High-Frequency Algorithmic Trading with Momentum and Order Imbalance

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Abstract

Robots in da skies.

Acknowledgements

And I would like to acknowledge ...

Contents

List of Figures

List of Tables

Todo list

Chapter 1

Introduction

Hi, my name is Stereo Mike.

Yeah, we got three tickets to the Bran Van concert this Monday night at the Pacific Pallisades. You can all dial in if you want to answer a couple of questions; namely, what is Todd's favorite cheese? Jackie just called up and said it was a form of Roquefort. We'll see about that.

Give us a ring ding ding, it's a beautiful day.

Yeah Todd, this is Liquid, ring-a-ding-a-dinging, I want those three Bran Van tickets, man. Whaddya think? Todd?

1.1 Algorithmic Trading

1.2 Limit Order Book

Limit order book imbalance is a ratio of limit order volumes between the bid and ask side, and can be calculated for example as $I(t) = \dfrac{V_b(t) - V_a(t)}{V_b(t) + V_a(t)} \in [-1, 1]$.

- We bin the bid/ask volume imbalances in the Limit Order Book into K bins, each being dubbed a "regime" of the limit order book.
- Z_t is a continuous-time Markov chain that tracks which regime we're in. Z_t takes values in

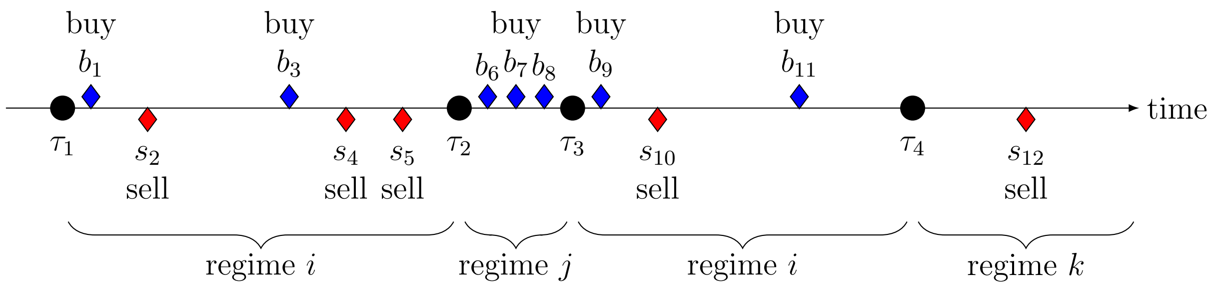


Figure 1.1: Hypothetical timeline of market orders arriving during changing order imbalance regimes.

$\{1, \ldots, K\}$, and has an infinitesimal generator matrix G.

- Conditional on being in some regime k, the arrival of buy and sell market orders follow independent Poisson processes with intensities $\lambda_k^{\pm}$.

We have observations of arrivals of buy/sell market orders and of regime switches occurring, all of which are timestamped. Pictorially, a timeline might look like:

1.3 ITCH Data Set

Chapter 2

Exploratory Data Analysis

2.1 Modeling Imbalance: Continuous Time Markov Chain

The aim of this research project is to utilize the LOB volume imbalance $I(t)$ in an algorithmic trading application; hence, a suitable choice of model for $I(t)$ must be made. Rather than modeling imbalance directly as a real-valued process, an alternative approach, and that which is utilized herein, is to discretize the imbalance value $I(t)$ into subintervals, or bins, and fit the resulting process to a continuous-time Markov chain.

The following definitions and properties are adapted from [Takahara, 2014]:

Definition 1. *A continuous-time stochastic process* $\{X(t) \mid t \geq 0\}$ *with state space* S *is called a* continuous-time Markov chain (CTMC) *if it has the Markov property; namely, that*

$$\mathbb{P}\left[X(t) = j \mid X(s) = i, X(t_{n-1} = i_{n-1}, \ldots, X(t_1) = i_1\right] = \mathbb{P}\left[X(t) = j \mid X(s) = i\right] \tag{2.1}$$

where for any integer $n \geq 1$, $0 \leq t_1 \leq \cdots \leq t_{n-1} \leq s \leq t$ *is any non-decreasing sequence of* $n+1$ *times, and* $i_1, \ldots, i_{n-1}, i, j \in S$ *are any* $n+1$ *states.*

Definition 2. *A CTMC* $X(t)$ *is* time homogeneous *if for any* $s \leq t$ *and any states* $i, j \in S$,

$$\mathbb{P}\left[X(t) = j \mid X(s) = i\right] = \mathbb{P}\left[X(t-s) = j \mid X(0) = i\right] \tag{2.2}$$

Definition 3. *The key quantities that determine a CTMC* $X(t)$ *are the* transition rates q_{ij}, *which specify the rate at which* X *jumps from state* i *to* j. *Conditional on leaving state* i, X *transitions* *to state j with* conditional transition probability p_{ij}*. The amount of time that X spends in state i, called the* holding time*, is exponentially distributed with rate v_i. These quantities are related by:*

$$v_i = \sum_{\substack{j \in S \\ j \neq i}} q_{ij} \tag{2.3}$$

$$q_{ij} = v_i \cdot p_{ij} \tag{2.4}$$

$$p_{ij} = \frac{q_{ij}}{v_i} \tag{2.5}$$

Definition 4. *A CTMC $X(t)$ has an* infinitesimal generator matrix $\boldsymbol{G}$*, whose entries are*

$$g_{ij} = q_{ij}, \qquad i \neq j \tag{2.6}$$

$$g_{ii} = -v_i \tag{2.7}$$

If $X(t)$ has transition probabilities $P_{ij}(t) = \mathbb{P}\left[X(t) = j \mid X(0) = i\right]$ and matrix $\boldsymbol{P}(t) = \{P_{ij}(t)\}$, then $\boldsymbol{P}(t)$ and $\boldsymbol{G}$ are related by

$$\dot{\boldsymbol{P}}(t) = \boldsymbol{G} \cdot \boldsymbol{P}(t) \tag{2.8}$$

$$\boldsymbol{P}(t) = e^{\boldsymbol{G}t} \tag{2.9}$$

Anton: confirm we're using left stochastic or righ stochastic matrices? a few formulas depend on this.

Conditional on $Z(t) = k$, we will assume the arrival of buy and sell market orders follow independent Poisson processes with intensities $\lambda_k^{\pm}$, where λ_k^{+} (λ_k^{-}) is the rate of arrivals of market buys (resp. sells). Such processes are hence called *Markov-modulated Poisson processes.*

In the sections that follow, I derive maximum likelihood estimations for the parameters of the CTMC, and evaluate the fit of the model to the data.

2.2 Maximum Likelihood Estimate of a Markov-modulated Poisson Process

2.2.1 Maximum Likelihood Estimation of G

Let G be the generator matrix for Z_t, so $G = \{q_{ij}\} \in \mathbb{R}^{K \times K}$ where q_{ij} are the transition rates from regime i to regime j for $i \neq j$, and $q_{ii} = -\sum_{j \neq i} q_{ij}$ so that the rows of G sum to 0.

When Z_t enters regime i, the amount of time it spends in regime i is exponentially distributed with rate $v_i = \sum_{j \neq i} q_{ij}$, and when it leaves regime i it will to go regime j with probability $p_{ij} = \dfrac{q_{ij}}{v_i}$.

From our observations we want to estimate the components of G. The holding time in a given regime i is exponentially distributed with pdf $f(t; v_i) = v_i e^{-v_i t}$. For the fictional events in the timeline given in Figure 1.1, the likelihood function (allowing for repetition of terms) would therefore be:

$$
\begin{aligned}
\mathcal{L}(G) &= (v_i e^{-v_i(\tau_2 - \tau_1)} p_{ij})(v_j e^{-v_j(\tau_3 - \tau_2)} p_{ji})(v_i e^{-v_i(\tau_4 - \tau_3)} p_{ik}) \dots && (2.10) \\
&= \prod_{i=1}^{K} \prod_{i \neq j} (v_i p_{ij})^{N_{ij}(T)} e^{-v_i H_i(T)} && (2.11) \\
&= \prod_{i=1}^{K} \prod_{i \neq j} (q_{ij})^{N_{ij}(T)} e^{-v_i H_i(T)} && (2.12)
\end{aligned}
$$

where:

$$
\begin{aligned}
N_{ij}(T) &\equiv \text{number of transitions from regime } i \text{ to } j \text{ up to time } T \\
H_i(T) &\equiv \text{holding time in regime } i \text{ up to time } T
\end{aligned}
$$

So that the log-likelihood becomes:

$$
\begin{aligned}
\ln \mathcal{L}(G) &= \sum_{i=1}^{K} \sum_{i \neq j} \left[N_{ij}(T) \ln(q_{ij}) - v_i H_i(T) \right] && (2.13) \\
&= \sum_{i=1}^{K} \sum_{i \neq j} \left[N_{ij}(T) \ln(q_{ij}) - \left(\sum_{i \neq k} q_{ik} H_i(T) \right) \right] && (2.14)
\end{aligned}
$$

To get a maximum likelihood estimate $\hat{q}_{ij}$ for transition rates and therefore the matrix G, we take the partial derivative of $\ln \mathcal{L}(G)$ and set it equal to zero:

$$\frac{\partial \ln \mathcal{L}(G)}{\partial q_{ij}} = \frac{N_{ij}(T)}{q_{ij}} - H_i(T) = 0 \tag{2.15}$$

$$\Rightarrow \boxed{\hat{q}_{ij} = \frac{N_{ij}(T)}{H_i(T)}} \tag{2.16}$$

2.2.2 Maximum Likelihood Estimation of $\lambda_k^{\pm}$

Now we want to derive an estimate for the intensity of the Poisson process of market order arrivals conditional on being in some bin k. We'll look first at just the market buys for some regime k. In the above timeline, the market order buy arrival times are indexed by b_i. Since we're assuming that the arrival process is Poisson with the same intensity throughout trials, we can consider the inter-arrival time of events conditional on being in regime k. Then the MLE derivation follows just as for the CTMC:

$$\mathcal{L}(\lambda_k^+; b_1, \ldots, b_N) = \prod_{i=2}^{N} \lambda_k^+ e^{-\lambda_k^+ (b_i - b_{i-1})} \tag{2.17}$$

$$= (\lambda_k^+)^{N_k^+(T)} e^{-\lambda_k^+ H_k(T)} \tag{2.18}$$

where:

$$N_k^+(T) \equiv \text{number of market order arrivals in regime } k \text{ up to time } T$$
$$H_k(T) \equiv \text{holding time in regime } k \text{ up to time } T$$

So that the log-likelihood becomes:

$$\ln \mathcal{L}(\lambda_k^+) = N_k^+(T) \ln(\lambda_k^+) - \lambda_k^+ H_k(T) \tag{2.19}$$

And the ML estimate for $\hat{\lambda}_k^+$ is:

$$\frac{\partial \ln \mathcal{L}}{\partial \lambda_k^+} = \frac{N_k^+(T)}{\lambda_k^+} - H_k(T) = 0 \tag{2.20}$$

$$\Rightarrow \boxed{\hat{\lambda}_k^+ = \frac{N_k^+(T)}{H_k(T)}} \tag{2.21}$$

2.3 Cross-Validation of CTMC Parameters

To cross-validate the CTMC calibration, we conduct a time-homogeneity test similar to that done in [Tan and Ylmaz, 2002] and . The null hypothesis is given by [Weibach and Walter, 2010]:

$$H_0 = ... \tag{2.22}$$

To test the hypothesis, we fix an imbalance averaging time Δt_I, number of imbalance bins, and calculate the MLE estimate of the infinitesimal generator matrix $\boldsymbol{G}$ on the full timeseries. For a chosen error threshold ϵ, we use the relationship in Equation (2.9) to calculate the number of timesteps n of size Δt_I such that $||\boldsymbol{P}\left((n+1)\Delta t_I\right) - \boldsymbol{P}\left(n\Delta t_I\right)|| < \epsilon$. This value n determines the size of the cross-validation timewindow in which to partition the full timeseries, yielding K equal subintervals of length n. For each "removed series" $k \in \{1, \ldots, K\}$, we recalibrate a CTMC generator matrix $\boldsymbol{G}_k$. Finally, we test whether the one-step transition probabilities p_{ij}^k contained in $\boldsymbol{P}_k\left(\Delta t_I\right)$ are statistically different from those of the full period. The asymptotically equivalent test statistic to the likelihood ratio test statistic is:

$$\phi = -2\ln(\mathcal{L}) = 2\sum_k \sum_{i,j} n_{i,j}^k \left[\ln(p_{ij}^k) - \ln(p_{ij})\right] \tag{2.23}$$

where n_{ij}^k is the number of observed transitions from state i to j in subinterval k.

Anton: this table is no longer relevant - rerun the cross-validation results

# **bins** averaging time	stationary n	window size (% of series)	err	$\boldsymbol{Err}$
#bins = 3				
100ms	478	47.8s (0.2%)	35.64%	644% - 11371%
500ms	144	72s (0.3%)	8.76%	236% - 985%
1000ms	89	89s (0.4%)	5.06%	150% - 480%
2000ms	57	114s (0.5%)	3.21%	122% - 725%
3000ms	45	135s (0.6%)	2.37%	98% - 552%
#bins = 5				
100ms	546	54.6s (0.2%)	16.27%	452% - 6785%
500ms	162	81s (0.3%)	4.62%	187% - 2590%
1000ms	100	100s (0.4%)	3.00%	136% - 2962%
2000ms	65	130s (0.6%)	1.73%	86% - 2141%
3000ms	52	156s (0.7%)	1.25%	87% - Inf%

Table 2.1: CTMC cross-validation results for $\epsilon = 10e - 5$.

The cross-validation results show a strong case for the rejection of the non-homogeneity assumption, and as such the assumption of homogeneity, despite its appeal, is disputable, and backtesting results may reflect this. Naturally, this suggests possible extensions to this research project, wherein the trading day is broken down into subintervals to better account for fluctuations and patterns in trading activity - perhaps early morning, mid-day, and final hour of trading.

2.4 2-dimensional CTMC

Next we considered a CTMC that jointly models the imbalance bin and the price change over a subsequent interval. That is, the CTMC modelled the joint distribution $(I(t), \Delta S(t))$ where $I(t) \in \{1, 2, \ldots, \#_{bins}\}$ is the bin corresponding to imbalance averaged over the interval $[t - \Delta t_I, t]$, and $\Delta S(t) = \text{sgn}(S(t + \Delta t_S) - S(t)) \in \{-1, 0, 1\}$. The pair $(I(t), \Delta S(t))$ was then reduced into one dimension with a simple encoding which we will denote $\varphi(I(t), S(t))$; for example, using 3 bins:

$Z(t)$	Bin $I(t)$	$\Delta S(t)$	$Z(t)$	Bin $I(t)$	$\Delta S(t)$	$Z(t)$	Bin $I(t)$	$\Delta S(t)$
1	Bin 1	< 0	4	Bin 1	0	7	Bin 1	> 0
2	Bin 2	< 0	5	Bin 2	0	8	Bin 2	> 0
3	Bin 3	< 0	6	Bin 3	0	9	Bin 3	> 0

Table 2.2: $\varphi(I(t), S(t))$: 1-Dimensional Encoding of 2-Dimensional CTMC

It is crucial to note that the value $\Delta S(t)$ contains the price change from time t over the *future* Δt_S seconds - hence in real-time one cannot know the state of the Markov Chain. However, the analytic results do prove enlightening: from the resulting timeseries we estimated a generator matrix $\boldsymbol{G}$, and transform it into a one-step transition probability matrix $\boldsymbol{P} = e^{\boldsymbol{G}\Delta t_I}$. The entries of $\boldsymbol{P}$ are the conditional probabilities

$$\boldsymbol{P}_{ij} = \mathbb{P}\left[\varphi(I_{[t-\Delta t_I, t]}, \Delta S_{[t, t+\Delta t_S]}) = j \mid \varphi(I_{[t-2\Delta t_I, t-\Delta t_I]}, \Delta S_{[t-\Delta t_I, t]}) = i\right] \tag{2.24}$$

which can be expressed semantically as

$$= \mathbb{P}\left[\varphi(\rho_{curr}, \Delta S_{future}) = j \mid \varphi(\rho_{prev}, \Delta S_{curr}) = i\right] \tag{2.25}$$

Since we can easily decode the 1-dimensional Markov state back into two dimensions, we can think of $\boldsymbol{P}$ as being four-dimensional and re-write its entries as

$$= \mathbb{P}\left[\rho_{curr} = i, \Delta S_{future} = j \mid \rho_{prev} = k\Delta S_{curr} = m\right] \tag{2.26}$$

$$= \mathbb{P}\left[\rho_{curr} = i, \Delta S_{future} = j \mid B\right] \tag{2.27}$$

where we're using the shorthand $B = (\rho_{n-1} \in k, \Delta S_{n-1} \in m)$ to represent the states in the previous timestep. Applying Bayes' Rule:

$$\mathbb{P}\left[\Delta S_n \in j \mid B, \rho_n \in i\right] = \frac{\mathbb{P}\left[\rho_n \in i, \Delta S_n \in j \mid B\right]}{\mathbb{P}\left[\rho_n \in i \mid B\right]} \tag{2.28}$$

where the right-hand-side numerator is each individual entry of the one-step probability matrix $\boldsymbol{P}$, and the denominator can be computed from $\boldsymbol{P}$ by:

$$\mathbb{P}\left[\rho_n \in i \mid B\right] = \sum_j \mathbb{P}\left[\rho_n \in i, \Delta S_n \in j \mid B\right] \tag{2.29}$$

This result is of great interest to us: the left-hand-side value is the probability of seeing a given price change over the immediate future time interval conditional on past imbalances and the most recent price change, and therefore allows us to predict future price moves. We'll denote by $\boldsymbol{Q}$ the matrix containing all values given by Equation (2.28).

The following results were obtained using data for `ORCL` from 2013-05-15, averaging imbalance timewindow $t_I = 1000\text{ms}$, $K = 3$ imbalance bins, and price change timewindow $t_S = 1000\text{ms}$:

Anton: these aren't the real results. re-run these, and format it better.

$$\boldsymbol{G}_{Z_{bid}} = \begin{bmatrix} -0.9928 & 0.0217 & 0 & 0.2826 & 0.5870 & 0.0870 & 0 & 0.0145 & 0 \\ 0.0118 & -0.9647 & 0 & 0.1412 & 0.5882 & 0.2000 & 0 & 0.0118 & 0.0118 \\ 0 & 0.0909 & -1.0000 & 0 & 0.3636 & 0.5455 & 0 & 0 & 0 \\ 0.0146 & 0.0005 & 0 & -0.0792 & 0.0562 & 0.0034 & 0.0036 & 0.0006 & 0.0003 \\ 0.0016 & 0.0052 & 0.0003 & 0.0435 & -0.0897 & 0.0300 & 0 & 0.0080 & 0.0011 \\ 0.0003 & 0.0025 & 0.0022 & 0.0053 & 0.0919 & -0.1277 & 0 & 0.0017 & 0.0237 \\ 0 & 0.0345 & 0 & 0.4138 & 0.4138 & 0.1034 & -1.0000 & 0.0345 & 0 \\ 0.0179 & 0.0179 & 0 & 0.2232 & 0.5536 & 0.1250 & 0.0089 & -0.9732 & 0.0268 \\ 0.0094 & 0.0189 & 0 & 0.1132 & 0.5189 & 0.3113 & 0 & 0.0094 & -0.9811 \end{bmatrix}$$

$\boldsymbol{Q}$:

$$\begin{array}{r} \Delta S_n<0\rightarrow \\ \Delta S_n=0\rightarrow \\ \Delta S_n>0\rightarrow \end{array} \left[\overbrace{\underbrace{\begin{matrix} .67 & .05 & .04 \\ .33 & .95 & .96 \\ .00 & .00 & .00 \end{matrix}}_{\Delta S_{n-1}<0} \underbrace{\begin{matrix} .01 & .03 & .04 \\ .99 & .97 & .96 \\ .00 & .00 & .00 \end{matrix}} \underbrace{\begin{matrix} .00 & .05 & .05 \\ .41 & .93 & .95 \\ .58 & .02 & .00 \end{matrix}}_{\Delta S_{n-1}>0}}^{\rho_n=1} \overbrace{\underbrace{\begin{matrix} .02 & .50 & .12 \\ .96 & .49 & .87 \\ .02 & .01 & .00 \end{matrix}} \underbrace{\begin{matrix} .01 & .00 & .02 \\ .98 & .99 & .97 \\ .01 & .01 & .01 \end{matrix}}_{\Delta S_{n-1}=0} \underbrace{\begin{matrix} .05 & .01 & .02 \\ .91 & .48 & .96 \\ .05 & .51 & .01 \end{matrix}}}^{\rho_n=2} \overbrace{\underbrace{\begin{matrix} .00 & .00 & .52 \\ .98 & .95 & .47 \\ .02 & .04 & .01 \end{matrix}} \underbrace{\begin{matrix} .00 & .01 & .00 \\ .95 & .96 & .93 \\ .05 & .03 & .02 \end{matrix}} \underbrace{\begin{matrix} .00 & .00 & .00 \\ .98 & .88 & .34 \\ .02 & .12 & .66 \end{matrix}}}^{\rho_n=3} \right]$$

Immediately evident from $\boldsymbol{Q}$ is that in most cases we are expecting no price change. In fact, the only cases in which the probability of a price change is > 0.5 show evidence of *momentum*; for example, the way to interpret the value in row 1, column 1 is: if $\rho_{prev} = \rho_{curr} = 1$ and previously we saw a downward price change, then we expect to again see a downward price change. In fact, the best way to summarize the matrix is:

$$\mathbb{P}\left[\Delta S_{curr} = \Delta S_{prev} \mid \rho_{prev} = \rho_{curr}\right] > 0.5 \tag{2.30}$$

2.5 In-Sample Backtesting of Naive Trading Strategies

Utilizing the key insight drawn from Equation (2.30), we backtested a number of naive trading strategies, descriptions of which follow:

2.5.1 Naive Trading Strategy

Using the conditional probabilities obtained from $\boldsymbol{P_C}$, we will execute a buy (resp. sell) market order if the probability of an upward (resp. downward) price change is > 0.5.

Algorithm 1 Naive Trading Strategy

$cash = 0$
$asset = 0$
for $t = 2$: `length`$(timeseries)$ **do**
    if $\mathbb{P}\left[\Delta S_{curr} < 0 \mid \rho_{curr}, \rho_{prev}, \Delta S_{prev}\right] > 0.5$ **then**
        $cash \mathrel{+}= data.BuyPrice(t)$
        $asset \mathrel{-}= 1$
    else if $\mathbb{P}\left[\Delta S_{curr} > 0 \mid \rho_{curr}, \rho_{prev}, \Delta S_{prev}\right] > 0.5$ **then**
        $cash \mathrel{-}= data.SellPrice(t)$
        $asset \mathrel{+}= 1$
    end if
end for
if $asset > 0$ **then**
    $cash \mathrel{+}= asset \times data.BuyPrice(t)$
else if $asset < 0$ **then**
    $cash \mathrel{+}= asset \times data.SellPrice(t)$
end if

2.5.2 Naive+ Trading Strategy

Extending the naive trading strategy, if we anticipate no change then we'll additionally keep limited orders posted at the touch, front of the queue. We'll track MO arrival, assume we always get executed, and immediately repost the limit orders.

Algorithm 2 Naive+ Trading Strategy

```
cash = 0
asset = 0
LO_posted = False
for t = 2 : length(timeseries) do
    if P[ΔS_curr < 0 | ρ_curr, ρ_prev, ΔS_prev] > 0.5 then
        cash += data.BuyPrice(t)
        asset −= 1
        LO_posted = False
    else if P[ΔS_curr > 0 | ρ_curr, ρ_prev, ΔS_prev] > 0.5 then
        cash −= data.SellPrice(t)
        asset += 1
        LO_posted = False
    else if P[ΔS_curr = 0 | ρ_curr, ρ_prev, ΔS_prev] > 0.5 then
        LO_posted = True
    end if
    if LO_posted then
        for MO ∈ ArrivedMarketOrders(t, t + 1) do
            if MO == Sell then
                cash −= data.BuyPrice(t)
                asset += 1
            else if MO == Buy then
                cash += data.SellPrice(t)
                asset −= 1
            end if
        end for
    end if
end for
if asset > 0 then
    cash += asset × data.BuyPrice(t)
else if asset < 0 then
    cash += asset × data.SellPrice(t)
end if
```

2.5.3 Naive++ Trading Strategy

We won't execute market orders or keep limit orders at the touch. Using the conditional probabilities obtained from $\boldsymbol{P_C}$, if we expect a downward (resp. upward) price change then we'll add a limit order to the sell (resp. buy) side, and hopefully pick up an agent who is executing a market order going against the price change momentum.

Algorithm 3 Naive++ Trading Strategy

```
cash = 0
asset = 0
LOBuy_posted = False
LOSell_posted = False
for t = 2 : length(timeseries) do
    if P[ΔS_curr < 0 | ρ_curr, ρ_prev, ΔS_prev] > 0.5 then
        LOBuy_posted = False
        LOSell_posted = True
    else if P[ΔS_curr > 0 | ρ_curr, ρ_prev, ΔS_prev] > 0.5 then
        LOBuy_posted = True
        LOSell_posted = False
    else if P[ΔS_curr = 0 | ρ_curr, ρ_prev, ΔS_prev] > 0.5 then
        LOBuy_posted = False
        LOSell_posted = False
    end if
    for MO ∈ ArrivedMarketOrders(t, t + 1) do
        if MO == Sell ∧ LOBuy_posted then
            cash −= data.BuyPrice(t)
            asset += 1
        else if MO == Buy ∧ LOSell_posted then
            cash += data.SellPrice(t)
            asset −= 1
        end if
    end for
end for
if asset > 0 then
    cash += asset × data.BuyPrice(t)
else if asset < 0 then
    cash += asset × data.SellPrice(t)
end if
```

Naive- Trading Strategy We additionally considered a trading strategy, for benchmark purposes, which used only current imbalance to predict future price change. But actually this predicted $\mathbb{P}[\Delta S_{curr} = 0] > 0.5$ at all times, so we could not run a strategy off it.

Backtesting these trading strategies required a choice of parameters for Δt_S, the price change observation period, Δt_I, the imbalance averaging period, and $\#_{bins}$, the number of imbalance bins. Through a brute force calibration technique we found that $\#_{bins} = 4$ provided the highest expected number of successful trades for most tickers, so this was chosen as a constant. Similarly, we empirically saw that calibration always yielded $\Delta t_S = \Delta t_I$, so this was taken as a given. Then

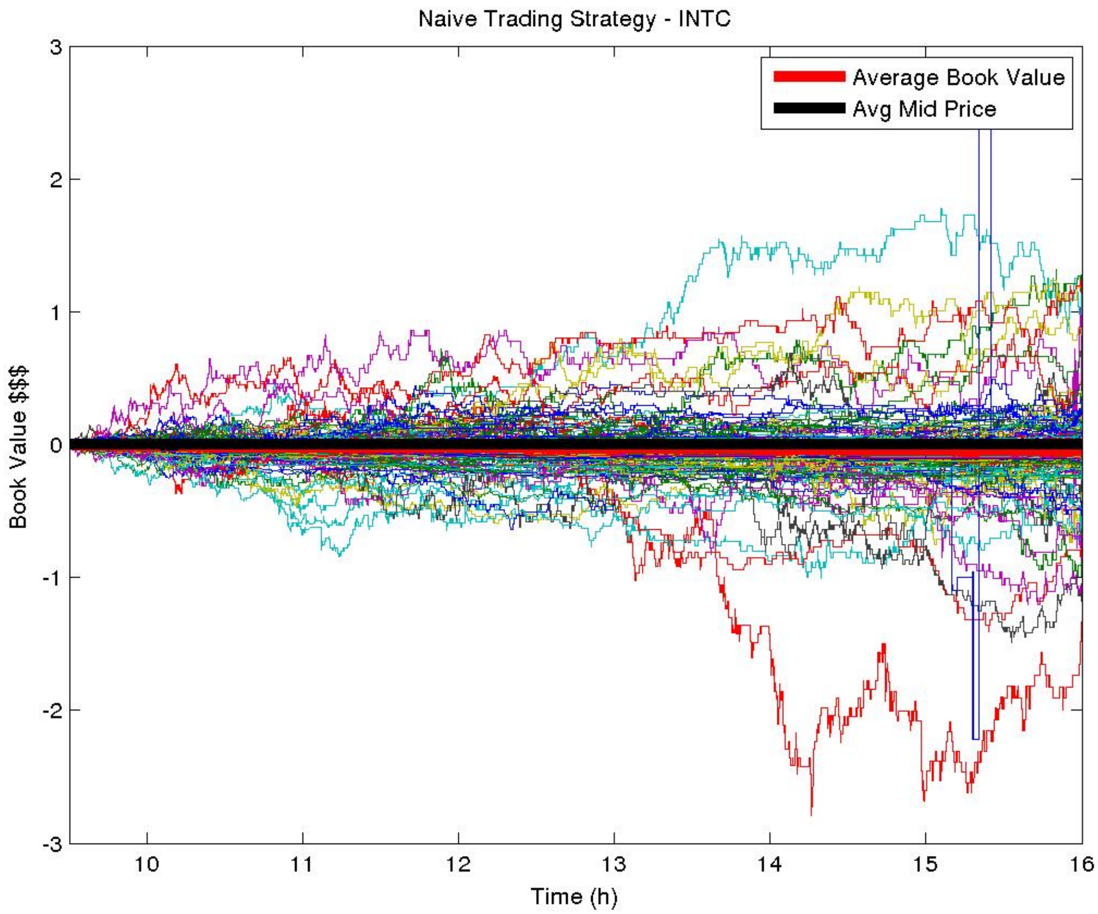


Figure 2.1: INTC: Book value against time of trading day.

each backtest consisted of first calibrating the value Δt_I from one day of data by maximizing the intra-day Sharpe ratio, then using the calibrated parameters to backtest the entire year.

2.6 Conclusions from Naive Trading Strategies

To properly compare the Naive trading strategies, it must be understood that the Naive+ strategy has the Naive built into it - thus it's actually the difference between the two that needs to be assessed to ascertain the effect of posting Limit Orders when no price change is predicted. As seen in Figure 2.4, the Naive trading strategy on average underperformed the average mid price, while the Naive+ (adding at-the-touch limit orders when no change was predicted) and Naive++ (adding limit orders to adversely selecting agents that traded against the price change momentum) strategies both on average generated revenue.

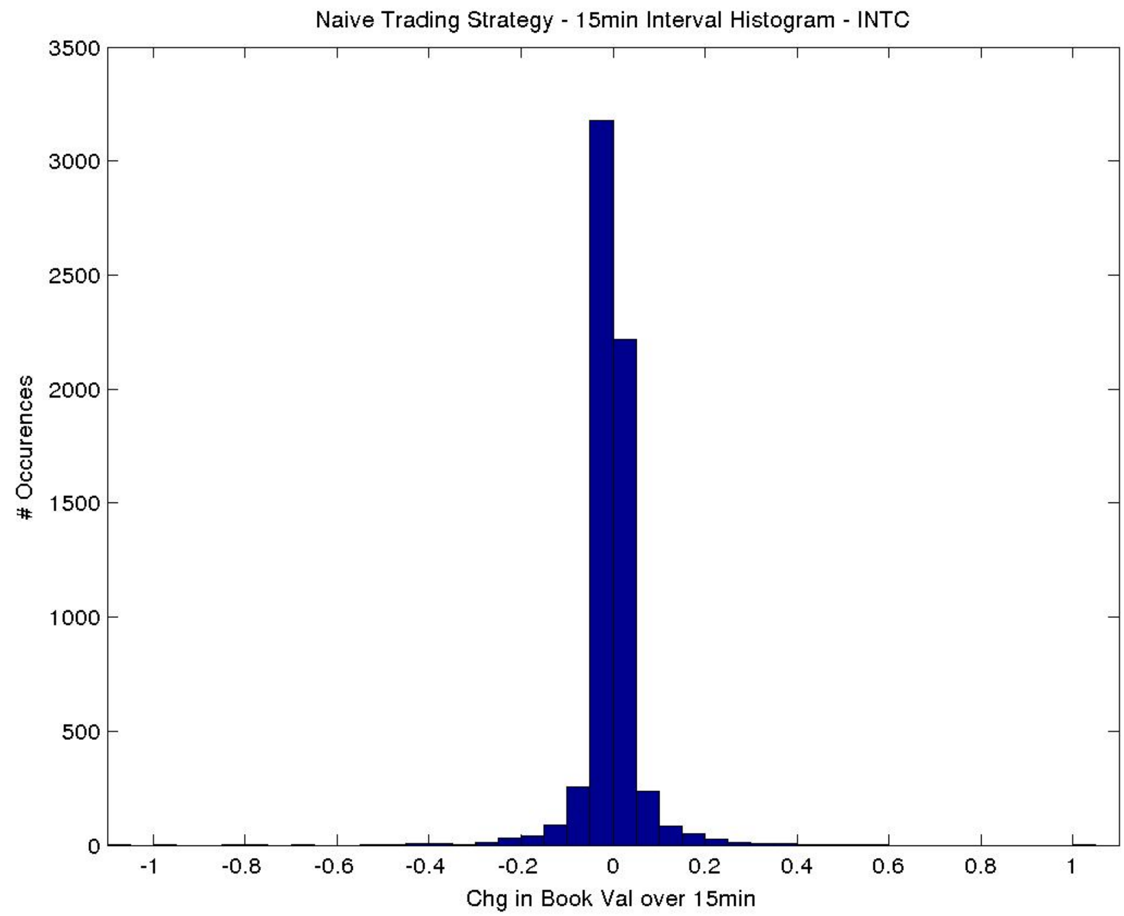


Figure 2.2: INTC: Histogram of 15min book value changes.

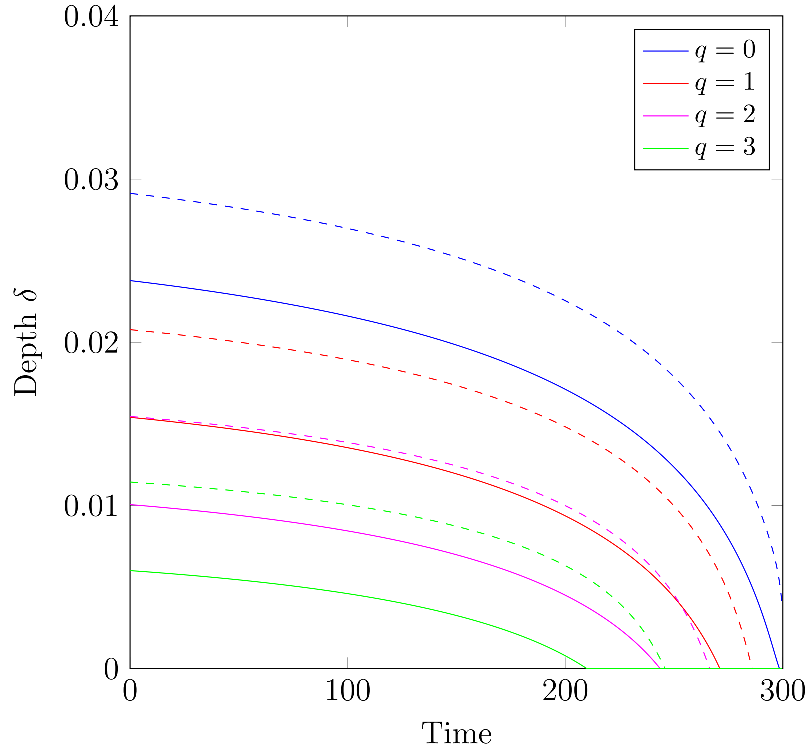
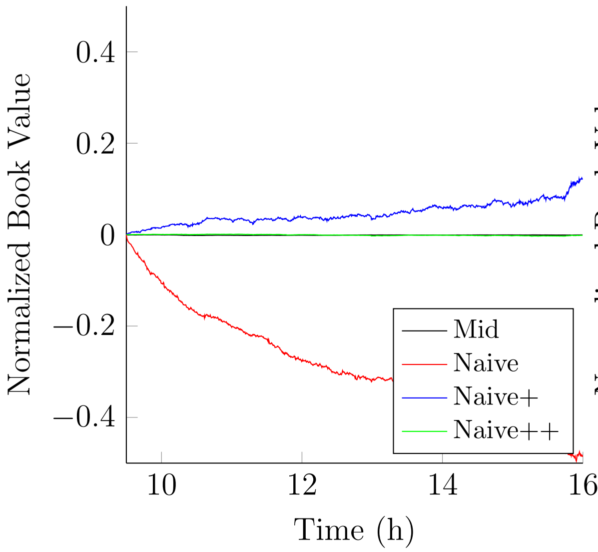


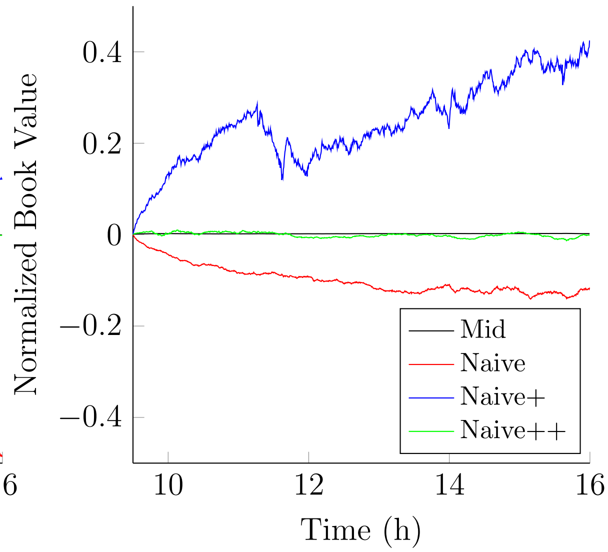
Figure 2.3: Optimal posting depth δ

Question 1 Why is the Naive strategy producing, on average, normalized losses? Especially so when considering that we are in-sample backtesting. On calibration, we see that our intra-day Sharpe ratio is around 0.01 or 0.02 when we choose our optimal parameters, so at the very least on the calibration date the strategy produces positive returns. The remainder of the calendar days are out-of-sample, as the parameters are (likely) not optimal. This suggests non-stationary data, and in particular not every day can be modelled by the same Markov chain. The problem may be exaggerated by the fact that we're calibrating on the first trading day of the calendar year, when we might expect reduced, or at least non-representative, trading activity. Further, we're currently obtaining the $\boldsymbol{P_C}$ probability matrix using only bid-side data, not sell-side or mid, and we're ignoring the bid-ask spread. Thus predicting a "price change" may be insufficient when considering a monetizable opportunity, as we won't be able to profit off a predicted increase followed by a predicted decrease unless the interim mid-price move is greater than the bid-ask spread (assuming constant spread). This suggests a potential straightforward modification to the strategy.

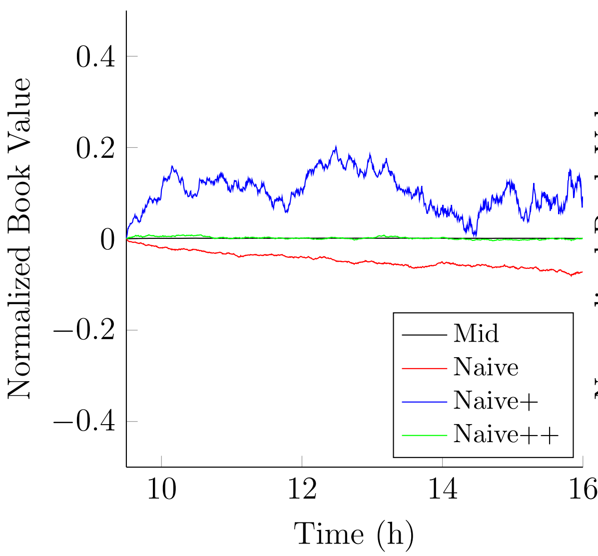
Question 2 Why do the Naive+ and ++ strategies outperform the Naive strategy? This is particularly interesting since the probabilities are being obtained from the same matrix. The



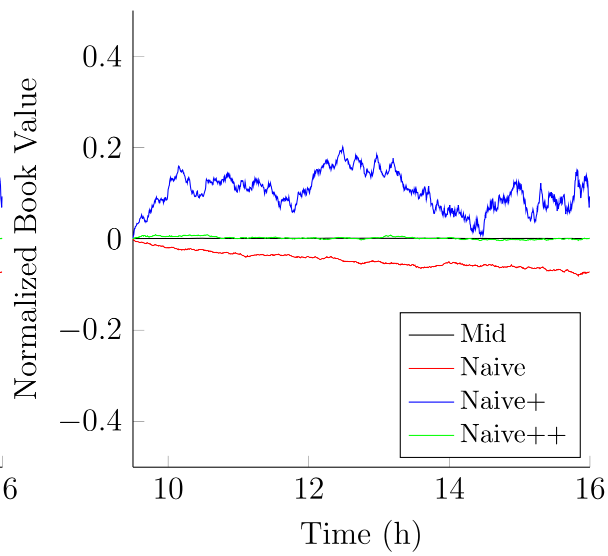
(a) FARO



(b) NTAP



(c) ORCL



(d) AAPL

Figure 2.4: Comparison of Naive (red), Naive+ (blue), and Naive++ (green) trading strategies, with benchmark Midprice (black). Plotted are book values against time of trading day, averaged across trading year.

obvious difference between the successful and unsuccessful strategies is that the former (a) uses limit orders, and (b) executes when we predict a zero change, whereas the latter uses (a) market orders, and (b) executes when we do predict non-zero change.

(a) obviously leads to a different transaction price being used: if I buy with a LO I'm paying the bid price, whereas buying with a MO I pay the ask price. If I value the stock using the mid price, and the mid price doesn't move as a result of my transaction, then with LO I'm buying the asset for less than I'm valuing it at, and with MO I'm paying more than its value.

(b) seems to be the largest flaw in the Naive strategy, to which there are two factors. One, we are not predicting the magnitude of the price change, only whether it is zero or non-zero. Two, from the probabilities presented above, *we will only predict a price change if we've already seen a price change*. Thus we're effectively reacting too late.

Here's how this works adversely. Suppose a stock has bid/ask quotes of \$9.99/\$10.01, for a bid-ask spread of \$0.02 and a mid of \$10.

1. Imbalance = 1 (pressure for upward price move). [$NPV = 0$]
2. Bid/ask goes up to \$10.00/\$10.02. [$NPV = 0$]
3. Imbalance = 1. We predict another > 0 price change. [$NPV = 0$]
4. We buy 1 share (at \$10.02). [$NPV = -0.01$]
5. Bid/ask goes up to \$10.01/\$10.03. [$NPV = 0$]
6. Imbalance = -1 (pressure for a downward move). [$NPV = 0$]
7. Bid/ask goes down to \$10.00/\$10.02. [$NPV = -0.01$]
8. Imbalance = -1. We predict another < 0 price change.
9. We sell 1 share (at \$10.00). [$NPV = -0.02$]
10. Bid/ask goes down to \$9.99/\$10.01. [$NPV = -0.02$]

In this example the price goes up and back down by two cents to return to where it started, and in the process we lost \$0.02. Now imagine what happens if we price goes up by one cent, up by one cent, then down by ten cents, down by one cent. In this case we lose \$0.11. We're unable to predict that initial upward or downward price change, and only react to it.

Chapter 3

Stochastic Optimal Control

Hello and welcome to this limited edition chapter on stochastic optimal control. Read on if you dare subject yourself to the infinite wisdom contained herein.

3.1 Continuous Time

Below we list the processes involved in the optimization problem:

Imbalance & Midprice Change	$\boldsymbol{Z}_t = (\rho_t, \Delta_t)$	CTMC with generator G
Imbalance	$\rho_t = \boldsymbol{Z}_t^{(1)}$	LOB imbalance at time t
Midprice	S_t	evolves according to CTMC
Midprice Change	$\Delta_t = \boldsymbol{Z}_t^{(2)} = S_t - S_{t-s}$	s a pre-determined interval
Bid-Ask half-spread	ξ_t	constant?
LOB Shuffling	N_t	Poisson with rate $\lambda(\boldsymbol{Z}_t)$
ΔPrice: LOB shuffled	$\{\eta_{0,z}, \eta_{1,z}, \dots\} \sim F_z$	i.i.d. with $z = (k, l)$, where $k \in \{\#\text{bins}\},\ l \in \{\Delta\$\}$
Other Agent MOs	$K_t^{\pm}$	Poisson with rate $\mu^{\pm}(\boldsymbol{Z}_t)$
LO posted depth	$\delta_t^{\pm}$	our $\mathcal{F}$-predictable controlled processes
Our LO fill count	$L_t^{\pm}$	$\mathcal{F}$-predictable, non-Poisson
Our MOs	$M_t^{\pm}$	our controlled counting process
Our MO execution times	$\boldsymbol{\tau}^{\pm} = \{\tau_k^{\pm} : k = 1, \dots\}$	increasing sequence of $\mathcal{F}$-stopping times
Cash	$X_t^{\boldsymbol{\tau},\delta}$	depends on our processes M and δ
Inventory	$Q_t^{\boldsymbol{\tau},\delta}$	depends on our processes M and δ

$L_t^{\pm}$ are counting processes (not Poisson) satisfying the relationship that if at time t we have a sell limit order posted at a depth δ_t^-, then our fill probability is $e^{-\kappa\delta_t^-}$ conditional on a buy market order arriving; namely:

$$\mathbb{P}[\,\mathrm{d}L_t^- = 1 \,|\, \mathrm{d}K_t^+ = 1] = e^{-\kappa\delta_t^-} \tag{3.1}$$
$$\mathbb{P}[\,\mathrm{d}L_t^+ = 1 \,|\, \mathrm{d}K_t^- = 1] = e^{-\kappa\delta_t^+} \tag{3.2}$$

The midprice S_t evolves according to the Markov chain and hence is Poisson with rate λ and jump size η, both of which depend on the state of the Markov chain. This Poisson process is all-inclusive in the sense that it accounts for any midprice change, be it from executions, cancellations, or order modifications with the LOB. Thus, the stock midprice S_t evolves according to the SDE:

$$\mathrm{d}S_t = \eta_{N_{t^-},Z_{t^-}}\,\mathrm{d}N_t \tag{3.3}$$

and additionally satisfies:

$$S_t = S_{t_0} + \int_{t_0+s}^{t} \Delta_u \,\mathrm{d}u \tag{3.4}$$

In executing market orders, we assume that the size of the MOs is small enough to achieve the best bid/ask price, and not walk the book. Hence, our cash process evolves according to:

$$\begin{aligned} \mathrm{d}X_t^{\boldsymbol{\tau},\delta} = &\underbrace{(S_t + \xi_t + \delta_t^-)\,\mathrm{d}L_t^-}_{\text{sell limit order}} - \underbrace{(S_t - \xi_t - \delta_t^+)\,\mathrm{d}L_t^+}_{\text{buy limit order}} \\ &+ \underbrace{(S_t - \xi_t)\,\mathrm{d}M_t^-}_{\text{sell market order}} - \underbrace{(S_t + \xi_t)\,\mathrm{d}M_t^+}_{\text{buy market order}} \end{aligned} \tag{3.5}$$

Based on our execution of limit and market orders, our inventory satisfies:

$$Q_0^{\boldsymbol{\tau},\delta} = 0, \qquad Q_t^{\boldsymbol{\tau},\delta} = L_t^+ + M_t^+ - L_t^- - M_t^- \tag{3.6}$$

We define a new variable for our net present value (NPV) at time t, call it $W_t^{\boldsymbol{\tau},\delta}$, and hence $W_T^{\boldsymbol{\tau},\delta}$ at terminal time T is our 'terminal wealth'. In algorithmic trading, we want to finish the trading day with zero inventory, and assume that at the terminal time T we will submit a market order (of a possibly large volume) to liquidate remaining stock. Here we do not assume that we can receive the best bid/ask price - instead, the price achieved will be $(S - \mathrm{sgn}(Q)\xi - \alpha Q)$, where

$\mathrm{sgn}(Q)\xi$ represents crossing the spread in the direction of trading, and αQ represents receiving a worse price linearly in Q due to walking the book. Hence, $W_t^{\boldsymbol{\tau},\delta}$ satisfies:

$$W_t^{\boldsymbol{\tau},\delta} = \underbrace{X_t^{\boldsymbol{\tau},\delta}}_{\text{cash}} + \underbrace{Q_t^{\boldsymbol{\tau},\delta}\left(S_t - \mathrm{sgn}(Q_t^{\boldsymbol{\tau},\delta})\xi_t\right)}_{\text{book value of assets}} - \underbrace{\alpha\left(Q_t^{\boldsymbol{\tau},\delta}\right)^2}_{\text{liquidation penalty}} \tag{3.7}$$

The set of admissible trading strategies is the product of the sets $\mathcal{T}$, the set of all $\mathcal{F}$-stopping times, and $\mathcal{A}$, the set of all $\mathcal{F}$-predictable, bounded-from-below depths δ. We only consider $\delta^{\pm} \geq 0$, since at $\delta = 0$ our fill probability is $e^{-\kappa\delta} = 1$, so we cannot increase the chance of our limit order being filled by posting any lower than at-the-touch; doing so would only diminish our profit.

For deriving an optimal trading strategy via dynamic programming, I will consider the performance criteria that maximizes terminal wealth. With the above notation, the performance criteria function can be written

$$H^{\boldsymbol{\tau},\delta}(t,x,s,\boldsymbol{z},q) = \mathbb{E}\left[W_T^{\boldsymbol{\tau},\delta}\right] \tag{3.8}$$

And the value function, in turn, is given by

$$H(t,x,s,\boldsymbol{z},q) = \sup_{\boldsymbol{\tau}\in\mathcal{T}_{[t,T]}} \sup_{\delta\in\mathcal{A}_{[t,T]}} H^{\boldsymbol{\tau},\delta}(t,x,s,\boldsymbol{z},q) \tag{3.9}$$

The following theorems establish the dynamic programming method we will utilize to solve this type of problem:

Theorem 5 ([Cartea et al., 2015])***. Dynamic Programming Principle for Optimal Stopping and Control.*** *If an agent's performance criteria for a given admissible control* $\boldsymbol{u}$ *and admissible stopping time* τ *are given by*

$$H^{\tau,\boldsymbol{u}}(t,\boldsymbol{x}) = \mathbb{E}_{t,\boldsymbol{x}}[G(X_\tau^{\boldsymbol{u}})]$$

and the value function is

$$H(t,\boldsymbol{x}) = \sup_{\tau\in\mathcal{T}_{[t,T]}} \sup_{\boldsymbol{u}\in\mathcal{A}_{[t,T]}} H^{\tau,\boldsymbol{u}}(t,\boldsymbol{x})$$

then the value function satisfies the Dynamic Programming Principle

$$H(t,\boldsymbol{x}) = \sup_{\tau\in\mathcal{T}_{[t,T]}} \sup_{\boldsymbol{u}\in\mathcal{A}_{[t,T]}} \mathbb{E}_{t,\boldsymbol{x}}\left[G(X_\tau^{\boldsymbol{u}})\mathbb{1}_{\tau<\theta} + H(\theta, X_\theta^{\boldsymbol{u}})\mathbb{1}_{\tau\geq\theta}\right] \tag{3.10}$$

for all $(t, \boldsymbol{x}) \in [0, T] \times \mathbb{R}^m$ *and all stopping times* $\theta \le T$.

Theorem 6 ([Cartea et al., 2015]). ***Dynamic Programming Equation for Optimal Stopping and Control.*** *Assume that the value function* $H(t, \boldsymbol{x})$ *is once differentiable in* t*, all second-order derivatives in* $\boldsymbol{x}$ *exist, and that* $G : \mathbb{R}^m \to \mathbb{R}$ *is continuous. Then* H *solves the quasi-variational inequality*

$$0 = \max\left\{\partial_t H + \sup_{\boldsymbol{u}\in\mathcal{A}_t} \mathcal{L}_t^{\boldsymbol{u}} H \; ; \; G - H\right\} \tag{3.11}$$

on $\mathcal{D}$*, where* $\mathcal{D} = [0, T] \times \mathbb{R}^m$.

3.1.1 Maximizing Terminal Wealth (Continuous)

In this section we solve the DPE that results from using the maximal terminal wealth performance criteria. The quasi-variational inequality in equation 3.11 can be interpreted as follows: the max operator is choosing between posting limit orders or executing market orders; the second term, $G - H$, is the stopping region and represents the value derived from executing a market order; and the first term is the continuation region, representing the value of posting limit orders. We'll use the shorthand $H(\cdot) = H(t, x, s, \boldsymbol{z}, q)$ and solve for $\mathrm{d}H$ inside the continuation region, hence $\mathrm{d}M^{\pm} = 0$, in order to then extract out the infinitesimal generator.

$$\begin{aligned}
\mathrm{d}H(t, x, s, \boldsymbol{z}, q) &= \sum_i \partial_{x_i} H \,\mathrm{d}x_i &(3.12)\\
&= \partial_t H \,\mathrm{d}t + \partial_{K^{\pm}} H \,\mathrm{d}K^{\pm} + \partial_{\boldsymbol{Z}} H \,\mathrm{d}\boldsymbol{Z} &(3.13)\\
&= \partial_t H \,\mathrm{d}t + \left\{e^{-\kappa\delta^-}\mathbb{E}\big[H(t, x + (s + \xi + \delta^-), s, \boldsymbol{z}, q - 1) - H(\cdot)\big]\right\}\mathrm{d}K^+ \\
&\qquad + \left\{e^{-\kappa\delta^+}\mathbb{E}\big[H(t, x - (s - \xi - \delta^+), s, \boldsymbol{z}, q + 1) - H(\cdot)\big]\right\}\mathrm{d}K^- &(3.14)\\
&\quad + \sum_{\mathbf{j}} \mathbb{E}\left[H(t, x, s + \eta_{0,\mathbf{j}}, \mathbf{j}, q) - H(\cdot)\right] \mathrm{d}Z_{\boldsymbol{z},\mathbf{j}}
\end{aligned}$$

Anton: section on compensated processes, especially for Markov Chain

Substitute in the following identities for the compensated processes

$$\mathrm{d}M^{\pm} = \mathrm{d}\tilde{K}^{\pm} + \mu^{\pm}(\boldsymbol{z})\,\mathrm{d}t \tag{3.15}$$

$$\mathrm{d}Z_{\boldsymbol{z},\mathbf{j}} = \mathrm{d}\tilde{Z}_{\boldsymbol{z},\mathbf{j}} + G_{\boldsymbol{z},\mathbf{j}}\,\mathrm{d}t \tag{3.16}$$

$$\begin{aligned}
&= \partial_t H \,\mathrm{d}t + \Big\{ \mu^+(\boldsymbol{z}) e^{-\kappa\delta^-}\, \mathbb{E}\big[H(t, x + (s + \xi + \delta^-), s, \boldsymbol{z}, q-1) - H(\cdot)\big] \\
&\qquad\qquad + \mu^-(\boldsymbol{z}) e^{-\kappa\delta^+}\, \mathbb{E}\big[H(t, x - (s - \xi - \delta^+), s, \boldsymbol{z}, q+1) - H(\cdot)\big] \\
&\quad + \sum_{\mathbf{j}} G_{\boldsymbol{z},\mathbf{j}} \mathbb{E}\left[H(t, x, s + \eta_{0,\mathbf{j}}, \mathbf{j}, q) - H(\cdot)\right] \Big\} \,\mathrm{d}t \\
&\qquad\qquad + \big\{ e^{-\kappa\delta^-}\, \mathbb{E}\big[H(t, x + (s + \xi + \delta^-), s, \boldsymbol{z}, q-1) - H(\cdot)\big] \big\}\, \mathrm{d}\tilde{K}^+ \\
&\qquad\qquad + \big\{ e^{-\kappa\delta^+}\mathbb{E}\big[H(t, x - (s - \xi - \delta^+), s, \boldsymbol{z}, q+1) - H(\cdot)\big] \big\}\, \mathrm{d}\tilde{K}^- \\
&\quad + \sum_{\mathbf{j}} \mathbb{E}\left[H(t, x, s + \eta_{0,\mathbf{j}}, \mathbf{j}, q) - H(\cdot)\right]\, \mathrm{d}\tilde{Z}_{\boldsymbol{z},\mathbf{j}}
\end{aligned} \tag{3.17}$$

From which we can see that the infinitesimal generator is given by

$$\begin{aligned}
\mathcal{L}_t^{\delta} H = {} & \mu^+(\boldsymbol{z}) e^{-\kappa\delta^-}\, \mathbb{E}\big[H(t, x + (s + \xi + \delta^-), s, \boldsymbol{z}, q-1) - H(\cdot)\big] \\
& + \mu^-(\boldsymbol{z}) e^{-\kappa\delta^+}\, \mathbb{E}\big[H(t, x - (s - \xi - \delta^+), s, \boldsymbol{z}, q+1) - H(\cdot)\big] \\
& + \sum_{\mathbf{j}} G_{\boldsymbol{z},\mathbf{j}} \mathbb{E}\left[H(t, x, s + \eta_{0,\mathbf{j}}, \mathbf{j}, q) - H(\cdot)\right]
\end{aligned} \tag{3.18}$$

Now, our DPE has the form

$$\begin{aligned}
0 = \max\Big\{ & \partial_t H + \sup_{\boldsymbol{u}\in\mathcal{A}_t} \mathcal{L}_t^{\boldsymbol{u}} H \;;\; H(t, x - (s + \xi), s, \boldsymbol{z}, q+1) - H(\cdot) \;; \\
& \qquad\qquad\qquad H(t, x + (s - \xi), s, \boldsymbol{z}, q-1) - H(\cdot) \Big\}
\end{aligned} \tag{3.19}$$

with boundary conditions

$$H(T, x, s, \boldsymbol{z}, q) = x + q(s - \mathrm{sgn}(q)\xi) - \alpha q^2 \tag{3.20}$$

$$H(t, x, s, \boldsymbol{z}, 0) = x \tag{3.21}$$

The three terms over which we are maximizing represent the continuation regions and stopping regions of the optimization problem. The first term, the continuation region, represents the limit order controls; the second and third terms, each a stopping region, represent the value gain from executing a buy market order and a sell market order, respectively.

Let's introduce the ansatz $H(\cdot) = x + q(s - \mathrm{sgn}(q)\xi) + h(t, \boldsymbol{z}, q)$. The first two terms are the wealth plus book value of assets, hence a mark-to-market of the current position, whereas the $h(t, \boldsymbol{z}, q)$

captures value due to the optimal trading strategy. The corresponding boundary conditions on h are

$$h(T, \boldsymbol{z}, q) = -\alpha q^2 \tag{3.22}$$
$$h(t, \boldsymbol{z}, 0) = 0 \tag{3.23}$$

Substituting this ansatz into equation 3.18, we get:

$$\begin{aligned}\mathcal{L}_t^\delta H = \mu^+(\boldsymbol{z})e^{-\kappa\delta^-}\big[&\delta^- + \xi[1 + \operatorname{sgn}(q-1) + q(\operatorname{sgn}(q) - \operatorname{sgn}(q-1))] \\ &+ h(t, \boldsymbol{z}, q-1) - h(t, \boldsymbol{z}, q)\big] \\ + \mu^-(\boldsymbol{z})e^{-\kappa\delta^+}\big[&\delta^+ + \xi[1 - \operatorname{sgn}(q+1) + q(\operatorname{sgn}(q) - \operatorname{sgn}(q+1))] \\ &+ h(t, \boldsymbol{z}, q+1) - h(t, \boldsymbol{z}, q)\big] \\ + \sum_{\mathbf{j}} G_{\boldsymbol{z},\mathbf{j}}\,&[q\mathbb{E}[\eta_{0,\mathbf{j}}] + h(t, \mathbf{j}, q) - h(t, \boldsymbol{z}, q)]\end{aligned} \tag{3.24}$$

We can further simplify the factors of ξ; for example, in the case of the δ^+ term, we can write

$$\begin{aligned}1 - \operatorname{sgn}(q+1) + q(\operatorname{sgn}(q) - \operatorname{sgn}(q+1)) &= 1 - (-\mathbb{1}_{q\leq -2} + \mathbb{1}_{q\geq 0}) + \mathbb{1}_{q=-1} \\ &= 1 + (\mathbb{1}_{q\leq -1} - \mathbb{1}_{q\geq 0}) \\ &= 2 \cdot \mathbb{1}_{q\leq -1}\end{aligned}$$

This gives us the simplified infinitesimal generator term

$$\begin{aligned}\mathcal{L}_t^\delta H &= \mu^+(\boldsymbol{z})e^{-\kappa\delta^-}\left[\delta^- + 2\xi \cdot \mathbb{1}_{q\geq 1} + h(t, \boldsymbol{z}, q-1) - h(t, \boldsymbol{z}, q)\right] \\ &\quad + \mu^-(\boldsymbol{z})e^{-\kappa\delta^+}\left[\delta^+ + 2\xi \cdot \mathbb{1}_{q\leq -1} + h(t, \boldsymbol{z}, q+1) - h(t, \boldsymbol{z}, q)\right] \\ &\quad + \sum_{\mathbf{j}} G_{\boldsymbol{z},\mathbf{j}}\,[q\mathbb{E}[\eta_{0,\mathbf{j}}] + h(t, \mathbf{j}, q) - h(t, \boldsymbol{z}, q)]\end{aligned} \tag{3.25}$$

In the DPE, the first term requires finding the supremum over all $\delta^\pm$ of the infinitesimal generator. For this we can set the partial derivatives with respect to both δ^+ and δ^- equal to zero to solve for the optimal posting depth, which we denote with a superscript asterisk. For δ^+ we get:

$$0 = \partial_{\delta^+}\left[e^{-\kappa\delta^{+*}}\left[\delta^{+*} + 2\xi \cdot \mathbb{1}_{q\leq -1} + h(t, \boldsymbol{z}, q+1) - h(t, \boldsymbol{z}, q)\right]\right] \tag{3.26}$$
$$= -\kappa e^{-\kappa\delta^{+*}}\left[\delta^{+*} + 2\xi \cdot \mathbb{1}_{q\leq -1} + h(t, \boldsymbol{z}, q+1) - h(t, \boldsymbol{z}, q)\right] + e^{-\kappa\delta^{+*}} \tag{3.27}$$
$$= e^{-\kappa\delta^{+}}\left[-\kappa\big(\delta^{+*} + 2\xi \cdot \mathbb{1}_{q\leq -1} + h(t, \boldsymbol{z}, q+1) - h(t, \boldsymbol{z}, q)\big) + 1\right] \tag{3.28}$$

Since $e^{-\kappa\delta^{+*}} > 0$, the term inside the square braces must be equal to zero:

$$0 = -\kappa\big(\delta^{+*} + 2\xi \cdot \mathbb{1}_{q\leq -1} + h(t,\boldsymbol{z},q+1) - h(t,\boldsymbol{z},q)\big) + 1 \tag{3.29}$$

$$\delta^{+*} = \frac{1}{\kappa} - 2\xi \cdot \mathbb{1}_{q\leq -1} - h(t,\boldsymbol{z},q+1) + h(t,\boldsymbol{z},q) \tag{3.30}$$

Recalling that our optimal posting depths are to be non-negative, we thus find that the optimal buy limit order posting depth can be written in feedback form as

$$\delta^{+*} = \max\left\{0\ ;\ \frac{1}{\kappa} - 2\xi \cdot \mathbb{1}_{q\leq -1} - h(t,\boldsymbol{z},q+1) + h(t,\boldsymbol{z},q)\right\} \tag{3.31}$$

We can follow similar steps to solve for the optimal sell limit order posting depth

$$\delta^{-*} = \max\left\{0\ ;\ \frac{1}{\kappa} - 2\xi \cdot \mathbb{1}_{q\geq 1} - h(t,\boldsymbol{z},q-1) + h(t,\boldsymbol{z},q)\right\} \tag{3.32}$$

Turning our attention to the stopping regions of the DPE, we can use the ansatz to simplify the expressions:

$$\begin{aligned}
&H(t, x-(s+\xi), s, \boldsymbol{z}, q+1) - H(\cdot) \\
&\quad = x - s - \xi + (q+1)(s - \mathrm{sgn}(q+1)\xi) + h(t,\boldsymbol{z},q+1) && (3.33)\\
&\qquad - \big[x + q(s - \mathrm{sgn}(q)\xi) + h(t,\boldsymbol{z},q)\big] \\
&\quad = -\xi\big[(q+1)\,\mathrm{sgn}(q+1) - q\,\mathrm{sgn}(q) + 1\big] + h(t,\boldsymbol{z},q+1) - h(t,\boldsymbol{z},q) && (3.34)\\
&\quad = -2\xi \cdot \mathbb{1}_{q\geq 0} + h(t,\boldsymbol{z},q+1) - h(t,\boldsymbol{z},q) && (3.35)
\end{aligned}$$

and similarly,

$$H(t, x+(s-\xi), s, \boldsymbol{z}, q-1) - H(\cdot) = -2\xi \cdot \mathbb{1}_{q\leq 0} + h(t,\boldsymbol{z},q-1) - h(t,\boldsymbol{z},q) \tag{3.36}$$

Substituting all these results and simplifications into the DPE, we find that h satisfies

$$
\begin{aligned}
0 = \max\Bigg\{ & \partial_t h + \mu^+(\boldsymbol{z}) e^{-\kappa \delta^{-*}} \left(\delta^{-*} + 2\xi \mathbb{1}_{q \geq 1} + h(t, \boldsymbol{z}, q-1) - h(t, \boldsymbol{z}, q) \right) \\
& \quad + \mu^-(\boldsymbol{z}) e^{-\kappa \delta^{+*}} \left(\delta^{+*} + 2\xi \cdot \mathbb{1}_{q \leq -1} + h(t, \boldsymbol{z}, q+1) - h(t, \boldsymbol{z}, q) \right) \\
& \quad + \sum_{\mathbf{j}} G_{\boldsymbol{z}, \mathbf{j}} \left[q l \mathbb{E}\left[\eta_{0, \mathbf{j}} \right] + h(t, \mathbf{j}, q) - h(t, \boldsymbol{z}, q) \right] \ ; \\
& - 2\xi \cdot \mathbb{1}_{q \geq 0} + h(t, \boldsymbol{z}, q+1) - h(t, \boldsymbol{z}, q) \ ; \\
& - 2\xi \cdot \mathbb{1}_{q \leq 0} + h(t, \boldsymbol{z}, q-1) - h(t, \boldsymbol{z}, q) \Bigg\}
\end{aligned} \tag{3.37}
$$

Looking at the simplified feedback form in the stopping region, we see that a buy market order will be executed at time τ_q^+ whenever

$$h(\tau_q^+, \boldsymbol{z}, q+1) - h(\tau_q^+, \boldsymbol{z}, q) = 2\xi \cdot \mathbb{1}_{q \geq 0} \tag{3.38}$$

and a sell market order whenever

$$h(\tau_q^+, \boldsymbol{z}, q-1) - h(\tau_q^+, \boldsymbol{z}, q) = 2\xi \cdot \mathbb{1}_{q \leq 0} \tag{3.39}$$

Consider than when our inventory is positive, we can purchase a stock at $s + \xi$, but it will be marked-to-market at $s - \xi$, resulting in a value difference of 2ξ. With negative inventory, we will still purchase at s_ξ, but will now also value at $s + \xi$ because our overall position is still negative, producing no value difference. In particular, with negative inventory, we will execute a buy market order so long as it does not change our value function; and with zero or positive inventory, only if it increases the value function by the value of the spread. The opposite holds for sell market orders. Together, these indicate a penchant for using market orders to drive inventory levels back toward zero when it has no effect on value, and using them to gain extra value only when the expected gain is equal to the size of the spread. This is reminiscent of what we saw in the exploratory data analysis: if a stock is worth S, we can purchase it at $S + \xi$ and immediately be able to sell it at $S - \xi$, at a loss of 2ξ; this was the most significant source of loss in the naive trading market order strategy. Hence we need to expect our value to increase by at least 2ξ when executing market orders for gain.

The variational inequality in Equation (3.37) yields that whilst in the continuation region, we instead have

$$h(\tau_q^+, \boldsymbol{z}, q+1) - h(\tau_q^+, \boldsymbol{z}, q) \leq 2\xi \cdot \mathbb{1}_{q \geq 0} \tag{3.40}$$

$$h(\tau_q^+, \boldsymbol{z}, q-1) - h(\tau_q^+, \boldsymbol{z}, q) \leq 2\xi \cdot \mathbb{1}_{q\leq 0} \tag{3.41}$$

Taken together, these inequalities yield

$$-2\xi \cdot \mathbb{1}_{q\geq 0} \leq h(t, \boldsymbol{z}, q) - h(t, \boldsymbol{z}, q+1) \leq 2\xi \cdot \mathbb{1}_{q\leq -1} \tag{3.42}$$

$$-2\xi \cdot \mathbb{1}_{q\leq 0} \leq h(t, \boldsymbol{z}, q) - h(t, \boldsymbol{z}, q-1) \leq 2\xi \cdot \mathbb{1}_{q\geq 1} \tag{3.43}$$

or alternatively,

$$h(t, \boldsymbol{z}, q) \leq h(t, \boldsymbol{z}, q+1) \leq h(t, \boldsymbol{z}, q) + 2\xi, \qquad q \geq 0 \tag{3.44}$$

$$h(t, \boldsymbol{z}, q) \leq h(t, \boldsymbol{z}, q-1) \leq h(t, \boldsymbol{z}, q) + 2\xi, \qquad q \leq 0 \tag{3.45}$$

Anton: insert the little bubbles with sell and buy at the inequatlity signs. sell, buy, buy, sell left right top down.

Recalling the boundary condition $h(t, \boldsymbol{z}, 0) = 0$, this tells us that the function h is non-negative everywhere. Furthermore, noting the feedback form of our optimal buy limit order depth given in equation Equation (3.31), together with the inequalities in Equation (3.42) and Equation (3.43), we obtain bounds on our posting depths given by

$$\delta^{+^*} = \frac{1}{\kappa} - 2\xi \cdot \mathbb{1}_{q\leq -1} - h(t, \boldsymbol{z}, q+1) + h(t, \boldsymbol{z}, q) \tag{3.46}$$

$$\geq \frac{1}{\kappa} - 2\xi \cdot \mathbb{1}_{q\leq -1} - 2\xi \cdot \mathbb{1}_{q\geq 0} \tag{3.47}$$

$$= \frac{1}{\kappa} - 2\xi \tag{3.48}$$

$$\delta^{+^*} \leq \frac{1}{\kappa} - 2\xi \cdot \mathbb{1}_{q\leq -1} + 2\xi \cdot \mathbb{1}_{q\leq -1} \tag{3.49}$$

$$= \frac{1}{\kappa} \tag{3.50}$$

Combined with the identical conditions on the sell depth, we have the conditions

$$\boxed{\frac{1}{\kappa} - 2\xi \leq \delta^{\pm^*} \leq \frac{1}{\kappa}} \tag{3.51}$$

A possible interpretation of the unexpected upper bound on the posting depth is that if the calculated buy (resp. sell) depth is 'sufficiently' large so as to indicate a disposition against buying (resp. selling), then it is actually optimal to sell (resp. buy) instead.

3.2 Discrete Time

Reminder of our processes (a little bit of abuse of notation going on):

$\boldsymbol{z}_k = (\rho_k, \Delta_k)$ - 2-D time-homogenous Markov Chain with transition probabilities $\boldsymbol{P}_{ij}$, where $\rho_k \in \Gamma$ and Γ represents the set of imbalance bins, and $\Delta_k = \text{sgn}(s_k - s_{k-1}) \in \{-1, 0, 1\}$.

$$\text{State } \vec{x}_k = \begin{pmatrix} x_k \\ s_k \\ \boldsymbol{z}_k \\ q_k \end{pmatrix} \qquad \begin{matrix} \text{cash} \\ \text{stock price} \\ \text{Markov chain state, as above} \\ \text{inventory} \end{matrix}$$

$$\text{Control } \vec{u}_k = \begin{pmatrix} \delta_k^+ \\ \delta_k^- \\ M_k^+ \\ M_k^- \end{pmatrix} \qquad \begin{matrix} \text{bid posting depth} \\ \text{ask posting depth} \\ \text{buy MO - binary control} \\ \text{sell MO - binary control} \end{matrix}$$

$$\text{Random } \vec{w}_k = \begin{pmatrix} K_k^+ \\ K_k^- \\ \omega_k \end{pmatrix} \qquad \begin{matrix} \text{other agent buy MOs - binary} \\ \text{other agent sell MOs - binary} \\ \text{random variable uniformly distributed on [0,1]} \end{matrix}$$

Following [Kwong, 2015], we'll write the evolution of the Markov chain as a function of the current state and a uniformly distributed random variable ω:

$$\boldsymbol{z}_{k+1} = T(\boldsymbol{z}_k, \omega_k) = \sum_{i=0}^{|\Gamma|} i \cdot \mathbb{1}_{\left(\sum_{j=0}^{i-1} \boldsymbol{P}_{\boldsymbol{z}_k, j}, \sum_{j=0}^{i} \boldsymbol{P}_{\boldsymbol{z}_k, j}\right]}(\omega_k) \tag{3.52}$$

Here $\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$, and hence Z_{k+1} is assigned to the value i for which ω_k is in the indicated interval of probabilities.

Our Markovian state evolution function f, given by $\vec{x}_{k+1} = f(\vec{x}_k, \vec{u}_k, \vec{w}_k)$, can be written explicitly as

$$\begin{pmatrix} x_{k+1} \\ s_{k+1} \\ \boldsymbol{z}_{k+1} \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ s_k + \eta_{k+1, T(\boldsymbol{z}_k, \omega_k)} \\ T(\boldsymbol{z}_k, \omega_k) \\ q_k \end{pmatrix} + \begin{pmatrix} s_k + \xi + \delta_k^- \\ 0 \\ 0 \\ -1 \end{pmatrix} L_k^- + \begin{pmatrix} -(s_k - \xi - \delta_k^+) \\ 0 \\ 0 \\ 1 \end{pmatrix} L_k^+ \tag{3.53}$$

The cash process at a subsequent timestep is equal to the cash at the previous step, plus the profits and costs of executing market and/or limit orders. At time k, if the agent posts a sell limit order that gets filled "between timesteps" k and $k+1$ (depending on the binary random variable L_k^-, itself depending on the binary random variable K_k^+), the revenue depends on the stock price at k. This is consistent with reality as with backtesting: while we are choosing to model the posting *depth*, in reality a submitted limit order has a specific price specified - thus once the order is submitted at k, the potential cash received is fixed.

Our impulse control at every time step is given by

$$\begin{pmatrix} x_k \\ s_k \\ \boldsymbol{z}_k \\ q_k \end{pmatrix} = \begin{pmatrix} x_k \\ s_k \\ \boldsymbol{z}_k \\ q_k \end{pmatrix} + \begin{pmatrix} s_k - \xi \\ 0 \\ 0 \\ -1 \end{pmatrix} M_k^- + \begin{pmatrix} -(s_k + \xi) \\ 0 \\ 0 \\ 1 \end{pmatrix} M_k^+ \tag{3.54}$$

Our market orders assume immediate execution, and are assumed to be sufficiently small in volume so as to not affect order imbalance or the midprice.

3.2.1 Dynamic Programming

The system formulation allows both continuous and impulse control to mimic what was done in the continuous time section, though in discrete time there is no *a priori* distinction between the two [Bensoussan, 2008]. The following theorem shows that in this case a quasi-variational inequality formulation does exist, and that it is equivalent to the standard dynamic programming formulation. The result is a simplified expression that mirrors the continuous time analysis.

Theorem 7 ([Bensoussan, 2008])**.** ***Dynamic Programming with Impulse Control in Discrete Time.*** *Consider a controlled Markov Chain with state space $X = \mathbb{R}^d$, transition probability $\pi(x, v, \mathrm{d}\eta)$, and positive, bounded, uniformly continuous cost function $l(x, v)$.*

Introduce an impulse control w. Define the extended cost function by $l(x, v, w) = l(x+w, v)+c(w)$, the extended transition probability by $\pi(x, v, w, \mathrm{d}\eta) = \pi(x+w, v, \mathrm{d}\eta)$ with the associated operator $\Phi^{v,w} f(x) = \int_{\mathbb{R}^d} f(\eta)\pi(x, v, w, \mathrm{d}\eta) = \Phi^v f(x + w)$.

Consider a decision rule V, W with associated probability $\mathbb{P}^{V,W,x}$ on $\Omega, \mathcal{A}$ for which $y_1 = x$ a.s.

Consider the pay-off function

$$J_x(V, W) = \mathbb{E}^{V,W,x} \left[\sum_{n=1}^{\infty} \alpha^{n-1} l(y_n, v_n, w_n) \right] \tag{3.55}$$

and the corresponding Bellman equation

$$u(x) = \inf_{\substack{v \in U \\ w \geq 0}} [l(x + w, v) + c(w) + \alpha \Phi^v u(x + w)] \tag{3.56}$$

Assume:

1. $\Phi^V \phi_v(x)$ *is continuous in* v, x *if* $\phi_v(x) = \phi(x, v)$ *is uniformly continuous and bounded in* x, v*;*

2. $c(w) = K \mathbb{1}_{w=0} + c_0(w), \quad c_0(0) = 0, \qquad c_0(w) \to \infty$ *as* $|w| \to \infty$*,* $c_0(w)$ *is sub-linear positive continuous;*

3. U *is compact.*

Then there exists a unique, positive, bounded solution of Equation (3.56) belonging to the space of uniformly continuous and bounded functions. Further, this solution is identical to that of

$$u(x) = \min \left\{ K + \inf_{w \geq 0} [c_0(w) + u(x + w)] \; ; \; \inf_{v \in U} [l(x, v) + \alpha \Phi^v u(x)] \right\} \tag{3.57}$$

3.2.2 Maximizing Terminal Wealth (Discrete)

Following the dynamic programming with impulse control programme, we introduce the value function $V_k^{\delta\pm}$. Here, as in the continuous-time formulation, our objective is to maximize the terminal wealth performance criteria given by

$$V_k^{\delta\pm}(x, s, \boldsymbol{z}, q) = \mathbb{E}\left[W_T^{\delta\pm}\right] = \mathbb{E}_{k,x,s,\boldsymbol{z},q}\left[X_T^{\delta\pm} + Q_T^{\delta\pm}(S_T - \text{sgn}(Q_T^{\delta\pm})\xi) - \alpha (Q_T^{\delta\pm})^2\right] \tag{3.58}$$

where, as before, the notation $\mathbb{E}_{k,x,s,\boldsymbol{z},q}[\,\cdot\,]$ represents the conditional expectation

$$\mathbb{E}[\,\cdot \mid X_k = x, S_k = s, \boldsymbol{Z}_k = \boldsymbol{z}, Q_k = q]$$

In this case, our dynamic programming equations (DPEs) are given by

$$V_T(x,s,\boldsymbol{z},q) = x + q(s - \operatorname{sgn}(q)\xi) - \alpha q^2 \tag{3.59}$$

$$\begin{aligned} V_k(x,s,\boldsymbol{z},q) = \max\Big\{ & \sup_{\delta^\pm}\big\{\mathbb{E}_{\mathbf{w}}\left[V_{k+1}(f((x,s,\boldsymbol{z},q),\boldsymbol{u},\mathbf{w}_k)\right]\big\} \; ; \\ & V_k(x + s_k - \xi, s_k, \boldsymbol{z}_k, q_k - 1) \; ; \\ & V_k(x - s_k - \xi, s_k, \boldsymbol{z}_k, q_k + 1)\Big\} \end{aligned} \tag{3.60}$$

where expectation is with respect to the random vector $\mathbf{w}_k$. Note that in this formulation we do not have per stage costs, as the cost of execution is bundled into the state x. Nevertheless, it is rather immediate that the execution costs could be disentangled from the system state and seen to satisfy the theorem assumptions. Hypothetically we could add the fourth case where $M^+ = M^- = 1$, though a quick substitution shows that it is always strictly 2ξ less in value than the case of only limit orders, where $M^+ = M^- = 0$. This should be evident, as buying and selling with market orders in a single timestep yields a guaranteed loss as the agent is forced to cross the spread.

To simplify the DPEs, we introduce a now familiar ansatz:

$$V_k(x,s,\boldsymbol{z},q) = x + q(s - \operatorname{sgn}(q)\xi) + h_k(\boldsymbol{z},q) \tag{3.61}$$

with boundary condition $h_k(\boldsymbol{z},0) = 0$ and terminal condition $h_T(\boldsymbol{z},q) = -\alpha q^2$. Substituting this ansatz into the Equation (3.60), we obtain

$$\begin{aligned} 0 = \max\Big\{ & \sup_{\delta^\pm}\big\{\mathbb{E}_{\mathbf{w}}\left[V_{k+1}(f((x,s,\boldsymbol{z},q),\boldsymbol{u},\mathbf{w}_k)\right] - V_k(x,s,\boldsymbol{z},q)\big\} \; ; \\ & V_k(x + s_k - \xi, s_k, \boldsymbol{z}_k, q_k - 1) - V_k(x,s,\boldsymbol{z},q) \; ; \\ & V_k(x - s_k - \xi, s_k, \boldsymbol{z}_k, q_k + 1) - V_k(x,s,\boldsymbol{z},q)\Big\} \end{aligned} \tag{3.62}$$

$$\begin{aligned} 0 = \max\Big\{ & \sup_{\delta^\pm}\Big\{\mathbb{E}_{\mathbf{w}}\Big[(s + \xi + \delta^-)L_k^- - (s - \xi - \delta^+)L_k^+ \\ & \quad + (L_k^+ - L_k^-)\big(s + \eta_{0,T(\boldsymbol{z},\omega)} - \operatorname{sgn}(q + L_k^+ - L_k^-)\xi\big) \\ & \quad + q\left(\eta_{0,T(\boldsymbol{z},\omega)} - \big(\operatorname{sgn}(q + L_k^+ - L_k^-) - \operatorname{sgn}(q)\big)\,\xi\right) \\ & \quad + h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^-) - h_k(\boldsymbol{z},q)\Big]\Big\} \; ; \\ & - 2\xi \cdot \mathbb{1}_{q \geq 0} + h_k(\boldsymbol{z}, q+1) \; ; \\ & - 2\xi \cdot \mathbb{1}_{q \leq 0} + h_k(\boldsymbol{z}, q-1)\Big\} \end{aligned} \tag{3.63}$$

We'll begin by concentrating on the first term in the quasi-variational inequality. Thus, we want to solve

$$\begin{aligned}
\sup_{\delta^\pm}\Bigg\{\mathbb{E}_{\mathbf{w}}\Bigg[&(s+\xi+\delta^-)L_k^- - (s-\xi-\delta^+)L_k^+ \\
&+ (L_k^+ - L_k^-)\big(s+\eta_{0,T(\boldsymbol{z},\omega)} - \operatorname{sgn}(q+L_k^+ - L_k^-)\xi\big) \\
&+ q\left(\eta_{0,T(\boldsymbol{z},\omega)} - \big(\operatorname{sgn}(q+L_k^+ - L_k^-) - \operatorname{sgn}(q)\big)\,\xi\right) \\
&+ h_{k+1}(T(\boldsymbol{z},\omega), q+L_k^+ - L_k^-) - h_k(\boldsymbol{z},q)\Bigg]\Bigg\}
\end{aligned} \tag{3.64}$$

As other agents' market orders as Poisson distributed, we have that $\mathbb{P}[K_k^+ = 0] = \frac{e^{-\mu^+(\boldsymbol{z})\Delta t}(\mu^+(\boldsymbol{z})\Delta t)^0}{0!} = e^{-\mu^+(\boldsymbol{z})\Delta t}$, and so the probability of seeing some positive number of market orders is

$$\mathbb{P}[K_k^+ > 0] = 1 - e^{-\mu^+(\boldsymbol{z})\Delta t} \tag{3.65}$$

Now we make the simplified assumption that the *aggregate* of the orders walks the limit order book to a depth of p_k, and if $p_k > \delta^-$, then our sell limit order is lifted. As in the continuous time section, we will assume that the probability of our order being lifted is $e^{-\kappa\delta^-}$. Thus we have the following preliminary results:

$$\mathbb{P}[L_k^- = 1 | K_k^+ > 0] = e^{-\kappa\delta^-} \tag{3.66}$$

$$\mathbb{P}[L_k^- = 0 | K_k^+ > 0] = 1 - e^{-\kappa\delta^-} \tag{3.67}$$

$$\begin{aligned}
\mathbb{E}[L_k^-] &= \mathbb{P}[L_k^- = 1 | K_k^+ > 0]\cdot\mathbb{P}[K_k^+ > 0] &(3.68)\\
&= (1 - e^{-\mu^+(\boldsymbol{z})\Delta t})e^{-\kappa\delta^-} &(3.69)
\end{aligned}$$

For ease of notation, we'll write the probability of the $L_k^- = 1$ event as $p(\delta^-)$. This gives us the additional results:

$$\mathbb{P}[L_k^- = 1] = p(\delta^-) = \mathbb{E}[L_k^-] \tag{3.70}$$

$$\mathbb{P}[L_k^- = 0] = 1 - p(\delta^-) \tag{3.71}$$

$$\partial_{\delta^-}\mathbb{P}[L_k^- = 1] = -\kappa p(\delta^-) \tag{3.72}$$

$$\partial_{\delta^-}\mathbb{P}[L_k^- = 0] = \kappa p(\delta^-) \tag{3.73}$$

Let's pre-compute some of the terms that we'll encounter in the supremum, namely the expectations of the random variables. To each we will assign an uppercase Greek letter as shorthand, as will be evident from the analysis.

$$\begin{aligned}
\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] &= \mathbb{P}[L_k^- = 1] \cdot \mathbb{P}[L_k^+ = 1] \cdot \operatorname{sgn}(q) \\
&\quad + \mathbb{P}[L_k^- = 1] \cdot \mathbb{P}[L_k^+ = 0] \cdot \operatorname{sgn}(q-1) \\
&\quad + \mathbb{P}[L_k^- = 0] \cdot \mathbb{P}[L_k^+ = 1] \cdot \operatorname{sgn}(q+1) \\
&\quad + \mathbb{P}[L_k^- = 0] \cdot \mathbb{P}[L_k^+ = 0] \cdot \operatorname{sgn}(q) \qquad (3.74)\\
&= p(\delta^-)p(\delta^+)\operatorname{sgn}(q) \\
&\quad + p(\delta^-)(1-p(\delta^+))\operatorname{sgn}(q-1) \\
&\quad + (1-p(\delta^-))p(\delta^+)\operatorname{sgn}(q+1) \\
&\quad + (1-p(\delta^-))(1-p(\delta^+))\operatorname{sgn}(q) \qquad (3.75)\\
&= \operatorname{sgn}(q)\big[1 - p(\delta^+) - p(\delta^-) + 2p(\delta^+)p(\delta^-)\big] \\
&\quad + \operatorname{sgn}(q-1)\big[p(\delta^-) - p(\delta^+)p(\delta^-)\big] \\
&\quad + \operatorname{sgn}(q+1)\big[p(\delta^+) - p(\delta^+)p(\delta^-)\big] \qquad (3.76)\\
&= \begin{cases} 1 & q \geq 2 \\ 1 - p(\delta^-)(1-p(\delta^+)) & q = 1 \\ p(\delta^+) - p(\delta^-) & q = 0 \\ -\big[1 - p(\delta^+)(1-p(\delta^-))\big] & q = -1 \\ -1 & q \leq -2 \end{cases} \qquad (3.77)\\
&= \Phi(q, \delta^+, \delta^-) \qquad (3.78)
\end{aligned}$$

Similarly:

$$\begin{aligned}
\mathbb{E}[L_k^+ \operatorname{sgn}(q + L_k^+ - L_k^-)] &= \mathbb{P}[L_k^- = 1] \cdot \mathbb{P}[L_k^+ = 1] \cdot \operatorname{sgn}(q) \\
&\quad + \mathbb{P}[L_k^- = 1] \cdot \mathbb{P}[L_k^+ = 0] \cdot 0 \operatorname{sgn}(q-1) \\
&\quad + \mathbb{P}[L_k^- = 0] \cdot \mathbb{P}[L_k^+ = 1] \cdot \operatorname{sgn}(q+1) \\
&\quad + \mathbb{P}[L_k^- = 0] \cdot \mathbb{P}[L_k^+ = 0] \cdot 0 \operatorname{sgn}(q) \qquad (3.79)\\
&= p(\delta^+)\big[p(\delta^-)\operatorname{sgn}(q) + (1 - p(\delta^-)\operatorname{sgn}(q+1)\big] \qquad (3.80)\\
&= p(\delta^+)\begin{cases} 1 & q \geq 2 \\ 1 & q = 1 \\ (1 - p(\delta^-)) & q = 0 \\ -p(\delta^-) & q = -1 \\ -1 & q \leq -2 \end{cases} \qquad (3.81)
\end{aligned}$$

$$= p(\delta^+)\Psi(q, \delta^-) \tag{3.82}$$

and

$$\mathbb{E}[L_k^- \operatorname{sgn}(q + L_k^+ - L_k^-)] = p(\delta^-)\big[p(\delta^+)\operatorname{sgn}(q) + (1 - p(\delta^+))\operatorname{sgn}(q-1)\big] \tag{3.83}$$

$$= p(\delta^-)\begin{cases} 1 & q \geq 2 \\ p(\delta^+) & q = 1 \\ -(1 - p(\delta^+)) & q = 0 \\ -1 & q = -1 \\ -1 & q \leq -2 \end{cases} \tag{3.84}$$

$$= p(\delta^-)\Upsilon(q, \delta^+) \tag{3.85}$$

We'll also require the partial derivatives of these expectations, which we can easily compute. Below we'll use the simplified notation Φ_+ to denote the function closely associated with the partial derivative of Φ with respect to δ^+.

$$\partial_{\delta^-}\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] = \partial_{\delta^-}\Phi(q, \delta^+, \delta^-) = \kappa p(\delta^-)\begin{cases} 0 & q \geq 2 \\ (1 - p(\delta^+)) & q = 1 \\ 1 & q = 0 \\ p(\delta^+) & q = -1 \\ 0 & q \leq -2 \end{cases} \tag{3.86}$$

$$= \kappa p(\delta^-)\Phi_-(q, \delta^+) \tag{3.87}$$

$$\partial_{\delta^+}\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] = \partial_{\delta^+}\Phi(q, \delta^+, \delta^-) = \kappa p(\delta^+)\begin{cases} 0 & q \geq 2 \\ -p(\delta^-) & q = 1 \\ -1 & q = 0 \\ -(1 - p(\delta^-)) & q = -1 \\ 0 & q \leq -2 \end{cases} \tag{3.88}$$

$$= \kappa p(\delta^+)\Phi_+(q, \delta^-) \tag{3.89}$$

$$\partial_{\delta^-}\mathbb{E}[L_k^+\,\mathrm{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^-}p(\delta^+)\Psi(q,\delta^-) = \kappa p(\delta^+)p(\delta^-)\begin{cases}0 & q\geq 2\\ 0 & q=1\\ 1 & q=0\\ 1 & q=-1\\ 0 & q\leq -2\end{cases} \tag{3.90}$$

$$= \kappa p(\delta^+)p(\delta^-)\Psi_-(q) \tag{3.91}$$

$$\partial_{\delta^+}\mathbb{E}[L_k^+\,\mathrm{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^+}p(\delta^+)\Psi(q,\delta^-) = -\kappa p(\delta^+)\Psi(q,\delta^-) \tag{3.92}$$

$$\partial_{\delta^-}\mathbb{E}[L_k^-\,\mathrm{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^-}p(\delta^-)\Upsilon(q,\delta^+) = -\kappa p(\delta^-)\Upsilon(q,\delta^+) \tag{3.93}$$

$$\partial_{\delta^+}\mathbb{E}[L_k^-\,\mathrm{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^+}p(\delta^-)\Upsilon(q,\delta^+) = \kappa p(\delta^+)p(\delta^-)\begin{cases}0 & q\geq 2\\ -1 & q=1\\ -1 & q=0\\ 0 & q=-1\\ 0 & q\leq -2\end{cases} \tag{3.94}$$

$$= \kappa p(\delta^+)p(\delta^-)\Upsilon_+(q) \tag{3.95}$$

Recalling that we have $\boldsymbol{P}$ the transition matrix for the Markov Chain $\boldsymbol{Z}$, with $\boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} = \mathbb{P}[\boldsymbol{Z}_{k+1} = \mathbf{j}|\boldsymbol{Z}_k = \boldsymbol{z}]$, then we can also write:

$$\begin{aligned}\mathbb{E}[h_{k+1}(T(\boldsymbol{z},\omega), q+L_k^+-L_k^-)] = \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[& h_{k+1}(\mathbf{j},q)\big[1-p(\delta^+)-p(\delta^-)+2p(\delta^+)p(\delta^-)\big]\\ & + h_{k+1}(\mathbf{j},q-1)\big[p(\delta^-)-p(\delta^+)p(\delta^-)\big]\\ & + h_{k+1}(\mathbf{j},q+1)\big[p(\delta^+)-p(\delta^+)p(\delta^-)\big]\Big]\end{aligned} \tag{3.96}$$

and its partial derivatives as

$$\begin{aligned}\partial_{\delta^-}\mathbb{E}[h_{k+1}(T(\boldsymbol{z},\omega), q+L_k^+-L_k^-)] = \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[& h_{k+1}(\mathbf{j},q)\big[\kappa p(\delta^-)-2\kappa p(\delta^+)p(\delta^-)\big]\\ & + h_{k+1}(\mathbf{j},q-1)\big[-\kappa p(\delta^-)+\kappa p(\delta^+)p(\delta^-)\big]\\ & + h_{k+1}(\mathbf{j},q+1)\big[\kappa p(\delta^+)p(\delta^-)\big]\Big]\end{aligned} \tag{3.97}$$

$$
\begin{aligned}
&= \kappa p(\delta^-) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q) \big[1 - 2p(\delta^+)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q-1) \big[-1 + p(\delta^+)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q+1) \big[p(\delta^+)\big] \Big]
\end{aligned} \tag{3.98}
$$

$$
\begin{aligned}
\partial_{\delta^+} \mathbb{E}[h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^-)] &= \kappa p(\delta^+) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q) \big[1 - 2p(\delta^-)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q-1) \big[p(\delta^-)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q+1) \big[-1 + p(\delta^-)\big] \Big]
\end{aligned} \tag{3.99}
$$

Now we tackle solving the supremum in equation 3.64 and thus finding the optimal posting depths, again denoted by a subscript asterisk. First we consider the first-order condition on δ^-, namely that the partial derivative with respect to it must be equal to zero.

$$
\begin{aligned}
0 = \partial_{\delta^-} \Big\{ &(s + \xi + \delta^{-*})\mathbb{E}[L_k^-] - (s - \xi - \delta^+)\mathbb{E}[L_k^+] \\
&+ \mathbb{E}[L_k^+] \left(s + \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}]\right) - \xi \mathbb{E}\left[L_k^+ \operatorname{sgn}(q + L_k^+ - L_k^-)\right] \\
&- \mathbb{E}[L_k^-] \left(s + \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}]\right) + \xi \mathbb{E}\left[L_k^- \operatorname{sgn}(q + L_k^+ - L_k^-)\right] \\
&+ q\mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}] - q\xi\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] + q\xi \operatorname{sgn}(q) \\
&+ \mathbb{E}\left[h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^-)\right] - h_k(\boldsymbol{z}, q) \Big\}
\end{aligned} \tag{3.100}
$$

$$
\begin{aligned}
= \partial_{\delta^-} \Big\{ &(s + \xi + \delta^{-*})\mathbb{E}[L_k^-] - \xi \mathbb{E}\left[L_k^+ \operatorname{sgn}(q + L_k^+ - L_k^-)\right] \\
&- \mathbb{E}[L_k^-] \left(s + \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}]\right) + \xi \mathbb{E}\left[L_k^- \operatorname{sgn}(q + L_k^+ - L_k^-)\right] \\
&- q\xi\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] + \mathbb{E}\left[h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^-)\right] \Big\}
\end{aligned} \tag{3.101}
$$

$$
\begin{aligned}
&= p(\delta^{-*}) - \kappa p(\delta^{-*})(s + \xi + \delta^{-*}) - \xi \kappa p(\delta^+) p(\delta^{-*}) \Psi_-(q) \\
&\quad + \kappa p(\delta^{-*}) \left(s + \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}]\right) - \xi \kappa p(\delta^{-*}) \Upsilon(q, \delta^+) - q\xi\kappa p(\delta^{-*}) \Phi_-(q, \delta^+) \\
&\quad + \kappa p(\delta^{-*}) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q) \big[1 - 2p(\delta^+)\big] + h_{k+1}(\mathbf{j}, q-1) \big[-1 + p(\delta^+)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q+1) \big[p(\delta^+)\big] \Big]
\end{aligned} \tag{3.102}
$$

Dividing through by $\kappa p(\delta^{-*})$, which is nonzero, and re-arranging, we find that the optimal sell posting depth is given by

$$\delta^{-*} = \frac{1}{\kappa} + \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}] - \xi\left(1 + p(\delta^+)\Psi_-(q) + \Upsilon(q,\delta^+) + q\Phi_-(q,\delta^+)\right)$$
$$+ \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q)\big[1-2p(\delta^+)\big] + h_{k+1}(\mathbf{j},q-1)\big[-1+p(\delta^+)\big] + h_{k+1}(\mathbf{j},q+1)\big[p(\delta^+)\big]\Big] \tag{3.103}$$

$$= \frac{1}{\kappa} + \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}] - 2\xi\left(\mathbb{1}_{q\geq 1} + p(\delta^+)\mathbb{1}_{q=0}\right)$$
$$+ \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q)\big[1-2p(\delta^+)\big] + h_{k+1}(\mathbf{j},q-1)\big[-1+p(\delta^+)\big] + h_{k+1}(\mathbf{j},q+1)\big[p(\delta^+)\big]\Big] \tag{3.104}$$

Recalling that we want $\delta^{\pm} \geq 0$, we find:

$$\begin{aligned}\delta^{-*} = \max\Bigg\{0\,;\ &\frac{1}{\kappa} + \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}] - 2\xi\mathbb{1}_{q\geq 1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\big[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q-1)\big] \\ &- p(\delta^+)\left(2\xi\mathbb{1}_{q=0} - \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\big[h_{k+1}(\mathbf{j},q-1) + h_{k+1}(\mathbf{j},q+1) - 2h_{k+1}(\mathbf{j},q)\big]\right)\Bigg\}\end{aligned} \tag{3.105}$$

And similarly, the optimal buy posting depth is given by:

$$\begin{aligned}\delta^{+*} = \max\Bigg\{0\,;\ &\frac{1}{\kappa} - \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}] - 2\xi\mathbb{1}_{q\leq -1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\big[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q+1)\big] \\ &- p(\delta^-)\left(2\xi\mathbb{1}_{q=0} - \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\big[h_{k+1}(\mathbf{j},q-1) + h_{k+1}(\mathbf{j},q+1) - 2h_{k+1}(\mathbf{j},q)\big]\right)\Bigg\}\end{aligned} \tag{3.106}$$

For ease of notation we'll write $\aleph(q) = \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\left[h_{k+1}(\mathbf{j},q-1) + h_{k+1}(\mathbf{j},q+1) - 2h_{k+1}(\mathbf{j},q)\right]$. Now, assuming we behave optimally on both the buy and sell sides simultaneously, we can substitute equation 3.106 into equation 3.105, while evaluating both at δ^{+*} and δ^{-*} to obtain the optimal posting depth in feedback form:

$$\begin{aligned}\delta^{-*} = &\frac{1}{\kappa} + \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}] - 2\xi\mathbb{1}_{q\geq 1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q-1)\Big] \\ &- (1 - e^{\mu^-(\boldsymbol{z})\Delta t})e^{-\kappa\max\left\{0\,;\ \frac{1}{\kappa} - \mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}] - 2\xi\mathbb{1}_{q\leq -1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\left[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q+1)\right] - (1-e^{\mu^+(\boldsymbol{z})\Delta t})e^{-\kappa\delta^{-*}}(2\xi\mathbb{1}_{q=0} - \aleph(q))\right\}}\left(2\xi\mathbb{1}_{q=0} - \aleph(q)\right)\end{aligned} \tag{3.107}$$

This equation will need to be solved numerically due to the difficulty in isolating δ^{-*} on one side of the equality. Once a solution has been obtained, the value can be substituted back into Equation (3.106) to solve for ${\delta^+}^*$.

3.2.3 Simplifying the DPE

We now turn to simplifying the DPE in Equation (3.63) by substituting in the optimal posting depths as written in recursive form: Equation (3.106) and Equation (3.105). In doing so we see a incredible amount of cancellation and simplification, and we obtain the rather elegant, and surprisingly simple form of the DPE:

$$
\begin{aligned}
h_k(\boldsymbol{z}, q) = \max\Bigg\{ & q\mathbb{E}[\eta_{0,T(\boldsymbol{z},\omega)}] + \frac{1}{\kappa}(p({\delta^+}^*) + p({\delta^-}^*)) + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} h_{k+1}(\mathbf{j}, q) \\
& \qquad + p({\delta^+}^*)p({\delta^-}^*) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \left[h_{k+1}(\mathbf{j}, q-1) + h_{k+1}(\mathbf{j}, q+1) - 2h_{k+1}(\mathbf{j}, q) \right] \; ; \\
& - 2\xi \cdot \mathbb{1}_{q \geq 0} + h_k(\boldsymbol{z}, q+1) \; ; \\
& - 2\xi \cdot \mathbb{1}_{q \leq 0} + h_k(\boldsymbol{z}, q-1) \Bigg\}
\end{aligned}
\tag{3.108}
$$

Anton: Similar commentary to the continuous case. Find the whole inequality thing, bounds on h, bounds on δ. Identical.

At terminal time T, we liquidate our position at a cost of $(s - xi\,\text{sgn}(q) - \alpha q)$ per share, whereas at $T-1$, we can liquidate at the regular cost of $(s - \xi\,\text{sgn}(q)$. It is thus never optimal to wait until maturity to liquidate the position, and instead we force liquidation one step earlier by setting $h(T-1, \boldsymbol{z}, q) = 0 \ \forall q$. This allows us to effectively ignore the terminal condition, and avoids a contradiction with the finding that $h \geq 0$.

We now have an explicit means of numerically solving for the optimal posting depths. Since we know the function h at the terminal timesteps T and $T-1$, we can take one step back to $T-2$ and solve for both the optimal posting depths. With these values we are then able to calculate the value function h_{T-2} using Equation (3.108), and in doing so determine whether to execute market orders in addition to posting limit orders. This process then repeats for each step backward.

Chapter 4

Results

Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Etiam lobortis facilisis sem. Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper.

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