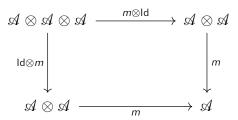
Hopf Algebras

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1. What is an algebra?

Let $(\mathcal{A}, +, \cdot)$ be a vector space over a field \mathbb{K} , endowed with a "multiplication" $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and a "unit" element $1_{\mathscr{A}}$, or equivalently, $u: \mathbb{K} \to \mathscr{A}$. Suppose the multiplication is associative:



and it is...

It is said that $(\mathcal{A}, +, \cdot, m, u)$ is an algebra (with unit) (or a \mathbb{K} -algebra).

2. The "dual" concept: coalgebras

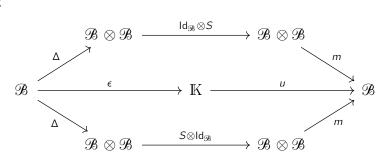
Let $(\mathscr{C},+,\,\cdot\,)$ be a vector space over a field $\mathbb K$ endowed with a "comultiplication" $\Delta:\mathscr{C}\to\mathscr{C}\otimes\mathscr{C}$ and a "counit" $\epsilon:\mathscr{C}\to\mathbb{K}$ It is said then that $(\mathcal{C}, +, \cdot, \Delta, \epsilon)$ is a co-algebra.

3. Bilinear algebras

4. Hopf algebras

Let $(\mathcal{B}, +, \cdot, m, u, \Delta, \epsilon)$ be a bilinear algebra.

Suppose there is $S: \mathcal{B} \to \mathcal{B}$ such that the following diagram commutes:



The map S is then called an antipode.

It is said then that $(\mathcal{B}, +, \cdot, m, u, \Delta, \epsilon, S)$ is a Hopf algebra.

In general, S is an antihomomorphism, so S^2 is a homomorphism, which is therefore an automorphism if S was invertible (as may be required).

Involutive Hopf algebras. If $S^2 = Id_{\Re}$, then it is said to be an involutive Hopf algebra.

The antipode is unique if it exists. If a bialgebra admits an antipode, then it is unique. Thus, the antipode does not pose any extra structure which we can choose: being a Hopf algebra is a property of a bialgebra.