric perspectives on knot theory, AMS central sectional meeting, Loyola university • Moab topology conference • AWM research symposium (poster presentation) • Brandeis university everytopic seminar

2014
Syracuse university topology seminar • Wesleyan university topology seminar • AMS special session on knot concordance and 4-manifolds, AMS central sectional meeting, University of Wisconsin-Eau Claire • Boston college topology seminar • Workshop on topology and invariants of smooth 4-manifolds, Simons center for geometry and physics • University of Wisconsin-Eau Claire STEM colloquium • AMS special session on knots and their invariants, joint mathematics meetings, Baltimore (contributed talk)

2013
AMS special session on geometric aspects of 3-manifold invariants, AMS central sectional meeting, Washington university at St. Louis • AMS special session on geometric topology in low dimensions, AMS central sectional meeting, Washington university at St. Louis • Rice university topology seminar (2 talks) • Lehigh university geometry and topology conference • SUNY Geneseo mathematics department colloquium

Teaching & mentorship

Postdocs mentored (MPIM)

- JungHwan Park — Jul 2017 – Jun 2018
- Paolo Aceto — Sep 2017 – Aug 2018
- Filip Misev — Mar 2018 – Feb 2020
- Sümeyra Sakallı — Sep 2018 – Aug 2020
- Alexandra Kjuchukova — Oct 2018 – Aug 2020
- Alberto Cavallo — Nov 2018 – Aug 2020
- Wenzhao Chen — Sep 2019 – Aug 2021
- Anthony Conway — Sep 2019 – Dec 2020
- Marco Marengon — Sep 2020 – Aug 2021
- Biji Wong — Sep 2020 – Jan 2022
- Tom Hockenhull — Sep 2020 – Aug 2022
- Patrick Orson — Sep 2021 – Aug 2022
- Abhishek Mallick — Sep 2021 – Aug 2022
- Katherine Raoux — Jul 2021 – Jun 2022
- Sarah Blackwell — Sep 2022 – Aug 2023

Students (MPIM/Universität Bonn)

- PhD thesis:
  - Benjamin Matthias Ruppik, Knotted surfaces, Casson-Whitney unknotting and deep slice disks — 2018-22
  - Hyeonhee Jin (joint with Peter Teichner) — 2021 – present
- PhD thesis committee member:
  - Tynan Kelly, Twisted linking numbers and Casson-Gordon invariants (Brandeis) — 2015
  - Katherine Raoux, Tau-invariants for knots in rational homology spheres (Brandeis) — 2016
  - Jason Joseph, Applications of Alexander ideals to knotted surfaces in 4-space (UGA) — 2020
  - Oliver Singh, Pseudo-isotopies and embedded surfaces in 4-manifolds (Durham) — 2022
- Masters thesis:
  - Didac Violan Aris, The proper h-cobordism theorem and the 4-dimensional Poincaré conjecture — 2018-19
  - Constanze Schwarz, Donaldson's theorem as a sliceness obstruction — 2020-21
  - Isacco Nonino, Smooth structures on non-compact 4-manifolds — 2021-22
  - Fadi Mezher, Kreck's modified surgery and applications to the classification of manifolds — 2021-22
  - Yikai Teng — 2022

  - Raphael Floris 2022-23
  - Agata Sienicka 2022-23
  - Magdalina von Wunsch-Rolshoven 2022-23

- Masters thesis, second evaluator:
  - Mihail Arabadji, Indeterminacy of triple linking numbers 2019-20
  - Freya Bretz, Surface graphs, surface bundles and spin structures 2020

- Bachelors thesis:
  - Miriam Ruß, Milnor's invariants of links 2018
  - Ekin Ergen, Fundamentals of 3-manifold topology 2019
  - Magdalina von Wunsch-Rolshoven, 3-manifolds as branched covers 2020
  - Arne Beines, The bridge number of knots 2020
  - Felix Bertram, Generalised Whitehead manifolds 2020
  - Frieda Kern, Universal branching links 2021
  - Yves Etienne Jäckle, The trace embedding lemma 2021
  - Line Uhe, The algebraic knot concordance group 2022
  - Maximilian Hans, Integer homology spheres 2022

- Bachelors thesis, second evaluator:
  - Florian Tecklenburg, Euler number of a map 2020
  - Pauline Kranenburg, Wirtinger-Präsentierung für Knotengruppen 2022

Lecture & seminar courses

- **Calculus II,** Rice University Summer 2011
- **Calculus II,** Rice University Fall 2011
- **Sequences and series,** Rice emerging scholars program, Rice University Summer 2012
- **Introduction to algebra, Part II,** Brandeis University Spring 2015
  (honours course)
- **Applied linear algebra,** Brandeis University Spring 2015
- **Topology I**, Brandeis University Fall 2015, Fall 2014
  (first-year PhD course)
- **Introduction to topology,** Brandeis University Fall 2015
  (honours course)
- **Introduction to proofs,** Brandeis University Spring 2017, Spring 2016
  (writing-intensive course)
- **Topology II**, Brandeis University Spring 2017
  (first-year PhD course)
- **Topology of 4-manifolds**, MPIM/Universität Bonn Winter 2018/19
  (masters-level course, co-taught with Peter Teichner)
- **Topological manifolds**, MPIM/Universität Bonn Winter 2020/21
  (masters-level course, co-taught with Mark Powell)
  offered online, 30+ regular viewers worldwide

- **Topological manifolds II**, MPIM/Universität Bonn — Summer 2021
  (masters-level seminar course, co-taught with Mark Powell)
- **Einführung in die Topologie und Geometrie**, Universität Bonn — Summer 2021
  (undergraduate course, in German, co-taught with Daniel Kasprowski)

Minicourses

- **The double suspension theorem,** 3 lectures, MPIM/Universität Bonn — Jan 2022
- **Slice knots and knot concordance,** 3 lectures, Winterbraids XI, Dijon — Dec 2021
- **Topological 4-manifolds: the disc embedding theorem and beyond,** 5 lectures, Tech topology summer school — Jul 2021
- **Exotic $\mathbb{R}^4$s,** 2 lectures, MPIM/Universität Bonn — Jan 2018

Reading courses (Brandeis University)

- **Abstract algebra**, Doreen Reuchsel — Spring 2015
  (post-baccalaureate student)
- **Mathematical biology**, Gabriel Pimentel — Summer 2015
  (undergraduate student, Brazil scientific mobility program)
- **4-manifolds**, Anthony Villafranca, McKee Krumpak, Langte Ma — Spring 2016
  (masters/PhD students)

Service & Outreach

- **Scientific committee**, MPIM, August 2017 – present; advise on selection of new guests at MPIM.
- **Gender equality officer**, MPIM, November 2020 – present.
- **Moderator,** MPIM Preprints, September 2020 – present.
- **Outreach event organisation**
  - Graduate research opportunities for women, a conference for undergraduate women in mathematics, Universität Bonn, March 2022, March 2023
  - Teatime with women in mathematics, event for PhD students and postdocs, Universität Bonn/MPIM, June 2021
  - Virtual screening, *Picture a scientist*, MPIM, March 2021
- **Outreach talks & event participation/moderation**
  - Moderator, discussion on *Picture a scientist*, Community-building in the Langlands program (CLAP), conference at Universität Bonn, August 2022
  - Lecturer, *Einführung in die Knotentheorie*, 2 lectures in German to high school students participating in Schüler*innenwoche 2022, Universität Bonn, August 2022
  - Participant, Communicating mathematics workshop, hybrid event at Cornell university, August 2022
  - Participant, Train the trainer workshop, Universität Heidelberg, March 2022
  - Speaker, *Welche Form hat unser Universum?*, Bonner Mathenacht, November 2021

- Invited guest, Ally day, Teatime for people in mathematics, Universität Bonn mathematics department, November 2021
- Speaker, *Knoten in der Mathematik*, Bonner Matheclub, June 2021
- Moderator, discussion on *Picture a scientist*, Universität Bonn mathematics department, May 2021
- Invited guest, Upstream lunch break, Universität Heidelberg, February 2021, February 2022
- Invited guest, Teatime for women in mathematics, Universität Bonn mathematics department, January 2021
- Judge, AWM essay contest: biographies of contemporary women in mathematics, 2016, 2017, 2018, 2020, 2021, 2022.
- Panelist, Increasing diversity in STEM and ways in which high school and middle school girls can make contributions: a panel discussion, Sonia Kovalevsky day, University of Wisconsin-Eau Claire, 2014
- Panelist, Women in grad school: a panel discussion, Women's resource center, Rice university, 2012
- Panelist, I know what you should do next summer: a panel discussion, Nebraska conference for undergraduate women in mathematics, 2009

• **Referee reports & quick opinions for:** *Inventiones Mathematicae*, *Journal of Differential Geometry*, *Geometry & Topology*, *Journal of Topology*, *Journal für die reine und angewandte Mathematik (Crelle's journal)*, *Compositio Mathematica*, *Advances in Mathematics*, *Journal of the European Mathematical Society*, *Algebraic & Geometric Topology*, *Forum of Mathematics, Sigma*, *International Mathematical Research Notices*, *Michigan Mathematical Journal*, *NYJM Monographs*, *Mathematical research letters*, *Communications in Analysis and Geometry*, *Bulletin of the London Mathematical Society*, *Journal of Knot Theory & its Ramifications*, *Topology Proceedings*, *Topology & its Applications.*

• **Reviewer,** Mathematical Reviews, Zentralblatt MATH.

• **Conference (co-)organisation:**

- MFO workshop: Morphisms in low dimensions, January 2023
- MATRIX-MFO tandem workshop: Invariants and structures in low-dimensional topology, September 2021
- AMS special session on women in topology, Joint mathematics meetings, January 2017, January 2019, January 2020
- Workshop on 4-manifolds, MPIM, September 2019
- Workshop on four-manifolds: confluence of high and low dimensions, Fields Institute, July 2019
- Workshop on smooth concordance classes of topologically slice knots, American institute for mathematics, June 2019
- AWM special session on 3- and 4-manifolds, AWM research symposium, Rice university, April 2019
- Conference on 4-manifolds and knot concordance, MPIM, October 2016

- **Seminar (co-)organisation:**
  - Topology seminar, MPIM, August 2017 – present
  - Low-dimensional topology seminar, MPIM, June 2020 – present
  - EveryTopic seminar, Brandeis university, Spring 2017, 2015–16
  - Spectral sequences seminar, Brandeis university, Fall 2014
  - Heegaard-Floer "computationar", Rice University, Spring 2014
  - 4-manifolds seminar, Rice university, Spring 2014
  - Characteristic classes seminar, Rice university, Summer 2012
  - Knot invariants seminar, Rice university, Summer 2011
- **Undergraduate advisor** (18 advisees), Brandeis university 2015–16
- Examining committee for a postdoctoral position, SISSA (Trieste)
- NSF merit review panel

Miscellaneous

- Languages:
  - English (near-native/spoken since early childhood)
  - German (C1)
  - Bengali (native)
  - Hindi (conversant)
- Citizenship: India

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Publication list

Edited volumes

[1] S. Behrens, B. Kalmár, M. H. Kim, M. Powell, and A. Ray, eds. *The disc embedding theorem.* Oxford University Press, 2021.

Book chapters

[2] S. Behrens, D. Kasprowski, M. Powell, and A. Ray. "Skyscrapers are standard: the details". In: *The disc embedding theorem.* Oxford University Press, 2021.

[3] S. Behrens, M. Powell, and A. Ray. "Context for the disc embedding theorem". In: *The disc embedding theorem.* Oxford University Press, 2021.

[4] M. H. Kim, P. Orson, J. Park, and A. Ray. "Good groups". In: *The disc embedding theorem.* Oxford University Press, 2021.

[5] M. H. Kim, P. Orson, J. Park, and A. Ray. "Open problems". In: *The disc embedding theorem.* Oxford University Press, 2021.

[6] D. McCoy, J. Park, and A. Ray. "Picture camp". In: *The disc embedding theorem.* Oxford University Press, 2021.

[7] J. Meier, P. Orson, and A. Ray. "Shrinking starlike sets". In: *The disc embedding theorem.* Oxford University Press, 2021.

[8] P. Orson, M. Powell, and A. Ray. "Surgery theory and the classification of simply connected 4-manifolds". In: *The disc embedding theorem.* Oxford University Press, 2021.

[9] P. Orson, M. Powell, and A. Ray. "The s-cobordism theorem, the sphere embedding theorem, and the Poincaré conjecture". In: *The disc embedding theorem.* Oxford University Press, 2021.

[10] W. Politarczyk, M. Powell, and A. Ray. "From immersed discs to capped gropes". In: *The disc embedding theorem.* Oxford University Press, 2021.

[11] M. Powell and A. Ray. "Basic geometric constructions". In: *The disc embedding theorem.* Oxford University Press, 2021.

[12] M. Powell and A. Ray. "Gropes, towers, and skyscrapers". In: *The disc embedding theorem.* Oxford University Press, 2021.

[13] M. Powell and A. Ray. "Intersection numbers and the statement of the disc embedding theorem". In: *The disc embedding theorem.* Oxford University Press, 2021.

[14] M. Powell and A. Ray. "The development of topological 4-manifold theory". In: *The disc embedding theorem.* Oxford University Press, 2021.

[15] A. Ray. "Outline of the upcoming proof". In: *The disc embedding theorem.* Oxford University Press, 2021.

Published and accepted papers

[16] A. Ray. "Slice knots which bound punctured Klein bottles". In: *Algebr. Geom. Topol.* 13.5 (2013), pp. 2713–2731.

[17] T. D. Cochran, C. W. Davis, and A. Ray. "Injectivity of satellite operators in knot concordance". In: *J. Topol.* 7.4 (2014), pp. 948–964.

[18] A. Ray. "Casson towers and filtrations of the smooth knot concordance group". In: *Algebr. Geom. Topol.* 15.2 (2015), pp. 1119–1159.

[19] A. Ray. "Satellite operators with distinct iterates in smooth concordance". In: *Proc. Amer. Math. Soc.* 143.11 (2015), pp. 5005–5020.

[20] T. D. Cochran and A. Ray. "Shake slice and shake concordant knots". In: *J. Topol.* 9.3 (2016), pp. 861–888.

[21] C. W. Davis and A. Ray. "Satellite operators as group actions on knot concordance". In: *Algebr. Geom. Topol.* 16.2 (2016), pp. 945–969.

[22] C. W. Davis and A. Ray. "A new family of links topologically, but not smoothly, concordant to the Hopf link". In: *J. Knot Theory Ramifications, memorial volume for T. Cochran* 26.2 (2017), pp. 1740002, 12.

[23] A. Ray and D. Ruberman. "Four-dimensional analogues of Dehn's lemma". In: *J. Lond. Math. Soc. (2)* 96.1 (2017), pp. 111–132.

[24] C. W. Davis, M. Nagel, J. Park, and A. Ray. "Concordance of knots in $S^1 \times S^2$". In: *J. Lond. Math. Soc. (2)* 98.1 (2018), pp. 59–84.

[25] J. Park and A. Ray. "A family of non-split topologically slice links with arbitrarily large smooth slice genus". In: *Proc. Amer. Math. Soc.* 146.1 (2018), pp. 439–448.

[26] P. Feller, J. Park, and A. Ray. "On the Upsilon invariant and satellite knots". In: *Math. Z.* 292.3-4 (2019), pp. 1431–1452.

[27] P. Aceto, M. H. Kim, J. Park, and A. Ray. "Pretzel links, mutation, and the slice-ribbon conjecture". In: *Math. Res. Lett.* 28.4 (2021), pp. 945–966.

[28] C. W. Davis, J. Park, and A. Ray. "Linear independence of cables in the knot concordance group". In: *Trans. Amer. Math. Soc.* 374.6 (2021), pp. 4449–4479.

[29] P. Feller, A. N. Miller, M. Nagel, P. Orson, M. Powell, and A. Ray. "Embedding spheres in knot traces". In: *Compositio Mathematica* 157.10 (2021), 2242–2279.

[30] S. Friedl, J. Park, B. Petri, J. Raimbault, and A. Ray. "On distinct finite covers of 3-manifolds". In: *Indiana Univ. Math. J.* 70.2 (2021), pp. 809–846.

[31] S. Baader, A. Kjuchukova, L. Lewark, F. Misev, and A. Ray. "Average four-genus of two-bridge knots". To appear: *Proceedings of AMS*. 2022+. arXiv: `1902.05721 [math.GT]`.

Preprints

[32] J. Meier, P. Orson, and A. Ray. "Null, recursively starlike-equivalent decompositions shrink". Submitted. 2019. arXiv: `1909.06165 [math.GT]`.

[33] P. Aceto, C. Bregman, C. W. Davis, J. Park, and A. Ray. "Isotopy and equivalence of knots in 3-manifolds". Submitted. 2020. arXiv: `2007.05796 [math.GT]`.

[34] M. Powell, A. Ray, and P. Teichner. "The 4-dimensional disc embedding theorem and dual spheres". Submitted. 2020. arXiv: `2006.05209 [math.GT]`.

[35] A. Kjuchukova, A. N. Miller, A. Ray, and S. Sakallı. "Slicing knots in definite 4-manifolds". Submitted. 2021. arXiv: `2112.14596 [math.GT]`.

[36] D. Kasprowski, M. Powell, and A. Ray. "Counterexamples in 4-manifold topology". Submitted. 2022. arXiv: `22203.13332 [math.GT]`.

[37] D. Kasprowski, M. Powell, and A. Ray. "Gluck twists on concordant or homotopic spheres". Submitted. 2022. arXiv: `22206.14113 [math.GT]`.

[38] D. Kasprowski, M. Powell, A. Ray, and P. Teichner. “Embedding surfaces in 4-manifolds”. Submitted. 2022. arXiv: `2201.03961 [math.GT]`.

Miscellaneous

[39] T. D. Cochran, C. W. Davis, and A. Ray. “Injectivity of satellite operators in knot concordance”. In: *OWR Reports* 9.12 (2012), pp. 1698–1701.

[40] A. Ray. *Casson Towers and Filtrations of the Smooth Knot Concordance Group*. Thesis (Ph.D.)–Rice University. ProQuest LLC, Ann Arbor, MI, 2014, p. 70.

[41] S. L. Harvey, J. Park, and A. Ray. “Pure braids, Whitney towers, and 0-solvability”. In: *OWR Reports* 14.4 (2017), pp. 3020–3021.

[42] M. Powell, A. Ray, and P. Teichner. “The 4-dimensional sphere embedding theorem”. In: *OWR Reports* 15.3 (2018), pp. 1888–1820.

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Statement of research interests

My research is focussed on 3- and 4-manifolds, via the lens of knots and links. Broadly speaking, I study 1- and 2-dimensional submanifolds of 3- and 4-dimensional spaces.

Low-dimensional topology is dedicated to the study of manifolds of dimension at most four. In a sense, dimension four is the boundary case between low and high dimensions; there is enough freedom for the manifold topology to exhibit complex behaviour, but not enough room for the usual tools to work. This phenomenon is exemplified by the following: Euclidean space $\mathbb{R}^n$, for $n \neq 4$, has a unique smooth structure, but the space $\mathbb{R}^4$ has *uncountably many* smooth structures [69]. Moreover, deep questions such as the Poincaré conjecture and the Schoenflies conjecture remain open in dimension four but are now settled in all other dimensions.

The key construction underpinning the success of high-dimensional topology is the *Whitney trick*, which allows algebraic intersection patterns between submanifolds to be realised geometrically; specifically the trick uses embedded 2-discs to remove algebraically cancelling intersection points in pairs. The Whitney trick works in high dimensions since maps of 2-discs in an ambient manifold of dimension at least five are generically embeddings. In ambient dimension four, however, we may only conclude that maps of 2-discs are generically immersed. Similarly, the surgery programme requires certain maps of spheres to be homotopic to embeddings, which in high dimensions can be arranged by transversality. In ambient dimension four, more work is needed.

The divergence between the topological and smooth settings for 4-manifolds was established by groundbreaking work of Freedman [25] and Donaldson [21]. Specifically, Freedman provided conditions under which maps of 2-discs may be replaced by flat embeddings, facilitating the Whitney trick. Donaldson showed that these conditions are not sufficient to find smooth embeddings. Broadly speaking, this established that simply connected 4-manifolds behave remarkably like high-dimensional manifolds in the topological category, but not in the smooth category. A central open problem in 4-manifold topology today is to understand the extent to which this phenomenon holds for 4-manifolds with possibly nontrivial fundamental group. From this perspective, the question of when a map of a surface to a 4-manifold is homotopic to an embedding is central to the study of 4-manifolds in general. Studying this question is the main thrust of my research programme.

A link in the 3-sphere S^3 is *slice* if the components bound a collection of embedded, disjoint discs in the 4-ball B^4, where $S^3 = \partial B^4$. If the discs are smoothly embedded, the link is *smoothly slice*, while it is *topologically slice* if the discs are merely flat. Every smoothly slice link is topologically slice, but infinitely many knots are topologically, but not smoothly, slice [28]. This phenomenon mirrors the disparity between the topological and smooth settings for 4-manifolds and also provides a direct connection: each topologically slice knot which is not smoothly slice produces a smooth structure on $\mathbb{R}^4$ which is distinct from the standard smooth structure [27].

Determining which knots and links are slice is the simplest version of the surface embedding problem in the relative setting, since e.g. every knot bounds an immersed disc in B^4, but not always a smoothly embedded disc. Incredibly, certain fundamental problems in 4-manifold topology admit equivalent formulations in terms of link slicing problems. For example, the surgery sequence applies to topological 4-manifolds irrespective of fundamental group if and only if every element of a certain family of *good boundary links* is topologically slice, so that the complement of the slicing discs has free fundamental group. Most known good boundary links are *satellite* knots. A prominent theme of my research is to study the sliceness of knots and links, with a focus on how this property interacts with the satellite construction.

1. Embedding surfaces in 4-manifolds

Question 1. Given a map of a surface to a 4-manifold, when is it homotopic to an embedding?

We restrict ourselves to embeddings and immersions that are *locally flat*, meaning that they are locally modelled on the inclusions $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$ and $\mathbb{R}^2_+ \hookrightarrow \mathbb{R}^4$. Even for maps of 2-spheres, Question 1 has been completely answered for only a handful of simple manifolds, such as S^4, $\mathbb{CP}^2$, and $S^2 \times S^2$. The primary tool for proving positive results is Freedman's celebrated *disc embedding theorem*, whose consequences include the exactness of the surgery sequence and the s-cobordism theorem for 4-manifolds with good fundamental group, both analogues of key tools in high dimensions, as well as the 4-dimensional topological Poincaré conjecture. Indeed all known disparities in the behaviour of 4-manifolds in the smooth and topological settings rely on the disc embedding theorem.

Theorem 2 (Disc embedding theorem, Freedman-Quinn, Powell-**Ray**-Teichner [25, 24, 62])**.** *Let M be a connected 4-manifold with good fundamental group. Consider an immersion*

$$F\colon (D^2, S^1) \looparrowright (M, \partial M)$$

that is an embedding on the boundary, has trivial self-intersection number, and admits a framed, algebraically dual sphere $G\colon S^2 \looparrowright M$. Then there exists an embedding

$$\overline{F}\colon (D^2, S^1) \hookrightarrow (M, \partial M)$$

such that $\overline{F}$ is homotopic to F, has the same framed boundary as F, and admits a framed, geometrically dual sphere $\overline{G}$, with $\overline{G}$ homotopic to G.

Above, two surfaces are said to be *algebraically dual* if their algebraic intersection number, valued in the group ring $\mathbb{Z}[\pi_1(M)]$, is 1. When $\pi_1(M) = 1$, this means that the signed sum of the intersection points between them is 1. On the other hand, two surfaces are said to be *geometrically dual* if they intersect precisely once, geometrically. The trivial group is *good*, as are groups of subexponential growth in general; it is a central open question to determine precisely which groups are good. Loosely, Theorem 2 says that the algebraic intersection behaviour between an immersed disc and a dual sphere can be realised geometrically, under a suitable fundamental group condition.

Theorem 2 was stated in [24] in the above form, but the geometrically dual sphere $\overline{G}$ was not constructed; Powell, Teichner, and I filled this gap in [62]. Numerous applications of the disc embedding theorem require these geometrically dual spheres. As an example, consider the surgery programme, which seeks to upgrade a given map $f\colon M^{2n} \to N^{2n}$, with certain properties, to a homotopy equivalence. This involves cut-and-paste operations on M to modify the groups $\pi_i(f)$: if $\pi_i(f)$ is trivial for $i \leq n$, then f is a homotopy equivalence. In the case of 4-manifolds, i.e. $n = 2$, it is straightforward to arrange that f induces an isomorphism on π_1. The kernel of the map on π_2 is represented by maps of spheres, and we use Theorem 2 to replace them, up to homotopy, by embedded spheres, which we subsequently surger along. Geometrically dual spheres ensure that the resulting map continues to induce an isomorphism on π_1, as required. Indeed these geometric duals are used throughout the second half of the book by Freedman-Quinn [24], which has served as the canonical, and often original, source material for topological 4-manifolds since its publication. The latter book is notoriously difficult to penetrate, explaining why this gap remained unnoticed in the 30 years since its publication. In general it has been difficult for new people to enter the field of topological 4-manifold topology due to the lack of an accessible introductory textbook. My recent expository work [4] seeks to correct this; see Section 3 for further details.

The disc embedding theorem was first proven for simply connected 4-manifolds. The proof has two steps. The first replaces a neighbourhood of a given immersed disc by an infinite, iterated 4-dimensional object called a *Casson handle*. These are morally approximations of discs, built using layers upon layers of immersed discs, each attached to kill the nontrivial π_1 of the previous layer. A finite truncation of a Casson handle is called a *Casson tower*. The second step of the proof uses techniques from a powerful, but somewhat outmoded branch of topology, called *decomposition space* *theory*, to show that a Casson handle is indeed homeomorphic to a thickened 2-disc, relative to its attaching region. Other notable successes of decomposition space theory include the topological Schoenflies theorem [7] and the double suspension theorem [8].

The extension of the disc embedding theorem to 4-manifolds with nontrivial fundamental group uses a different construction using *gropes*, another iterated 4-dimensional construction, but now consisting of layers upon layers of compact surfaces, with each layer attached to a basis for H_1 of the previous layer. The following theorem establishes a link between the two constructions, and was a key ingredient in Cha-Powell's proof [9], extending the work of Freedman [25], that any Casson tower of height four contains an embedded disc bounded by its attaching circle.

Theorem 3 (**Ray** [65])**.** *Let $n \geq 1$. Every Casson tower of height n contains a grope of height n with the same attaching circle.*

Returning to Question 1, in [34] Kasprowski, Powell, Teichner, and I extended the disc embedding theorem (Theorem 2) to all compact surfaces with algebraically dual spheres, in any connected 4-manifold with good fundamental group. In particular, we no longer assume that the algebraically dual spheres have a framed normal bundle. In this general setting, a secondary invariant appears, which we call the *Kervaire-Milnor invariant*. Briefly, a surface F in a 4-manifold has trivial self-intersection number if its self-intersections can be paired up by maps of discs, called *Whitney discs*. The latter discs may intersect one another, as well as the surface F. The Kervaire-Milnor invariant counts the mod 2 intersection number of the Whitney discs with the surface F.

Theorem 4 (Surface embedding theorem, Kasprowski-Powell-**Ray**-Teichner [34])**.** *Let M be a connected 4-manifold with good fundamental group. Let Σ be a connected, compact surface. Consider an immersion*

$$F\colon (\Sigma, \partial\Sigma) \looparrowright (M, \partial M)$$

that is an embedding on the boundary, has trivial self-intersection number, and admits an algebraically dual sphere $G\colon S^2 \looparrowright M$. Then there exists an embedding

$$\overline{F}\colon (\Sigma, \partial\Sigma) \hookrightarrow (M, \partial M)$$

such that $\overline{F}$ is homotopic to F, has the same framed boundary as F, and admits a geometrically dual sphere $\overline{G}$, with $\overline{G}$ homotopic to G, if and only if the Kervaire-Milnor invariant $\mathrm{km}(F) \in \mathbb{Z}/2$ *vanishes.*

A naïve strategy to prove Theorem 4 would embed the 1-skeleton of Σ using transversality, and then apply the disc embedding theorem to the remaining 2-cell of Σ in the complement of the embedding already constructed. This strategy does not succeed; that is, the embedding obstruction may be nontrivial for the 2-cell even though it is trivial for Σ, as we show in [34]. Moreover, we characterise when the Kervaire-Milnor invariant does not depend on the choice of Whitney discs for the self-intersections of F, providing broad conditions under which a map can be replaced by an embedding, such as the following.

Corollary 5 (Kasprowski-Powell-**Ray**-Teichner [34])**.** *Let M be a simply connected 4-manifold and let Σ be a connected, compact surface, either orientable with positive genus or non-orientable. Then any immersion $F\colon (\Sigma, \partial\Sigma) \looparrowright (M, \partial M)$ with vanishing self-intersection number and an algebraically dual sphere is homotopic, relative to $\partial\Sigma$, to an embedding.*

Theorems 2 and 4 both require algebraically dual spheres in the hypothesis. In the absence of such spheres, one may approach Question 1 with the techniques of topological surgery theory; as mentioned above, this is possible in 4-manifolds with good fundamental group by the disc embedding theorem (Theorem 2). Successes of this strategy include work of Lee-Wilczyński [39, 40] and Hambleton-Kreck [29], who described the minimal genus of an embedded surface in a fixed homology class, in any given simply connected, closed 4-manifold, assuming that the fundamental group of the complement is abelian. In [22] we extend this work to a large class of 4-manifolds, called *knot traces*. Given a knot $K \subseteq S^3$ and an integer n, the n-framed knot trace $X_n(K)$ is the smooth 4-manifold built from D^4 by attaching a 2-handle $D^2 \times D^2$ to a tubular neighbourhood νK of K with framing coefficient n, and smoothing corners. Note that $X_n(K)$ is homotopy equivalent to S^2, irrespective of the choice of K and n, and therefore $\pi_2(X_n(K)) \cong \mathbb{Z}$. The boundary $\partial X_n(K)$ is the n-framed Dehn surgery on S^3 along K, denoted by $S^3_n(K)$.

Theorem 6 (Feller-Miller-Nagel-Orson-Powell-**Ray** [22]). *Let $K \subseteq S^3$ be a knot and $n \geq 1$. A generator of $\pi_2(X_n(K)) \cong \mathbb{Z}$ can be represented by an embedded 2-sphere whose complement has abelian fundamental group if and only if*

(1) the group $H_1(S^3_n(K); \mathbb{Z}[\mathbb{Z}/n]) = 0$;
(2) the Arf invariant $\mathrm{Arf}(K) = 0$; and
(3) the signature function $\sigma_K(\xi) = 0$ for every $\xi \in S^1$ such that $\xi^n = 1$.

The restriction on the fundamental group of the complement of the embedded sphere, as in the aforementioned work of Lee-Wilczyński [39, 40] and Hambleton-Kreck [29], arises since topological surgery is only known to apply for good fundamental groups. Nevertheless the above theorem provides effective and usable obstructions for this particular class of embedding problems. In the constructive direction, it provides straightforward conditions under which an embedded sphere can be found.

1.1. **Topological classifications of 4-manifolds.** As a striking consequence of the disc embedding theorem (Theorem 2), topological 4-manifolds can sometimes be classified up to homeomorphism, while a diffeomorphism classification remains far out of reach. More generally, there is active interest in studying 4-manifolds up to other equivalence relations, such as homotopy equivalence, stable homeomorphism/diffeomorphism, and smooth/topological h-cobordism. In [32], Kasprowski, Powell, and I give a detailed survey of the implications between these notions and known counterexamples, as well as supplying new results and open questions, such as the following.

Theorem 7 (Kasprowski-Powell-**Ray** [32]). *For every $n \geq 1$, there is a collection of closed, orientable 4-manifolds $\{N_i\}_{i=1}^n$, that are all simple homotopy equivalent and h-cobordant, but which are pairwise not s-cobordant.*

In a different direction, in [33] we study the effect of the Gluck twisting operation on 4-manifolds. Given an embedded sphere $S \hookrightarrow M$ in a 4-manifold M, with trivial normal bundle, the result of the *Gluck twist* on M along S is given by $M_S := (M \smallsetminus \nu S) \cup_G (D^2 \times S^2)$. Here $G\colon S^1 \times S^2 \to S^1 \times S^2$ is the diffeomorphism given by $(\theta, s) \mapsto (\theta, \rho_\theta(s))$, where ρ_θ is the rotation by angle θ. A classical question compares the diffeomorphism type of M and M_S for a sphere $S \subseteq M$, most commonly in the case of $M = S^4$. One usually computes that when $S \subseteq M$ is null-homologous and $\pi_1(M) = 1$, the intersection form of M_S agrees with that of M, and then appeals to Freedman's classification result for closed, simply connected 4-manifolds. Since full classification results are relatively sparse, it is useful to provide more general criteria where M and M_S can be compared.

Theorem 8 (Kasprowski-Powell-**Ray** [33]). *Let $S, T \subseteq M$ be embedded 2-spheres with trivial normal bundle in a compact 4-manifold. If S and T are concordant, then M_S and M_T are s-cobordant. If $\pi_1(M)$ is a good group, it follows that M_S and M_T are homeomorphic.*

If S and T are homotopic, then M_S and M_T are simple homotopy equivalent.

Using the surgery sequence directly rather than appealing to classification results, we give many new conditions under which M_S and M_T are homeomorphic. Moreover we produce new examples where Gluck twisting produces manifolds that are homotopy equivalent but not homeomorphic, and examples which are homeomorphic but not diffeomorphic. For further ongoing and future work towards classifying 4-manifolds, see Section 4.1.

2. Slicing knots and links

Two links are *smoothly or topologically concordant* if the components cobound smooth (resp. locally flat) embedded, disjoint annuli in $S^3 \times [0, 1]$. Knots modulo concordance, in either category, under the operation of connected sum, form an abelian group called the *concordance group*, denoted by $\mathcal{C}$ in the smooth setting, and $\mathcal{C}_{\text{top}}$ in the topological setting. The identity element consists precisely of the collection of smoothly (resp. topologically) slice knots.

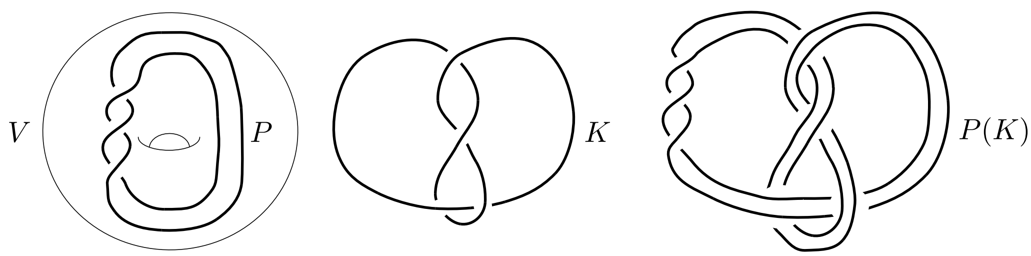


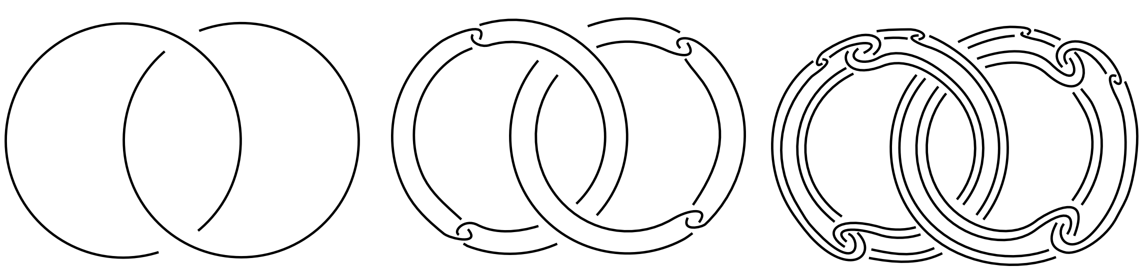
Figure 1. The satellite construction. The pattern P lies in a solid torus V. When V is mapped onto a tubular neighbourhood of the knot K, the image of P is the satellite knot $P(K)$. Here the winding number of P is 2 and $P(K)$ is the $(2, 3)$-cable of K, denoted by $K_{2,3}$.

2.1. The action of satellite operators on knot concordance.

A *pattern* P is a knot in a solid torus V; the *satellite knot* $P(K)$ is obtained by tying the solid torus V into the knot K, in an appropriately untwisted manner, as shown in Figure 1. This operation is compatible with concordance, and yields well-defined functions on the concordance groups, called *satellite operators*. The algebraic intersection number of P with a generic meridional disc of V is called the *winding number* of P.

Problem 9. Investigate the properties of satellite operators.

As previously mentioned, the sliceness of satellite knots and links is intimiately related to deep questions about 4-manifold topology. For example, if the link shown in Figure 2 (c) is not topologically slice, then the *surgery conjecture* is false. The link in question is a satellite of the Hopf link, which is not slice, in either category. Therefore, we are interested in whether satellite operators are injective, and in particular, whether a satellite of a non-slice link can ever be slice.



(a) (b) (c)

Figure 2. (a) The Hopf link. (b) Bing double of the Hopf link. (c) Whitehead double of the Bing double of the Hopf link, an example of a *good boundary link.*

The relevant satellites for the surgery conjecture have winding number zero. For all other winding numbers, we give broad conditions under which they are injective.

Theorem 10 (Cochran-Davis-**Ray**, **Ray** [64, 11])**.** *All patterns with* strong winding number ± 1 *induce injective maps on $\mathcal{C}_{top}$ and $\mathcal{C}$, the latter modulo the smooth 4-dimensional Poincaré conjecture.*

In general, all patterns with non-zero winding number induce injective maps on knots modulo certain homological versions of concordance.

'Strong' winding number is a homotopy analogue of the winding number. In particular, each of the infinitely many embeddings of the trivial knot in the solid torus with winding number ± 1, such as in Figure 3, is a strong winding number ± 1 pattern.

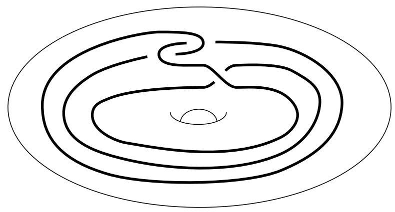


FIGURE 3. A strong winding number one pattern called the *Mazur pattern.*

Theorem 10 was generalised by Cochran and Harvey in [10], where it was shown that winding number ± 1 satellite operators are quasi-isometries with respect to certain natural metrics on $\mathcal{C}$. Theorem 10 also provides strong evidence for a *fractal* structure on $\mathcal{C}$, as conjectured in [13], i.e. that there exist *self-similarities at arbitrarily small scales.* Our result shows that infinite classes of winding number ± 1 satellite operators are self-similarities for $\mathcal{C}_{\text{top}}$ and $\mathcal{C}$, the latter modulo the Poincaré conjecture. The question of 'scale' is related to the dynamics of iterated satellite operators, which I addressed in the following theorem.

Theorem 11 (**Ray** [66])**.** *For infinitely many strong winding number ± 1 patterns P, including the Mazur pattern from Figure 3, infinitely many knots K have non-periodic orbits, i.e. $P^i(K) = P^j(K)$ in $\mathcal{C}$ if and only if $i = j$.*

Moreover, A. Levine showed in [41] that each iterate M^i of the Mazur satellite operator M is non-surjective on $\mathcal{C}$, for $i \geq 1$. By contrast, in joint work with Feller and J. Park [23], we show that the image of each M^i is still large, even when restricted to topologically slice knots, as follows.

Theorem 12 (Feller-Park-**Ray** [23])**.** *There is an infinite family of topologically slice knots whose image under any iterate of the Mazur satellite operator generates an infinite rank subgroup of $\mathcal{C}$.*

While the Mazur satellite operator is non-surjective on $\mathcal{C}$ [41], in joint work with C. Davis [19], we gave a complete characterisation of patterns inducing surjective satellite operators and constructed an infinite family of strong winding number ± 1 patterns, distinct from connected sum patterns, which induce bijections on the concordance groups. Further examples of bijective satellite operators were given by Miller-Piccirillo [49] and Piccirillo [53].

Since a knot and its satellite under a slice winding number ± 1 pattern share several classical invariants, we may ask what we obtain when such satellites are identified. In [15], Cochran and I showed that the equivalence relation on knots generated by concordance together with setting $K \sim P(K)$ for all K and all slice winding number one patterns P is the same as the equivalence relation generated by a generalisation of concordance, called *shake concordance*. This is a relative version of *shake sliceness*: a knot K is 0-*shake slice* if a generator of $\pi_2(X_0(K)) \cong \mathbb{Z}$ is represented by an embedded sphere, for $X_0(K)$ the 0-framed knot trace from Section 1 (as usual there are parallel notions in the smooth and topological settings). There are no known examples of knots that are 0-shake slice but not slice, in either category. However, Cochran and I gave a complete characterisation of shake concordance in terms of concordance and satellite operators, with the following consequence in the relative case.

Theorem 13 (Cochran-**Ray** [15]). *Two knots K and J are smoothly* 0*-shake concordant if and only if there are slice winding number* ± 1 *patterns P and Q with $P(K)$ smoothly concordant to $Q(J)$.*

As a consequence, there are infinitely many topologically slice knots which are pairwise smoothly 0*-shake concordant, but distinct in smooth concordance.*

Finally, another strategy for studying the knot concordance groups via satellite operators is to consider the image of a fixed knot under natural families of satellite operators, such as *cabling*. Here, for relatively prime integers p, q, the (p, q)-cable of a knot K, denoted by $K_{p,q}$, is the satellite with respect to the pattern given by the (p, q)-torus knot, as in Figure 1.

Theorem 14 (Davis-Park-**Ray** [18]). *There exist infinitely many knots K such that the set of cables $\{K_{p,1}\}_{p\geq 1}$ is linearly independent in $\mathcal{C}_{top}$ and $\mathcal{C}$.*

We focus on $(p, 1)$-cabling since for K a slice knot, the cables $K_{p,1}$ are slice. In Theorem 14 we may choose K to be arbitrarily close to slice, as measured by the solvable and bipolar filtrations of $\mathcal{C}$ and $\mathcal{C}_{\text{top}}$ [14, 12]. As a consequence, Theorem 14 cannot be reached by any combination of algebraic concordance invariants, Casson-Gordon invariants, and Heegaard-Floer invariants.

2.2. **Knot theory in general manifolds.** We now generalise our perspective and consider knots in an arbitrary 3-manifold and/or surfaces bounded by knots in arbitrary spanning 4-manifolds. Knots are generally considered up to two natural notions of equivalence. Two knots in a fixed oriented 3-manifold M are said to be *equivalent* if there is an orientation preserving homeomorphism of M taking one to the other, while they are said to be *isotopic* if they are related by a 1-parameter family of embeddings in M. Since the mapping class group of S^3 is trivial, the two notions coincide for knots in S^3. In [1] we consider the extent to which this holds in general 3-manifolds.

Theorem 15 (Aceto-Bregman-Davis-Park-**Ray** [1]). *Any two equivalent knots in a prime, closed, oriented* 3*-manifold M are isotopic if and only if the mapping class group $Mod^+(M)$ is trivial.*

Stated differently, if an orientation preserving homeomorphism of M is *not* isotopic to the identity, this is detected by the isotopy class of some knot. The proof consists of showing that if an orientation preserving homeomorphism $f\colon M \to M$ for an irreducible, closed, oriented 3-manifold M preserves every conjugacy class of $\pi_1(M)$, then f_* is an inner automorphism of $\pi_1(M)$. A group has Grossman's *Property A* if every automorphism which preserves conjugacy classes is inner. Extending the work of many authors, we prove that for M a closed, orientable 3-manifold, the fundamental group $\pi_1(M)$ has Property A.

While the classical study of sliceness of knots and links focusses on B^4, recent work [42, 43, 44] indicates that slicing knots in more general 4-manifolds may answer long-standing questions about the existence of exotic smooth structures. As an example, [44] provided a list of 23 knots, such that if any of them bounds a smooth, properly embedded, null-homologous disc in $(\#^m\mathbb{CP}^2) \smallsetminus \mathring{B}^4$, for some m, then there exists an exotic smooth structure on $\#^m\mathbb{CP}^2$. In [38], we define the $\mathbb{CP}^2$*-slicing number* of a knot K, denoted by $u_{\mathbb{CP}^2}(K)$, to be either the smallest $m \geq 0$ such that K bounds a smooth, properly embedded, null-homologous disc in $(\#^m\mathbb{CP}^2) \smallsetminus \mathring{B}^4$, or ∞ if no such m exists. We define the $\overline{\mathbb{CP}^2}$-slicing number analogously and denote it by $u_{\overline{\mathbb{CP}^2}}(K)$. We find the first examples of knots for which both $u_{\mathbb{CP}^2}$ and $u_{\overline{\mathbb{CP}^2}}$ are finite and arbitrarily large.

Theorem 16 (Kjuchukova-Miller-**Ray**-Sakallı [38]). *For any $n \geq 0$, there exists a knot K such that $n \leq u_{\mathbb{CP}^2}(K) < \infty$ and $n \leq u_{\overline{\mathbb{CP}^2}}(K) < \infty$.*

A knot K as above must have trivial signature function, and therefore Theorem 16 is inaccessible through bounds on the signature function. Additionally, while tools from gauge theory and Khovanov homology can be used to obstruct sliceness in smooth, compact, oriented, definite 4-manifolds, these are generally only in terms of the *signs* of the invariants, not their values, so they do not provide bounds for $u_{\mathbb{CP}^2}$ or $u_{\overline{\mathbb{CP}^2}}$.

Given a closed 4-manifold M, let M° denote the punctured manifold $M \smallsetminus \mathring{D}^4$. The *$M$-genus* of a knot $K \subseteq S^3 = \partial M^\circ$, denoted by $g_M(K)$, is the minimal genus of an embedded orientable surface bounding K in M°. If M is smooth, we also consider the analogous *smooth M-genus*. The following is a corollary of Theorem 4.

Corollary 17 (Kasprowski-Powell-**Ray**-Teichner [34]). *For every knot $K \subseteq S^3$,*

(1) $g_M(K) = 0$ for every simply connected 4-manifold M homeomorphic to neither S^4 nor $\mathbb{CP}^2$;
(2) $g_{\mathbb{CP}^2}(K) \leq 1$ and $g_{\mathbb{CP}^2}(\#^3 T(2,3)) = 1$.

In contrast to Corollary 17 (2), we show in [48] that the smooth $\mathbb{CP}^2$-genus of knots can be arbitrarily high. In [16], we also study knots in $S^1 \times S^2$ and surfaces bounded by them in $D^2 \times S^2$.

3. Expository work

The remarkable disparity between the smooth and topological categories for 4-manifolds was established by the revolutionary advances of Freedman on topological manifolds [25] and Donaldson on smooth manifolds [21], both in the early 1980s. While the intervening years have seen substantial improvements in the understanding of smooth 4-manifolds, progress has been comparatively scant for topological 4-manifolds, apart from the notable contributions of Quinn, Teichner, and Krushkal. Quinn in particular masterfully combined Freedman's work with subtle tools of controlled topology and generalised manifold theory to establish foundational results for topological 4-manifolds, such as topological transversality, the existence of normal bundles for locally flat submanifolds, and smoothing results for noncompact 4-manifolds [63]. These are analogues of the high-dimensional work of Kirby-Siebenmann [37], and it is no exaggeration to say that any work on topological 4-manifolds inevitably hinges on these results. The canonical source material for the work of Freedman and Quinn is their book [24], published in 1990. The latter is notoriously difficult to penetrate; it contains substantial original material, as well as errors, the most recently detected of which I discussed in Section 1.

In the 1980s mathematicians learned the proofs of Freedman and Quinn through seminars and personal discussions, but as the founders switched focus to other topics, and lacking an accessible means to learn the fundamental tools, few newcomers have ventured into the field. Facing the real prospect that the details of this beautiful mathematics could be lost to time, Kreck and Teichner organised a semester at the Max Planck Institute for Mathematics in Bonn, Germany, in 2013, centred around a lecture series by Freedman on his proof of the topological 4-dimensional Poincaré conjecture. This semester catalysed the creation of a new textbook [4], published by Oxford University Press in 2021, with editors Behrens, Kalmár, Kim, Powell, and myself. I am also an author of 15 of its 28 chapters [47, 57, 5, 60, 36, 58, 35, 59, 6, 61, 67, 45, 54, 50, 51].

The book features 20 contributing authors in total, and the primary goal is to give a detailed, but accessible, proof of the disc embedding theorem (Theorem 2). While some chapters are based closely on lectures by Freedman in 2013, and later by Powell and others in 2016, several were developed independently. This includes an explanation of topological surgery theory in dimension four and its applications to the classification of 4-manifolds [51] and a detailed account of the open problems surrounding the work of Freedman-Quinn [36]. The final chapter on the decomposition space theory in the proof of the disc embedding theorem explicates several steps for the first time in full detail [5]. Unlike the traditional role of editors of a collected volume, we served as *de facto* referees, ensuring both the correctness of each chapter as well as that the chapters comprise a complete proof of Theorem 2.

Virtually all of the 20 contributing authors were postdocs or junior faculty at the time of writing, and now form a new cohort of researchers in topological 4-manifold theory. Several recent papers and current projects have used the tools and techniques learned by its authors while writing the book, including [62, 34, 46, 22, 17, 55, 56, 31, 32].

4. Selected current and future projects

4.1. **New applications of Freedman-Quinn technology.** The main focus of my current and upcoming work is on the embedding of surfaces in topological 4-manifolds, and the resulting consequences, for example related to classifying 4-manifolds up to homeomorphism and subtle phenomena such as the existence of pairs of manifolds that are homotopy equivalent but not homeomorphic.

Note that while the disc embedding theorem (Theorem 2) holds for all 'good' groups, there do not yet exist classification results for 4-manifolds with arbitrary good fundamental group. Indeed, the disc embedding theorem is only one of the many necessary ingredients in a classification scheme, and new challenges arise as the fundamental group becomes more complicated. Currently classification results are known primarily for cyclic and Baumslag-Solitar fundamental groups, in general only in the closed, oriented case. In upcoming work with Kasprowski and Powell [31], we classify closed, oriented 4-manifolds with fundamental group $\mathbb{Z}/2*\mathbb{Z}/2$, up to homeomorphism. As a tool we develop a new homeomorphism invariant of topological 4-manifolds, inspired by the Kervaire-Milnor invariant from Section 1, but incorporating the data of the fundamental group elements associated to certain double points. As a consequence, we construct new examples of low complexity 4-manifolds that decompose topologically as connected sums, but not smoothly. For many fundamental groups, we also classify closed, oriented 4-manifolds up to homotopy equivalence, unifying and extending previous work of Hambleton and Kreck.

In joint work with Kasprowski, Powell, and Teichner, I am also investigating *star partners* of 4-manifolds. Given a 4-manifold M, a star partner $*M$ is a manifold such that $M\# *\mathbb{CP}^2 \cong *M\#\mathbb{CP}^2$, where $*\mathbb{CP}^2$ is the *Chern manifold*, namely the unique 4-manifold homotopy equivalent, but not homeomorphic to $\mathbb{CP}^2$. Teichner [70] found a 4-manifold with two distinct star partners. Kasprowski, Powell, Teichner, and I have extended this to show that there exist manifolds with arbitrarily many star partners. We are interested in finding manifolds with infinitely many star partners, or proving this is impossible. Constructing such an example appears to require a novel approach. In joint work with Kasprowski and Powell, we leverage the star construction to produce new examples of smooth 4-manifolds that are homotopy equivalent but not homeomorphic.

Recent work of Gabai and Budney-Gabai has also reinvigorated interest in the mapping class groups of 4-manifolds. In ongoing work with Gompf and Orson, I am studying the smooth mapping class groups of noncompact 4-manifolds homeomorphic to $\mathbb{R}^4$.

4.2. **Slicing knots and links.** As previously mentioned, slicing problems for knots and links are closely related to 4-manifold topology in general. A motivating question follows.

Question 18. Does there exist a knot $K \subseteq S^3$ which is slice in a compact 4-manifold M with $H_*(M;\mathbb{Z}) \cong H_*(B^4;\mathbb{Z})$, but which is not slice in B^4?

All known invariants of topological concordance fail to detect such knots. Hom, Levine, and Lidman [30] used subtle invariants from Heegaard-Floer homology to show that there exist closed 3-manifolds Y and knots $K \subseteq Y$, such that Y is smoothly integer homology cobordant to S^3 but the knot K is not smoothly concordant to any knot in S^3 in any smooth integer homology cobordism from Y to S^3. In ongoing work with Chen, Miller, and Raoux, we have extended their result by showing there exist closed 3-manifolds Y and links $L \subseteq Y$ with unknotted components, such that Y is smoothly integer homology cobordant to S^3 but L is not smoothly concordant to any link in S^3 with unknotted components, in any smooth integer homology cobordism from Y to S^3. We also study the partial order on integer homology 3-spheres given by setting $Y \leq Y'$ if every knot in Y is smoothly concordant to some knot in Y' in a smooth homology cobordism. My long-term focus is rather on the topological analogue, specifically on the following conjecture.

Conjecture 19. Given any closed 3-manifold Y with $H_*(Y;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$ and any knot $K \subseteq Y$, there is a cobordism W with $\partial W = Y^3 \sqcup S^3$ and $H_*(W;\mathbb{Z}) \cong H_*(S^3 \times [0,1];\mathbb{Z})$, within which K is topologically concordant to some knot in S^3.

In a similar vein, several recent results on satellite operators are limited to the smooth category and the topological analogues remain open. In particular, I am interested in the following problem.

Question 20. Let P be a winding number one pattern with $P(U)$ unknotted, for U the unknot. Does P induce a surjective map on $\mathcal{C}_{\text{top}}$?

As previously mentioned, A. Levine answered the above in the negative in the smooth category [41]. In the topological category Theorem 10 shows that the induced map is injective. The Mazur pattern provides a natural test case, in that it appears related to the Whitehead doubling pattern, to which Freedman-Quinn technology can be directly applied.

4.3. **Exotic smooth structures on $\mathbb{R}^4$.** There exist infinitely many topologically slice knots that are not smoothly slice [28]. Each such knot gives rise to an exotic smooth structure on $\mathbb{R}^4$. Specifically, a smooth 4-manifold is called an $\mathbb{R}^4$-*homeomorph* if it is homeomorphic to $\mathbb{R}^4$. For a knot K, let $\mathcal{R}_K$ denote the collection of $\mathbb{R}^4$-homeomorphs admitting a smooth embedding of the 0-framed knot trace $X_0(K)$. Gompf showed that $\mathbb{R}^4_{\text{std}} \in \mathcal{R}_K$ if and only if K is smoothly slice [27], and that there exists K for which $\mathcal{R}_K$ is uncountable. Not much more is known.

Problem 21. Find a sequence of knots $\{K_i\}$ so that $\mathcal{R}_{K_i} \neq \mathcal{R}_{K_j}$ whenever $i \neq j$.

Note that if a knot K does not bound a smooth, properly embedded, null-homologous disc in $(\#^m\mathbb{CP}^2) \smallsetminus \mathring{B}^4$, then the knot trace $X_0(K)$, and consequently every element of $\mathcal{R}_K$, does not embed in $\#^m\mathbb{CP}^2$. Following a construction of Gompf one can find a knot K, so that elements of $\mathcal{R}_{nK}$ embed in $\#^n\mathbb{CP}^2$ for each $n \geq 1$. Therefore, a complete answer to Problem 21 would follow from an appropriate generalisation of Theorem 16, due to Kjuchukova, Miller, Sakallı, and myself, as follows.

Problem 22. Find a topologically slice knot K such that $u_{\mathbb{CP}^2}(nK) = n$. Preferably, show this is true for K some iterated, positive clasped Whitehead double.

4.4. **Expository work.** The study of topological manifolds rests on the foundational work of Kirby-Siebenmann [37] and Quinn [63]. The work of Quinn builds on Freedman's disc embedding theorem, to which my recently completed project [4] provides an accessible introduction. Quinn's results also require substantial input from other fields, such as controlled topology and generalised manifold theory. The corresponding chapters of the book of Freedman-Quinn [24] contain much original material that has never been refereed. Similarly, the book of Kirby-Siebenmann [37] assumes prior knowledge of techniques which were perhaps well-known at the time of publication, but are no longer so. Kirby and Kister began to write a comprehensive account of the work of Kirby-Siebenmann in the 1970s, but this had not been completed when Kister passed away in 2018.

In an attempt to make the foundations of the field both more solid and more accessible, Powell and I taught a lecture course at the University of Bonn in the winter semester of 2020-21 focussed on the work of Kirby-Siebenmann. Due to the Covid-19 pandemic, lectures were held online and participants included PhD students, postdocs, and some faculty from across Europe and the United States. The lecture notes from the course, and a subsequent seminar course, now form the core material for a long-term book project. The current version has 20 contributing authors and exceeds 400 pages; we are also in contact with Kirby, and will eventually subsume the incomplete Kirby-Kister draft. We are in the process of incorporating new explanations of the work of Quinn.

The new project is envisioned in the same format as [4] i.e. with many contributing authors and a handful of editors. The advantages of this paradigm are two-fold. On the one hand, each individual contributor has a manageable task, so the project is more likely to be completed. On the other hand, the contributors themselves form a new generation of researchers well-versed in these classical techniques, and ideally begin to apply them in their own work, as has already happened in the case of [4]. Further, I intend to use this opportunity to add valuable new tools to my arsenal, with which to attack long-standing open problems in the field.

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