Statistical Arbitrage Using Limit-Order Book Imbalance

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Roadmap

Roadmap

Background Information

Exploratory Data Analysis

Maximizing Wealth via Continuous-Time Stochastic Optimal Control

Maximizing Wealth via Discrete-Time Stochastic Optimal Control

Results

Future Work

Roadmap

Modelling Imbalance

Incorporating Price Change

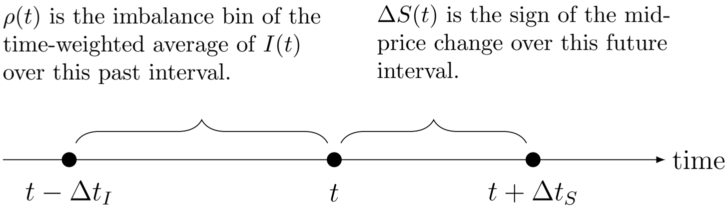
Next, consider a two-dimensional CTMC $Z(t)$ that jointly models imbalance bin $\rho(t)$ and price change $\Delta S(t)$, where

$$\rho(t) \in \{1, 2, \ldots, \#_{bins}\}$$

is the bin corresponding to imbalance averaged over the interval $[t - \Delta t_I, t]$, and

$$\Delta S(t) = \operatorname{sgn}(S(t + \Delta t_S) - S(t)) \in \{-1, 0, 1\}$$

is the *sign* of the change in midprice of the *future* time interval Δt_S.



Transition Matrix

Using MLE, we obtain a generator matrix $\boldsymbol{G}$ for the CTMC. The transition matrix over a step of size Δt_I is given by

$$\boldsymbol{P}(\Delta t_I) = [p_{ij}(\Delta t_I)] = e^{\boldsymbol{G}\Delta t_I}$$

called our *one-step transition probability matrix*. Matrix entries give the probability of transition from one (imbalance, price change) pair to another over the time interval Δt_I. This can be written semantically as

$$p_{ij} = \mathbb{P}\left[\varphi(\rho_{\text{curr}}, \Delta S_{\text{future}}) = j \mid \varphi(\rho_{\text{prev}}, \Delta S_{\text{curr}}) = i\right]$$

Predicting Future Price Change

Using Bayes' Rule, we can transform the $\boldsymbol{P}$ matrix to

$$\mathbb{P}\left[\Delta S_{\text{future}} = j \mid B, \rho_{\text{curr}} = i\right] = \frac{\mathbb{P}\left[\rho_{\text{curr}} = i, \Delta S_{\text{future}} = j \mid B\right]}{\mathbb{P}\left[\rho_{\text{curr}} = i \mid B\right]}$$

This allows us to predict future price moves. We'll call the collection of these probabilities the $\boldsymbol{Q}$ matrix.

Predicting Future Price Change

Sample $\boldsymbol{Q}$ matrix.

	$\Delta S_{\text{curr}} < 0$			$\Delta S_{\text{curr}} = 0$			$\Delta S_{\text{curr}} > 0$		
	$\rho_{curr} = 1$	2	3	1	2	3	1	2	3
$\Delta S_{\text{future}} < 0$									
$\rho_{\text{prev}} = 1$	**0.53**	0.15	0.12	0.05	0.10	0.14	0.08	0.13	0.14
$\rho_{\text{prev}} = 2$	0.10	**0.58**	0.14	0.07	0.04	0.10	0.13	0.06	0.12
$\rho_{\text{prev}} = 3$	0.08	0.12	**0.52**	0.09	0.06	0.03	0.11	0.10	0.05
$\Delta S_{\text{future}} = 0$									
$\rho_{\text{prev}} = 1$	0.41	0.75	0.78	**0.91**	0.84	0.79	0.42	0.79	0.77
$\rho_{\text{prev}} = 2$	0.79	0.36	0.71	0.83	**0.92**	0.82	0.75	0.37	0.78
$\rho_{\text{prev}} = 3$	0.79	0.74	0.40	0.81	0.83	**0.91**	0.70	0.76	0.39
$\Delta S_{\text{future}} > 0$									
$\rho_{\text{prev}} = 1$	0.06	0.10	0.09	0.04	0.06	0.07	**0.50**	0.09	0.09
$\rho_{\text{prev}} = 2$	0.10	0.06	0.15	0.10	0.04	0.08	0.12	**0.57**	0.10
$\rho_{\text{prev}} = 3$	0.13	0.14	0.08	0.10	0.11	0.05	0.19	0.14	**0.56**

Trading Strategies Informed by the Q Matrix

- **Naive** Use market orders to buy (sell) if price is predicted to move up (down).
- **Naive+** Post at-the-touch limit orders when zero price change is predicted.
- **Naive++** Post a limit order to buy (sell) is price is predicted to move up (down).

Calibration of Naive Trading Strategies

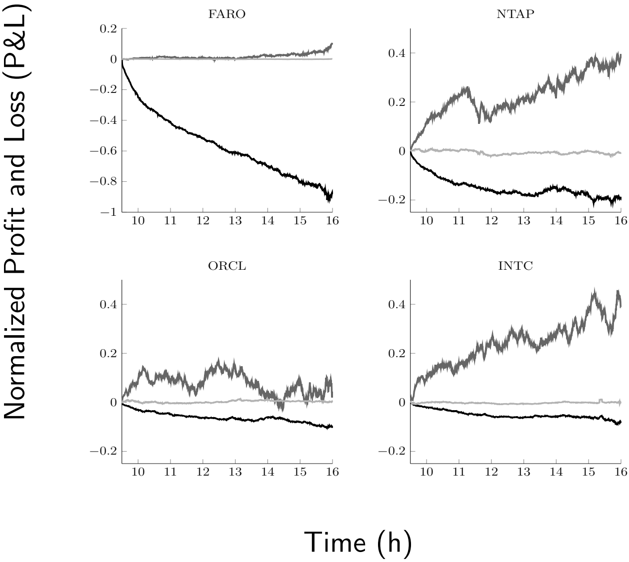
Need to select:

- price change observation period Δt_S
- imbalance averaging period Δt_I
- number of imbalance bins $\#_{bins}$

Calibration done on the first day of the trading year, same parameters used for all days.

Brute-force search of parameter space, using max Sharpe ratio criterion, found that $\Delta t_S = \Delta t_I = 1 sec$, and $\#_{bins} = 4$

Results of Naive Trading Strategies



Conclusions from Naive Trading Strategies

Why is the Naive strategy producing, on average, normalized losses?

- Backtest is out-of-sample; evidence to reject time-homogeneity
- Calibration is done on first trading day; likely nonrepresentative of trading activity
- Price change probability matrix $\boldsymbol{Q}$ obtained using midprices, ignoring bid-ask spread; $\operatorname{sgn}(\Delta S)$ may be insufficient for create profit, especially on `FARO`

Conclusions from Naive Trading Strategies

Why do the Naive+ and Naive++ strategies outperform the Naive strategy?

- LOs vs MOs means different transaction price is being used (only MO loses value)
- Naive only executes when predicting non-zero price change
    - Only sign, not magnitude
    - Only *if one was already seen*

Roadmap

System Description

- Imbalance Averaging Time Δt_I
  A constant, specifying the time window over which the imbalance ratio $I(t)$ will be averaged.
- Price Change Time Δt_S
  A constant, specifying the time window over which price changes will be computed.
- Number of Imbalance Bins $\#_{bins}$
  A constant, specifying the number of bins (spaced by percentiles, symmetric around zero) into which $I(t)$ will be sorted.
- Imbalance ρ_t
  The finite, discrete stochastic process that results from sorting $I(t)$ into the imbalance bins $\{1, \ldots, \#_{bins}\}$, and which evolves in accordance with the CTMC $\boldsymbol{Z}$.

System Description

- Midprice S_t
  Stochastic process, evolves according to CTMC $\boldsymbol{Z}$.
- Midprice Change $\Delta S_t = \mathrm{sgn}(S_t - S_{t-\Delta t_S})$
- Imbalance & Midprice Change $\boldsymbol{Z}_t = (\rho_t, \Delta S_t)$
  Continuous-time Markov chain with generator $\boldsymbol{G}$.
- Bid-Ask Half-Spread ξ
  Assumed constant. 2ξ is equal to the bid-ask spread.
- Midprice Change $\{\eta_{0,\boldsymbol{z}}, \eta_{1,\boldsymbol{z}}, \dots\} \sim F_{\boldsymbol{z}}$
  i.i.d. RVs, with distribution dependent on the Markov chain state.

System Description

- Other Agent Market Orders $K_t^{\pm}$
  Poisson processes with rate $\mu^{\pm}(\boldsymbol{Z}_t)$. K^+ represents the arrival of another agent's buy market order.
- Our Limit Order Posting Depth $\delta_t^{\pm}$
  One of our controlled $\mathcal{F}$-predictable processes. δ^+ dictates how deep on the buy side we will post our buy limit order; $\delta^+ = 0$ implies at-the-touch.
- Our Limit Order Fill Count $L_t^{\pm}$
  Counting processes (not Poisson), satisfying

$$\mathbb{P}[\,\mathrm{d}L_t^{\pm} = 1 \,|\, \mathrm{d}K_t^{\mp} = 1] = e^{-\kappa\delta_t^{\pm}}$$

System Description

- Fill Probability Constant κ
  Fitted to satisfy the above relation, by considering the avg vol available at the first few depths relative to distribution of volumes of incoming market orders
- Our Market Orders $M_t^{\pm}$
  M^+ represents our buy market order. Assume we achieve the best bid/ask price.
- Our Market Order Execution Times
  $\boldsymbol{\tau}^{\pm} = \{\tau_k^{\pm} : k = 1, \dots\}$
  An increasing sequence of $\mathcal{F}$-stopping times.

System Description

- Cash $X_t^{\tau,\delta}$
  A stochastic variable representing our cash, initially zero, that evolves according to

$$\mathrm{d}X_t^{\tau,\delta} = \underbrace{(S_t + \xi + \delta_t^-)\,\mathrm{d}L_t^-}_{\text{sell limit order}} - \underbrace{(S_t - \xi - \delta_t^+)\,\mathrm{d}L_t^+}_{\text{buy limit order}} + \underbrace{(S_t - \xi)\,\mathrm{d}M_t^-}_{\text{sell market order}} - \underbrace{(S_t + \xi)\,\mathrm{d}M_t^+}_{\text{buy market order}}$$

- Inventory $Q_t^{\tau,\delta}$
  A stochastic process representing our assets, initially zero, that satisfies

$$Q_0^{\tau,\delta} = 0, \qquad Q_t^{\tau,\delta} = L_t^+ + M_t^+ - L_t^- - M_t^-$$

Terminal Wealth

Call $W_t^{\tau,\delta}$ our net present value (NPV) at time t. Hence $W_T^{\tau,\delta}$ at terminal time T is our 'terminal wealth.'

At T, we:

- finish each trading day with zero inventory (avoid overnight positional risk)
- submit a market order (of a possibly large volume) to liquidate remaining stock
- price achieved will be $S - \xi \operatorname{sgn} Q - \alpha Q$
  - $\xi \operatorname{sgn} Q$ represents crossing the spread
  - α is a penalty constant
  - αQ represents receiving a worse price linearly in Q due to walking the book

Hence, $W_t^{\tau,\delta}$ satisfies:

$$W_t^{\tau,\delta} = \underbrace{X_t^{\tau,\delta}}_{\text{cash}} + \underbrace{Q_t^{\tau,\delta}\left(S_t - \xi \operatorname{sgn}(Q_t^{\tau,\delta})\right)}_{\text{book value of assets}} - \underbrace{\alpha\left(Q_t^{\tau,\delta}\right)^2}_{\text{liquidation penalty}}$$

Dynamic Programming Value Function

Our performance criterion will be to maximize terminal wealth:

$$H^{\boldsymbol{\tau},\delta}(t,x,s,\boldsymbol{z},q) = \mathbb{E}_{t,x,s,z,q}\left[W_T^{\boldsymbol{\tau},\delta}\right]$$

The *value function* is given by

$$H(t,x,s,\boldsymbol{z},q) = \sup_{\boldsymbol{\tau}\in\mathcal{T}_{[t,T]}} \sup_{\delta\in\mathcal{A}_{[t,T]}} H^{\boldsymbol{\tau},\delta}(t,x,s,\boldsymbol{z},q)$$

Admissible trading strategies is the product of the set $\mathcal{T}$ of all $\mathcal{F}$-stopping times, with the set $\mathcal{A}$ of all $\mathcal{F}$-predictable, bounded-from-below depths $\delta \geq 0$.

- recognizing τ does not require knowledge of the future
- δ cannot 'see into the future'; measurable with respect to information at an earlier time

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