GPU Applications for Modern Large Scale Asset Management

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I) Entropy Approach

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Parallel Metropolis Hastings on GPUs

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Portfolio Construction

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- Markowitz mean variance portfolio optimization
  - Optimally balance risk and performance

$$w_\lambda^* = \underset{w,\, \mathbf{1}^T w = b}{\arg\max}\; U_\lambda(w), \quad U_\lambda(w) = \mu^\top w - \lambda w^\top \Sigma w \tag{1}$$

- Risk budgeting
  - Construct portfolio so that risk contributions satisfy $RC_i(w) = b_i$ for given risk budgets b_i
  - For volatility risk measure we obtain

$$RC_i(w) = w_i \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \tag{2}$$

- For both approaches we need means μ and covariances Σ of asset returns

Asset Return Distributions

Two steps

1. Prior asset return distribution from historical data or inverse optimization principle
2. Posterior asset return distribution should reflect expert views and beliefs
   - Views are distortions of a prior distribution
   - Views translate into constraints
   - Linear constraints on mean return pioneered by Black and Litterman

We discuss two approaches

- Entropy based approach
- Bayesian methods

Entropy Approach

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- Minimum discrimination principle (Kullback and Leibler 1951) infers a posterior distribution such that it
  - Satisfies a set of moments constraints
  - Is as hard as possible to differentiate from given prior distribution
- Minimum discrimination principle suitable methodology to incorporate constraints on a prior distribution which are
  - Linear
  - Quadratic
  - Convex

Nonparametric Setup

- Sample the probability distributions
  - Discrete sample space $\mathcal{X} = \{x_1, \dots, x_m\}$
  - Probabilities elements of the standard simplex

$$p = (p_1, \dots, p_m)^\top \in \Delta^m \subset \mathbb{R}^m$$

- Implement views as distortions of prior probabilities p_i
- Express views as feature functions $f_j : \mathcal{X} \to \mathbb{R}$, $j = 1, \dots, n$
- Identify feature vector $\mathbf{f} = (f_1, \dots, f_n)^\top : \mathcal{X} \to \mathbb{R}^n$ with matrix $F = (f_{ji}) \in \mathbb{R}^{n \times m}$, with $f_{ji} = f_j(x_i)$

Linear Constraints on Features

- Expectation of feature $f : \mathcal{X} \to \mathbb{R}$ w.r.t measure $q \in \Delta^m$ is

$$E_q[f] = \sum q_i f(x_i) = q^\top f$$

- Minimum discrimination principle defines probability p^* as

$$p^* = \arg\min_{q \in \Delta^m} q^\top (\log(q) - \log(p)) \tag{9}$$

subject to constraints

$$\mathbb{E}_q[f_j] \leq c_j, j = 1, \ldots, n, \quad \mathbb{E}_q[\tilde{f}_j] = \tilde{c}_j, j = 1, \ldots, \tilde{n}. \tag{10}$$

Optimization Problem

- Expectation views (10) lead to linear constraints

$$p^* = \underset{\substack{q \in \mathbb{R}^m_+, \mathbf{1}^\top q = 1 \\ Fq \leq c, \tilde{F} q = \tilde{c}}}{\arg\min} \; q^\top (\log(q) - \log(p)) \tag{11}$$

- Need large number of samples for accurate representation of distributions with $m \simeq 10^6$
- With linear constraints: Lagrange duality to solve the optimal solution
  - Strong duality, i.e. no duality gap
  - Converting problem of dimension m to much smaller problems of dimension $n + \tilde{n}$ to find optimal Lagrange multipliers

Convex Constraints

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- How to extend to variance and covariance constraints?
- Lower variance constraint $q^\top f^2 - (q^\top f)^2 \geq c$ convex but dual problem is ill-conditioned
- Upper variance constraint $q^\top f^2 - (q^\top f)^2 \leq c$ not even convex
- Approximation by linearization $q^\top f^2 \leq c + (p^\top f)^2$ to get a second moment constraint
- But .... linearized version often not accurate enough

Large Scale Convex Optimization

- For convex constraints solve directly the primal problem
- Several accelerated first order methods (FOM) for convex optimization algorithms have been developed
- Modern algorithms adjust to the problem's geometry
- Examples
  - Mirror descent (MD) algorithm
  - Level bundle methods
  - Bundle mirror descent
  - Many different variations

Exploiting Geometry

- Gradient methods regularized with quadratic proximity term

$$y = \arg\min_{x \in \mathcal{X}} \{\gamma \langle \xi, \nabla f(x) \rangle + c \|\xi - x\|^2\}$$

- Replace $\| \cdot \|^2$ with some distance like function $D(\cdot, \cdot)$ that better exploits the geometry of $\mathcal{X}$ and is simple to calculate
- Candidates: Bregman type distance based on kernel $\omega : \mathcal{X} \to \mathbb{R}$

$$D(x, y) = D_\omega(x, y) \equiv \omega(x) - \omega(y) - \langle x - y, \omega'(y) \rangle$$

Bregman Distances

- Using $\omega(u) = \frac{1}{2}\|u\|^2$ gives Euclidian norm distance

$$D_\omega(x, y) = \frac{1}{2}\|x - y\|^2$$

- The entropy $\omega(x) = x^\top \log(x)$ on $\mathcal{X} = \Delta^m$ leads to

$$D_\omega(x, y) = x^\top \log\left(\frac{x}{y}\right)$$

which is 1-strongly convex with respect to the L_1 norm

$$D_\omega(x, y) \geq \frac{1}{2}\|x - y\|_1^2 \quad \forall x, y \in \Delta^m$$

Constrained Mirror Descent Setup

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- Convex problem $\min_{x\in\mathcal{X}} f_0(x)$ subject to $f_1(x) \leq 0$
- $\mathcal{X} \subset E$, closed convex subset of f.d. vector space E
- $\|\cdot\|$ norm on E, $\|\cdot\|_*$ dual norm on E^*
- Duality pairing $\langle\cdot,\cdot\rangle : E^* \times E \to \mathbb{R}$
- Bregman distance $D_\omega(x,y)$ from kernel $\omega : \mathcal{X} \to \mathbb{R}$
- Prox mapping $\mathrm{Prox}_x : E^* \to \mathcal{X}^\circ$

$$\mathrm{Prox}_x(\xi) = \arg\min_{u\in\mathcal{X}}\{\langle\xi,u\rangle + D_\omega(u,x)\}$$

Constrained Mirror Descent

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- Mirror descent with constraints is a simple FOM
- Initial search point $x_1 = x_\omega = \arg\min_{x \in \mathcal{X}} \omega(x)$
- For $t = 1, \ldots, N$ do
  - If $f_1(x_t) \leq tol$ then $i(t) = 0$ else $i(t) = 1$
  - Select step size γ_t
  - Update search point $x_{t+1} = \mathrm{Prox}_{x_t}(\gamma_t f'_{i(t)}(x_t))$
- Approximate solution after N steps

$$\hat{x}_N = \underset{x \in \{x_t | i(t)=0\}}{\arg\min} \; f_0(x)$$

- Possible choice $tol = \gamma \|f'_1(x_t)\|_*$ and $\gamma_t = \gamma \|f'_{i(t)}(x_t)\|_*^{-1}$

Geometry on Simplex

- Optimization problem (11) formulated on simplex $\mathcal{X} = \Delta^m$
- Norm $\|\cdot\|_1$ with dual norm $\|\cdot\|_\infty$
- Prox mapping allows analytical expression

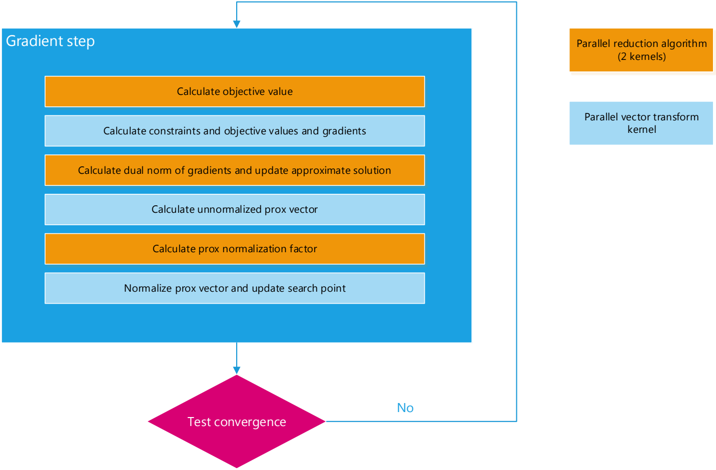
$$\mathrm{Prox}_x(\xi) = \left(x^\top e^{-\xi}\right)^{-1} x \bullet e^{-\xi}$$

- Initial point $x_\omega = n^{-1}\mathbf{1}_n$ the uniform distribution

Constrained Mirror Descent on GPU

- GPU implementation of constrained mirror descent algorithms for very large n
  - Parallel coordinate-wise calculation of gradients for objective function and constraints
  - Vector reductions to calculate objective value, norms and scalar products
  - Parallel coordinate-wise calculation of prox vector, search points and execution of gradient step
- Gradient steps can be further parallelized by coordinate aggregation or randomization methods

Schematic Gradient Step



Tuning with Kernel Fusion

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- Usually kernel launch time ($\approx 50\mu s$) not an issue
- For algorithms with many iterations, this becomes crucial
- Use kernel fusion technique to reduce number of kernel calls per gradient step
- Minimize number of arguments to pass to each kernel

Bayesian Approach

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- Bayesian methods treat the model parameters as random variables
- Views go into the prior distributions of the model parameters
- Historical data is used to update the prior distribution
- Not as flexible as entropy approach
- Does not rely on optimization hence less fragile
- Work horse is a parallel Metropolis Hastings Markov chain Monte Carlo sampler

Parallel Metropolis Hastings

- Parallelization of Metropolis Hastings through multiple parallel chains
- Further parallelization through pre-fetching
- Require efficient parallel random number generation and branch free methods to sample from proposal distribution

Parallel Metropolis Hastings

GPU Metropolis Hastings sampler generation from generic target and proposal distribution expressions with F# and Alea.cuBase

```
let inline mhsamples (targetpdf : Expr<'T -> 'T>)
                     (proppdf : Expr<'T -> 'T -> 'T -> 'T>)
                     (proprnd : Expr<'T -> 'T -> 'T -> 'T>) = cuda {

    let! proppdf = proppdf |> Compiler.DefineInlineFunction
    let! proprnd = proprnd |> Compiler.DefineInlineFunction
    let! targetpdf = targetpdf |> Compiler.DefineInlineFunction

    let! kernel =
        <@ fun (steps:int) (scale:'T) (initial:deviceptr<'T>)
               (numbers:deviceptr<'T>) (result:deviceptr<'T>) ->
            let idx = blockIdx.x * blockDim.x + threadIdx.x
            let nchains = gridDim.x * blockDim.x

    . . .
```

Parallel Metropolis Hastings

Expressions for target distribution and proposal distribution are compiled directly into kernel

```
/// Dynamically generate expressions for MH at runtime
let targetpdf =
    <@ fun x ->
        exp(-x*x)*(2G + sin(5G*x) + sin(2G*x)) @>
let proppdf =
    <@ fun sigma x y ->
        exp(-(y - x)*(y - x)/(2G*sigma*sigma)) / (sigma*__sqrt2pi()) @>
let proprng =
    <@ fun sigma x u -> x + sigma*u @>

/// Compiler integrated into F#, callable through API
let mh = mhsamples targetpdf proppdf proprng |> Util.load Worker.Default
```

Example

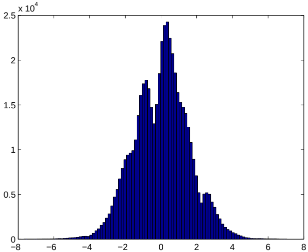

Figure : Sample histogram for target density $e^{-x^2}(2+\sin(5x)+\sin(2x))$

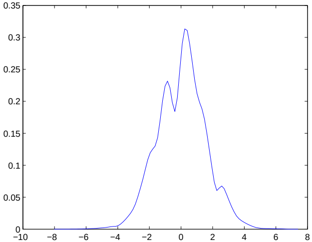

Figure : Kernel density smoothed sample pdf for $e^{-x^2}(2+\sin(5x)+\sin(2x))$

Conclusion

- We have shown two applications of GPUs for quantitative asset management
- More applications in strategy calibration, back-testing and MIP
- Large scale convex optimization occurs in other fields such as
  - Machine learning such as e.g. deep believe network training in a big data context
  - Network analysis
  - Truss topology design
  - Compressed sensing
- Markov chain Monte Carlo methods require flexible ways to specify various parts such as target pdf, the proposal distribution and a dynamic way to derive tuning parameters, for which our Alea.cuBase F# GPU compiler is ideal