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**High-Frequency Algorithmic Trading
with Momentum and Order Imbalance**

My goal is to establish and solve the stochastic optimal control problem that captures the momentum and order imbalance dynamics of the Limit Order Book (LOB). The solution will yield an optimal trading strategy that will permit statistical arbitrage of the underlying stock, which will then be backtested on historical data.

Progress Timeline

DATE	THESIS	STA4505
~~Dec 2014~~	~~Complete CTMC calibration~~	
~~Dec 2014~~	~~Backtest naive strategies based on CTMC~~	
~~Jan-May~~	~~Study stochastic controls: ECE1639, STA4505~~	
~~Jun 5~~	~~Establish models~~	~~Exam Study~~
~~Jun 12~~	~~Establish performance criteria~~	~~Exam Study~~
~~Jun 15~~	~~Derive DPP/DPE~~	~~EXAM~~
~~Jun 19~~	~~Derive DPP/DPE~~	
~~Jun 26~~	~~Derive continuous-time equations~~	
~~Jul 3~~	~~Derive discrete-time equations~~	
~~Jul 10~~	~~Set up MATLAB numerical integration~~	
Jul 17	Integrate functions and plot dynamics	~~Integrate and analyze~~
Jul 24	More dynamics, and calib/choose parameters	
Jul 31	Backtest on historical data	~~Simulate results~~
Aug 7	More backtesting, comparing with previous	
Aug 14	Dissertation writeup / buffer	Project writeup
Aug 21	Dissertation writeup / buffer	
Aug 28	Dissertation writeup	Presentation

Whiteboard Inspirational Quote of the Week

> *I'm sorry to say that the subject I most disliked was mathematics. I have thought about it. I think the reason was that mathematics leaves no room for argument. If you made a mistake, that was all there was to it.*
> - Malcolm X

For Our Readers in the Middle East...

snap crackle and pop. the sound of breakfast, the sound of success. actually every now and again my urine smells potently of honey nut cheerios and it really sends me right back to the good ol days, the days before I understood the first thing about nutrition, the days of cereal for breakfast and cereal after school and cereal before bed. a time of having the livejournal account cereal_time. things were simpler back then... purer...

well currently you're in the mountains of Oman in the horn of Africa, and if not THE horn then at least one of the horns, and regardless we can probably deduce you're having a real horny time. pretty spectacular that you find yourself at this moment in time, in life, in the fucking thick of the desert and arabs and islam. for the many years that you spent as a non-traveller, and really it's still remarkable that you have barely stepped foot in Europe, you've really taken the bull by the horns (of Africa) in recent years. sunny Bermuda, sunny Israel, sunny Vancouver... and now a place where the sun really doesn't stop shining, *but a place of darkness nonetheless.*

Anyway, T-6 days, the next TMW will find you on home turf and that's pretty unreal. i used to have this distinct feeling that time was flying, even accelerating in its flight, this was probably around the 4th and 5th years university, and it was like shit where is it going. anyway, i think we've reached terminal velocity (but not escape velocity! yay!), and i think what i'm really trying to express here is a certain difficult-to-peg contentedness at the passage of time. and anyway, to tie this paragraph together, i guess you really have spent a considerable amount of time with sandniggas.

okay so check one two, i had this thought the other day, what if i came down to Duke on a student visa (that i created in microsoft paint) and chilled hard with you for a little bit while working like crazy on getting this thesis together? like i'm looking at the progress timeline, from mid-August onward it's really disseration writing time, which is just going to be a heavy LaTeXtime and i know you're into LaTeXand leather. i dunno, presumably i'd crash on your bed and you could sleep on the floor, and we could have lots of beans for breakfast all the time. oh and i'd bring the ol Oster coffee maker obviously, it would just be constantly maintained by one of your roommates that we've bullied into submission / you secretly paid off and didn't tell me when the bullying wasn't working. anyway, i suppose a true motivating factor is that who knows when we'll get to chill again, cause there are so many question marks in my dry erase notebook timeline as well as your business admistration future.

last night i went to the Pan Am opening ceremonies dress rehearsal with my mom and met up with my friend Ashley there too. my mom had gotten a free ticket by virtue of every pan am volunteer receiving one, and had obtained another for me from a fellow volunteer, as the general consensus was that it was silly to give everyone just one if they wanted to go with, you know, a family member or friend or something, not just other volunteers. anyway. so we had nonadjacent seats up in the 500s, and after i took them to guest services we walked out with adjacent seats in the 100s, as well as having managed to get three of us in with only two tickets. big logistical success. but that's not what i'm here to talk about ... man, Cirque Du Soleil. they were basically the entire ceremony, aside from the politician preamble and the marching in of the athletes. at first, prior to the march-in, they did a tiny little number featuring a native american hoop dancer and a fuck ton of dancers that really just seemed like one hoop dancer in the mist of an enormous rave with rave music going, it was strange and anticlimactic but obviously still good in its own right. but then came the actual Cirque show after. interestingly I haven't really seen much acrobatics since i began the whole gymnastics thing, and it's really Cirque that sparked my desire to get into it in the first place. i did see their production Kurios in Sept, but it really greatly underplayed acrobatics and was more like, a production, with cool sets and dancing and stuff, but not the bread and butter acrobatics. anyway, man, i'm sitting there at the ceremony, watching the Cirque show really begin...and never more have I just burned to be part of that. it's just so phantasmal, so mystical, defying all reality and norms, full of the most magical characters, where nothing but impossible feats exist and pure simple powerful love stories thrive, a world that just beckons but of course doesn't exist. I don't know. above all I could just watch and watch circus and every moment pine to take the plunge in. nothing, I don't think, has the same capacity to enthral, and draw in. yeah. in a fucking heartbeat i'd be there.

The Academic Week in Review

Recall we are trying to solve the first case of the DPE, namely:

$$\begin{aligned}\sup_{\delta^\pm}\Bigg\{\mathbb{E}_{\mathbf{w}}\Bigg[&(s+\pi+\delta^-)L_k^- - (s-\pi-\delta^+)L_k^+ \\ &+ (L_k^+ - L_k^-)\big(s+\eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)} - \operatorname{sgn}(q+L_k^+-L_k^-)\pi\big) \\ &+ q\left(\eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)} - \big(\operatorname{sgn}(q+L_k^+-L_k^-) - \operatorname{sgn}(q)\big)\,\pi\right) \\ &+ h_{k+1}(T(\boldsymbol{z},\omega), q+L_k^+-L_k^-) - h_k(\boldsymbol{z},q)\Bigg]\Bigg\}\end{aligned} \tag{1}$$

You might recall that things got pretty messy pretty fast. Previously we had set up the problem such that at each timestep k there can be multiple other agents' market orders (K_k^+) arriving, and these were Poisson distributed. For each arriving order, the probability of our posted limit order being filled was $e^{-\kappa\delta^-}$. We're going to modify this slightly. Keeping the market orders as Poisson distributed, we have that $\mathbb{P}[K_k^+ = 0] = \frac{e^{-\lambda(\boldsymbol{z})\Delta t}(\lambda(\boldsymbol{z})\Delta t)^0}{0!} = e^{-\lambda(\boldsymbol{z})\Delta t}$, and so the probability of seeing some positive number of market orders is

$$\mathbb{P}[K_k^+ > 0] = 1 - e^{-\lambda(\boldsymbol{z})\Delta t} \tag{2}$$

Now we make the simplified assumption that the *aggregate* of the orders walks the limit order book to a depth of p_k, and if $p_k > \delta^-$, then our sell limit order is lifted. Thus we have the following preliminary results:

$$
\begin{aligned}
\mathbb{P}[L_k^- = 1 | K_k^+ > 0] &= e^{-\kappa\delta^-} \\
\mathbb{P}[L_k^- = 0 | K_k^+ > 0] &= 1 - e^{-\kappa\delta^-} \\
\mathbb{E}[L_k^-] &= \mathbb{P}[L_k^- = 1 | K_k^+ > 0] \cdot \mathbb{P}[K_k^+ > 0] \\
&= (1 - e^{-\lambda(\boldsymbol{z})\Delta t}) e^{-\kappa\delta^-}
\end{aligned}
$$

For ease of notation, we'll write the probability of the $L_k^- = 1$ event as $p(\delta^-)$. This gives us the additional results:

$$
\begin{aligned}
\mathbb{P}[L_k^- = 1] &= p(\delta^-) = \mathbb{E}[L_k^-] \\
\mathbb{P}[L_k^- = 0] &= 1 - p(\delta^-) \\
\partial_{\delta^-}\mathbb{P}[L_k^- = 1] &= -\kappa p(\delta^-) \\
\partial_{\delta^-}\mathbb{P}[L_k^- = 0] &= \kappa p(\delta^-)
\end{aligned}
$$

Contributing to last week's frustration, a la our man Malcolm X, was an error in calculation. I had mistakenly calculated $\mathbb{E}[L_k^- \operatorname{sgn}(q + L_k^+ - L_k^-)] = \mathbb{E}[L_k^-]\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)]$, as if to say they are independent, but of course they are not. This is fixed below. On the contrary, $\mathbb{E}[L_k^- \eta_{0,\boldsymbol{z}} T(\boldsymbol{z}, \omega)^{(2)}]$ was correctly treated as the product of independent random variables. We have that $\mathbb{E}[L_k^-] = (1 - e^{-\lambda(\boldsymbol{z})\Delta t})e^{-\kappa\delta^-}$ is clearly dependent on $\boldsymbol{z}$, but these expectations are over the vector of random variables $\mathbf{w} = (K^+, K^-, \omega)$ and are evaluated at a given point $\boldsymbol{z}$.

Let's pre-compute some of the terms that we'll encounter in the supremum, namely the expectations of the random variables.

$$
\begin{aligned}
\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] &= \mathbb{P}[L_k^- = 1] \cdot \mathbb{P}[L_k^+ = 1] \cdot \operatorname{sgn}(q) \\
&\quad + \mathbb{P}[L_k^- = 1] \cdot \mathbb{P}[L_k^+ = 0] \cdot \operatorname{sgn}(q - 1) \\
&\quad + \mathbb{P}[L_k^- = 0] \cdot \mathbb{P}[L_k^+ = 1] \cdot \operatorname{sgn}(q + 1) \\
&\quad + \mathbb{P}[L_k^- = 0] \cdot \mathbb{P}[L_k^+ = 0] \cdot \operatorname{sgn}(q) \\
&= p(\delta^-)p(\delta^+) \operatorname{sgn}(q) \\
&\quad + p(\delta^-)(1 - p(\delta^+)) \operatorname{sgn}(q - 1) \\
&\quad + (1 - p(\delta^-))p(\delta^+) \operatorname{sgn}(q + 1) \\
&\quad + (1 - p(\delta^-))(1 - p(\delta^+)) \operatorname{sgn}(q) \\
&= \operatorname{sgn}(q) \big[1 - p(\delta^+) - p(\delta^-) + 2p(\delta^+)p(\delta^-)\big] \\
&\quad + \operatorname{sgn}(q - 1) \big[p(\delta^-) - p(\delta^+)p(\delta^-)\big] \\
&\quad + \operatorname{sgn}(q + 1) \big[p(\delta^+) - p(\delta^+)p(\delta^-)\big]
\end{aligned}
$$

$$= \begin{cases} 1 & q \geq 2 \\ 1 - p(\delta^-)(1 - p(\delta^+)) & q = 1 \\ p(\delta^+) - p(\delta^-) & q = 0 \\ -\big[1 - p(\delta^+)(1 - p(\delta^-))\big] & q = -1 \\ -1 & q \leq -2 \end{cases} \tag{3}$$

$$= \Phi(q, \delta^+, \delta^-) \tag{4}$$

Similarly:

$$\begin{aligned} \mathbb{E}[L_k^+ \operatorname{sgn}(q + L_k^+ - L_k^-)] &= \mathbb{P}[L_k^- = 1] \cdot \mathbb{P}[L_k^+ = 1] \cdot \operatorname{sgn}(q) \\ &\quad + \mathbb{P}[L_k^- = 1] \cdot \mathbb{P}[L_k^+ = 0] \cdot 0 \operatorname{sgn}(q - 1) \\ &\quad + \mathbb{P}[L_k^- = 0] \cdot \mathbb{P}[L_k^+ = 1] \cdot \operatorname{sgn}(q + 1) \\ &\quad + \mathbb{P}[L_k^- = 0] \cdot \mathbb{P}[L_k^+ = 0] \cdot 0 \operatorname{sgn}(q) \\ &= p(\delta^+)\big[p(\delta^-) \operatorname{sgn}(q) + (1 - p(\delta^-) \operatorname{sgn}(q + 1)\big] \end{aligned}$$

$$= p(\delta^+) \begin{cases} 1 & q \geq 2 \\ 1 & q = 1 \\ (1 - p(\delta^-)) & q = 0 \\ -p(\delta^-) & q = -1 \\ -1 & q \leq -2 \end{cases} \tag{5}$$

$$= p(\delta^+)\Psi(q, \delta^-) \tag{6}$$

and

$$\mathbb{E}[L_k^- \operatorname{sgn}(q + L_k^+ - L_k^-)] = p(\delta^-)\big[p(\delta^+) \operatorname{sgn}(q) + (1 - p(\delta^+)) \operatorname{sgn}(q - 1)\big]$$

$$= p(\delta^-) \begin{cases} 1 & q \geq 2 \\ p(\delta^+) & q = 1 \\ -(1 - p(\delta^+)) & q = 0 \\ -1 & q = -1 \\ -1 & q \leq -2 \end{cases} \tag{7}$$

$$= p(\delta^-)\Upsilon(q, \delta^+) \tag{8}$$

We'll also require the partial derivatives of these expectations, which we can easily compute. Below we'll use the simplified notation Φ_+ to denote the function closely associated with the partial derivative of Φ with respect to δ^+.

$$\partial_{\delta^-} \mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] = \partial_{\delta^-} \Phi(q, \delta^+, \delta^-) = \kappa p(\delta^-) \begin{cases} 0 & q \geq 2 \\ (1 - p(\delta^+)) & q = 1 \\ 1 & q = 0 \\ p(\delta^+) & q = -1 \\ 0 & q \leq -2 \end{cases} \tag{9}$$

$$= \kappa p(\delta^-)\Phi_-(q,\delta^+) \tag{10}$$

$$\partial_{\delta^+}\mathbb{E}[\operatorname{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^+}\Phi(q,\delta^+,\delta^-) = \kappa p(\delta^+)\begin{cases} 0 & q\geq 2 \\ -p(\delta^-) & q=1 \\ -1 & q=0 \\ -(1-p(\delta^-)) & q=-1 \\ 0 & q\leq -2 \end{cases} \tag{11}$$

$$= \kappa p(\delta^+)\Phi_+(q,\delta^-) \tag{12}$$

$$\partial_{\delta^-}\mathbb{E}[L_k^+\operatorname{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^-}p(\delta^+)\Psi(q,\delta^-) = \kappa p(\delta^+)p(\delta^-)\begin{cases} 0 & q\geq 2 \\ 0 & q=1 \\ 1 & q=0 \\ 1 & q=-1 \\ 0 & q\leq -2 \end{cases} \tag{13}$$

$$= \kappa p(\delta^+)p(\delta^-)\Psi_-(q) \tag{14}$$

$$\partial_{\delta^+}\mathbb{E}[L_k^+\operatorname{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^+}p(\delta^+)\Psi(q,\delta^-) = -\kappa p(\delta^+)\Psi(q,\delta^-) \tag{15}$$

$$\partial_{\delta^-}\mathbb{E}[L_k^-\operatorname{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^-}p(\delta^-)\Upsilon(q,\delta^+) = -\kappa p(\delta^-)\Upsilon(q,\delta^+) \tag{16}$$

$$\partial_{\delta^+}\mathbb{E}[L_k^-\operatorname{sgn}(q+L_k^+-L_k^-)] = \partial_{\delta^+}p(\delta^-)\Upsilon(q,\delta^+) = \kappa p(\delta^+)p(\delta^-)\begin{cases} 0 & q\geq 2 \\ -1 & q=1 \\ -1 & q=0 \\ 0 & q=-1 \\ 0 & q\leq -2 \end{cases} \tag{17}$$

$$= \kappa p(\delta^+)p(\delta^-)\Upsilon_+(q) \tag{18}$$

Recalling that we have $\boldsymbol{P}$ the transition matrix for the Markov Chain $\boldsymbol{Z}$, with $\boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} = \mathbb{P}[\boldsymbol{Z}_{k+1} = \mathbf{j}|\boldsymbol{Z}_k = \boldsymbol{z}]$, then we can also write:

$$\begin{aligned}\mathbb{E}[h_{k+1}(T(\boldsymbol{z},\omega),q+L_k^+-L_k^-)] = \sum_{\mathbf{j}}\boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[&h_{k+1}(\mathbf{j},q)\big[1-p(\delta^+)-p(\delta^-)+2p(\delta^+)p(\delta^-)\big] \\ &+h_{k+1}(\mathbf{j},q-1)\big[p(\delta^-)-p(\delta^+)p(\delta^-)\big] \\ &+h_{k+1}(\mathbf{j},q+1)\big[p(\delta^+)-p(\delta^+)p(\delta^-)\big]\Big]\end{aligned} \tag{19}$$

and its partial derivatives as

$$\begin{aligned}\partial_{\delta^-}\mathbb{E}[h_{k+1}(T(\boldsymbol{z},\omega),q+L_k^+-L_k^-)] = \sum_{\mathbf{j}}\boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[&h_{k+1}(\mathbf{j},q)\big[\kappa p(\delta^-)-2\kappa p(\delta^+)p(\delta^-)\big] \\ &+h_{k+1}(\mathbf{j},q-1)\big[-\kappa p(\delta^-)+\kappa p(\delta^+)p(\delta^-)\big] \\ &+h_{k+1}(\mathbf{j},q+1)\big[\kappa p(\delta^+)p(\delta^-)\big]\Big]\end{aligned} \tag{20}$$

$$
\begin{aligned}
&= \kappa p(\delta^-) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q)\big[1 - 2p(\delta^+)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q-1)\big[-1 + p(\delta^+)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q+1)\big[p(\delta^+)\big] \Big]
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\partial_{\delta^+}\mathbb{E}[h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^-)] &= \kappa p(\delta^+) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q)\big[1 - 2p(\delta^-)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q-1)\big[p(\delta^-)\big] \\
&\qquad + h_{k+1}(\mathbf{j}, q+1)\big[-1 + p(\delta^-)\big] \Big]
\end{aligned} \tag{22}
$$

Now we tackle solving the supremum in equation 1. First we consider the first-order condition on δ^-, namely that the partial derivative with respect to it must be equal to zero.

$$
\begin{aligned}
0 = \partial_{\delta^-} \Big\{ &(s + \pi + \delta^-)\mathbb{E}[L_k^-] - (s - \pi - \delta^+)\mathbb{E}[L_k^+] \\
&+ \mathbb{E}[L_k^+]\left(s + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}]\right) - \pi\mathbb{E}\left[L_k^+ \operatorname{sgn}(q + L_k^+ - L_k^-)\right] \\
&- \mathbb{E}[L_k^-]\left(s + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}]\right) + \pi\mathbb{E}\left[L_k^- \operatorname{sgn}(q + L_k^+ - L_k^-)\right] \\
&+ q\mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - q\pi\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] + q\pi \operatorname{sgn}(q) \\
&+ \mathbb{E}\left[h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^-)\right] - h_k(\boldsymbol{z}, q) \Big\}
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
= \partial_{\delta^-} \Big\{ &(s + \pi + \delta^-)\mathbb{E}[L_k^-] - \pi\mathbb{E}\left[L_k^+ \operatorname{sgn}(q + L_k^+ - L_k^-)\right] \\
&- \mathbb{E}[L_k^-]\left(s + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}]\right) + \pi\mathbb{E}\left[L_k^- \operatorname{sgn}(q + L_k^+ - L_k^-)\right] \\
&- q\pi\mathbb{E}[\operatorname{sgn}(q + L_k^+ - L_k^-)] + \mathbb{E}\left[h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^-)\right] \Big\}
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
&= p(\delta^-) - \kappa p(\delta^-)(s + \pi + \delta^-) - \pi\kappa p(\delta^+)p(\delta^-)\Psi_-(q) \\
&\quad + \kappa p(\delta^-)\left(s + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}]\right) - \pi\kappa p(\delta^-)\Upsilon(q, \delta^+) - q\pi\kappa p(\delta^-)\Phi_-(q, \delta^+) \\
&\quad + \kappa p(\delta^-) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q)\big[1 - 2p(\delta^+)\big] + h_{k+1}(\mathbf{j}, q-1)\big[-1 + p(\delta^+)\big] \\
&\qquad\qquad + h_{k+1}(\mathbf{j}, q+1)\big[p(\delta^+)\big] \Big]
\end{aligned} \tag{25}
$$

Dividing through by $\kappa p(\delta^-)$, which is nonzero, and re-arranging, we find that the optimal sell posting depth is given by

$$\delta^{-*} = \frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - \pi\left(1 + p(\delta^+)\Psi_-(q) + \Upsilon(q,\delta^+) + q\Phi_-(q,\delta^+)\right) \\ + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q)\big[1-2p(\delta^+)\big] + h_{k+1}(\mathbf{j},q-1)\big[-1+p(\delta^+)\big] + h_{k+1}(\mathbf{j},q+1)\big[p(\delta^+)\big]\Big] \tag{26}$$

$$= \frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\left(\mathbb{1}_{q\geq 1} + p(\delta^+)\mathbb{1}_{q=0}\right) \\ + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q)\big[1-2p(\delta^+)\big] + h_{k+1}(\mathbf{j},q-1)\big[-1+p(\delta^+)\big] + h_{k+1}(\mathbf{j},q+1)\big[p(\delta^+)\big]\Big] \tag{27}$$

$$\delta^{-*} = \frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\geq 1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q-1)\Big] \\ - p(\delta^+)\left(2\pi\mathbb{1}_{q=0} - \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q-1) + h_{k+1}(\mathbf{j},q+1) - 2h_{k+1}(\mathbf{j},q)\Big]\right) \tag{28}$$

And similarly, for the optimal buy posting depth:

$$\delta^{+*} = \frac{1}{\kappa} - \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\leq -1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q+1)\Big] \\ - p(\delta^-)\left(2\pi\mathbb{1}_{q=0} - \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q-1) + h_{k+1}(\mathbf{j},q+1) - 2h_{k+1}(\mathbf{j},q)\Big]\right) \tag{29}$$

For ease of notation we'll write $\aleph = \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\left[h_{k+1}(\mathbf{j},q-1) + h_{k+1}(\mathbf{j},q+1) - 2h_{k+1}(\mathbf{j},q)\right]$. Now, assuming we behave optimally on both the buy and sell sides simultaneously, we can substitute equation 29 into equation 28, and vice versa, while evaluating both at δ^{+*} and δ^{-*} to obtain the optimal posting depths in feedback form:

$$\boxed{\begin{aligned}\delta^{-*} &= \frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\geq 1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q-1)\Big] \\ &\quad - (1-e^{\lambda(\boldsymbol{z})\Delta t})e^{-\kappa\left(\frac{1}{\kappa} - \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\leq -1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\left[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q+1)\right]\right)} \\ &\quad \times e^{\kappa(1-e^{\lambda(\boldsymbol{z})\Delta t})e^{-\kappa\delta^{-*}}(2\pi\mathbb{1}_{q=0}-\aleph)}\left(2\pi\mathbb{1}_{q=0} - \aleph\right)\end{aligned}} \tag{30}$$

$$\boxed{\begin{aligned}\delta^{+*} &= \frac{1}{\kappa} - \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\leq -1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q+1)\Big] \\ &\quad - (1-e^{\lambda(\boldsymbol{z})\Delta t})e^{-\kappa\left(\frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\geq 1} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\left[h_{k+1}(\mathbf{j},q) - h_{k+1}(\mathbf{j},q-1)\right]\right)} \\ &\quad \times e^{\kappa(1-e^{\lambda(\boldsymbol{z})\Delta t})e^{-\kappa\delta^{+*}}(2\pi\mathbb{1}_{q=0}-\aleph)}\left(2\pi\mathbb{1}_{q=0} - \aleph\right)\end{aligned}} \tag{31}$$

DPE Cases 2 and 3

We have derived the optimal LO posting depths for the first case of the DPE, where no market orders are placed. The analysis of the other two cases, where $M_k^+ = 1$ and $M_k^- = 1$, respectively, proceeds almost identically. Looking first at the case $M_k^+ = 1$, we try to solve:

$$\begin{aligned}\sup_{\delta^\pm}\Bigg\{ \mathbb{E}_{\mathbf{w}}\Bigg[&(s+\pi+\delta^-)L_k^- - (s-\pi-\delta^+)L_k^+ - (s+\pi) \\ &+ (L_k^+ - L_k^- + 1)\big(s + \eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)} - \operatorname{sgn}(q + L_k^+ - L_k^- + 1)\pi\big) \\ &+ q\left(\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)} - \big(\operatorname{sgn}(q+L_k^+ - L_k^- + 1) - \operatorname{sgn}(q)\big)\pi\right) \\ &+ h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^- + 1) - h_k(\boldsymbol{z}, q)\Bigg]\Bigg\}\ ;\end{aligned} \tag{32}$$

We find that the functions Φ, Ψ, Υ are identical but evaluated at $q+1$. Thus, for example, we have:

$$\Phi(q+1,\delta^+,\delta^-) = \begin{cases} 1 & q \geq 1 \\ 1 - p(\delta^-)(1-p(\delta^+)) & q = 0 \\ p(\delta^+) - p(\delta^-) & q = -1 \\ -\big[1 - p(\delta^+)(1-p(\delta^-))\big] & q = -2 \\ -1 & q \leq -3 \end{cases} \tag{33}$$

In a similar fashion to the previous analysis, we obtain the posting depths for the second case (indicated with a subscript 2)

$$\begin{aligned}\delta_2^{-*} = &\frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi \mathbb{1}_{q\geq 0} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j}, q+1) - h_{k+1}(\mathbf{j}, q)\Big] \\ &- p(\delta^+)\left(2\pi\mathbb{1}_{q=-1} - \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q) + h_{k+1}(\mathbf{j},q+2) - 2h_{k+1}(\mathbf{j},q+1)\Big]\right)\end{aligned} \tag{34}$$

$$\begin{aligned}\delta_2^{+*} = &\frac{1}{\kappa} - \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi \mathbb{1}_{q\leq -2} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j}, q+1) - h_{k+1}(\mathbf{j}, q+2)\Big] \\ &- p(\delta^-)\left(2\pi\mathbb{1}_{q=-1} - \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j},q) + h_{k+1}(\mathbf{j},q+2) - 2h_{k+1}(\mathbf{j},q+1)\Big]\right)\end{aligned} \tag{35}$$

Again, substitution equations 34 and 35 into one another and evaluating both at the optimal posting depths, we obtain

$$\boxed{\begin{aligned}\delta_2^{-*} = &\frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi \mathbb{1}_{q\geq 0} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\Big[h_{k+1}(\mathbf{j}, q+1) - h_{k+1}(\mathbf{j}, q)\Big] \\ &- (1 - e^{\lambda(\boldsymbol{z})\Delta t})e^{-\kappa\left(\frac{1}{\kappa} - \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\leq -2} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}}\left[h_{k+1}(\mathbf{j},q+1) - h_{k+1}(\mathbf{j},q+2)\right]\right)} \\ &\times e^{\kappa(1-e^{\lambda(\boldsymbol{z})\Delta t})e^{-\kappa\delta_2^{-*}}\left(2\pi\mathbb{1}_{q=-1} - \aleph_{(+1)}\right)}\left(2\pi\mathbb{1}_{q=-1} - \aleph_{(+1)}\right)\end{aligned}} \tag{36}$$

$$
\boxed{\begin{aligned}\delta_2^{+*} &= \frac{1}{\kappa} - \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\leq -2} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q+1) - h_{k+1}(\mathbf{j}, q+2)\Big] \\ &\quad - (1-e^{\lambda(\boldsymbol{z})\Delta t}) e^{-\kappa\left(\frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\geq 0} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \left[h_{k+1}(\mathbf{j},q+1) - h_{k+1}(\mathbf{j},q)\right]\right)} \\ &\qquad \times e^{\kappa(1-e^{\lambda(\boldsymbol{z})\Delta t}) e^{-\kappa \delta_2^{+*}} \left(2\pi\mathbb{1}_{q=-1} - \aleph_{(+1)}\right)} \left(2\pi\mathbb{1}_{q=-1} - \aleph_{(+1)}\right)\end{aligned}} \tag{37}
$$

Likewise, in the third DPE case where $M_k^+ = 0, M_k^- = 1$, we get

$$
\boxed{\begin{aligned}\delta_3^{-*} &= \frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\geq 2} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q-1) - h_{k+1}(\mathbf{j}, q-2)\Big] \\ &\quad - (1-e^{\lambda(\boldsymbol{z})\Delta t}) e^{-\kappa\left(\frac{1}{\kappa} - \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\leq 0} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \left[h_{k+1}(\mathbf{j},q-1) - h_{k+1}(\mathbf{j},q)\right]\right)} \\ &\qquad \times e^{\kappa(1-e^{\lambda(\boldsymbol{z})\Delta t}) e^{-\kappa \delta_3^{-*}} \left(2\pi\mathbb{1}_{q=1} - \aleph_{(-1)}\right)} \left(2\pi\mathbb{1}_{q=1} - \aleph_{(-1)}\right)\end{aligned}} \tag{38}
$$

$$
\boxed{\begin{aligned}\delta_3^{+*} &= \frac{1}{\kappa} - \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\leq 0} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \Big[h_{k+1}(\mathbf{j}, q-1) - h_{k+1}(\mathbf{j}, q)\Big] \\ &\quad - (1-e^{\lambda(\boldsymbol{z})\Delta t}) e^{-\kappa\left(\frac{1}{\kappa} + \mathbb{E}[\eta_{0,\boldsymbol{z}} T(\boldsymbol{z},\omega)^{(2)}] - 2\pi\mathbb{1}_{q\geq 2} + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \left[h_{k+1}(\mathbf{j},q-1) - h_{k+1}(\mathbf{j},q-2)\right]\right)} \\ &\qquad \times e^{\kappa(1-e^{\lambda(\boldsymbol{z})\Delta t}) e^{-\kappa \delta_3^{+*}} \left(2\pi\mathbb{1}_{q=1} - \aleph_{(-1)}\right)} \left(2\pi\mathbb{1}_{q=1} - \aleph_{(-1)}\right)\end{aligned}} \tag{39}
$$

The boxed equations for optimal depth in feedback form will need to be solved numerically due to the difficulty in isolating $\delta^{\pm *}$ on one side of the equality.

Simplifying the DPE

For reference, we repeat here the DPE we are attempting to solve:

$$
\begin{aligned}
0 = \max\Bigg\{ & \sup_{\delta^\pm}\Bigg\{\mathbb{E}_{\mathbf{w}}\Bigg[(s+\pi+\delta^-)L_k^- - (s-\pi-\delta^+)L_k^+ \\
&\quad + (L_k^+ - L_k^-)\big(s + \eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)} - \operatorname{sgn}(q+L_k^+ - L_k^-)\pi\big) \\
&\quad + q\left(\eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)} - \big(\operatorname{sgn}(q+L_k^+ - L_k^-) - \operatorname{sgn}(q)\big)\,\pi\right) \\
&\quad + h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^-) - h_k(\boldsymbol{z},q)\Bigg]\Bigg\}\ ; \\
& \sup_{\delta^\pm}\Bigg\{\mathbb{E}_{\mathbf{w}}\Bigg[(s+\pi+\delta^-)L_k^- - (s-\pi-\delta^+)L_k^+ - (s+\pi) \\
&\quad + (L_k^+ - L_k^- + 1)\big(s + \eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)} - \operatorname{sgn}(q+L_k^+ - L_k^- + 1)\pi\big) \\
&\quad + q\left(\eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)} - \big(\operatorname{sgn}(q+L_k^+ - L_k^- + 1) - \operatorname{sgn}(q)\big)\,\pi\right) \\
&\quad + h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^- + 1) - h_k(\boldsymbol{z},q)\Bigg]\Bigg\}\ ; \\
& \sup_{\delta^\pm}\Bigg\{\mathbb{E}_{\mathbf{w}}\Bigg[(s+\pi+\delta^-)L_k^- - (s-\pi-\delta^+)L_k^+ + (s-\pi) \\
&\quad + (L_k^+ - L_k^- - 1)\big(s + \eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)} - \operatorname{sgn}(q+L_k^+ - L_k^- - 1)\pi\big) \\
&\quad + q\left(\eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)} - \big(\operatorname{sgn}(q+L_k^+ - L_k^- - 1) - \operatorname{sgn}(q)\big)\,\pi\right) \\
&\quad + h_{k+1}(T(\boldsymbol{z},\omega), q + L_k^+ - L_k^- - 1) - h_k(\boldsymbol{z},q)\Bigg]\Bigg\}\Bigg\}
\end{aligned}
\tag{40}
$$

We now turn to simplifying our DPE by substituting in the optimal posting depths as written in recursive form (e.g. 29 and 28). In doing so we see a incredible amount of cancellation and simplification, and we obtain the rather elegant, and surprisingly simple form of the DPE:

$$
h_k(\boldsymbol{z},q) = \max\left\{ \begin{aligned} & q\mathbb{E}[\eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)}] + \frac{1}{\kappa}(p(\delta^{+*}) + p(\delta^{-*})) - 2\pi p(\delta^{+*})p(\delta^{-*})\mathbb{1}_{q=0} \\ & \quad + p(\delta^{+*})p(\delta^{-*}) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \left[h_{k+1}(\mathbf{j},q-1) + h_{k+1}(\mathbf{j},q+1) - 2h_{k+1}(\mathbf{j},q) \right] \\ & \quad + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} h_{k+1}(\mathbf{j},q) \; ; \\ & (q+1)\mathbb{E}[\eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)}] + \frac{1}{\kappa}(p(\delta_2^{+*}) + p(\delta_2^{-*})) - 2\pi p(\delta_2^{+*})p(\delta_2^{-*})\mathbb{1}_{q=-1} \\ & \quad + p(\delta_2^{+*})p(\delta_2^{-*}) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \left[h_{k+1}(\mathbf{j},q) + h_{k+1}(\mathbf{j},q+2) - 2h_{k+1}(\mathbf{j},q+1) \right] \\ & \quad + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} h_{k+1}(\mathbf{j},q+1) \; ; \\ & (q-1)\mathbb{E}[\eta_{0,\boldsymbol{z}}T(\boldsymbol{z},\omega)^{(2)}] + \frac{1}{\kappa}(p(\delta_3^{+*}) + p(\delta_3^{-*})) - 2\pi p(\delta_3^{+*})p(\delta_3^{-*})\mathbb{1}_{q=1} \\ & \quad + p(\delta_3^{+*})p(\delta_3^{-*}) \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} \left[h_{k+1}(\mathbf{j},q-2) + h_{k+1}(\mathbf{j},q) - 2h_{k+1}(\mathbf{j},q-1) \right] \\ & \quad + \sum_{\mathbf{j}} \boldsymbol{P}_{\boldsymbol{z},\mathbf{j}} h_{k+1}(\mathbf{j},q-1) \end{aligned} \right\} \tag{41}
$$

TODO: commentary on this final form.

We now have an explicit means of numerically solving for the optimal posting depths. Since we know the function h at the terminal timestep T, we can take one step back to $T-1$ and solve for each of the optimal posting depths in each of the cases. With these values we are then able to calculate the value function h_{T-1}, and in doing so determine whether to execute market orders in addition to posting limit orders (essentially taking the arg max instead of the max in equation 41). This process then repeats for each step backward.

Looking Ahead

Holy fucking shamwow. Well... I mean I don't know whether this is obvious, but I hope that you can see ... we're absolutely flying here. I wish you could see all the fucking scrap note calculations I had to do over the course of this analysis, it's absolutely nuts, especially when insanely fucked up functions simplify to like $2\pi \cdot \mathbb{1}_{q=0}$. It's nuts and brings a smile to my face each time. Okay so equally so, we're a few weeks ahead with the course project. Really this is because I used the numerical integration/simulation as a proof of concept to make sure I was doing the right thing, as I had results to cross-validate with. And, thumbs up, we sure fucking are doing the right thing! So ultimately, with the course project, I can either just end it here and do the write up, or there're a few possible 'extensions' Sebastian and I had talked about doing, which I've done zero work on. I'm inclined to completely shelf it for now and see how the thesis work goes, pick it back up if time allows. Or if I get my exam mark back and I failed and really need the high marks. That being said, if I *do* do the extensions, it'll make considerably more sense to bundle the course project straight into the dissertation ... okay fuck it, this is all just talk, let's shelf it and see how things go. But yeah, a la the timeline and falling behind last week ... I'm back!