Hopf Algebras

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1. What is an algebra?

Let $(\mathscr{A}, +, \cdot)$ be a vector space over a field $\mathbb{K}$,

endowed with a "multiplication" $m : \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$

and a "unit" element $1_{\mathscr{A}}$, or equivalently, $u : \mathbb{K} \to \mathscr{A}$.

Suppose the multiplication is associative:

$$
\begin{tikzcd}
\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A} \arrow[r, "m \otimes \mathrm{Id}"] \arrow[d, "\mathrm{Id} \otimes m"'] & \mathscr{A} \otimes \mathscr{A} \arrow[d, "m"] \\
\mathscr{A} \otimes \mathscr{A} \arrow[r, "m"'] & \mathscr{A}
\end{tikzcd}
$$

and it is...

$\cdots$

It is said that $(\mathscr{A}, +, \cdot, m, u)$ is an algebra (with unit) (or a $\mathbb{K}$-algebra).

2. The "dual" concept: coalgebras

Let $(\mathscr{C}, +, \cdot)$ be a vector space over a field $\mathbb{K}$

endowed with a "comultiplication" $\Delta : \mathscr{C} \to \mathscr{C} \otimes \mathscr{C}$

and a "counit" $\epsilon : \mathscr{C} \to \mathbb{K}$

It is said then that $(\mathscr{C}, +, \cdot, \Delta, \epsilon)$ is a co-algebra.

3. Bilinear algebras

4. Hopf algebras

Let $(\mathscr{B}, +, \cdot, m, u, \Delta, \epsilon)$ be a bilinear algebra.

Suppose there is $S : \mathscr{B} \to \mathscr{B}$ such that the following diagram commutes:

$$\begin{tikzcd}
& \mathscr{B} \otimes \mathscr{B} \arrow[rr, "\mathrm{Id}_{\mathscr{B}} \otimes S"] & & \mathscr{B} \otimes \mathscr{B} \arrow[dr, "m"] & \\
\mathscr{B} \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "u"] & & \mathscr{B} \\
& \mathscr{B} \otimes \mathscr{B} \arrow[rr, "S \otimes \mathrm{Id}_{\mathscr{B}}"] & & \mathscr{B} \otimes \mathscr{B} \arrow[ur, "m"'] &
\end{tikzcd}$$

The map S is then called an antipode.

It is said then that $(\mathscr{B}, +, \cdot, m, u, \Delta, \epsilon, S)$ is a Hopf algebra.

In general, S is an antihomomorphism, so S^2 is a homomorphism, which is therefore an automorphism if S was invertible (as may be required).

Involutive Hopf algebras. If $S^2 = \mathrm{Id}_{\mathscr{B}}$, then it is said to be an involutive Hopf algebra.

The antipode is unique if it exists. If a bialgebra admits an antipode, then it is unique. Thus, the antipode does not pose any extra structure which we can choose: *being a Hopf algebra is a property of a bialgebra*.