Aruzhan Kakibay
1996005

Assignment 2.

Definition of the 2-norm of a matrix A: $\|A\|_2 = \sigma_{max}(A)$, *where* $\sigma_{max}(A)$

Represents the largest singular value of matrix A.

Theorem

Let $A = U\Sigma V^T$ be the singular value decomposition of $A \in \mathbb{R}^{m\times n}$, and let $U = (u^1, \dots, u^m)$ and $V = (v^1, \dots, v^n)$.

Then for $k < n$

$$A_k := \sum_{j=1}^{k} \sigma_j u^j (v^j)^T$$

is the best approximation of A with $\text{rank}(A_k) = k$ with respect to the spectral norm, and it holds that

$$\|A - A_k\|_2 = \sigma_{k+1}.$$

It holds that

$$\begin{aligned} \|A - A_k\|_2 &= \|\sum_{j=k+1}^{n} \sigma_j u^j (v^j)^T\|_2 \\ &= \|U\text{diag}\{0, \dots, 0, \sigma_{k+1}, \dots, \sigma_n\} V^T\|_2 = \sigma_{k+1}, \end{aligned}$$

and it remains to show, that there does not exist a matrix of rank k, the distance to A of which is less than σ_{k+1}.

Let B be any matrix with $\text{rank}(B) = k$. Then the dimension of the null space of B is $n - k$. The dimension of $\text{span}\{v^1, \dots, v^{k+1}\}$ is $k + 1$, and therefore the intersection of these two spaces contains a nontrivial vector w with $\|w\|_2 = 1$.

Hence,

$$\begin{aligned} \|A - B\|_2^2 &\geq \|(A - B)w\|_2^2 = \|Aw\|_2^2 \\ &= \|U\Sigma V^T w\|_2^2 = \|\Sigma(V^T w)\|_2^2 \\ &\geq \sigma_{k+1}^2 \|V^T w\|_2^2 = \sigma_{k+1}^2. \qquad \square \end{aligned}$$

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1. $\|A_k\|_F^2 = \sum_{i=1}^{k} \sigma_i^2$
2. $\|A - A_k\|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$
3. $\|A_k\|_2^2 = \sigma_i^2$
4. $\|A - A_k\|_2^2 = \sigma_{k+1}^2$

Assignment 3.

1. Prove $\sigma_k \leq \frac{\|A\|_F}{\sqrt{k}}$
   As we considered above, we will use $\|A_k\|_F^2 = \sum_{i=1}^{k} \sigma_i^2$.

   $$\frac{\|A\|_F}{\sqrt{k}} \geq \sqrt{\frac{\sum_{i=1}^{k} \sigma_i^2}{k}} \geq \sqrt{\frac{k\sigma_k^2}{k}} = \sigma_k$$

   $$\sigma_k \leq \frac{\|A\|_F}{\sqrt{k}} \blacksquare$$

2. Prove that there exists a matrix B of rank at most k such that $\|A - B\|_2 \leq \frac{\|A\|_F}{\sqrt{k}}$

Here we will use $\|A - A_k\|_2^2 = \sigma_{k+1}^2$, as it is known A_k with rank k, so we can say by using property $\sigma_{k+1} \leq \sigma_k$ that:

$$\|A - B\|_2 \leq \sigma_{k+1} \leq \sigma_k \leq \frac{\|A\|_F}{\sqrt{k}} \blacksquare$$