

High-Frequency Algorithmic Trading with Momentum and Order Imbalance

My goal is to establish and solve the stochastic optimal control problem that captures the momentum and order imbalance dynamics of the Limit Order Book (LOB). The solution will yield an optimal trading strategy that will permit statistical arbitrage of the underlying stock, which will then be backtested on historical data.

Progress Timeline

DATE	THESIS	STA4505
Dec 2014	Complete CTMC calibration	
Dec 2014	Backtest naive strategies based on CTMC	
Jan-May	Study stochastic controls: ECE1639, STA4505	
Jun 5	Establish models	Exam Study
Jun 12	Establish performance criteria	Exam Study
Jun 15	Derive DPP/DPE	EXAM
Jun 19	Derive DPP/DPE	
Jun 26	Derive DPE/Solve PDEs	
Jul 3	Solve PDEs	
Jul 10	Solve PDEs and implement numerical solution	Numerical solution
Jul 31	Backtest solution on historical data	Implement simulations
Aug 15	Dissertation Writeup	Project Writeup

For Our Readers in the Middle East...

I just saw a big black guy, short dreads, one of those pastafarian ball caps with a bit of poof to them, big headphones, crossing the street in the Forest Hill Village. He had on an over-sized purple t-shirt with huge light blue writing on it:

IT
WAS
ALL
A
DREAM

In other news, this morning I was awakened to first my alarm, which I took the kitchen to scan the QR code in order to make it stfu so I could go back to sleep, and then about ten minutes later a doorbell. Fuck, the cleaning lady! My cleaning lady arrived today, nice older Jamaican woman with an unpronounceable name and a penchant for country music. I tried pretty hard to get KX96fm working on the stereo, but for some reason the reception was total shit. I sweated for a while trying to adjust the plugs on the back of the tuner (I'd forgotten in the process that the stack of machinery was actually holding up the shelf above it, so when I removed the cassette deck thinking that I should probably chuck that thing anyway, your vinyl player and my vinyls nearly came crashing down; back went the cassette deck player...), ultimately failed, and I'm still not really sure why the knobs are so finicky. Maybe because the set is older than us. Maybe because it's your set and I still don't know how it works despite now having it on loan for probably longer than you ever had it. Anyway, I snuck out for a while to drive up to the armory where it turns out I had won a door prize at some fundraiser I attended for about 10 minutes. I won a Starbucks gift package: a hot tumbler, cold tumbler, a box of instant vanilla latte and a box of instant iced caramel coffee. The boxes contain 4 and 5 sachets each, respectively!! The fuck is that?? Probably costs like 20 bucks, you're basically paying full price for a powdered down version. Total bullshit; I'm outraged even though it was free.

So how're we doing academically, you must be wondering? First let me tell you how much pizza I've eaten this week. Okay but no, really, things are moving along nicely, as hopefully you'll be able to glimpse from the below. I'm basically where I want to be on the timeline, working away at the DPE and just barely beginning to solve the PDEs, so maybe even a bit ahead. Summer officially starts in two days. Somehow I thought that was going to be a relevant comment to lead into something deeper, but nope, that's about it.

Ok not gonna lie bud, I've missed these days. Just me and old faithful, chilling at the neighborhood bucks. A girl with a really beautiful ass just waltzed past me, straight to – use the shitter.

Yesterday I had an absolute waste of a day. At 645am I got myself out of bed to go to yet another boot camp class with my fat friend Cyrus. Again I made it on time and he didn't. This time the instructor was a big sweaty black guy named Geoff, who immediately had us go out for a warm up run. I tells him I says: I can't run bra, ankle sprain. And he's incredulous, how could I do

boot camp if I can't do high impact workouts, blah blah, is talking down at me like I don't know shit about shit, I'm thinking fuck you I've actually DONE boot camp you twat, anyway, he has the front-desk cutie freeze my account till I heal up ('give it a week', he says, as if sprains heal in a week), and that's that. So it's 8am and I'm a free man, downtown, unshowered, unfed. Great start to the day. Between 1-5pm I was out having lunch with Gabe and a couple of my UTIAS labmates out in the Kingsway area, at an Italian place where Gabe has a pizza named after him. As a way of showing solidarity for the beginning of Ramadan, I ate my second personal pizza in less than 24h. Anyway, nice but a giant waste of time. And from there I booked it over to Meera's to meet her (potentially, unconfirmed as of yet) new dog named Echo, about to be renamed Alfie, and I spent the whole time calling him Alphagetti [side note: this punk rock drummer died a few weeks ago in Seattle, so in his honor they decided to realize one of his life dreams: he always wanted a Rush-themed Italian restaurant, so for one night only they opened up Spageddy Lee], meanwhile this little shih tzu spent the whole time alternately barking at me and licking his own balls or dick. He chilled out by the end, but I'm not sure whether we're still friends. Anyway, I'm leaving Meera's, I'm walking back to my bike on Cumberland, keys in the ignition, fire her up, and before I hop on, this girl on the sidewalk standing outside one of those beauty stores offers me a moisturizer sample. She's hot, got on a nice black dress, showing tons of her hazot gdolot. I'm like cool, I'll take it. We start chatting, she asks me something like "what do you use on your beautiful face?", so obviously when she suggests I follow her inside the shop for a second to see something else I'm like yes you beautiful thing. We do the exchange of names (hers is Tom, wtf right?), where you from, turns out she's Israeli, I tell her I'm Jewish and she says 'wow where have you been all my life', which oddly enough is an expression I've heard other Israeli girls say, I don't know why they love that expression so much considering how actually strong and non-casual that expression is, at least in my mind? Anyway, I want to bang her senseless, suddenly I realize she's trying to sell me \$299 exfoliant, and I'm like ughhhh okay I'll just take the moisturizer sample, seeee ya. Anyway, might have to babysit that little pooch on Monday.

The Academic Week in Review

The Dissertation

Okay at this point I'd like to cycle back and re-jig the problem setup that we've done thus far. Largely this is due to Tue morning's meeting with Sebastian at the cafe: whereas I was convinced I was dealing with a **delayed** stochastic differential equation, which would require sifting through another textbook to understand, turns out this is far from the truth. To recap: previously I was concerned that I needed $Z_t = (\rho_t, \Delta_t)$ and $\zeta_t = Z_{t-s} = (\rho_{t-s}, \Delta_{t-s})$ terms in my value function, because in the exploratory data analysis we had derived some conditional probabilities that informed us that **if** we had seen a price change, and the imbalance now is the same as the imbalance before, **then** we would expect another equal price change. But! These conditional probabilities are actually *embedded within the CTMC generator G* , which is for the couplet at time t , it doesn't require a time-delay. Hence, we don't need this lookback/delay ζ_t

term at all, and so it's smooth sailing to proceed.

Below we list the processes involved in the optimization problem:

Imbalance & Midprice Change	$\mathbf{Z}_t = (\rho_t, \Delta_t)$	CTMC with generator G
Imbalance	$\rho_t = \mathbf{Z}_t^{(1)}$	LOB imbalance at time t
Midprice	S_t	evolves according to CTMC
Midprice Change	$\Delta_t = \mathbf{Z}_t^{(2)} = S_t - S_{t-s}$	s a pre-determined interval
Bid-Ask half-spread	π_t	constant?
LOB Shuffling	N_t	Poisson with rate $\lambda(\mathbf{Z}_t)$
Δ Price: LOB shuffled	$\{\eta_{0,z}, \eta_{1,z}, \dots\} \sim F_z$	i.i.d. with $z = (k, l)$, where $k \in \{\#\text{bins}\}$, $l \in \{\Delta\$\}$
Other Agent MOs	K_t^\pm	Poisson with rate $\mu^\pm(\mathbf{Z}_t)$
LO posted depth	δ_t^\pm	our \mathcal{F} -predictable controlled processes
Our LO fill count	L_t^\pm	\mathcal{F} -predictable, non-Poisson
Our MOs	M_t^\pm	our controlled counting process
Our MO execution times	$\tau^\pm = \{\tau_k^\pm : k = 1, \dots\}$	increasing sequence of \mathcal{F} -stopping times
Cash	$X_t^{\tau, \delta}$	depends on our processes M and δ
Inventory	$Q_t^{\tau, \delta}$	depends on our processes M and δ

L_t^\pm are counting processes (not Poisson) satisfying the relationship that if at time t we have a sell limit order posted at a depth δ_t^- , then our fill probability is $e^{-\kappa\delta_t^-}$ conditional on a buy market order arriving; namely:

$$\begin{aligned}\mathbb{P}[\mathrm{d}L_t^- = 1 \mid \mathrm{d}K_t^+ = 1] &= e^{-\kappa\delta_t^-} \\ \mathbb{P}[\mathrm{d}L_t^+ = 1 \mid \mathrm{d}K_t^- = 1] &= e^{-\kappa\delta_t^+}\end{aligned}$$

The midprice S_t evolves according to the Markov chain and hence is Poisson with rate λ and jump size η , both of which depend on the state of the Markov chain. This Poisson process is all-inclusive in the sense that it accounts for any midprice change, be it from executions, cancellations, or order modifications with the LOB. Thus, the stock midprice S_t evolves according to the SDE:

$$\mathrm{d}S_t = \eta_{N_t^-, Z_t^-} \mathrm{d}N_t \quad (1)$$

and additionally satisfies:

$$S_t = S_{t_0} + \int_{t_0}^t \Delta_u \mathrm{d}u \quad (2)$$

In executing market orders, we assume that the size of the MOs is small enough to achieve the best bid/ask price, and not walk the book. Hence, our cash process evolves according to:

$$\begin{aligned}\mathrm{d}X_t^{\tau, \delta} &= \underbrace{(S_t + \pi_t + \delta_t^-) \mathrm{d}L_t^-}_{\text{sell limit order}} - \underbrace{(S_t - \pi_t - \delta_t^+) \mathrm{d}L_t^+}_{\text{buy limit order}} \\ &\quad + \underbrace{(S_t - \pi_t) \mathrm{d}M_t^-}_{\text{sell market order}} - \underbrace{(S_t + \pi_t) \mathrm{d}M_t^+}_{\text{buy market order}}\end{aligned} \quad (3)$$

Based on our execution of limit and market orders, our inventory satisfies:

$$Q_0^{\tau,\delta} = 0, \quad Q_t^{\tau,\delta} = L_t^+ + M_t^+ - L_t^- - M_t^- \quad (4)$$

We define a new variable for our net present value (NPV) at time t , call it $W_t^{\tau,\delta}$, and hence $W_T^{\tau,\delta}$ at terminal time T is our ‘terminal wealth’. In algorithmic trading, we want to finish the trading day with zero inventory, and assume that at the terminal time T we will submit a market order (of a possibly large volume) to liquidate remaining stock. Here we do not assume that we can receive the best bid/ask price - instead, the price achieved will be $(S - \text{sgn}(Q)\pi - \alpha Q)$, where $\text{sgn}(Q)\pi$ represents crossing the spread in the direction of trading, and αQ represents receiving a worse price linearly in Q due to walking the book. Hence, $W_t^{\tau,\delta}$ satisfies:

$$W_t^{\tau,\delta} = \underbrace{X_t^{\tau,\delta}}_{\text{cash}} + \underbrace{Q_t^{\tau,\delta} \left(S_t - \text{sgn}(Q_t^{\tau,\delta})\pi_t \right)}_{\text{book value of assets}} - \underbrace{\alpha \left(Q_t^{\tau,\delta} \right)^2}_{\text{liquidation penalty}} \quad (5)$$

The set of admissible trading strategies \mathcal{A} is the set of all \mathcal{F} -stopping times and \mathcal{F} -predictable, bounded-from-below depths δ .

For deriving an optimal trading strategy, I will consider three separate performance criteria, which allow us to evaluate the performance of a given strategy:

1. Profit: $H^{\tau,\delta}(\cdot) = \mathbb{E} \left[W_T^{\tau,\delta} \right]$
2. Profit with risk aversion: $H^{\tau,\delta}(\cdot) = \mathbb{E} \left[W_T^{\tau,\delta} - \gamma \mathbf{1}_{W_T^{\tau,\delta} < 0} \cdot W_T^{\tau,\delta} \right]$
3. Profit with running inventory penalty: $H^{\tau,\delta}(\cdot) = \mathbb{E} \left[W_T^{\tau,\delta} - \varphi \int_t^T (Q_u^{\tau,\delta})^2 du \right]$

In each of the cases, the value function is given by

$$H(t, x, s, \mathbf{z}, q) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \sup_{\delta \in \mathcal{A}_{[t,T]}} H^{\tau,\delta}(t, x, s, \mathbf{z}, q) \quad (6)$$

Dynamic Programming Principle for Optimal Stopping and Control

Theorem 1. *If an agent's performance criteria for a given admissible control \mathbf{u} and admissible stopping time τ are given by*

$$H^{\tau,\mathbf{u}}(t, \mathbf{x}) = \mathbb{E}_{t,\mathbf{x}}[G(X_\tau^{\mathbf{u}})]$$

and the value function is

$$H(t, \mathbf{x}) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \sup_{\mathbf{u} \in \mathcal{A}_{[t,T]}} H^{\tau,\mathbf{u}}(t, \mathbf{x})$$

then the value function satisfies the Dynamic Programming Principle

$$H(t, \mathbf{x}) = \sup_{\tau \in \mathcal{T}_{[t, T]}} \sup_{\mathbf{u} \in \mathcal{A}_{[t, T]}} \mathbb{E}_{t, \mathbf{x}} [G(X_\tau^{\mathbf{u}}) \mathbf{1}_{\tau < \theta} + H(\theta, X_\theta^{\mathbf{u}}) \mathbf{1}_{\tau \geq \theta}] \quad (7)$$

for all $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^m$ and all stopping times $\theta \leq T$.

Dynamic Programming Equation for Optimal Stopping and Control

Theorem 2. Assume that the value function $H(t, \mathbf{x})$ is once differentiable in t , all second-order derivatives in \mathbf{x} exist, and that $G : \mathbb{R}^m \rightarrow \mathbb{R}$ is continuous. Then H solves the quasi-variational inequality

$$\max \left\{ \partial_t H + \sup_{\mathbf{u} \in \mathcal{A}_t} \mathcal{L}_t^{\mathbf{u}} H ; G - H \right\} = 0 \quad (8)$$

on \mathcal{D} , where $\mathcal{D} = [0, T] \times \mathbb{R}^m$.

Maximizing Profit

Ok let's get to business. We need to solve the DPE that results from using the first performance criteria in our value function. So our G function is exactly our NPV term W , and really the work comes from finding the infinitesimal generator for the processes. Let's get on it.

The quasi-variational inequality in equation 8 can be interpreted as follows: the max operator is choosing between posting limit orders or executing market orders; the second term, $G - H$, is the stopping region and represents the value derived from executing a market order; and the first term is the continuation region, representing the value of posting limit orders. We'll use the shorthand $H(\cdot) = H(t, x, s, \mathbf{z}, q)$ and solve for dH inside the continuation region, hence $dM^\pm = 0$, in order to then extract out the infinitesimal generator.

$$\begin{aligned} dH(t, x, s, \mathbf{z}, q) &= \sum_i \partial_{x_i} H dx_i \\ &= \partial_t H dt + \partial_{K^\pm} H dK^\pm + \partial_{\mathbf{Z}} H d\mathbf{Z} \\ &= \partial_t H dt + \left\{ e^{-\kappa \delta^-} \mathbb{E} [H(t, x + (s + \pi + \delta^-), s, \mathbf{z}, q - 1) - H(\cdot)] \right\} dK^+ \\ &\quad + \left\{ e^{-\kappa \delta^+} \mathbb{E} [H(t, x - (s - \pi - \delta^+), s, \mathbf{z}, q + 1) - H(\cdot)] \right\} dK^- \\ &\quad + \sum_{k \in P} \sum_{l \in \{-1, 0, 1\}} \mathbb{E} [H(t, x, s + \text{sgn}(l)\eta_{0, \mathbf{z}}, \mathbf{z} + (k, l), q) - H(\cdot)] dZ_{\mathbf{z}, (k, l)} \end{aligned}$$

Substitute in the following identities for the compensated processes

$$\begin{aligned} dM^\pm &= d\tilde{K}^\pm + \mu^\pm(\mathbf{z}) dt \\ dZ_{\mathbf{z}, (k, l)} &= d\tilde{Z}_{\mathbf{z}, (k, l)} + G_{\mathbf{z}, (k, l)} dt \end{aligned}$$

$$\begin{aligned}
&= \partial_t H \, dt + \left\{ \mu^+(\mathbf{z}) e^{-\kappa \delta^-} \mathbb{E}[H(t, x + (s + \pi + \delta^-), s, \mathbf{z}, q - 1) - H(\cdot)] \right. \\
&\quad + \mu^-(\mathbf{z}) e^{-\kappa \delta^+} \mathbb{E}[H(t, x - (s - \pi - \delta^+), s, \mathbf{z}, q + 1) - H(\cdot)] \\
&\quad + \sum_{k \in P} \sum_{l \in \{-1, 0, 1\}} G_{\mathbf{z}, (k, l)} \mathbb{E}[H(t, x, s + \text{sgn}(l) \eta_{0, \mathbf{z}}, \mathbf{z} + (k, l), q) - H(\cdot)] \Big\} dt \\
&\quad + \{ e^{-\kappa \delta^-} \mathbb{E}[H(t, x + (s + \pi + \delta^-), s, \mathbf{z}, q - 1) - H(\cdot)] \} d\tilde{K}^+ \\
&\quad + \{ e^{-\kappa \delta^+} \mathbb{E}[H(t, x - (s - \pi - \delta^+), s, \mathbf{z}, q + 1) - H(\cdot)] \} d\tilde{K}^- \\
&\quad + \sum_{k \in P} \sum_{l \in \{-1, 0, 1\}} \mathbb{E}[H(t, x, s + \text{sgn}(l) \eta_{0, \mathbf{z}}, \mathbf{z} + (k, l), q) - H(\cdot)] d\tilde{Z}_{\mathbf{z}, (k, l)}
\end{aligned}$$

From which we can see that the infinitesimal generator is given by

$$\begin{aligned}
\mathcal{L}_t^\delta H &= \mu^+(\mathbf{z}) e^{-\kappa \delta^-} \mathbb{E}[H(t, x + (s + \pi + \delta^-), s, \mathbf{z}, q - 1) - H(\cdot)] \\
&\quad + \mu^-(\mathbf{z}) e^{-\kappa \delta^+} \mathbb{E}[H(t, x - (s - \pi - \delta^+), s, \mathbf{z}, q + 1) - H(\cdot)] \\
&\quad + \sum_{k \in P} \sum_{l \in \{-1, 0, 1\}} G_{\mathbf{z}, (k, l)} \mathbb{E}[H(t, x, s + \text{sgn}(l) \eta_{0, \mathbf{z}}, \mathbf{z} + (k, l), q) - H(\cdot)]
\end{aligned} \tag{9}$$

Now, our DPE has the form

$$\max \left\{ \partial_t H + \sup_{\mathbf{u} \in \mathcal{A}_t} \mathcal{L}_t^{\mathbf{u}} H ; H(t, x - (s + \pi), s, \mathbf{z}, q + 1) - H(\cdot) ; \right. \\
\left. H(t, x + (s - \pi), s, \mathbf{z}, q - 1) - H(\cdot) \right\} = 0 \tag{10}$$

with boundary conditions

$$H(T, x, s, \mathbf{z}, q) = x + q(s - \text{sgn}(q)\pi) - \alpha q^2 \tag{11}$$

$$H(T, x, s, \mathbf{z}, 0) = x \tag{12}$$

The three terms over which we are maximizing represent the continuation regions and stopping regions of the optimization problem. The first term, the continuation region, represents the limit order controls; the second and third terms, each a stopping region, represent the value gain from executing a buy market order and a sell market order, respectively.

Let's introduce the ansatz $H(\cdot) = x + q(s - \text{sgn}(q)\pi) + h(t, \mathbf{z}, q)$. The first two terms are the wealth plus book value of assets, hence a mark-to-market of the current position, whereas the $h(t, \mathbf{z}, q)$ captures value due to the optimal trading strategy. The corresponding boundary conditions on h are

$$h(T, \mathbf{z}, q) = -\alpha q^2 \tag{13}$$

$$h(T, \mathbf{z}, 0) = 0 \tag{14}$$

Substituting this ansatz into equation 9, we get:

$$\begin{aligned}\mathcal{L}_t^\delta H = & \mu^+(\mathbf{z})e^{-\kappa\delta^-} [\delta^- + \pi[1 + \text{sgn}(q-1) + q(\text{sgn}(q) - \text{sgn}(q-1))]] + h(t, \mathbf{z}, q-1) - h(t, \mathbf{z}, q)] \\ & + \mu^-(\mathbf{z})e^{-\kappa\delta^+} [\delta^+ + \pi[1 - \text{sgn}(q+1) + q(\text{sgn}(q) - \text{sgn}(q+1))]] + h(t, \mathbf{z}, q+1) - h(t, \mathbf{z}, q)] \\ & + \sum_{k \in P} \sum_{l \in \{-1, 0, 1\}} G_{\mathbf{z}, (k, l)} [ql\mathbb{E}[\eta_{0, \mathbf{z}}] + h(t, (k, l), q) - h(t, \mathbf{z}, q)]\end{aligned}$$

Looking Ahead

Okay time soon to give El Cheesy a ride on the bike, then climb together, and then afterward I'll ride and climb –