Limit Order Book Dynamics

An agent wants to liquidate a position by time T using limit orders (LOs) posted at an arbitrary depth δ_t at time t, which is the process that we control. The following processes are at play:

Order Imbalance	$Z = (Z_t)_{0 \leq t \leq T} \in \{-1, 0, +1\}$	a Markov chain with generator G
Arrival of MOs	$M^{\pm} = (M_t^{\pm})_{0 \leq t \leq T}$	Poisson with rate $\lambda^{\pm}(Z_t)$
Arrival of LOB Shuffling	$J^{\pm} = (J_t^{\pm})_{0 \leq t \leq T}$	Poisson with rate $\gamma^{\pm}(Z_t)$
ΔPrice: MO arrives	$\{\epsilon_{0,k}^{\pm}, \epsilon_{1,k}^{\pm}, \dots\} \sim F_k^{\pm}$	i.i.d. with $k \in \{-1, 0, 1\}$
ΔPrice: LOB shuffled	$\{\eta_{0,k}^{\pm}, \eta_{1,k}^{\pm}, \dots\} \sim L_k^{\pm}$	i.i.d. with $k \in \{-1, 0, 1\}$
Midprice	$S = (S_t)_{0 \leq t \leq T}$	
	$\mathrm{d}S_t = \epsilon_{M_{t-}^+, Z_{t-}}^+ \, \mathrm{d}M_t^+ - \epsilon_{M_{t-}^-, Z_{t-}}^- \, \mathrm{d}M_t^- + \eta_{J_{t-}^+, Z_{t-}}^+ \, \mathrm{d}J_t^+ - \eta_{J_{t-}^-, Z_{t-}}^- \, \mathrm{d}J_t^-$	
LO posted depth	$\delta = (\delta_t)_{0 \leq t \leq T}$	
LO fill count	$N^{\delta} = (N_t^{\delta})_{0 \leq t \leq T}$	
Cash	$X^{\delta} = (X_t^{\delta})_{0 \leq t \leq T}$	
	$\mathrm{d}X_t^{\delta} = (S_t + \delta_t) \, \mathrm{d}N_t^{\delta}$	
Inventory	$Q_t = Q_0 - N_t^{\delta}$	

Note that N_t^{δ} is NOT a Poisson process - it is a jump process satisfying the relationship that if at time t we have a sell limit order posted at a depth δ_t, then our fill probability is $e^{-\kappa \delta_t}$ conditional on a buy market order arriving; namely:

$$\mathbb{P}[\, \mathrm{d}N_t = 1 \,|\, \mathrm{d}M_t^+ = 1] = e^{-\kappa \delta_t}$$

For our liquidation optimization problem, we stop trading either when we liquidate all our shares, or the trading day ends. Therefore, our stopping time τ can be expressed as

$$\tau = T \wedge \min\{t : Q_t^{\delta} = 0\}$$

We measure the performance of our LO posting strategy δ according to the *performance criteria function H^{δ}*:

$$H^{\delta}(t, x, s, z, q) = \mathbb{E}[X_{\tau}^{\delta} + Q_{\tau}^{\delta}(S_{\tau} - \alpha Q_{\tau}^{\delta}) - \varphi \int_t^{\tau} (Q_u^{\delta})^2 \, \mathrm{d}u \,|\, X_{t-}^{\delta} = x, S_{t-}^{\delta} = s, Z_{t-} = z, Q_{t-}^{\delta} = q]$$

$$= \mathbb{E}_{t,x,s,z,q}[X_{\tau}^{\delta} + Q_{\tau}^{\delta}(S_{\tau} - \alpha Q_{\tau}^{\delta}) - \varphi \int_t^{\tau} (Q_u^{\delta})^2 \, \mathrm{d}u]$$

We seek to maximize this performance criteria, and thereby attain the *value function H*:

$$H(t, x, s, z, q) = \sup_{\delta \in \mathcal{A}} H^{\delta}(t, x, s, z, q)$$

where $\mathcal{A}$ is the set of $\mathcal{F}$-predictable, bounded from below processes – note I actually still have no idea what this means.

I've canonically shown that we can apply the dynamic programming principle (DPP) to H. This means that instead of solving for the optimal strategy δ for the entire trading day, we instead only have to solve for it over a small time interval h, and after that time interval we 'behave optimally'. In other words, we can express H as:

$$H(t, x, s, z, q) = \sup_{\delta \in \mathcal{A}} \mathbb{E}_{t,x,s,z,q} \left[-\varphi \int_t^{t+h} (Q_u^\delta)^2 \, \mathrm{d}u + H(t + h, X_{t+h}^\delta, S_{t+h}^\delta, Z_{t+h}, Q_{t+h}^\delta) \right]$$

Now again, rather canonically, we can take this DPP and write it in infinitesimal form, which is called the dynamic programming equation (DPE). Writing it so, we end up with the following:

$$\begin{cases} \partial_t H(t, x, s, z, q) + \sup_{\delta \in \mathcal{A}} \left\{ \mathcal{L}_t^\delta H(t, x, s, z, q) + F(t, x, s, z, q, \delta) \right\} = 0 \\ \\ \hspace{6cm} H(\tau, x, s, z, q) = G(x, s, z, q) \end{cases} \tag{1}$$

where $F(t, x, s, z, q, \delta) = -\varphi q^2$, and $H(\tau, x, s, z, q) = x - q(s - \alpha q)$, both of which are read directly off of our performance criteria function H^δ, and $\mathcal{L}_t^\delta$ is the so-called 'infinitesimal generator' for our vector of random processes (x, s, z, q).

To solve for $\mathcal{L}_t^\delta$ we will determine $\mathrm{d}H$. In order to do so, we need to present some results on Ito's formula for Poisson processes and for Markov chains. (I haven't had time to write this out yet.)

Consider our Markov chain Z with generator G. Define $K_l(t)$ to be the number of jumps with $Z_s - Z_{s^-} = l$ up to time t, and $\beta_l(x) = G_{x,x+l}$. Then $\tilde{K}_l(t) = K_l(t) - \int_0^t \beta_l(Z_s) \, \mathrm{d}s$ is a martingale.

Now we solve for $\mathrm{d}H$, using the shorthand $H(\cdot) = H(t, x, s, z, q)$:

$$\begin{aligned} \mathrm{d}H(t, x, s, z, q) &= \partial_t H \, \mathrm{d}t + \partial_x H \, \mathrm{d}x + \partial_s H \, \mathrm{d}s + \partial_q H \, \mathrm{d}q + \partial_z H \, \mathrm{d}z \\ &= \partial_t H \, \mathrm{d}t + \partial_{M^+} H \, \mathrm{d}M^+ + \partial_{M^-} H \, \mathrm{d}M^- + \partial_{J^+} H \, \mathrm{d}J^+ + \partial_{J^-} H \, \mathrm{d}J^- \\ &\quad + \sum_l \left[H(t, x, s, z + l, q) - H(\cdot) \right] \mathrm{d}K_l \\ &= \partial_t H \, \mathrm{d}t + \left\{ e^{-\kappa\delta} \left(\mathbb{E} \left[H(t, x + (s + \delta), s + \epsilon_{0,z}^+, z, q - 1) \right] - H(\cdot) \right) \right. \\ &\quad \left. + (1 - e^{-\kappa\delta}) \left(\mathbb{E} \left[H(t, x, s + \epsilon_{0,z}^+, z, q) \right] - H(\cdot) \right) \right\} \mathrm{d}M^+ \\ &\quad + \left\{ \mathbb{E} \left[H(t, x, s - \epsilon_{0,z}^-, z, q) \right] - H(\cdot) \right\} \mathrm{d}M^- \\ &\quad + \left\{ \mathbb{E} \left[H(t, x, s + \eta_{0,z}^+, z, q) \right] - H(\cdot) \right\} \mathrm{d}J^+ \\ &\quad + \left\{ \mathbb{E} \left[H(t, x, s - \eta_{0,z}^-, z, q) \right] - H(\cdot) \right\} \mathrm{d}J^- \\ &\quad + \sum_l \left[H(t, x, s, z + l, q) - H(\cdot) \right] \mathrm{d}K_l \end{aligned}$$

Substitute in the compensated identifies

$$
\begin{aligned}
\mathrm{d}M^{\pm} &= \mathrm{d}\tilde{M}^{\pm} + \lambda^{\pm}(z)\,\mathrm{d}t \\
\mathrm{d}M^{\pm} &= \mathrm{d}\tilde{M}^{\pm} + \gamma^{\pm}(z)\,\mathrm{d}t \\
\mathrm{d}K_l &= \mathrm{d}\tilde{K}_l + \beta_l(z)\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
= \Bigg\{ & \partial_t H + \lambda^+(z) \left\{ e^{-\kappa\delta} \left(\mathbb{E}\left[H(t, x + (s+\delta), s + \epsilon_{0,z}^+, z, q-1) \right] - H(\cdot) \right) \right. \\
& \left. + (1 - e^{-\kappa\delta}) \left(\mathbb{E}\left[H(t, x, s + \epsilon_{0,z}^+, z, q) \right] - H(\cdot) \right) \right\} \\
& + \lambda^-(z) \left\{ \mathbb{E}\left[H(t, x, s - \epsilon_{0,z}^-, z, q) \right] - H(\cdot) \right\} \\
& + \gamma^+(z) \left\{ \mathbb{E}\left[H(t, x, s + \eta_{0,z}^+, z, q) \right] - H(\cdot) \right\} \\
& + \gamma^-(z) \left\{ \mathbb{E}\left[H(t, x, s - \eta_{0,z}^-, z, q) \right] - H(\cdot) \right\} \\
& + \sum_l \beta_l(z) \left[H(t, x, s, z+l, q) - H(\cdot) \right] \Bigg\} \mathrm{d}t \\
& + \left\{ e^{-\kappa\delta} \left(\mathbb{E}\left[H(t, x + (s+\delta), s + \epsilon_{0,z}^+, z, q-1) \right] - H(\cdot) \right) \right. \\
& \left. + (1 - e^{-\kappa\delta}) \left(\mathbb{E}\left[H(t, x, s + \epsilon_{0,z}^+, z, q) \right] - H(\cdot) \right) \right\} \mathrm{d}\tilde{M}^+ \\
& + \left\{ \mathbb{E}\left[H(t, x, s - \epsilon_{0,z}^-, z, q) \right] - H(\cdot) \right\} \mathrm{d}\tilde{M}^- \\
& + \left\{ \mathbb{E}\left[H(t, x, s + \eta_{0,z}^+, z, q) \right] - H(\cdot) \right\} \mathrm{d}\tilde{J}^+ \\
& + \left\{ \mathbb{E}\left[H(t, x, s - \eta_{0,z}^-, z, q) \right] - H(\cdot) \right\} \mathrm{d}\tilde{J}^- \\
& + \sum_l \left[H(t, x, s, z+l, q) - H(\cdot) \right] \mathrm{d}\tilde{K}_l
\end{aligned}
$$

From which we can see that the infinitesimal generator is given by

$$
\begin{aligned}
\mathcal{L}_t^{\delta} = \; & \lambda^+(z) \left\{ e^{-\kappa\delta} \left(\mathbb{E}\left[H(t, x + (s+\delta), s + \epsilon_{0,z}^+, z, q-1) \right] - H(\cdot) \right) \right. \\
& \left. + (1 - e^{-\kappa\delta}) \left(\mathbb{E}\left[H(t, x, s + \epsilon_{0,z}^+, z, q) \right] - H(\cdot) \right) \right\} \\
& + \lambda^-(z) \left\{ \mathbb{E}\left[H(t, x, s - \epsilon_{0,z}^-, z, q) \right] - H(\cdot) \right\} \\
& + \gamma^+(z) \left\{ \mathbb{E}\left[H(t, x, s + \eta_{0,z}^+, z, q) \right] - H(\cdot) \right\} \\
& + \gamma^-(z) \left\{ \mathbb{E}\left[H(t, x, s - \eta_{0,z}^-, z, q) \right] - H(\cdot) \right\} \\
& + \sum_l \beta_l(z) \left[H(t, x, s, z+l, q) - H(\cdot) \right]
\end{aligned}
\tag{2}
$$

In our case, the Markov chain Z can only be in three possible states $-1, 0, 1$, so summing over the possible jump sizes l is identical (up to reordering) of summing over the possible destination states k with the corresponding transition rates. Thus, plugging this back into our DPE equation

, we obtain:

$$
\begin{aligned}
\varphi q^2 = {} & \partial_t H(t,x,s,z,q) + \lambda^+(z)\sup_{\delta\in\mathcal{A}}\left\{ e^{-\kappa\delta}\left(\mathbb{E}\left[H(t,x+(s+\delta),s+\epsilon^+_{0,z},z,q-1)\right] - H(\cdot)\right)\right. \\
& \left. + (1-e^{-\kappa\delta})\left(\mathbb{E}\left[H(t,x,s+\epsilon^+_{0,z},z,q)\right] - H(\cdot)\right)\right\} \\
& + \lambda^-(z)\left\{\mathbb{E}\left[H(t,x,s-\epsilon^-_{0,z},z,q)\right] - H(\cdot)\right\} \\
& + \gamma^+(z)\left\{\mathbb{E}\left[H(t,x,s+\eta^+_{0,z},z,q)\right] - H(\cdot)\right\} \\
& + \gamma^-(z)\left\{\mathbb{E}\left[H(t,x,s-\eta^-_{0,z},z,q)\right] - H(\cdot)\right\} \\
& + \sum_{k=-1,0,1} G_{z,k}\left[H(t,x,s,k,q) - H(\cdot)\right]
\end{aligned}
\tag{3}
$$

where expectations are over the random variables $\epsilon^\pm_{0,z}$ and $\eta^\pm_{0,z}$. We also obtain two boundary conditions from the two possible stopping times:

$$
\begin{cases}
H(T,x,s,z,q) = x + q(s - \alpha q) & \text{if } \tau = T \\
H(t,x,s,z,0) = x & \text{if } q = 0 \text{ (inventory has been liquidated by } \tau < T)
\end{cases}
$$

The terminal and boundary conditions suggest a familiar ansatz for H:

$$
H(t,x,s,z,q) = x + qs + h(t,z,q)
\tag{4}
$$

where

$$
h(T,z,q) = -\alpha q^2
$$
$$
h(t,z,0) = 0
$$

Substituting the ansatz 4 back into the PDE 3, we obtain:

$$
\begin{aligned}
\varphi q^2 = {} & \partial_t h(t,z,q) + \lambda^+(z)\sup_{\delta\in\mathcal{A}}\left\{ e^{-\kappa\delta}\mathbb{E}\left[x+(s+\delta)+(q-1)(s+\epsilon^+_{0,z}) + h(t,z,q-1)\right.\right. \\
& \hspace{6cm} \left. - x - qs - h(t,z,q)\right] \\
& \hspace{3cm} + (1-e^{-\kappa\delta})\mathbb{E}\left[x+q(s+\epsilon^+_{0,z}) + h(t,z,q)\right. \\
& \hspace{6cm} \left.\left. - x - qs - h(t,z,q)\right]\right\} \\
& + q\left(-\lambda^-(z)\mathbb{E}\left[\epsilon^-_{0,z}\right] + \gamma^+(z)\mathbb{E}\left[\eta^+_{0,z}\right] - \gamma^-(z)\mathbb{E}\left[\eta^-_{0,z}\right]\right) \\
& + \sum_{k=-1,0,1} G_{z,k}\left[h(t,k,q) - h(t,z,q)\right] \\
= {} & \partial_t h(t,z,q) + \lambda^+(z)\sup_{\delta\in\mathcal{A}}\left\{ e^{-\kappa\delta}\left(\delta - \mathbb{E}\left[\epsilon^+_{0,z}\right] + h(t,z,q-1) - h(t,z,q)\right)\right\} \\
& + \mu(z)q + \sum_{k=-1,0,1} G_{z,k}\left[h(t,k,q) - h(t,z,q)\right]
\end{aligned}
$$

where

$$
\mu(z) = \lambda^+(z)\mathbb{E}\left[\epsilon^+_{0,z}\right] - \lambda^-(z)\mathbb{E}\left[\epsilon^-_{0,z}\right] + \gamma^+(z)\mathbb{E}\left[\eta^+_{0,z}\right] - \gamma^-(z)\mathbb{E}\left[\eta^-_{0,z}\right]
$$

To find the supremum over δ, consider the first-order constraint:

$$0 = \partial_\delta \left\{ e^{-\kappa\delta} \left(\delta - \mathbb{E}\left[\epsilon_{0,z}^+\right] + h(t,z,q-1) - h(t,z,q) \right) \right\}$$
$$= -\kappa e^{-\kappa\delta} \left(\delta - \mathbb{E}\left[\epsilon_{0,z}^+\right] + h(t,z,q-1) - h(t,z,q) \right) + e^{-\kappa\delta}$$
$$= e^{-\kappa\delta} \left[-\kappa \left(\delta - \mathbb{E}\left[\epsilon_{0,z}^+\right] + h(t,z,q-1) - h(t,z,q) \right) + 1 \right]$$

which implies the term inside the square brackets must equal zero, and hence the optimal depth δ^* is given by:

$$\delta^* = \frac{1}{\kappa} + \Delta_q h + \mathbb{E}\left[\epsilon_{0,z}^+\right] \tag{5}$$

where $\Delta_q h = h(t,z,q) - h(t,z,q-1)$. Substituting this optimal depth back into the PDE, we obtain:

$$0 = -\varphi q^2 + \partial_t h(t,z,q) + \frac{1}{\kappa}\lambda^+(z)e^{-\kappa\left(\frac{1}{\kappa} + \Delta_q h + \mathbb{E}\left[\epsilon_{0,z}^+\right]\right)}$$
$$+ \mu(z)q + \sum_{k=-1,0,1} G_{z,k}\left[h(t,k,q) - h(t,z,q)\right] \tag{6}$$

With these results we run into the issue that δ can become negative. This could be interpreted as an urgency to liquidate so great that we post prices at a loss, or namely execute a market order, but alternatively this can be deemed a failure to specify the model correctly. We can rather simply overcome this by restricting the permissible processes to the subset $\mathcal{A}^+ = \{\delta \geq 0\} \subset \mathcal{A}$. The analysis proceeds identically, and the resulting optimal depth is given by:

$$\delta^* = \max\left\{ \frac{1}{\kappa} + \Delta_q h + \mathbb{E}\left[\epsilon_{0,z}^+\right] \; ; \; 0 \right\} \tag{7}$$

Substituting this feedback control back into the $\mathcal{A}^+$-restricted PDE, we obtain:

$$0 = -\varphi q^2 + \partial_t h(t,z,q) + \mu(z)q + \sum_{k=-1,0,1} G_{z,k}\left[h(t,k,q) - h(t,z,q)\right]$$
$$+ \lambda^+(z)\left[-(\Delta_q h + \mathbb{E}\left[\epsilon_{0,z}^+\right])\mathbb{1}_{\delta^*=0} + \frac{1}{\kappa}e^{-\kappa\left(\frac{1}{\kappa}+\Delta_q h + \mathbb{E}\left[\epsilon_{0,z}^+\right]\right)}\mathbb{1}_{\delta^*>0} \right] \tag{8}$$

where

$$\delta^* = 0 \iff -\Delta_q h \geq \frac{1}{\kappa} + \mathbb{E}\left[\epsilon_{0,z}^+\right]$$

and $\delta^* > 0$ otherwise.