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REPORT

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1 task

Write the basic equations for modeling the disease. Name the possible forms of disease dynamics and their classification.

Answer:

As we know at present time the infectious diseases continue to be a serious problem. To have victory over disease we need to do experimentation and for this we use the mathematical model of processes. Let's consider the mathematical model of infection disease of Marchuk which describes immune response system. It is known that the predecessors of immune-component cells are produced in marrow of bone as the predecessors of blood cells.

Let's make some denote:

$T - lymphocytes$;

$T_h - helper\ lymphocytes$;

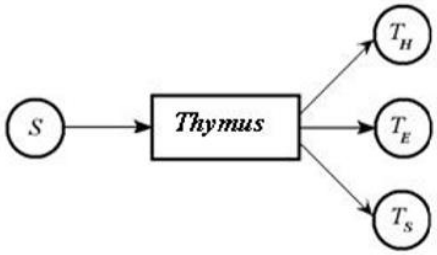
$T_e - effector\ (killer)\ lymphocytes$;

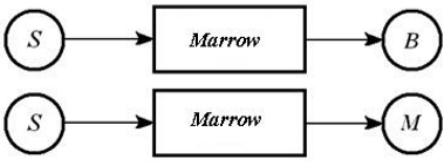
$T_c - suppressing\ lymphocytes$;

$S - all\ of\ blood\ and\ immunocomponent\ cells$;

$B - lymphocytes$;

$M - macrophages$;





T lymphocytes take major role in the immune process. T_h helper lymphocytes encourage to transformation of B cells in plasma cells and interact with specific antigen. T_e effector lymphocytes have role to keep clean cells of organism. T_c suppressors regulate immune response.

In this work we will consider immune response of organism, when the number of T-lymphocytes and number of macrophages is continuous and immunology process is built on interaction of antibody and antigen. For this, need to build mathematical models, which describes general process of diseases.

Now, let's consider the mathematical models and classification of immune response to antigen which can be described by mathematical models. So, we will consider next classifications: **chronic form, subclinical form, acute form with recovery, and acute form with lethal outcome**.

First, let's call main factors value before describing mathematical models of disease.

V(t) – it contents of pathogenic reproducing antigens;

F(t) – it contents of antibodies;

C(t) –is plasma cells;

m(t) – is a feature of an affected organ, which is defined in rate [0, 1];

C^*- is normal level of immunocompetent cells in relation to this antigen in a healthy body, $C^* = 0$ then the body is tolerant to this antigen;

α – is a coefficient taking into account the probability of meeting antigen-antibody;

β – is a multiplication factor of antigens;

γ – coefficient associated with the probability of neutralization of the antigen by antibodies;

η – the number of antibodies required to neutralize a single antigen;

τ – is the time during which the formation of cascade of plasma cells;

ρ – is the rate of antibody production by single plasma cell;

Б – is a some constant, its own for each disease;

t – time;

We assume that the increment of antigens over the time interval dt is described by the formula

$$dV = \beta V dt - \gamma F V dt$$

Here,

dV- in an increasing of antigen number;
$\beta V dt$ – antigen reproduction coefficient to V.
By using plasmatic cells cascade population, we can arrive to:

$$d(C - C^*) = dC = Q(t - \tau)dt$$
$$Q(t) = \alpha FV$$
$$dC = \alpha F(t - \tau)V(t - \tau) - \mu_c(C - C^*)$$

Here,
dC- is the generation of plasmatic cells;
μ_c-is a coefficient which equals to the value of their life;
All other parts we described above.
The balance of the antibodies reacting with antigen:

$$dF = \rho C dt - (\mu_f + \eta\gamma V)F dt$$

Here,
dF-is a generation of antibodies;
μ_f- coefficient reciprocally proportional to the antibody;
Characteristic feature:

$$dm = 6\,\mathrm{V}\mathrm{d}t - \mu_\mathrm{m} m dt$$

Non-linear ordinary differential equation:

$$\begin{aligned} \frac{dV}{dt} &= (\beta - \gamma F)V, \\ \frac{dC}{dt} &= \xi(m)\alpha V(t - \tau)F(t - \tau) - \mu_c(C - C^*), \\ \frac{dF}{dt} &= \rho C - \left(\mu_f + \eta\gamma V\right)F, \\ \frac{dm}{dt} &= 6\mathrm{V} - \mu_\mathrm{m} m. \end{aligned} \quad (1)$$

Here $V(t) \equiv 0\ at\ t < t^0. V(t_0) = V_0, C(t_0) = C_0, F(t_0) = F_0, m(t_0) = m_0.$ (2)

(1) and (2) can be called as ***the simplest mathematical model of diseases***. Denote that the value of coefficients must be positive.

So, we described mathematical model of diseases and we know what part responsible for what. Next, we will consider the forms of dynamic of diseases and its classification.

If in an organism was got antigens and started to hit the organs, we will consider four forms of diseases dynamic. First form, let's see **lethal form**:

It can be happened, if organs cannot be recovered which means that $\mu_m = 0$, or when human has very weak immune system, especially it can happen often with elderly people and it can be known as $\alpha\rho > \mu_c\eta\gamma$. Let's see the program of this form of diseases in MATLAB.

Code:

```
alpha=0.000001;%alpha*rho>mu*eta*gamma
betta=0.9;%multiplication factor of antigens;
gamma=0.00001;%coefficient associated with the probability of neutralization of the antigen
by antibodies
etta=1;%the number of antibodies required to neutralize a single antigen;
sigma=0.0001;%some constant, its own for each disease
tau=3;%the time during which the formulation of cascade of plasma cells;
rho=3;%the rate of antibody production by a single plasma cell
t0=0;%initial time
tn=5;%time
dt=1/24;%taking time as days
n=(tn-t0)/dt;
T_cells=7;%life Time Plasma Cells
miu_c=1/T_cells;
T_antib = 1;%antibodies Decay Time
miu_f=1/T_antib;
Recovery= 30;%organ Recovery Period 20 for organism is needed 3 0 days for recoverying
miu_m=0.0;
C_st=814;%making new cells very low so it is cause of death
F_st=rho*C_st/miu_f;
```

```
    F=zeros(1,n);%antibody concentration;
    V=zeros(1,n);%concentration of pathogenic propagating antigens;
    C=zeros(1,n);%plasma cell concentration;
    m=zeros(1,n);%relative characteristic of target organ damage, which is [0,1] defined;
    C(1)=C_st;
    V(1)=1000;
    F(1)=F_st;
    m(1)=0;
    for i=2:n
        C(i)=C(i-1)+dt*(sin(m(i-1)*pi/2)*alpha*V(i-1)*(dt*i-tau)*F(i-1)*(dt*i-tau)-
miu_c*(C(i-1)-C_st));
        V(i)=V(i-1)+dt*((betta-gamma*F(i-1))*V(i-1));
        F(i)=F(i-1)+dt*(rho*C(i-1) - (miu_f+etta*gamma*V(i-1))*F(i-1));
        m(i)=m(i-1)+dt*(sigma*V(i-1)-miu_m*m(i-1));
        iter=i;
        if ((m(i) >= 1 || m(i) <= 0 || V(i) <= 0 || C(i) <= 0 || F(i) <= 0))
            break
       end
    end
    time=dt*(1:n);
    fig=figure();
    plot(time, C, 'b-.', 'LineWidth',2)
    hold on
    plot(time, V, 'r.', 'LineWidth', 2)
    hold on
    plot(time, F, 'g--', 'LineWidth', 2)
    set(gca, 'FontSize', 14)
    set(fig, 'color', 'white')
    grid on
    xlabel ('time[days]')
    ylabel ('development scale')
    legend('C(plasma-cell)', 'V(antigen)', 'F(antibody)');
    fig=figure();
    plot(time, m, 'b--', 'LineWidth', 2)
    set(gca, 'FontSize', 14)
    set(fig, 'color', 'white')
    xlabel('time[days]')
    ylabel('part')
    legend('m(degree of damage to organ)');
```

Results:

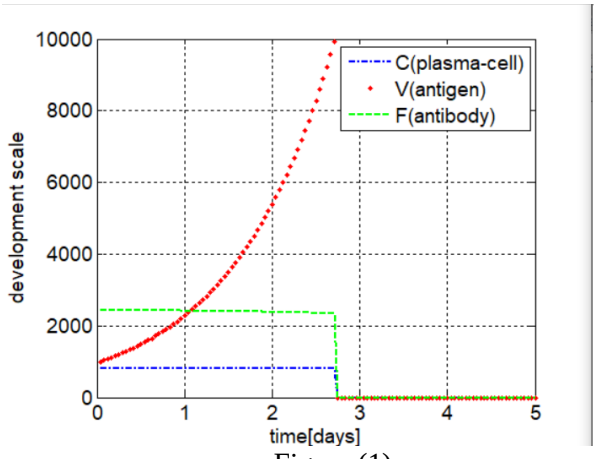


Figure (1)

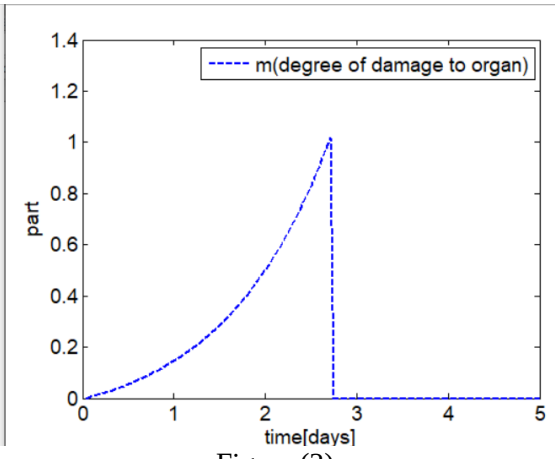


Figure (2)

Here, in figure (2) you see the death of organism in 3rd day; however, it reached 1, the maximum rate of damage of organism. Why? Because, antibody cannot destroy the antigen, we can see it in figure (1), and recovery coefficient is zero. So, antigen is getting destroy the organism.

Now, let us consider next form, **subclinical form** where $\beta < \gamma F^*$, this form of disease proceeds stealthily and is not connected with physiological disorder of organism, that means it is being hit with known antigens and immune system is being able to discharge it. So, it is better to understand and explain with real example in MATLAB, here program for subclinical form of disease.

Code:

```
alpha=0.000001;%alpha*rho<>mu*eta*gamma
betta=0.9;%multiplication factor of antigens;
gamma=0.0005;%coefficient associated with the probability of neutralization of the antigen
by antibodies
etta=1;%the number of antibodies required to neutralize a single antigen;
sigma=0.0001;%some constant, its own for each disease
tau=3;%the time during which the formulation of cascade of plasma cells;
rho=3;%the rate of antibody production by a single plasma cell
t0=0;%initial time
tn=30;%time
dt=1/24;%taking time as days
n=(tn-t0)/dt;
T_cells=7;%life Time Plasma Cells
miu_c=1/T_cells;
T_antib = 1;%antibodies Decay Time
miu_f=1/T_antib;
Recovery= 30;%organ Recovery Period 20 for organism is needed 3 0 days for recoverying
miu_m=1/Recovery;
C_st=1000;
F_st=rho*C_st/miu_f;
F=zeros(1,n);%antibody concentration;
V=zeros(1,n);%concentration of pathogenic propagating antigens;
C=zeros(1,n);%plasma cell concentration;
m=zeros(1,n);%relative characteristic of target organ damage, which is [0,1] defined;
C(1)=C_st;
V(1)=1000;
F(1)=F_st;
m(1)=0;
for i=2:n
    C(i)=C(i-1)+dt*(sin(m(i-1)*pi/2)*alpha*V(i-1)*(dt*i-tau)*F(i-1)*(dt*i-tau)-
miu_c*(C(i-1)-C_st));
    V(i)=V(i-1)+dt*((betta-gamma*F(i-1))*V(i-1));
    F(i)=F(i-1)+dt*(rho*C(i-1) - (miu_f+etta*gamma*V(i-1))*F(i-1));
    m(i)=m(i-1)+dt*(sigma*V(i-1)-miu_m*m(i-1));
end
time=dt*(1:n);
fig=figure();
plot(time, C, 'b-.', 'LineWidth',2)
hold on
plot(time, V, 'r.', 'LineWidth', 2)
hold on
plot(time, F, 'g--', 'LineWidth', 2)
set(gca, 'FontSize', 14)
set(fig, 'color', 'white')
grid on
xlabel ('time[days]')
ylabel ('development scale')
legend('C(plasma-cell)', 'V(antigen)', 'F(antibody)');
fig=figure();
plot(time, m, 'r--', 'LineWidth', 2)
set(gca, 'FontSize', 14)
set(fig, 'color', 'white')
xlabel('time[days]')
ylabel('part')
legend('m(degree of damage to organ)');
```

Results:

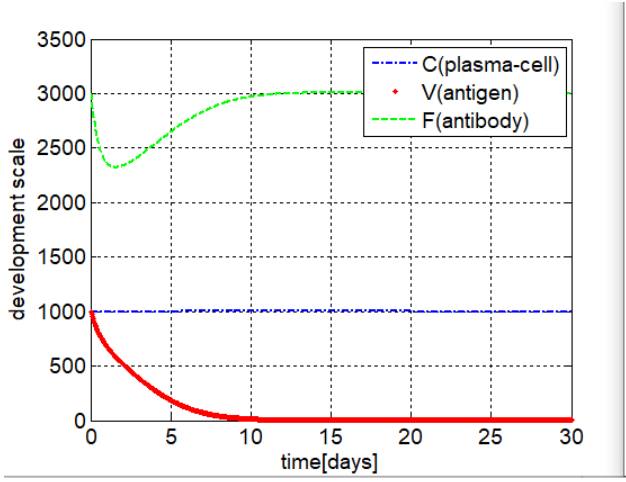


Figure (3)

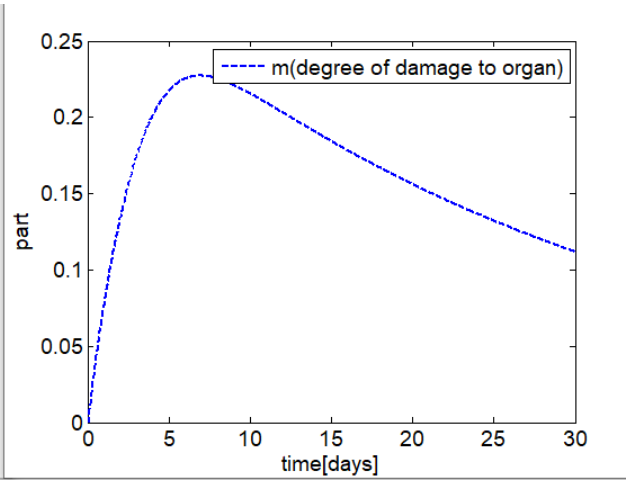


Figure (4)

As we see in Figure (4) the organism got to the damage like 22.5% and after 7 days pass, the organism is getting well, also to recover it needs more time. To remind we considered case where immune system has effective response to antigens and it can be written as $\alpha\rho < \mu_c\eta\gamma$.

Let's make longer the time period to see its recoveries. Let us take it as 40 days, result is as follows:

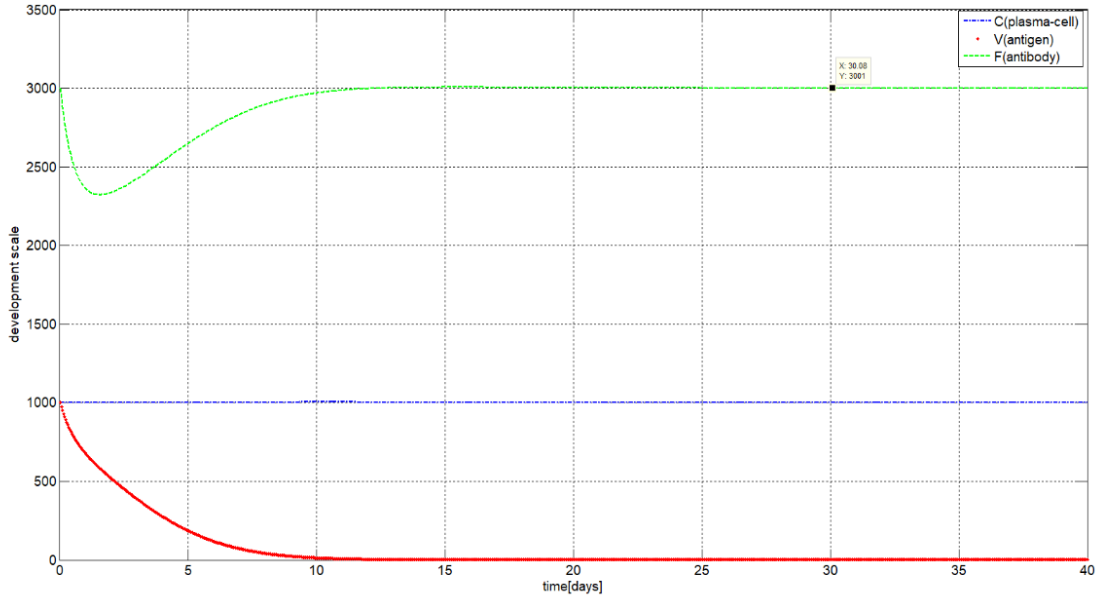


Figure (5)

As we can see the organism recovered fully in 30 days, it is shown in Figure (5). Here also in this figure is shown the fight between antigen and antibody, after antibody destroys antigen.

Next form is **the acute form** of disease, where long time period disease and the number of concentrations of antigen is increasing. Let us consider the acute form with full recovery and with efficient immune system of organism, then the case will be as $\beta < \gamma F^*$ which means that there is no immunological barrier to pathogens. Let's see the program code in MATLAB and describe the dynamic of this form.

Code:

```
alpha=0.0000013;%alpha*rho<>mu*eta*gamma
betta=0.4;%multiplication factor of antigens;
gamma=0.0001;%coefficient associated with the probability of neutralization of the antigen
by antibodies
etta=0.2;%the number of antibodies required to neutralize a single antigen;
sigma=0.00006;%some constant, its own for each disease
tau=15;%the time during which the formulation of cascade of plasma cells;
rho=1;%the rate of antibody production by a single plasma cell
t0=0;%initial time
tn=20;%time
dt=1/24;%taking time as days
n=(tn-t0)/dt;
T_cells=7;%life Time Plasma Cells
miu_c=1/T_cells;
T_antib = 3.5;%antibodies Decay Time
miu_f=1/T_antib;
Recovery= 20;%organ Recovery Period 20 for organism is needed 3 0 days for recoverying
miu_m=1/Recovery;
C_st=1000;
F_st=rho*C_st/miu_f;
F=zeros(1,n);%antibody concentration;
V=zeros(1,n);%concentration of pathogenic propagating antigens;
C=zeros(1,n);%plasma cell concentration;
m=zeros(1,n);%relative characteristic of target organ damage, which is [0,1] defined;
C(1)=C_st;
V(1)=1000;
F(1)=F_st;
m(1)=0;

for i=2:n
    C(i)=C(i-1)+dt*(sin(m(i-1)*pi/2)*alpha*V(i-1)*(dt*i-tau)*F(i-1)*(dt*i-tau)-
miu_c*(C(i-1)-C_st));
    V(i)=V(i-1)+dt*((betta-gamma*F(i-1))*V(i-1));
    F(i)=F(i-1)+dt*(rho*C(i-1) - (miu_f+etta*gamma*V(i-1))*F(i-1));
    m(i)=m(i-1)+dt*(sigma*V(i-1)-miu_m*m(i-1));
    iter=i;
    if ((m(i) >= 1 || m(i) <= 0 || V(i) <= 0 || C(i) <= 0 || F(i) <= 0))
        break
  end
end
time=dt*(1:n);
fig=figure();
plot(time, C, 'b-.', 'LineWidth',2)
hold on
plot(time, V, 'r.', 'LineWidth', 2)
hold on
plot(time, F, 'g--', 'LineWidth', 2)
set(gca, 'FontSize', 14)
set(fig, 'color', 'white')
grid on
xlabel ('time[days]')
ylabel ('development scale')
legend('C(plasma-cell)', 'V(antigen)', 'F(antibody)');
fig=figure();
plot(time, m, 'b--', 'LineWidth', 2)
set(gca, 'FontSize', 14)
set(fig, 'color', 'white')
xlabel('time[days]')
ylabel('part')
legend('m(degree of damage to organ)');

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```
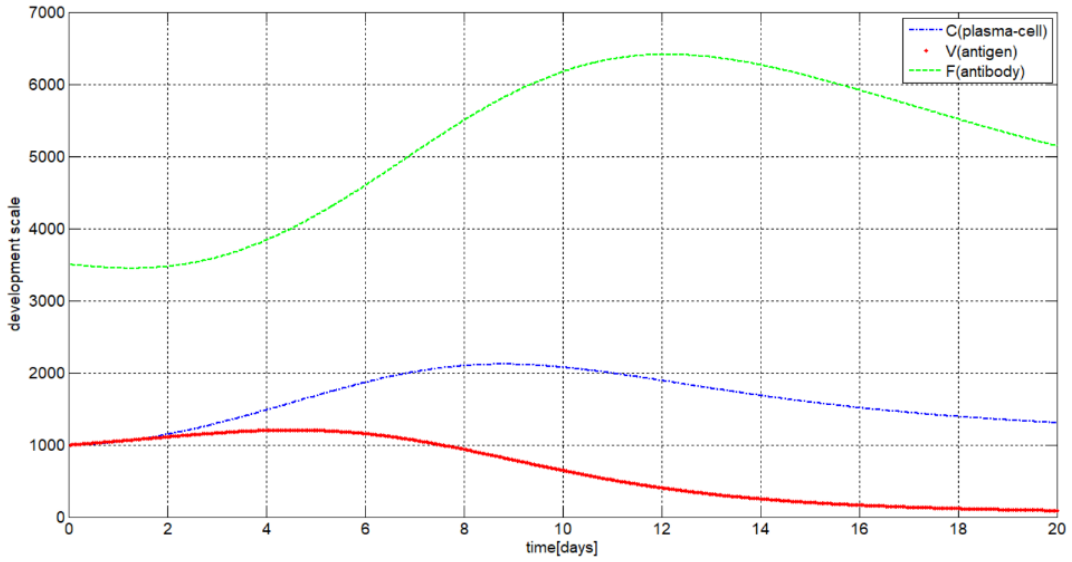

$\beta > \gamma F^*$

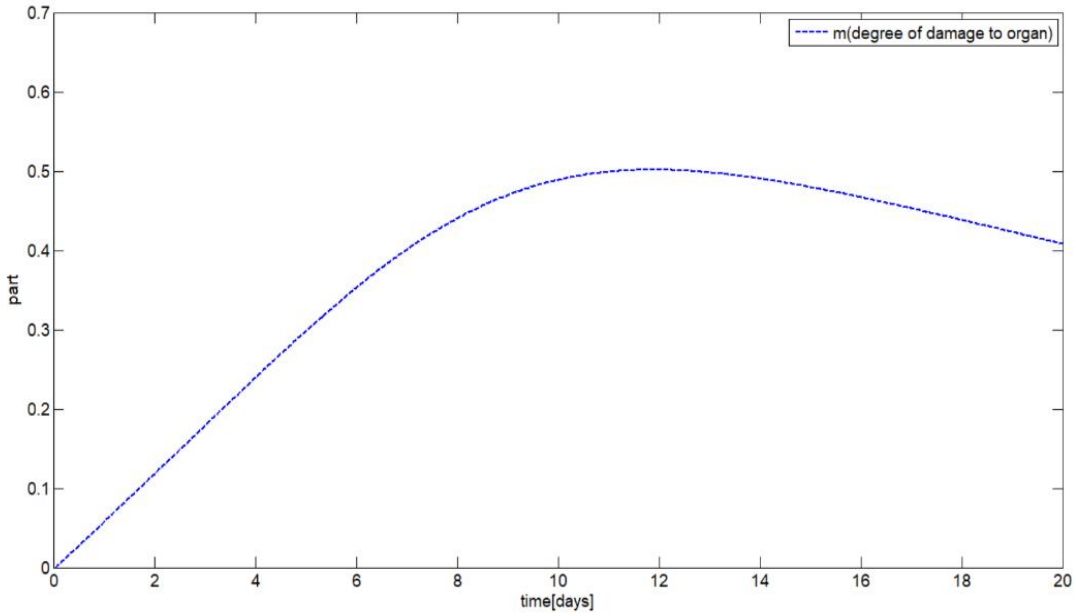
betta=0.4, gamma=0.0001, F^*=3500.

0.4>0.35

Results:



(a)



(b)

Figure (7)

In case $\beta > \gamma F^*$, was expected all conditions of building of model and there is no immunological barrier to the causes of the disease. For acute forms of the disease, a rapid increase in the number of antigens in the body and no less rapid elimination of antigens from the body are characteristic.

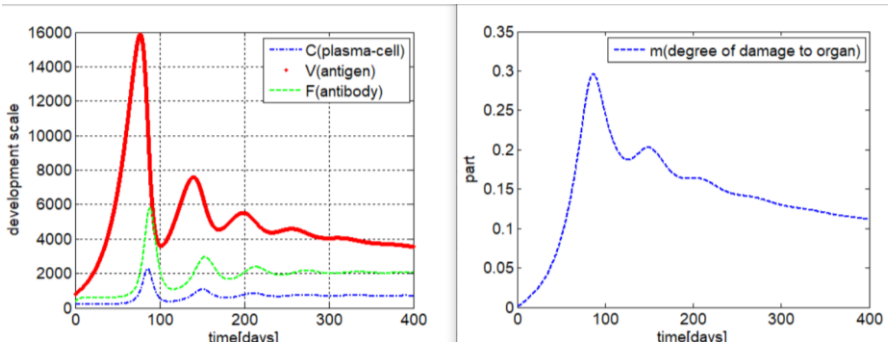
The last form we will consider **the chronic form** of the diseases, for more learning of this for, let's see the dependence on the infection dose V(0), and attempt different rate of reproduction of antigen β ($\beta_1 < \beta_2 < \beta_3$). So, here the program code with conditions as in above.

Code:

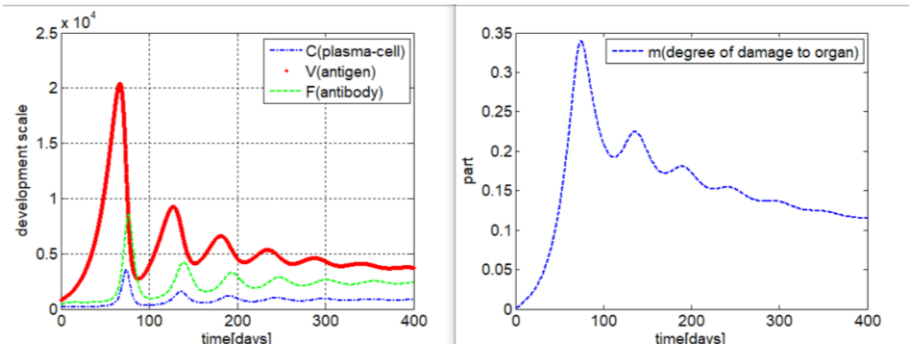
```
alpha=0.000001;%alpha*rho<>mu*eta*gamma
betta=0.06;%multiplication factor of antigens;
gamma=0.00003;%coefficient associated with the probability of neutralization of the antigen
by antibodies
etta=0.1;%the number of antibodies required to neutralize a single antigen;
sigma=0.000001;%some constant, its own for each disease
tau=3;%the time during which the formulation of cascade of plasma cells;
rho=1;%the rate of antibody production by a single plasma cell
t0=0;%initial time
tn=400;%time
dt=1/24;%taking time as days
n=(tn-t0)/dt;
T_cells=1;%life Time Plasma Cells
miu_c=1/T_cells;
T_antib = 3;%antibodies Decay Time
miu_f=1/T_antib;
Recovery= 30;%organ Recovery Period 20 for organism is needed 30 days for recoverying
miu_m=1/Recovery;
C_st=200;
F_st=rho*C_st/miu_f/2;
F=zeros(1,n);%antibody concentration;
V=zeros(1,n);%concentration of pathogenic propagating antigens;
C=zeros(1,n);%plasma cell concentration;
m=zeros(1,n);%relative characteristic of target organ damage, which is [0,1] defined;
C(1)=C_st;
V(1)=800;
F(1)=F_st;
m(1)=0;
for i=2:n
    C(i)=C(i-1)+dt*(sin(m(i-1)*pi/2)*alpha*V(i-1)*(dt*i-tau)*F(i-1)-miu_c*(C(i-1)-C_st));
    V(i)=V(i-1)+dt*((betta-gamma*F(i-1))*V(i-1));
    F(i)=F(i-1)+dt*(rho*C(i-1) - (miu_f+etta*gamma*V(i-1))*F(i-1));
    m(i)=m(i-1)+dt*(sigma*V(i-1)-miu_m*m(i-1));
    iter=i;
    if ((m(i) >= 1 || m(i) <= 0 || V(i) <= 0 || C(i) <= 0 || F(i) <= 0))
        break
   end
end
time=dt*(1:n);
fig=figure();
plot(time, C, 'b-.', 'LineWidth',2)
hold on
plot(time, V, 'r.', 'LineWidth', 2)
hold on
plot(time, F, 'g--', 'LineWidth', 2)
set(gca, 'FontSize', 14)
set(fig, 'color', 'white')
grid on
xlabel ('time[days]')
ylabel ('development scale')
legend('C(plasma-cell)', 'V(antigen)', 'F(antibody)');
fig=figure();
plot(time, m, 'b--', 'LineWidth', 2)
set(gca, 'FontSize', 14)
set(fig, 'color', 'white')
xlabel('time[days]')
ylabel('part')
legend('m(degree of damage to organ)');
```

Results:

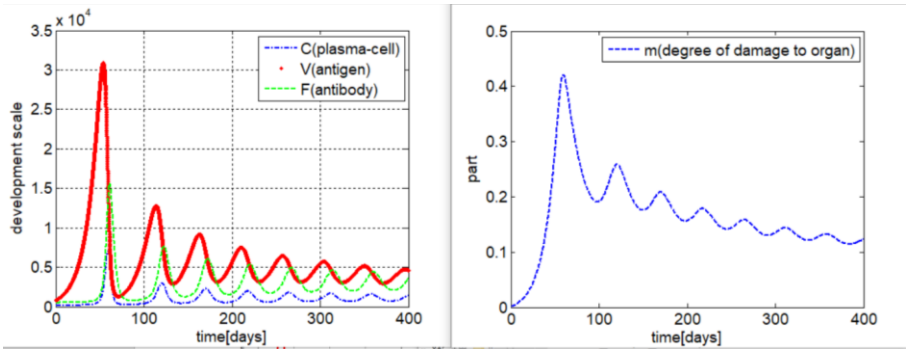
When betta=0.06



When betta=0.07

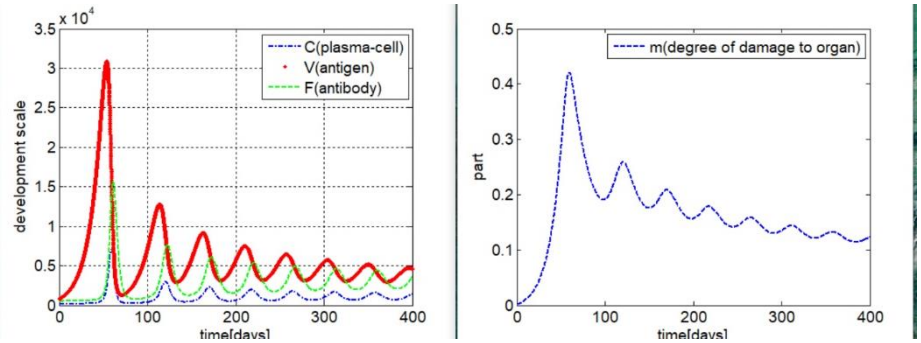


When betta=0.09



These results show us how can be chronic disease in different rates of reproduction of the antigen.

When V0=800:



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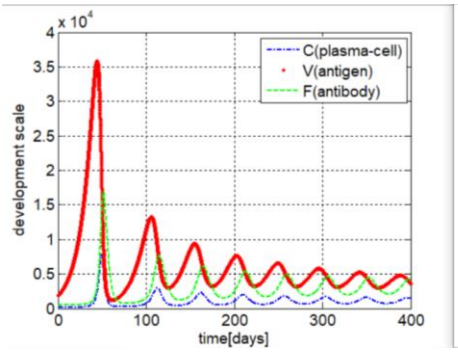
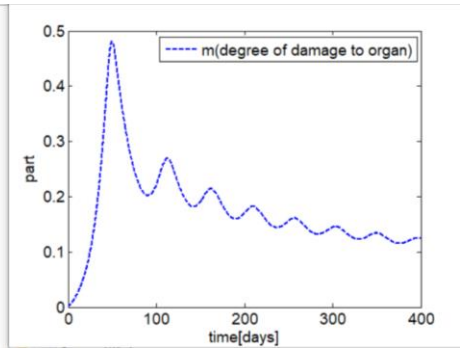
When V0=1800:





Clear to see that the disease depending on the infection dose V (0) can grow faster if give to body more dose of infection, you can see difference in two results how can be changed.

The chronic form is characterized by the presence in the body of a non-zero population of antigens with sluggish dynamics. This form is associated with insufficient stimulation of the immune system. As a rule, this chronicity is the result of reduced activity of the body's immune system. In this model, all prescribed conditions were modeled by relatively weak plasma cell multiplication rates and low pathogenicity of antigen and antibody.

Conclusion for task 1:

We have considered the basic equations for modeling the disease and named the possible forms of disease dynamics and their classification. We learnt to distinguish the forms of diseases and how to work with coefficients to get desirable results, in case to show the dynamic of diseases.

Task 2

Determine the model of interaction between the two types using known (own version) coefficients. Find the coordinates of special points. Determine the type of each stationary state found. Draw a phase portrait (Matlab).

$$\begin{cases} \frac{dx}{dt} = x(\varepsilon_x - \gamma_{xy} y - \delta_x x) \\ \frac{dy}{dt} = y(\gamma_{yx} x - \varepsilon_y - \delta_y y) \end{cases}$$

Where $\varepsilon_x = 2, \gamma_{xy} = 1, \delta_x = 1, \gamma_{yx} = 3, \varepsilon_y = 5,\ \delta_y = 1.$

Answer:

First, let us give some definitions:

1. Isocline is a curve on the phase plane which has following property, that all tangential to the integral curve which are built in points of intersect of isoclines, are parallel. Which means that all integral curves intersect isocline by the same angle.

a) For isocline of horizontal tangents define as: $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = tg\ 0 = 0, where\ Q(x,y) = 0$

b) For vertical isocline defines as: $\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} = tg\frac{\pi}{2} = \infty, where\ P(x,y) = 0$

2. Types of special points phase portrait:

a) Saddle – is a type where characteristic numbers have different sign.

b) Node – is a type where characteristic numbers have same sign. By the sign can be defined as “stable” and “not stable”.

c) Center – is a type where characteristic numbers are purely imaginary and all trajectories near to special point are closed.

d) Focus – is a type where characteristic numbers are complex conjugate with a non-zero real part. By real part can be defined as “stable” and “not stable”.

3. A stationary state is called asymptotically stable if small deviations from it decrease over time.
4. An unstable state is called if small deviations increase over time.

$$\begin{cases} \frac{dx}{dt} = x(2-y-x) \\ \frac{dy}{dt} = y(3x-5-y) \end{cases}$$

$$\begin{cases} \frac{dx}{dt} = 2x - xy - x^2 \\ \frac{dy}{dt} = 3xy - 5y - y^2 \end{cases} \quad (1)$$

Let's to make some substitution (2) and linearize the system (1):

$$\begin{cases} x = \bar{x} + \xi \\ y = \bar{y} + \eta \end{cases} \quad (2)$$

After we get system (3):

$$\begin{cases} \frac{d\xi}{dt} = 2(\bar{x}+\xi) - (\bar{x}+\xi)(\bar{y}+\eta) - (\bar{x}+\xi)^2 \\ \frac{d\eta}{dt} = 3(\bar{x}+\xi)(\bar{y}+\eta) - 5(\bar{y}+\eta) - (\bar{y}+\eta)^2 \end{cases} \quad (3)$$

$$\begin{cases} \frac{d\xi}{dt} = 2\bar{x} + 2\xi - (\bar{x}\bar{y} + \bar{x}\eta + \xi\bar{y} + \xi\eta) - \bar{x}^2 - 2\bar{x}\xi - \xi^2 \\ \frac{d\eta}{dt} = 3(\bar{x}\bar{y} + \bar{x}\eta + \xi\bar{y} + \xi\eta) - 5\bar{y} - 5\eta - \bar{y}^2 - 2\bar{y}\eta - \eta^2 \end{cases}$$

$$\begin{cases} \frac{d\xi}{dt} = 2\bar{x} + \bar{x}\bar{y} - \bar{x}^2 + \xi(2 - \bar{y} - 2\bar{x}) - \bar{x}\eta - \xi^2 - \xi\eta \\ \frac{d\eta}{dt} = 3\bar{x}\bar{y} - 5\bar{y} - \bar{y}^2 + 3\xi\bar{y} + \eta(3\bar{x} - 5 - 2\bar{y}) - \eta^2 + 3\xi\eta \end{cases}$$

$$\begin{cases} 2\bar{x} - \bar{x}\bar{y} - \bar{x}^2 = 0 \\ 3\bar{x}\bar{y} - 5\bar{y} - \bar{y}^2 = 0 \end{cases}$$

After by derivatives find a, b, c, d:

$$P'_x(\bar{x},\bar{y}) = a = 2 - \bar{y} - 2\bar{x}$$
$$P'_y(\bar{x},\bar{y}) = b = -\bar{x}$$
$$Q'_x(\bar{x},\bar{y}) = c = 3\bar{y}$$
$$Q'_y(\bar{x},\bar{y}) = d = 3\bar{x} - 5 - 2\bar{y}$$

Cut out non-linear part.

$$\begin{cases} \frac{d\xi}{dt} = a\xi + b\eta \\ \frac{d\eta}{dt} = c\xi + d\eta \end{cases} \quad (4)$$

Let's find four special points from (1) and equating it to zero:

$$\begin{cases} 2x - xy - x^2 = 0 \\ 3xy - 5y - y^2 = 0 \end{cases}$$

$$\begin{cases} x(2-y-x) = 0 \\ y(3x-5-y) = 0 \end{cases}$$

For this we need to take as a first value zero for x and y, after one by one take as zero, and final point take as non-zero.

1. $x_1 = 0, y_1 = 0$
2. $x_2 = 0, y_2 = -\frac{5}{2}$
3. $x_3 = 2, y_3 = 0$
4. $\begin{cases} 2 - y - x = 0 \\ 3x - 5 - 2y = 0 \end{cases}$

$$\begin{cases} x = 2 - y \\ 3(2-y) - 5 - 2y = 0 \end{cases}$$

$x_4 = \frac{9}{5}, y_4 = \frac{1}{5}$

Let's put points to (4) system and find coefficients a, b, c, d.

1. $(0,0)$

$a = 2, b = 0, c = 0, d = -5.$

$$\begin{cases} \frac{d\xi}{dt} = 2\xi + 0\eta \\ \frac{d\eta}{dt} = 0\xi - 5\eta \end{cases}$$

$$D = (a+d)^2 - 4(ad - bc) = (2-5)^2 - 4(-10) = 49$$

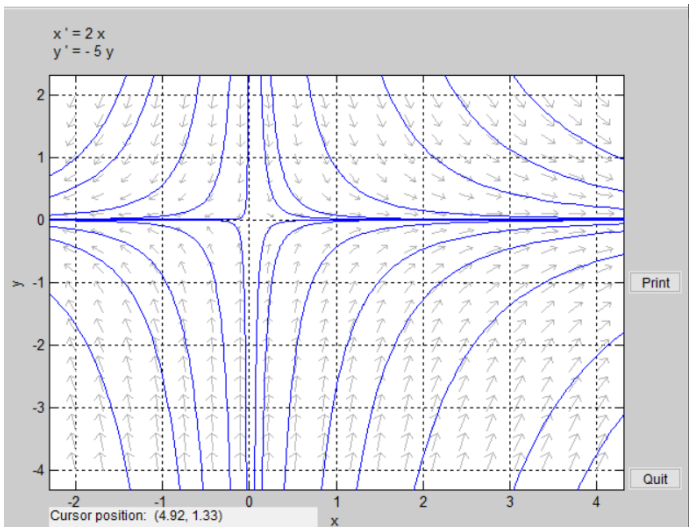
$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-3 \pm 7}{2} = 2;\ -5$$

Here λ_1, λ_2 are characteristic numbers, and are natural numbers.

By these results we can say type of phase portrait, it is saddle because $\lambda_1 * \lambda_2 < 0$.

Horizontal isocline: $\frac{d\xi}{dt} = 0, \xi = 0$

Vertical isocline: $\frac{d\eta}{dt} = 0, \eta = 0$



Saddle

2. $(0; -\frac{5}{2})$

$a = \frac{9}{2}, b = 0, c = -\frac{15}{2}, d = 0.$

$$\begin{cases} \frac{d\xi}{dt} = \frac{9}{2}\xi \\ \frac{d\eta}{dt} = -\frac{15}{2}\xi \end{cases}$$

$$D = (a+d)^2 - 4(ad - bc) = \left(\frac{9}{2}\right)^2 = \frac{81}{4}$$

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{\frac{9}{2} \pm \frac{9}{2}}{2} = \frac{9}{2};\ 0$$

By these results we cannot say exactly the type of phase portrait, we some searching to define it.

Let`s from system (4) putting coefficients a, b, c, d take matrix A:

$$A = \begin{pmatrix} \frac{9}{2} & 0 \\ -\frac{15}{2} & 0 \end{pmatrix}$$

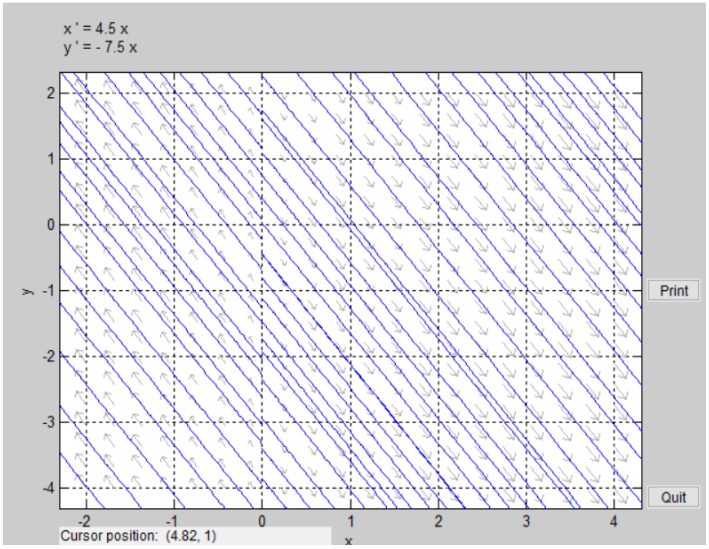
Find determinant of this matrix A and rank of matrix A.

$$detA = 0, rankA = 1$$

Then we can say that it is singular matrix.

Horizontal isocline: $\frac{d\xi}{dt} = 0, \xi = 0$

Vertical isocline: $\frac{d\eta}{dt} = 0, \xi = 0$



Singular matrix

3. (2,0)

$a = -2, b = -2, c = 0, d = 1.$

$$\begin{cases} \frac{d\xi}{dt} = -2\xi - 2\eta \\ \frac{d\eta}{dt} = \eta \end{cases}$$

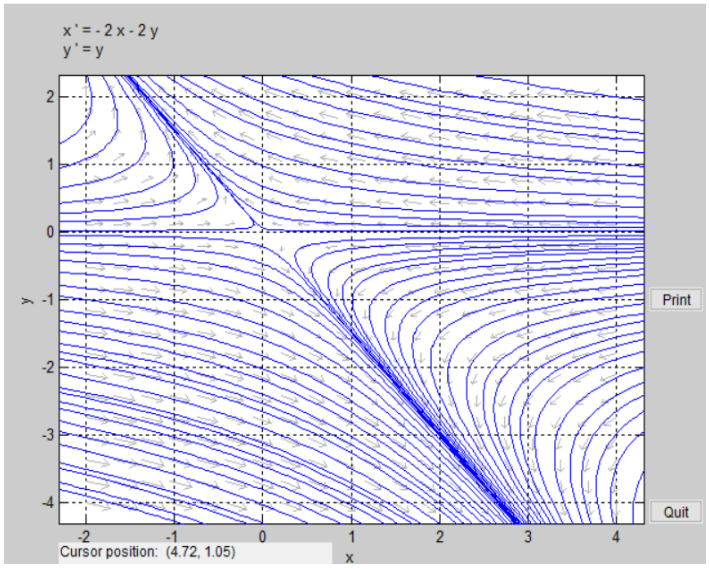
$$D = (a+d)^2 - 4(ad - bc) = (-2+1)^2 - 4(-2) = 9$$

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-1 \pm 3}{2} = 1;\ -2$$

By these results we can say type of phase portrait, it is saddle because $\lambda_1 * \lambda_2 < 0$.

Horizontal isocline: $\frac{d\xi}{dt} = 0,\ -2\xi = 2\eta,\ \xi = -\eta$

Vertical isocline: $\frac{d\eta}{dt} = 0,\ \eta = 0$



Saddle

4. $\left(\frac{9}{5}, \frac{1}{5}\right)$

$a = -\frac{9}{5}, b = -\frac{9}{5}, c = \frac{3}{5}, d = 0.$

$$\begin{cases} \frac{d\xi}{dt} = -\frac{9}{5}\xi - \frac{9}{5}\eta \\ \frac{d\eta}{dt} = \frac{3}{5}\xi \end{cases}$$

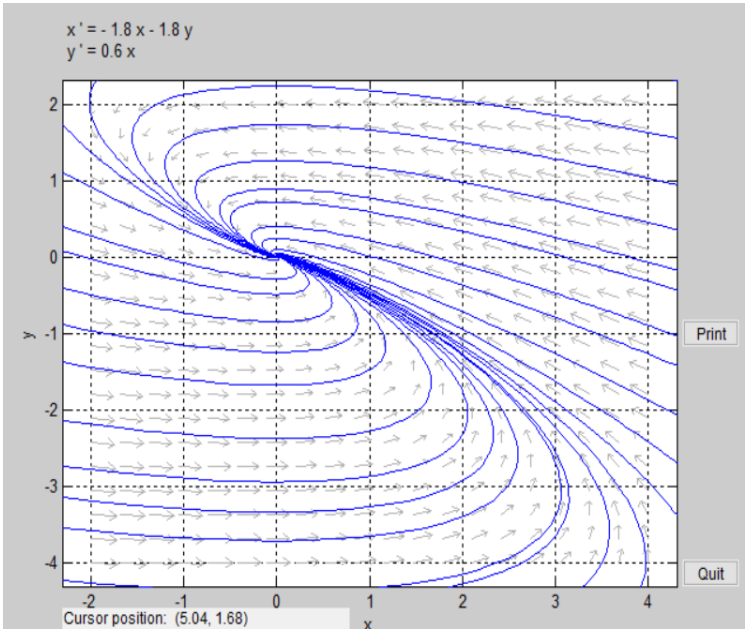
$$D = (a+d)^2 - 4(ad - bc) = \left(-\frac{9}{5}\right)^2 - 4\left(\frac{9}{5}\frac{3}{5}\right) = -\frac{27}{25}$$

$$\lambda_{1,2} = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-\frac{9}{5} \pm \sqrt{\frac{27}{25}}i}{2} = \frac{-9 \pm \sqrt{27}i}{10}$$

By these results we can say type of phase portrait, it is stable focus, because has an imaginary number in root.

Horizontal isocline: $\frac{d\xi}{dt} = 0, \eta = -\xi$

Vertical isocline: $\frac{d\eta}{dt} = 0, \xi = 0$



Stable focus

Conclusion for task 2:

We have searched for stationary states given by second-order nonlinear differential equations, and for each stationary point we drawn phase portrait and named the types of them. Moreover, we found isoclines of stationary points. By doing this task we learnt to describe the behavior of the phase trajectories of the systems.

References:

1. Marchuk, G. (1991). *Mathematical Models in Immunology. Computational Methods and Experiments.*[Математические модели в иммунологии. Вычислительные методы и эксперименты]. 3rd ed. Moscow: Science.