Surfaces in 4-manifolds

Grenoble
June 2024.

Lecture 1, June 17, 2024 (1hr)

Outline:
1. Constructions & definitions (some diagrams)
2. Existence (in a given hom. class or w. a given ∂)
3. Uniqueness (up to isotopy)
4. Other equivalence relations
   e.g. concordance, stabilisation, ...

HW available at: tinyurl.com/aruray-problems
Feedback on minicourse at: tinyurl.com/aruray-feedback
(but also ask questions / talk to me directly ofc)
Tutorial sessions: Room 15. $\boxed{1630-1730}$

<u>Def$\underline{n}$</u>: A smooth <u>knotted surface</u>, or sm <u>surface knot</u> is a sm embedding $K: \Sigma \hookrightarrow S^4$, where Σ is a <u>closed</u> surface.
(closed: compact, $\partial = \emptyset$)

- If $\Sigma = S^2$, K is called a <u>2-knot</u>
- Often we conflate F and its image $F(\Sigma)$.
- Generalisations:
  - surfaces in arbitrary 4-mflds.
  - "properly embedded" surfaces, ie.

$$\begin{tikzcd} \Sigma \arrow[r, hook] & M \\ \partial\Sigma \arrow[u, hook, "\partial"] \arrow[r, hook] & \partial M \arrow[u, hook, "\partial"'] \end{tikzcd}$$

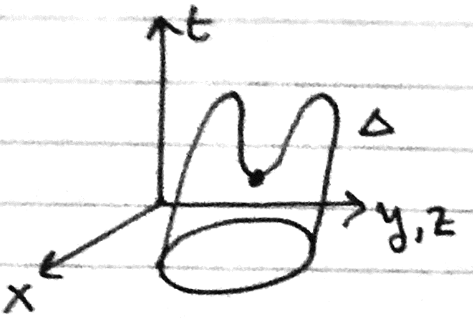
<u>Def$\underline{n}$</u>: A knot $K: S^1 \hookrightarrow S^3$ is said to be <u>smoothly</u> <u>slice</u> if there exists a sm. $\Delta: D^2 \hookrightarrow B^4$.

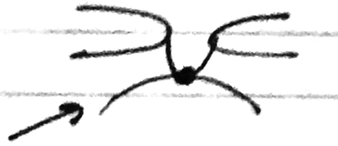
$$\begin{tikzcd} D^2 \arrow[r, hook, "\Delta"] & B^4 \\ S^1 \arrow[u, hook, "\partial"] \arrow[r, hook, "K"'] & S^3 \arrow[u, hook, "\partial"'] \end{tikzcd}$$

How can we construct or visualise such surfaces inside 4D ambient space?

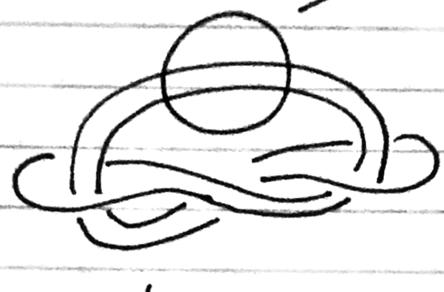
Method 1: Movies in B^4 or S^4

- Let's begin with slicing knots $B^4 = S^3 \times [0,1] / S^3 \times 1$.

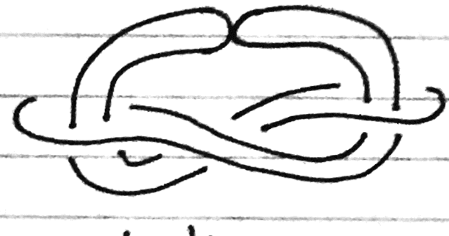




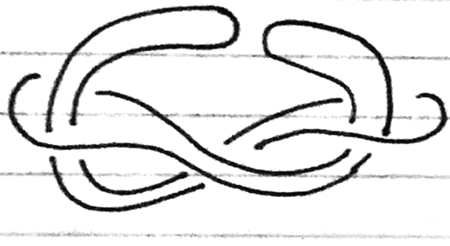
saddle singularity.



$t=0$

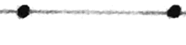


$t = 1/4$



$t = 1/2$

$t = 3/4$



$t = 1$

Let Δ denote a sm. slice disc.
Then note $\Delta \cup -\Delta \subseteq B^4 \cup -B^4 = S^4$ is a sm. 2-knot.
This is one reason for the terminology of "slice" knot.

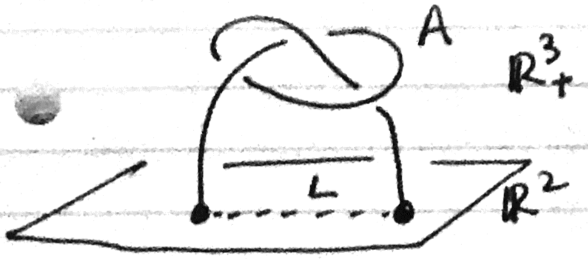
Method 2: Spinning to build 2-knots in $S^4 \supseteq \mathbb{R}^4$

Consider $\mathbb{R}^3_+ := \{(x_1, x_2, x_3, 0) \mid x_3 \geq 0\} \subseteq \mathbb{R}^4$
& $\mathbb{R}^2 := \{(x_1, x_2, 0, 0)\} \subseteq \mathbb{R}^4$.

Given $x = (x_1, x_2, x_3, 0) \in \mathbb{R}^3_+$ and $\theta \in [0, 2\pi]$,
define $x_\theta := (x_1, x_2, x_3 \cos\theta, x_4 \sin\theta)$.

For $X \subseteq \mathbb{R}^3_+$ define $spin(X) := \{x_\theta \mid x \in X, \theta \in [0, 2\pi$

- Note $spin(\mathbb{R}^3_+) = \mathbb{R}^4$.



Let $A \subseteq \mathbb{R}^3_+$ be an arc in $\mathbb{R}^3_+$ with endpts on $\mathbb{R}^2$.

Then spin(A) is a 2-knot

Let $L \subseteq \mathbb{R}^2$ join the endpts of A.
Note $A \cup L$ is a knot in $\mathbb{R}^3$.

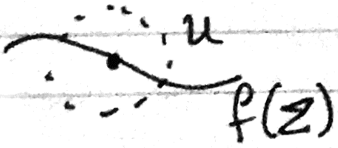
We usually say spin(A) is the <u>spin of $A \cup L$</u>.

$\longrightarrow$

There is a second notion of surfaces/embeddings in 4-mflds

i.e. homeo onto image (Topological)

<u>Definition</u>: An embedding $f : \Sigma^2 \hookrightarrow M^4$ (with $\partial\Sigma \hookrightarrow \partial M$) is said to be <u>locally flat</u> if $\forall p \in \Sigma$, $\exists$ nbd $\mathcal{U} \ni f(x)$ s.t.



$(\mathcal{U}, f(\Sigma) \cap \mathcal{U}) \xrightarrow{\approx} (\mathbb{R}^4, \mathbb{R}^2)$ if $p \in \mathrm{Int}(\Sigma)$

$(\mathcal{U}, f(\Sigma) \cap \mathcal{U}) \xrightarrow{\approx} (\mathbb{R}^4_+, \mathbb{R}^2_+)$ if $p \in \partial\Sigma$.

[Quinn '80s] Locally flat submanifolds of 4-mflds have normal vector bundles.

$$\begin{tikzcd} \Sigma^2 \arrow[r, hook] \arrow[d] & M^4 \\ \nu\Sigma \arrow[ur] & \end{tikzcd}$$

$\Rightarrow$ If $f : \Sigma^2 \hookrightarrow M^4$ is a loc. fl. emb. we have an extension of f to the total space of some D^2-bundle over Σ.

<u>Note</u>: Smooth embeddings are loc. flat, but not vice versa!

Def$^{\underline{n}}$: A knot $K \subseteq S^3$ is <u>top</u> slice if $\exists\, \Delta :$

$$\begin{tikzcd} D^2 \arrow[r, hook, "\text{loc. flat}"] & B^4 \\ S^1 \arrow[u, hook, "\partial"] \arrow[r, hook, "K"] & S^3 \arrow[u, hook, "\partial"'] \end{tikzcd}$$

[Casson, Akbulut, using Freedman, Donaldson]

$\exists$ TOP slice knots that are not sm slice e.g. $Wh^+(RHT)$

- Beware: not all TOP embs are loc. flat.
  e.g. given $K \subseteq S^3$ consider $\mathrm{cone}(K) \subseteq \mathrm{cone}(S^3) = B^4$

cone(K) is loc. flat $\Leftrightarrow$ K unknotted.

Finally why are surfaces in 4-mflds interesting?

- Operations on surfaces

Suppose $\exists T: S^1 \times S^1 \hookrightarrow M^4$ with trivial normal bundle
i.e. $(S^1 \times S^1) \times D^2 \hookrightarrow M^4$

Then we can either: $M' := M \setminus (T^2 \times D^2) \cup_\varphi T^2 \times D^2$ gen. log. transform
or
$M^4_K := M \setminus (T^2 \times D^2) \cup_{\varphi'} (S^3 \setminus \nu K) \times S^1$ knot surgery (arb choice $K \neq 0$)

- Under certain conditions, one can arrange that M', M^4_K are homeo to M, but not diffeomorphic.
[use Morgan-Mrowka-Szabo for log transform or Fintushel-Stern formula for knot surgery]

e.g. can produce infinitely many distinct sm. structures on the K3 surface using knot surgery.

- Similarly, suppose $K: S^2 \hookrightarrow S^4$ is a 2-knot
  then we have an embedding $K \times D^2 \subseteq S^4$.

We have $\mathrm{Gluck}(S^4, K) := S^4 \setminus (S^2 \times D^2) \cup_{\varphi: S^2 \times S^1 \circlearrowleft} S^2 \times D^2$

with $\varphi: S^1\times S^2 \to S^1\times S^2$ given by $(\theta, x) \mapsto (\theta, R_\theta(x))$

rotation by θ

Check: $\mathrm{Gluck}(S^4, K) \approx S^4 \quad \forall K$.

Question: Can we construct an exotic S^4 using Gluck twisting?

Exotic sm. structures on $\mathbb{R}^4$

Every $K \subseteq S^3$ which is topologically slice, but not sm. slice, gives rise to an exotic sm str. on $\mathbb{R}^4$.

Uncountably many sm. structures on $\mathbb{R}^4$ arise this way.

[Gompf, using Quinn, Freedman, Donaldson, ...]

4D surgery conjecture

Recall that the surgery sequence is an extremely powerful tool in high-dim manifold topology.

It is a very important question to what extent it holds in dimension four.

[Donaldson] seq. not exact in dim 4 for sm. mflds

[Freedman-Quinn] seq. exact for TOP 4-mfds with "good" π_1

[Casson-Freedman] The surgery seq. is exact for all TOP 4-mfds

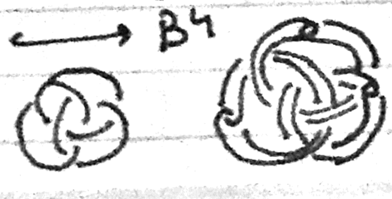
iff

every good boundary link is freely TOP slice

i.e. $L: \sqcup S^1 \hookrightarrow S^3$ s.t. $\pi_1(B^4 \setminus \nu \sqcup \Delta^2) =$ free gp (gen. by meridians)

$$\begin{array}{ccc} \sqcup S^1 & \hookrightarrow & S^3 \\ \downarrow\cap & & \downarrow\cap \\ \sqcup \Delta^2 & \hookrightarrow & B^4 \end{array}$$

e.g. $\mathrm{Wh}(\mathrm{Bor})$



Lecture 2, June 18, 2024 (1.5 hr)

- Today we'll focus on existence questions

Theorem: Let M^4 be a closed oriented 4-mfld.
$\forall x \in H_2(M;\mathbb{Z})$ $\exists \Sigma^2 \subseteq M^4$ a submanifold s.t. $[\Sigma] = x$.
i.e. every class in H_2 can be represented by an embedded surface.

Proof I'll use a lot of smooth terminology. but the result also holds for TOP 4-mflds

- $x \in H_2(M;\mathbb{Z}) \rightsquigarrow PD(x) \in H^2(M;\mathbb{Z}) \rightsquigarrow f_x : M \to \mathbb{CP}^\infty = K(\mathbb{Z},2)$

[used that $M \simeq$ CWcx]

By cellular approx. thm, assume $f_x : M \to \mathbb{CP}^2$

Make f_x transverse to $\mathbb{CP}^1 \subseteq \mathbb{CP}^2$

Then $\Sigma = f_x^{-1}(\mathbb{CP}^1)$ is a 2-dim submfld of M

Check that $[\Sigma] = x$. $\square$

- Similarly, every $K \subseteq S^3$ bounds an emb (or.) sm surface in B^4

In other words, we can always solve the simplest existence question

Note: the analogue does not hold in full generality. Some hom. classes are not even realised by maps from mflds. c.f. Steenrod problem

[Thom] '54 Class $[K(\mathbb{Z}/3 \oplus \mathbb{Z}/3, 1)] \in H_7(X^{14})$

[$\exists$ exist 7-dim $(n-3)$ classes in 10-mflds $n \geq 10$

c.f. Kreck, Bohr-Hanke-Kotschick

- So in general weak questions about minimality.

Minimal genus problem

- Given $x \in H_2(M^4;\mathbb{Z})$, M^4 closed.

What is the least genus of a $\substack{\text{sm} \\ \text{loc.flat}}$ surface $\Sigma \subseteq M$ w. $[\Sigma] = x$?

Organise as the genus function $g_M^{\text{sm/top}} : H_2(M;\mathbb{Z}) \to \mathbb{Z}_{\geq 0}$.

- Given $K \subseteq S^3$ a 1-knot, what is the least genus of an $\substack{\text{sm} \\ \text{loc. flat}}$ orientable Σ s.t.

$$\begin{tikzcd} K : S^1 \arrow[r, hook] \arrow[d, hook, "\partial"'] & S^3 \arrow[d, hook, "\partial"] \\ \Sigma \arrow[r, hook, "\text{sm or loc.flat}"'] & B^4 \end{tikzcd}$$

This is called the $\substack{\text{sm} \\ \text{top.}}$ slice genus of K, denoted $g_4^{\text{sm/top}}(K)$

- obvious generalisation: surfaces with given ∂ in arb 4-mflds, fixed homology class.

  - Constructions $\begin{cases} \text{sm} & \text{– this first, then break} \\ \text{top} & \text{– if time} \end{cases}$
  - Obstructions $\begin{cases} \text{sm} & \text{– cf. gauge thy, HF, Khovanov, ... not from me.} \\ \text{top} & \text{– after break} \end{cases}$

M^4 closed. $\mathring{M} := M \setminus \mathring{B}^4$. $K \subseteq S^3$ 1-knot

$$g_M^{\text{sm/top}}(K) := \min \{ g(\Sigma) \mid \Sigma \xhookrightarrow[\text{sm/top}]{} \mathring{M} \text{ s.t. } \partial\Sigma = K \}$$

[Kronheimer-Mrowka '94] $g_{\mathbb{CP}^2}^{\text{sm}} : H_2(\mathbb{CP}^2;\mathbb{Z}) \to \mathbb{Z}_{\geq 0}$

Seiberg-Witten

$$\begin{aligned} H_2(\mathbb{CP}^2;\mathbb{Z}) &\cong \mathbb{Z} \\ d &\mapsto \frac{(|d|-1)(|d|-2)}{2} \end{aligned}$$

[Lee-Wilczynski '97] $g^{\text{top}}_{\mathbb{CP}^2}(d) \leq \lfloor \frac{1}{4}d^2 + 1 \rfloor < \frac{(|d|-1)(|d|-2)}{2}$ for $|d| \geq 4$ ($d > 1$)

$=$ for d even.

- g^{sm}_M is known for $M = S^2 \times S^2, \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ [Ruberman] (and S^4 of course)

  but virtually nothing else.

  e.g. $g_{\mathbb{CP}^2 \# \mathbb{CP}^2}$ unknown

- $g^{top}_{\mathbb{CP}^2}$ still not known for odd classes!

- $g^{top}_{\mathbb{CP}^2}(K) \leq 1$ $\forall K$ [Kasprowski-Powell-R.-Teichner]

  $g^{top}_{\mathbb{CP}^2}(3T_{2,3}) = 1$

- $g^{sm}_{\mathbb{CP}^2}(K)$ can be arb. large [Marengon-Miller-R.-Stipsicz]

$\pi_1 M = 1$, M closed

$g^{top}_M(K) \leq 1$ $\forall M \neq S^4$

$g^{top}_M(K) = 0$ $\forall M \neq S^4, \mathbb{CP}^2, \overline{\mathbb{CP}^2}, *\overline{\mathbb{CP}^2}, *\mathbb{CP}^2$.

$g^{sm}_{\mathbb{CP}^2 \# \mathbb{CP}^2}(K)$ pretty unknown.

<u>Question</u>: Is every $K \subseteq S^3$ slice in the K3 surface?

$\longrightarrow$ [Marengon-Mihajlović] yes for $u(K) \leq 21$

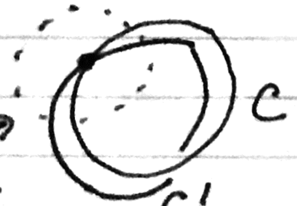
<u>Sm construction</u>: $\mathbb{CP}^1 \subseteq \mathbb{CP}^2$ generates $H_2(\mathbb{CP}^2; \mathbb{Z})$

$\uparrow$

this is a 2-sphere embedded with euler # +1 normal bundle

$d[\mathbb{CP}^1]$ represents $d \in \mathbb{Z} \cong H_2(\mathbb{CP}^2; \mathbb{Z})$

Consider $d = 2$.

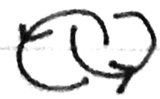


zoom in at transverse int point.

$\partial B = S^3$

$\partial B \cap (C \cup C')$ = Hopf link.

[exercise]



<u>Resolve</u> by adding annulus

For higher d, get $\binom{d}{2}$ intersections.

Result: d spheres
$\binom{d}{2}$ tubes

$$\text{genus: } \binom{d}{2} - (d-1) = \frac{(d-1)(d-2)}{2}.$$

→ Break ←

Now we switch over to the top setting.

First a TOP obstruction to sliceness in rel. setting (easier)

<u>Def$^{\underline{n}}$</u>: Given $K: S^1 \hookrightarrow S^3$, $\exists$ compact, oriented surface $\Sigma \hookrightarrow S^3$ with $\partial \Sigma = K$: <u>Seifert surface</u>

LHT (I think)

Seifert form:
$$H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) \to \mathbb{Z}$$
$$\alpha, \beta \longmapsto \mathrm{lk}(\alpha, \beta^+)$$
pushoff in +ve direction

$$V = \begin{array}{c|cc} & \alpha^+ & \beta^+ \\ \hline \alpha & 1 & 1 \\ \beta & 0 & 1 \end{array}$$
Seifert matrix

The signature of K, $\sigma(K)$, is
$$\sigma(K) := \sigma(V + V^T)$$

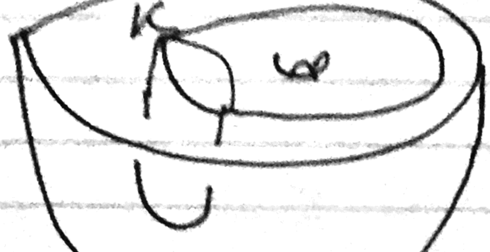
$$\sigma(K) = \sigma\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 2$$

By the way, $\Delta_K(t) := \det(V - tV^T)$ well defined up to mult by $\pm t^k$.

Theorem: If $K \subseteq S^3$ is TOP slice then $\sigma(K) = 0$
[Murasugi]

More generally, $\sigma(K) \leq g_4^{top}(K)$.

Proof: Let Σ be a Seifert surface for K in S^3



Let Δ be a slice disc.

$F := \Sigma \cup \Delta^+$ is a closed surface with a bicollar

i.e. we have $F \times [-1,1] \subseteq B^4$.

Define $F \times [-1,1] \xrightarrow{\text{proj.}} [-1,1] \to S^1$

Extend to $S^3 \setminus \nu K \to S^1$

Use obstr. theory to extend to $f: B^4 \setminus \nu\Delta \to S^1$

Perturb (rel ∂) to be transverse to $0 \in S^1$.

Then $M := f^{-1}(0)$ is a bicollared 3-mfld w. $\partial M = \Sigma \cup \Delta^+$

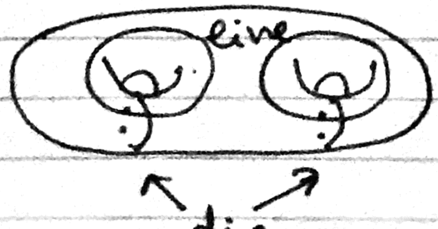
Note: $H^1(B^4 \setminus \nu\Delta) \cong \mathbb{Z}$. So we are just taking a map rep. the generator.
$\quad = [B^4 \setminus \nu\Delta, S^1]$

The key is to make it have the right boundary

½ lives ½ dies Let M be a compact, orientable 3-mfld with ∂M = closed surface of genus g.

Let $i: \partial M \to M$ be the inclusion.

Then $\operatorname{Ker}(i_*: H_1(\partial M; \mathbb{Q}) \to H_1(M; \mathbb{Q}))$ has dimension g.



So, $\exists$ basis $[x_1], \ldots, [x_g], [y_1], \ldots, [y_g]$ of $H_1(\partial M; \mathbb{Z})$ s.t. $[x_1], \ldots, [x_g] \in \operatorname{Ker} i_*$

Assume x_i curves lying in F.

$i_*[x_i] = 0 \Rightarrow \exists n_i \in \mathbb{Z}$ s.t. $n_i [x_i] = 0$ in $H_1(M; \mathbb{Z})$
$n_i \neq 0$

i.e. $n_i x_i = \partial X_i$
$\quad$ 2-chain in M.

M bicollared, so x_i^+ bounds the pushoff X_i^+ in M

Then $n_i n_j \, lk(x_i, x_j^+) = lk(n_i x_i, n_j x_j^+)$
$\qquad = X_i \pitchfork X_j^+ = 0$.

$\Rightarrow lk(x_i, x_j^+) = 0 \ \forall i, j \leq g$.

$\Rightarrow$ Seifert matrix $\begin{array}{c} \begin{bmatrix} 0 & A \\ B & C \end{bmatrix} \\ \underbrace{\quad}_{g \times g \text{ block}} \end{array} \Rightarrow \sigma(K) = 0$ alg. fact.

Lecture 3 June 19, 2024 (1.5 hr)
Finally we get to TOP constructions

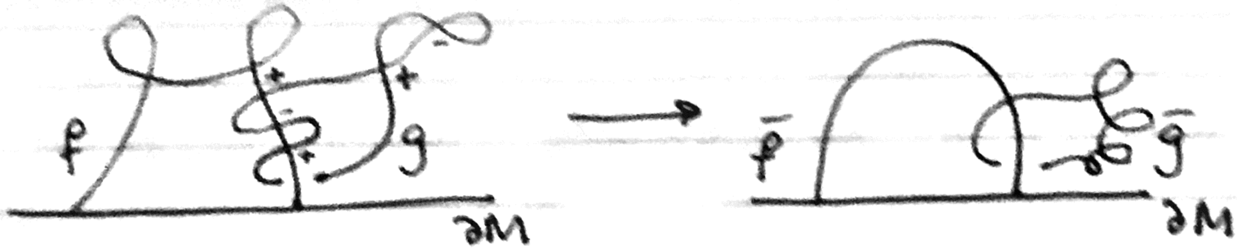
[Freedman-Quinn] Let $K \subseteq S^3$ be a 1-knot with $\Delta_K(t) \doteq 1$
Then K is TOP slice.

Key tool: Disc embedding theorem [Freedman-Quinn]

Let M^4 be a TOP 4-mfld with $\underbrace{\pi_1 M = 1 \text{ or } \mathbb{Z}}$
also works for other gps, but let's not think abt that

Suppose we have
gen. immersions
$f: D^2 \looparrowright M^4$ and $g: S^2 \looparrowright M^4$ such that
$\partial D^2 \hookrightarrow \partial M$ loc. flat
(i) alg intersections $\lambda(f, g) = 1$
(ii) alg self-int $\mu(g) = 0$
(iii) g has trivial normal bundle

Then f is (regularly) homotopic rel. ∂ to $\bar{f}$, g is htpic to $\bar{g}$ s.t.
$\bar{f}$ is a loc. flat emb., and $\bar{f} \pitchfork \bar{g}$ = single pt



Several applications (but not in S^4 or B^4 easily)

E.g.

Theorem (Lee-Wilczynski '97, Kasprowski-Powell-R. Teichner '22)
Every primitive hom. class in a simply connected 4-mfld is rep. by a loc. flat torus.

- Let's prove $\Delta_K(t) = 1$ knots are TOP slice.

Proof [Garoufalidis-Teichner]

Let's focus on $Wh(J)$.

Seifert matrix

$$\begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a^+ & b^+ \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

Push int F into B^4. call it F^+.
Note: $\pi_1(B^4 \setminus \nu F^+) \cong \mathbb{Z}$ $\quad N := B^4 \setminus \nu F^+$.
$\exists$ discs A and B with $\partial A = a$
$\partial B = b$

Assume $\partial A \subseteq \partial N$.

Goal:  Surger F^+ along emb discs bounded by a alg.

We have almost everything! Just need dual spheres

Let $T := b \times S^1$ boundary of tubular nbd
$\subseteq N$

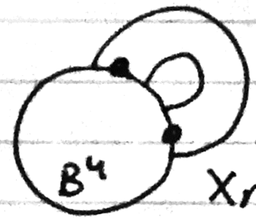
- Then $T \pitchfork A$ = single pt because $\mathrm{lk}(a, b^+) = 1$.

$g :=$ surger T along two copies of L.

then $\lambda(g, A) = 1 + \mathrm{lk}(b, a^+)(1-t) \in \mathbb{Z}[t^{\pm 1}]$
$= 1.$

$\square$

$\boxed{\text{Earlier}}$



$\leftarrow$ 2-handle
$D^2 \times D^2$
$\nu K \hookleftarrow \partial D^2 \times D^2$
$X_n(K)$
$\simeq S^2$

A.N.
[Feller-Miller.
Nagel-Orson-Powell
-R.]
'20

- Generator of $\pi_2(X_n(K))$ rep. by loc. flat sphere S
with $\pi_1(X_n(K) \setminus S)$ abelian

iff

$\leftarrow \mathbb{Z}/n$ cover

$\xi \in S^1$ with

(i) $\mathrm{Arf}(K) = 0$ (ii) $H_1(S^3_n(K); \mathbb{Z}) = 0$ (iii) $\sigma_K(\xi) = 0 \;\; \forall \xi^n = 1$

Goeritz formula $\Updownarrow$ $n \neq 0$

$\prod_{\{\xi \mid \xi^n = 1\}} \Delta_K(\xi) = 1$

$\sigma((1-\xi)V + (1-\bar{\xi})V^T)$
V Seifert matrix

- Before the break, let me highlight one very important <u>open</u> question re: existence.

Q: Does there exist $\Sigma \subseteq S^4$ closed which is locally flat, but not smoothable?
i.e. not iso to a sm. emb?

Remark: [Lashof '71] $\exists$ nonsmoothable $S^3 \hookrightarrow S^5$

- [Torres] $\exists$ nonsmoothable nullhtpic 2-sphere in $4\mathbb{CP}^2$.

Let's move on to uniqueness questions.

- Surfaces $K_1, K_2 : \Sigma_g \hookrightarrow M^4$ are <u>isotopic</u> if $\exists$ ambient isotopy
$$\underset{=:\Phi_t}{\Phi} : M \times [0,1] \to M$$
(i.e. $\Phi(-,t)$ is a diffeo $\forall t$)
s.t. $\Phi_0 = id$, $\Phi_1 \circ K_1 = K_2$.

Alternatively: $\exists$ 1 parameter family of embeddings taking K_1 to K_2.
– equivalent because of the isotopy ext. thm

- Note: notions makes sense for both the sm. & top settings.

Let Σ_g be closed, orientable

<u>Def^{n}</u>: A surface $K : \Sigma_g \to M^4$ is sm/top <u>unknotted</u> if it bounds a genus g 3-dim handlebody ($\text{ie} \natural_g S^1 \times D^2$) in M. $\boxed{\text{sm/top}}$ embedded

ie. for $K : S^2 \hookrightarrow M^4$, K is unknotted if it bounds a D^3 in M^4.

-

Recall (from classical knot thy)

If $k : S^1 \hookrightarrow S^3$ is a 1-knot.
k is unknotted $\Longleftrightarrow \pi_1(S^3 \setminus k) \cong \mathbb{Z}$.
(uses Dehn's lemma)

Q: what about the 4D analogue?

-

closed, orientable

Let Σ be a loc. flat surface in S^4.
with $\pi_1(S^3 \setminus \Sigma)$ cyclic.

TOP unknotting conj: Σ is TOP unknotted.

True for (i) $\Sigma = S^2$ [Freedman-Quinn '90]
(ii) Σ orientable, genus ≥ 3
[Conway-Powell '23]

c.f. -Hillman-Kawauchi '95

(iii) Σ non-orientable. $\#_h \mathbb{RP}^2$

note $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z}/2$ in this case

$h=1$ [Lawson '84]
$h=2,3$ [Conway-Orson-Powell '23]
$h=4,5$ [Pencovitch '24]
\+ other cases related to euler # [COP'23]
$|e(\Sigma)| < 2h$

skip

Remaining cases are open.
e.g. orientable, genus $= 1, 2$.

Smooth unknotting problem.

Q: Suppose Σ is a sm. oriented surface in S^4 with $\pi_1(S^4 \setminus \Sigma)$ cyclic
Is Σ unknotted?

Note: $\exists$ "exotically knotted" non-orientable surfaces in S^4.

i.e. TOP unknotted but not sm.

e.g. [Finashin-Kreck-Viro '87] $\#_{10} \mathbb{RP}^2$

[Miyazawa '23] $\mathbb{RP}^2$.

More generally, does there exist a pair of closed, sm. orientable surfaces in S^4 that are TOP isotopic but not Sm isotopic?

$\exists$ many <u>exotic</u> embeddings in other 4-mflds
This is kind of an industry now with many tools (HF, Khovanov, ...)

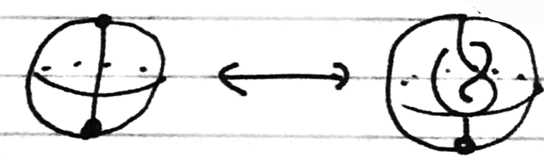
First examples were due to Fintushel-Stern '97.

"Rim surgery"

Σ

$\leftarrow$ annulus.

$(S^1\times B^3, S^1\times[0,1])$
replaced by
$(S^1\times B^3, S^1\times A)$ Result is Σ_K



Under certain conditions $(\pi_1(X\setminus\Sigma)=1)$, the Sm embs are $\overset{TOP}{\wedge}$ isotopic

if $\exists\ (X,\Sigma_{K_1})\cong(X,\Sigma_{K_2})$ diffeo of pairs
then $\Delta_{K_1}(t)=\Delta_{K_2}(t)$.

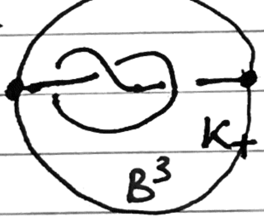
Lecture 4 (1hr)

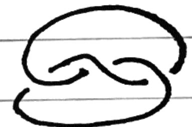
- Finishing uniqueness discussion
  - rim surgery
  - $\pi_1(S^4 \setminus \Sigma) \cong \mathbb{Z} \Rightarrow \Sigma$ unknotted for $\Sigma = S^2$ [Freedman Quinn '90]

Then we'll have a short summary to end.

<u>Rim surgery</u> [Fintushel-Stern '97]

Let $\Sigma \subseteq M^4$ be a loc. flat emb surface, M closed.
$C \subseteq \Sigma$ an o.p. circle
$K \subseteq S^3$ arb 1-knot.





replace
$(S^1 \times B^3, S^1 \times [0,1])$
$\wr$
$(S^1 \times B^3, S^1 \times K_+)$

Call the result $\Sigma_{K,C}$.

Fact: If $\pi_1(M \setminus \Sigma) = 1$ then (M, Σ) & $(M, \Sigma_{K,C})$ are homeomorphic [Boyer]

[Note: for M^4 closed, $\pi_1 M = 1$, if two homeos induce the same isom. on int. form then they are isotopic]

Thm [Fintushel-Stern] M^4 symplectic, $\pi_1 M = 1$
$\Sigma \subseteq M$ symplectic, $\pi_1(X \setminus \Sigma) = 1$
$g(\Sigma) > 0$, $\Sigma \cdot \Sigma \geq 0$

If $\exists$ diffeo $(M, \Sigma_{K_1,C}) \xrightarrow{\cong} (M, \Sigma_{K_2,C})$ then $\Delta_{K_1}(t) = \Delta_{K_2}(t)$.
e.g. $M = K3$, Σ = generic elliptic fibre

- Alternative definition: Consider the rim torus $C \times \partial D^2$.
$\uparrow$
meridional disc to Σ.

$M \setminus (C \times \partial D^2 \times D^2) \cup (S^3 \setminus \nu K) \times S^1$

$\partial D^2 \longleftarrow \mu_K$
$\partial D^2 \longleftarrow \lambda_K$ $\quad C \longleftarrow S^1$

In other words, perform knot surgery along the rim tori.

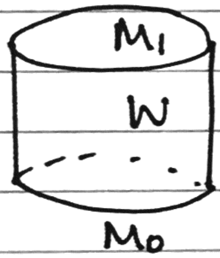
- This operation has been generalised in numerous ways

$\longrightarrow$

<u>Theorem</u> [Freedman-Quinn '90] Let $K: S^2 \hookrightarrow S^4$ be a 2-knot with $\pi_1(S^4 \setminus K) \cong \mathbb{Z}$

Then K is TOP unknotted.

<u>Key tool</u>: The s-cobordism theorem.



- Let $(W; M_0, M_1)$ be a cobordism ie. $\partial W = M_0 \sqcup M_1$.

We say $(W; M_0, M_1)$ is an <u>h-cobordism</u> if $\iota_i: M_i \hookrightarrow W$ is a homotopy equivalence

It is an <u>s-cobordism</u> if the inclusions are <u>simple</u> homotopy equivalences

<u>Fact</u>: If $\pi_1 W = 1$ or $\mathbb{Z}$ then h-cobs are s-cobs.

- Theorem [Freedman-Quinn '80s] Let $(W^5; M_0, M_1)$ be an h-cob with $\pi_1 W = 1$ or $\mathbb{Z}$.
4D s-cobordism theorem.

Then $W \approx M_0 \times [0,1]$. (homeo)

As w. the DET, the theorem holds more broadly for other π_1.

The DET is the key tool for proving the s-cob thm

In high dims this is a highly powerful tool.

s-cob: Barden, Mazur, Stallings   c.f. Smale, Poincaré conj.

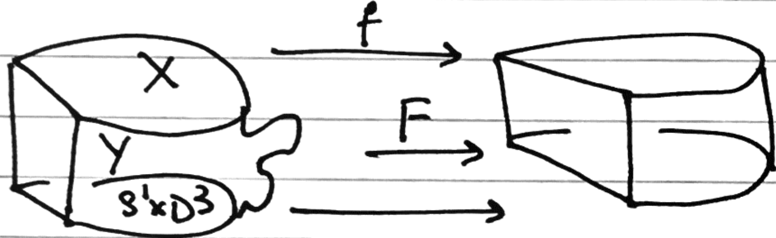
- <u>Proof (sketch)</u>: It will suffice to show that $S^4 \setminus \nu K \approx S^1 \times D^3$.

Step 1: Show that $X = S^4 \setminus \mathring{\nu} K \xrightarrow[f]{\simeq} S^1 \times D^3$

We get a map to S^1 since $H^1(S^4 \setminus \mathring{\nu} K) \cong \mathbb{Z}$ inducing an iso on π_1.

Then we have to compute the higher htpy groups

Step 2: Show there is a "normal bordism



$S^1 \times D^3 \times [0,1]$

To do this, need to show that $[f] = [id]$ in the relevant bordism group.

Step 3: (Try to) Surger Y to an s-cobordism.

The obstruction to such a surgery being possible lies in $L_5(\mathbb{Z}[\mathbb{Z}]) \cong 8\mathbb{Z}$ detected by signature.

The generator of this gp is realised by a map.

$S^1 \times E_8 \to S^1 \times S^4$

Connect sum this "along loops" to arrange that our surgery ob. is zero.

Then can surger Y to an s-cob W.

Step 4: Apply s-cobordism theorem $\square$.

What we skipped (many things, here are just a few)

1. Concordance

$F', F : \Sigma \hookrightarrow M$

Two surface links $\Sigma = \sqcup \Sigma_i$

in a 4-mfld M^4 are said to be sm/top <u>concordant</u> if

- they cobound a sm/top emb. $F: \Sigma \times [0,1] \to M \times [0,1]$.

$$F|_{\Sigma \times 0} = f_0 \quad \& \quad F|_{\Sigma \times 1} = f_1$$

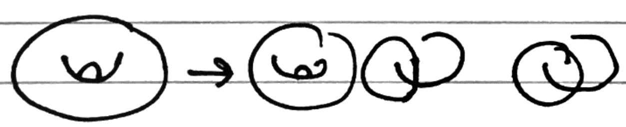
Theorem [Sunukjian'15, Kervaire'65]

Let M be a simply conn 4-mfld.
Any two sm/top emb. <u>connected</u> surfaces $F_0, F_1 \subseteq M$ are sm/top conc. iff they have the same genus and the same homology class.

<u>Question</u>: Are all 2-links slice?
$L: \sqcup S^2 \hookrightarrow S^4$

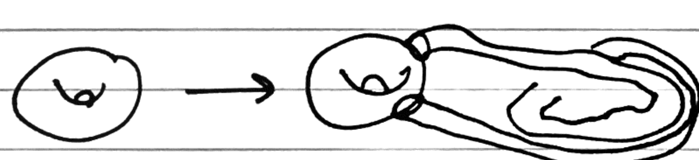
<u>2 Stabilisation</u>

Three flavours:



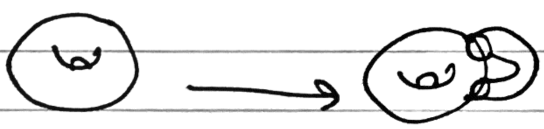
external stabs

$\Sigma \subseteq M \longrightarrow \Sigma \subseteq M \# k S^2 \times S^2$



weak internal stab.

$\Sigma \subseteq M \longrightarrow \Sigma \cup$ emb 1h $\subseteq M$



$\Sigma \subseteq M \qquad \Sigma \# \mathcal{U}_1 \subseteq M \# S^4 = M.$

↑ unknotted genus 1 surface

[Baykur-Sunukjian'15, Hosokawa-Kawauchi'79]

- Any pair of homologous, sm embedded surfaces in a closed, oriented 4-mfld become sm. isotopic after enough <u>weak, internal</u> stabilisations

[Galvin'24] $\pi_1 M = 1$. If $\Sigma, \Sigma' \subseteq M$ are TOP isotopic
M smooth then they are sm. isotopic after enough <u>ext. stabs</u>.

cf. Galvin Orson-Powell for others.

<u>Question</u>: Suppose M sm,
$\Sigma, \Sigma' \subseteq M$ are TOP isotopic.

Are Σ, Σ' sm isotopic after enough strong internal stabs?
