Algorithmic Trading as a Markov Decision Problem

Institute for Aerospace Studies
UNIVERSITY OF TORONTO

Anton Rubisov
Space Robotics Lab
2014 10 23

A few definitions to begin...

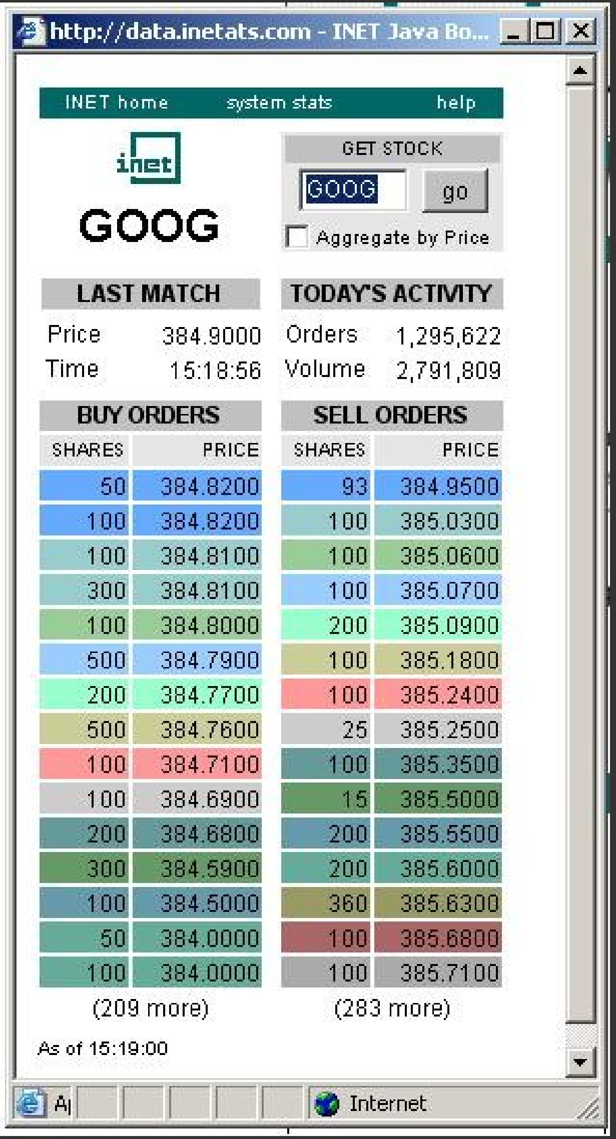
- Algorithmic / High-Frequency Trading:

  - rapid (< 1s) transactions
  - algorithmic execution
  - maintain no inventory at day's end

A few definitions to begin...

Limit Order Book (LOB):

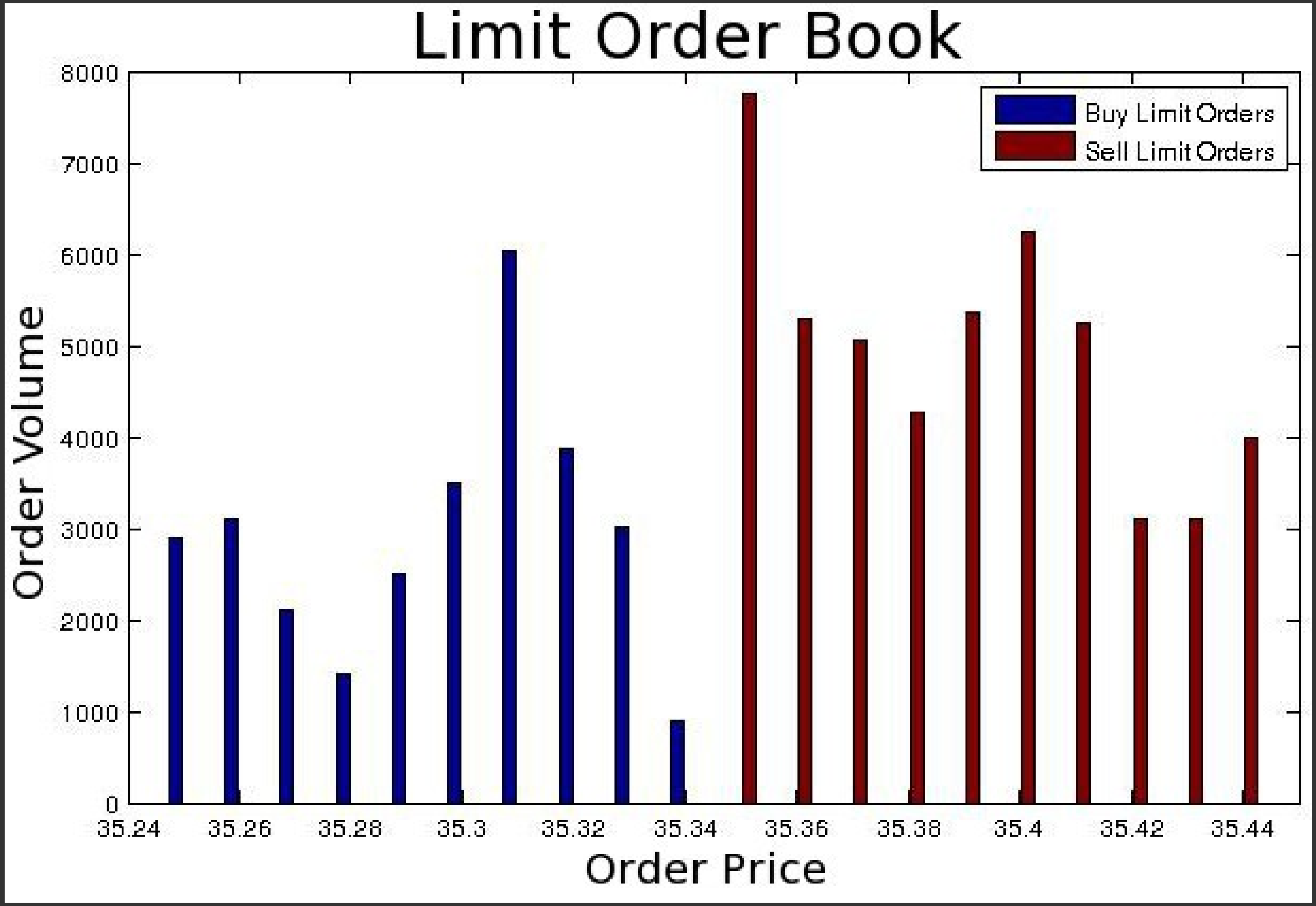
- a system (in the dynamics sense)
- tracks arrival of buy and sell orders of a given asset
- executes trades when a match is made



BUY ORDERS SHARES	BUY ORDERS PRICE	SELL ORDERS SHARES	SELL ORDERS PRICE
50	384.8200	93	384.9500
100	384.8200	100	385.0300
100	384.8100	100	385.0600
300	384.8100	100	385.0700
100	384.8000	200	385.0900
500	384.7900	100	385.1800
200	384.7700	100	385.2400
500	384.7600	25	385.2500
100	384.7100	100	385.3500
100	384.6900	15	385.5000
200	384.6800	200	385.5500
300	384.5900	200	385.6000
100	384.5000	360	385.6300
50	384.0000	100	385.6800
100	384.0000	100	385.7100
(209 more)		(283 more)	

A few definitions to begin...

LOB transformed into:



Research Goal

Our goal is to use the dynamics of the Limit Order Book (LOB) as an indicator for high-frequency stock price movement, thus enabling statistical arbitrage. Formally, we will the study limit order book imbalance process, $I(t)$, and the stock price process, $S(t)$, and attempt to establish a stochastic relationship $\dot{S} = f(S, I, t)$. We will then attempt to derive an optimal trading strategy based on the observed relationship.

Research Programme

1. Modeling LOB dynamics

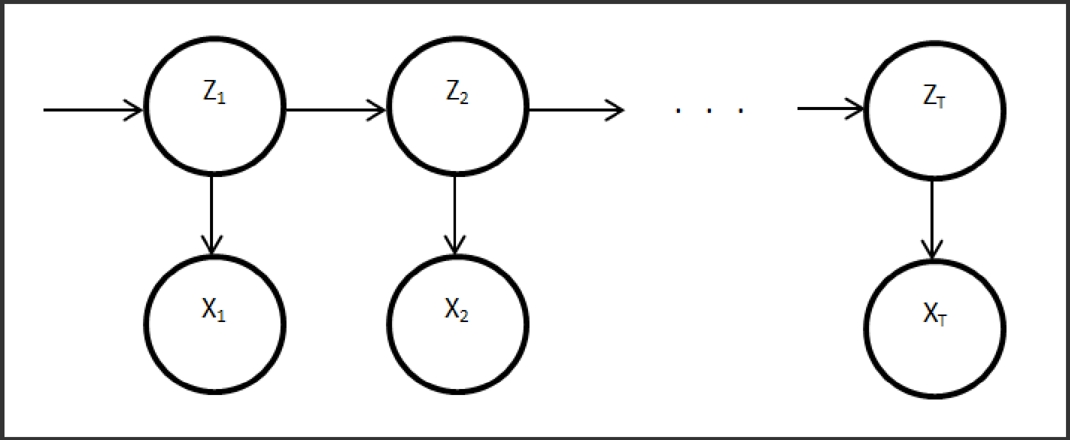
2. Relating order imbalance and price change

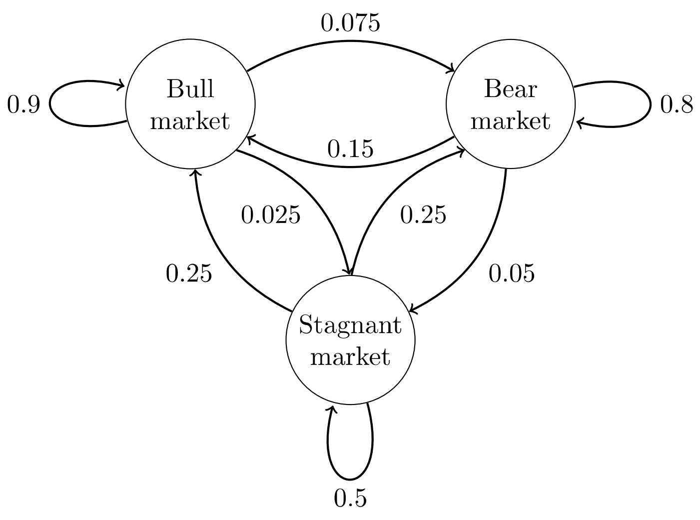
3. Exploratory data analysis

4. Deriving an optimal trading strategy

Modeling LOB Dynamics

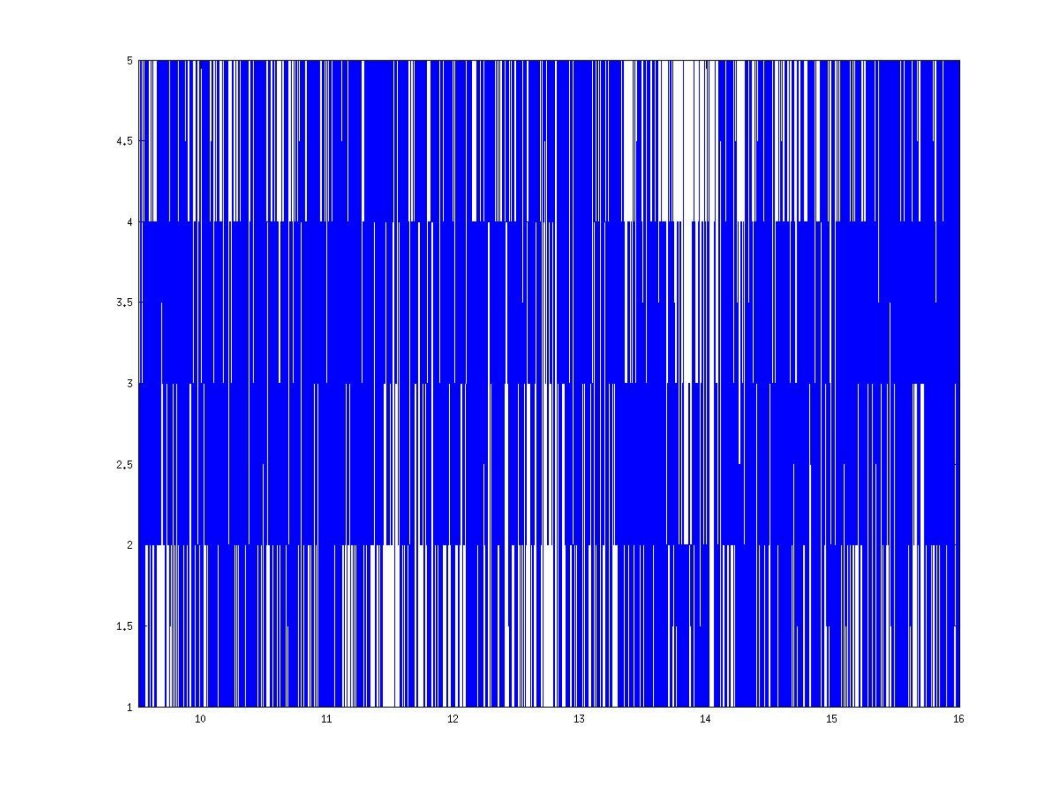
- To achieve goal, we need a suitable model
- Markov condition considered appropriate
- Hence, options:
  - Markov Chain, Continuous-Time Markov Chain, Hidden Markov Model, …





Modeling LOB Dynamics

- The imbalance process is very noisy...
  - So we average using a fixed interval...
  - And put it into bins



Modeling LOB Dynamics

- Given the interval averaging and binning, we could use discrete-time
- But, continuous-time is more powerful

- Continuous-Time Markov Chain:
  - Contains an embedded discrete-time Markov chain
  - Therefore an obvious and simple choice

Research Programme

1. Modeling LOB dynamics
2. Relating order imbalance and price change
3. Exploratory data analysis
4. Deriving an optimal trading strategy

Imbalance and Price Change

- Consider joint distribution for $(I(t), \Delta S(t))$
- Reduced to 1-dimension and estimated CTMC generator matrix $\boldsymbol{G}$, satisfying:

$$\dot{\boldsymbol{P}}(t) = \boldsymbol{P}(t)\boldsymbol{G} \;\Rightarrow\; \boldsymbol{P}(t) = e^{t\boldsymbol{G}}$$

- Elements of ***P*** are:

$$\begin{aligned} \boldsymbol{P}_{ij} &= \mathbb{P}\left[Z_n \in j \mid Z_{n-1} \in i\right] \\ &= \mathbb{P}\left[(\rho_n, \Delta S_n) \in j \mid (\rho_{n-1}, \Delta S_{n-1}) \in i\right] \end{aligned}$$

- Rewritten: $\mathbb{P}\left[\rho_n \in i, \Delta S_n \in j \mid \rho_{n-1} \in k, \Delta S_{n-1} \in m\right]$

Imbalance and Price Change

- $$\mathbb{P}\left[\rho_n \in i, \Delta S_n \in j \mid \rho_{n-1} \in k, \Delta S_{n-1} \in m\right]$$
  $$= \mathbb{P}\left[\rho_n \in i, \Delta S_n \in j \mid B\right]$$ (shorthand)

- Using Bayes' Rule, can solve:

$$\mathbb{P}\left[\Delta S_n \in j \mid B, \rho_n \in i\right] = \frac{\mathbb{P}\left[\rho_n \in i, \Delta S_n \in j \mid B\right]}{\mathbb{P}\left[\rho_n \in i \mid B\right]}$$

Probability of current price change conditional on current imbalance, previous imbalance, and previous price change

Imbalance and Price Change

Using 3 bins, 1000ms imbalance smoothing, and 500ms price change, we computed:

$$
\begin{array}{r}
\Delta S_n{<}0\rightarrow \\
\Delta S_n{=}0\rightarrow \\
\Delta S_n{>}0\rightarrow
\end{array}
\left[
\overbrace{
\underbrace{\begin{matrix} .67 & .05 & .04 \\ .33 & .95 & .96 \\ .00 & .00 & .00 \end{matrix}}_{\Delta S_{n-1}<0}
\underbrace{\begin{matrix} .01 & .03 & .04 \\ .99 & .97 & .96 \\ .00 & .00 & .00 \end{matrix}}
\underbrace{\begin{matrix} .00 & .05 & .05 \\ .41 & .93 & .95 \\ .58 & .02 & .00 \end{matrix}}_{\Delta S_{n-1}>0}
}^{\rho_n=1}
\overbrace{
\underbrace{\begin{matrix} .02 & .50 & .12 \\ .96 & .49 & .87 \\ .02 & .01 & .00 \end{matrix}}
\underbrace{\begin{matrix} .01 & .00 & .02 \\ .98 & .99 & .97 \\ .01 & .01 & .01 \end{matrix}}_{\Delta S_{n-1}=0}
\underbrace{\begin{matrix} .05 & .01 & .02 \\ .91 & .48 & .96 \\ .05 & .51 & .01 \end{matrix}}
}^{\rho_n=2}
\overbrace{
\underbrace{\begin{matrix} .00 & .00 & .52 \\ .98 & .95 & .47 \\ .02 & .04 & .01 \end{matrix}}
\underbrace{\begin{matrix} .00 & .01 & .00 \\ .95 & .96 & .93 \\ .05 & .03 & .02 \end{matrix}}
\underbrace{\begin{matrix} .00 & .00 & .00 \\ .98 & .88 & .34 \\ .02 & .12 & .66 \end{matrix}}
}^{\rho_n=3}
\right]
$$

Imbalance and Price Change

Using 3 bins, 1000ms imbalance smoothing, and 500ms price change, we computed:

$$
\begin{array}{r|ccccccccccccccccccccccccccc|}
\multicolumn{1}{r}{} & \multicolumn{9}{c}{\rho_n = 1} & \multicolumn{9}{c}{\rho_n = 2} & \multicolumn{9}{c}{\rho_n = 3} \\
\Delta S_n<0\to & \boxed{.67} & .05 & .04 & .01 & .03 & .04 & .00 & .05 & .05 & .02 & \boxed{.50} & .12 & .01 & .00 & .02 & .05 & .01 & .02 & .00 & .00 & \boxed{.52} & .00 & .01 & .00 & .00 & .00 & .00 \\
\Delta S_n=0\to & .33 & .95 & .96 & .99 & .97 & .96 & .41 & .93 & .95 & .96 & .49 & .87 & .98 & .99 & .97 & .91 & .48 & .96 & .98 & .95 & .47 & .95 & .96 & .93 & .98 & .88 & .34 \\
\Delta S_n>0\to & .00 & .00 & .00 & .00 & .00 & .00 & \boxed{.58} & .02 & .00 & .02 & .01 & .00 & .01 & .01 & .01 & .05 & \boxed{.51} & .01 & .02 & .04 & .01 & .05 & .03 & .02 & .02 & .12 & \boxed{.66} \\
\multicolumn{1}{r}{} & \multicolumn{3}{c}{\Delta S_{n-1} < 0} & \multicolumn{3}{c}{} & \multicolumn{3}{c}{\Delta S_{n-1} > 0} & \multicolumn{3}{c}{} & \multicolumn{3}{c}{\Delta S_{n-1} = 0} & \multicolumn{12}{c}{}
\end{array}
$$

Interpretation: there is price change momentum when staying in same imbalance bin

Research Programme

1. Modeling LOB dynamics

2. Relating order imbalance and price change

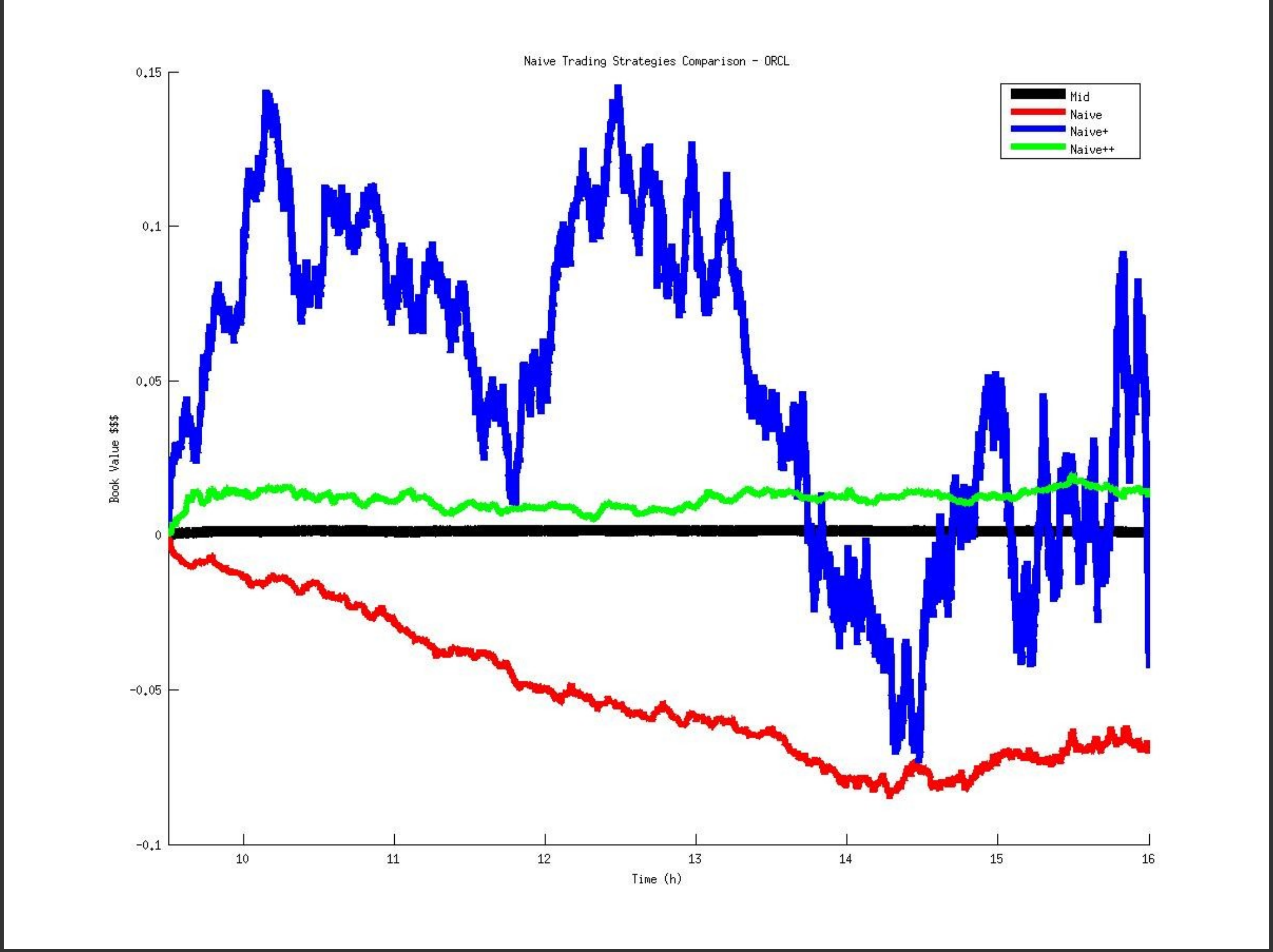
3. Exploratory data analysis

4. Deriving an optimal trading strategy

Exploratory Data Analysis

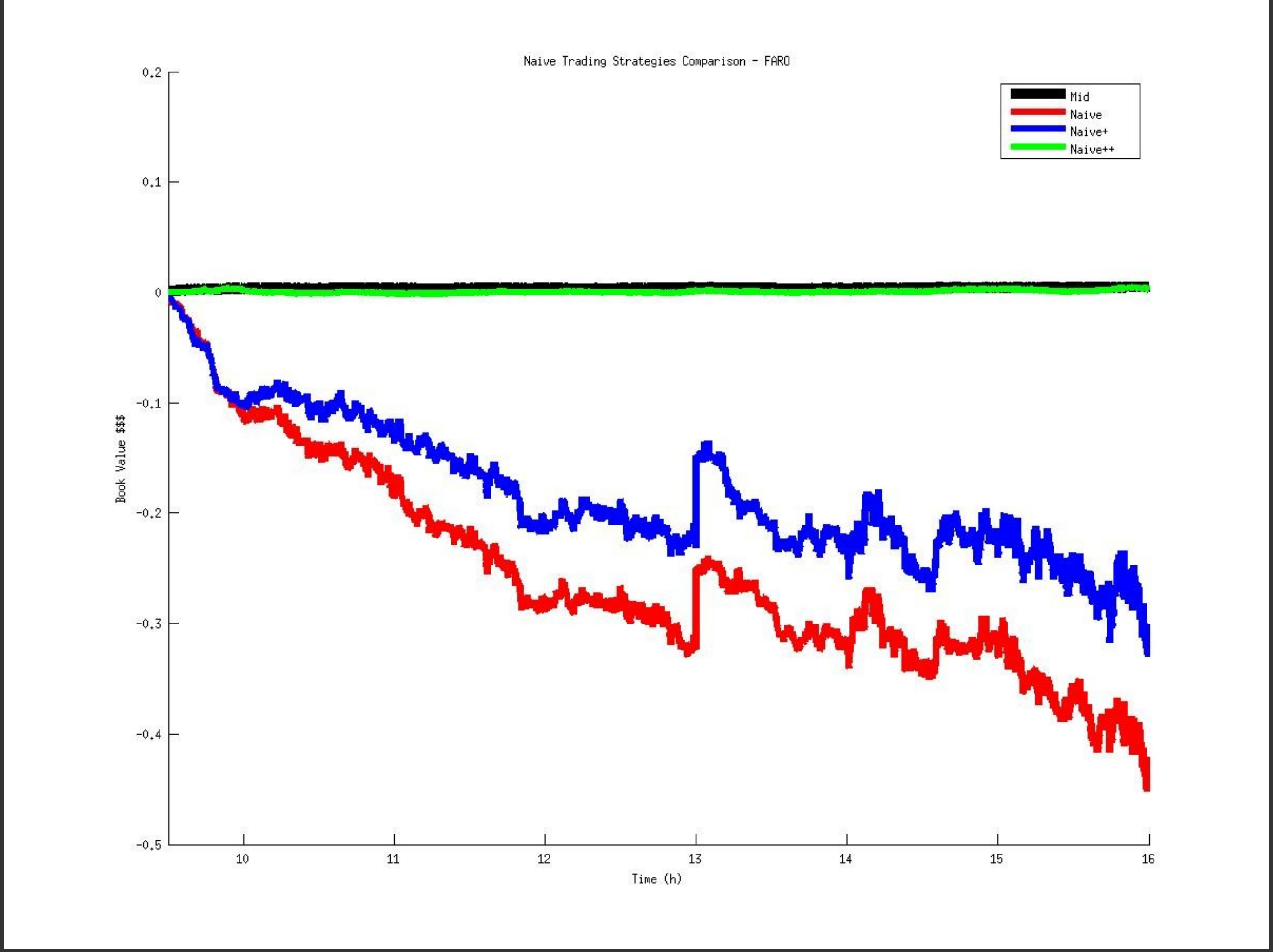
- Using the observed price change momentum, we explore a few trading strategies:
  - Naive: execute a buy (resp. sell) market order if the probability of an upward (resp. downward) price change is > 0.5
  - Naive+: extend the Naive strategy to additionally keep limit orders posted at-the-touch if the probability of a price change is < 0.5
  - Naive++: Like the Naive strategy, but using LOs instead of MOs

Exploratory Data Analysis



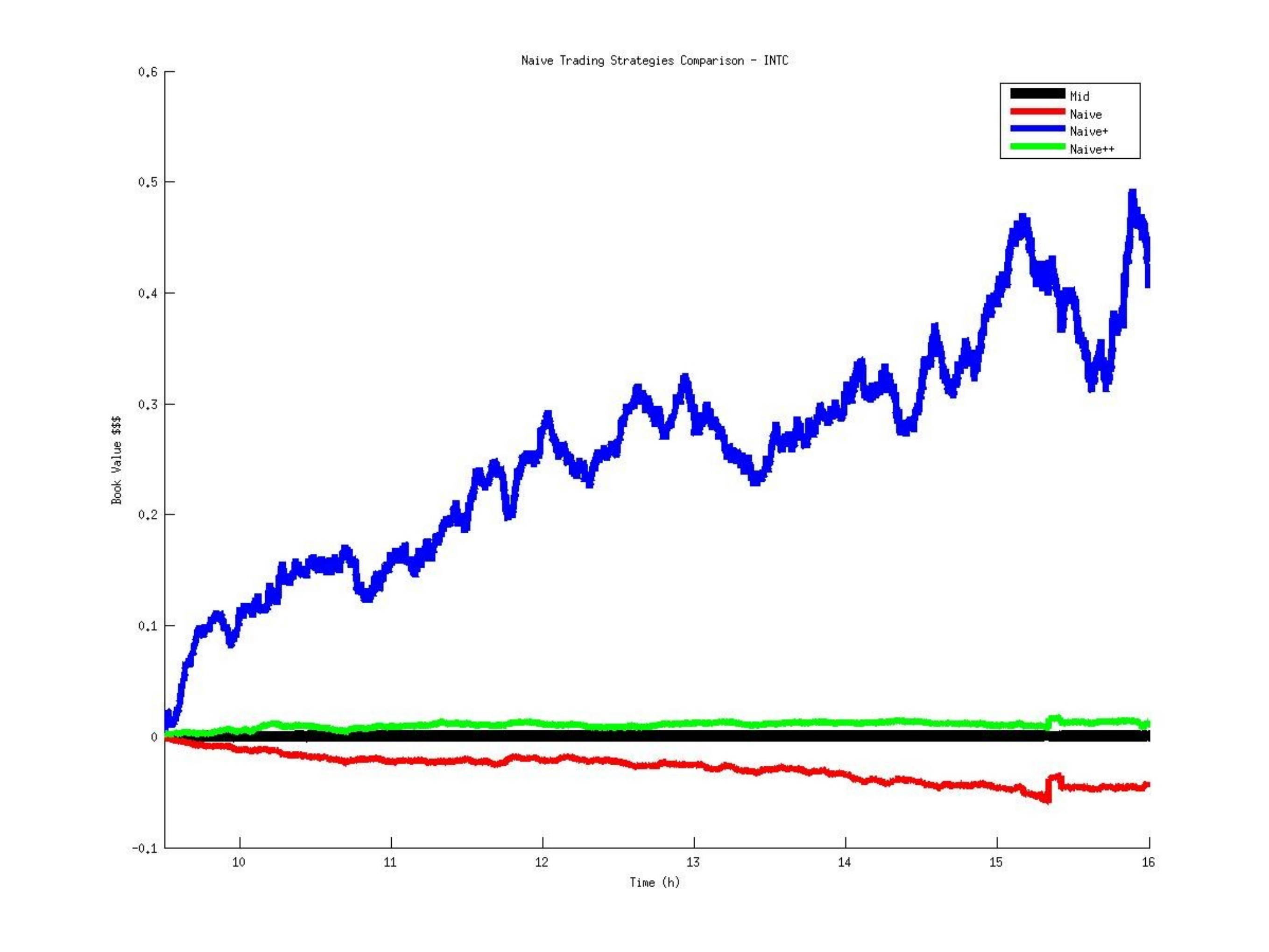
ORCL

Exploratory Data Analysis



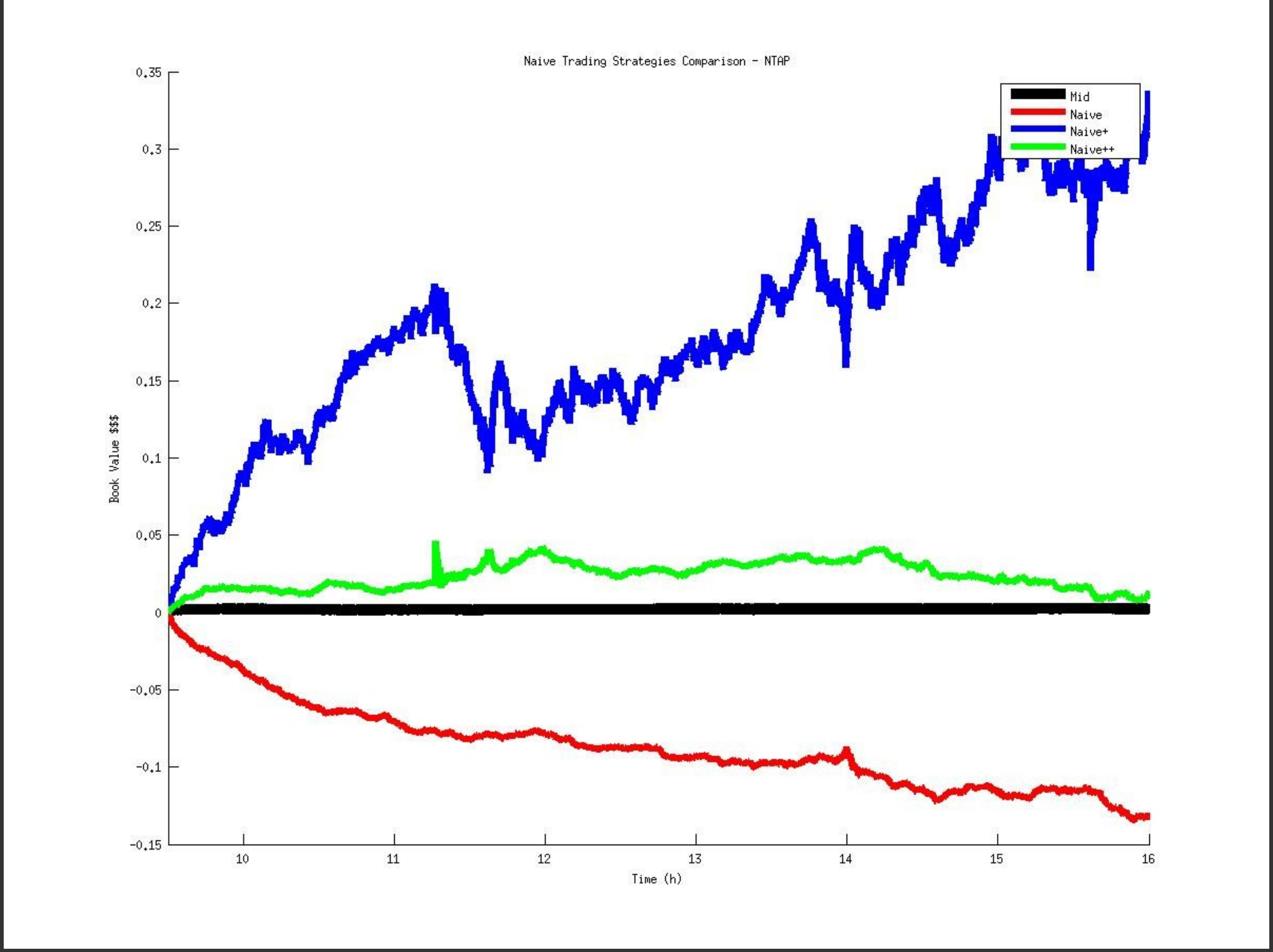
FARO

Exploratory Data Analysis



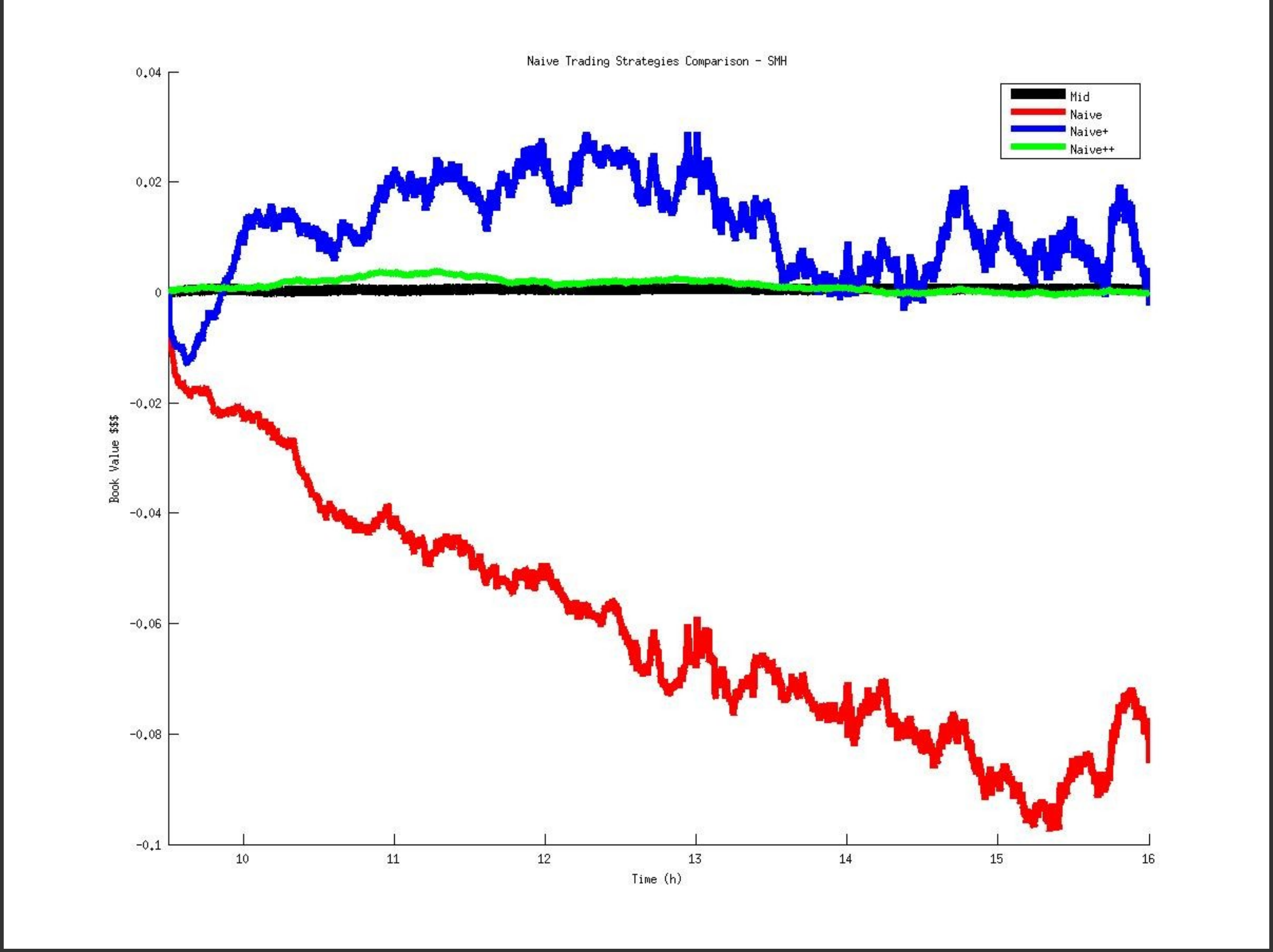
INTC

Exploratory Data Analysis



NTAP

Exploratory Data Analysis



SMH

Exploratory Data Analysis

- Conclusion: posting at-the-touch LOs on average produces positive returns

- Other considerations still being investigated:
  - How do we calibrate parameters?
  - How do we compute imbalance?

Research Programme

1. Modeling LOB dynamics
2. Relating order imbalance and price change
3. Exploratory data analysis
4. Deriving an optimal trading strategy

Optimal Trading Strategy

m - market orders strategy

X_t^m - cash at time *t* given orders *m*

S_t - stock price at time *t*

Δ_t - bid/ask spread at time *t*

m_t^+ - buy MO at time *t*

m_t^- - sell MO at time *t*

$q_t^m = m_t^+ - m_t^-$ - # shares held at time *t*

Cash stream:

$$\mathrm{d}X_t^m = -(S_t + \Delta_t)\mathrm{d}m_t^+ + (S_t - \Delta_t)\mathrm{d}m_t^-$$

Optimal Trading Strategy

m - market orders strategy

X_t^m - cash at time *t* given orders *m*

S_t - stock price at time *t*

Δ_t - bid/ask spread at time *t*

m_t^+ - buy MO at time *t*

m_t^- - sell MO at time *t*

$q_t^m = m_t^+ - m_t^-$ - # shares held at time *t*

Cash stream:

$$\mathrm{d}X_t^m = -(S_t + \Delta_t)\mathrm{d}m_t^+ + (S_t - \Delta_t)\mathrm{d}m_t^-$$

Optimal Trading Strategy

Terminal (end of day) wealth:

$$W_T^m = X_T + q_T^m(S_T - \mathrm{sign}(q_T^m)\Delta_T)$$

How do we measure performance or constrain possible strategies *m*?

Optimal Trading Strategy

1. $\max \mathbb{E}[W_T]$ Maximize profit

2. $\max \mathbb{E}[W_T | W_T < 0]$ Minimize loss

3. $\max \mathbb{E}[W_T - \gamma \cdot \mathbf{1}_{W_T<0} \cdot W_T]$ Risk aversion

4. $\max \dfrac{\mathbb{E}[W_T]}{\mathrm{var}(W_T)}$ Maximize Sharpe ratio

5. $\max \dfrac{\mathbb{E}[W_T]}{\mathrm{var}[W_T | W_T < 0]}$ Maximize Sortino ratio

Next Steps

- Include limit orders into stochastic framework
- Derive optimal trading strategy based on Markov Decision Processes
  - dynamic programming
-