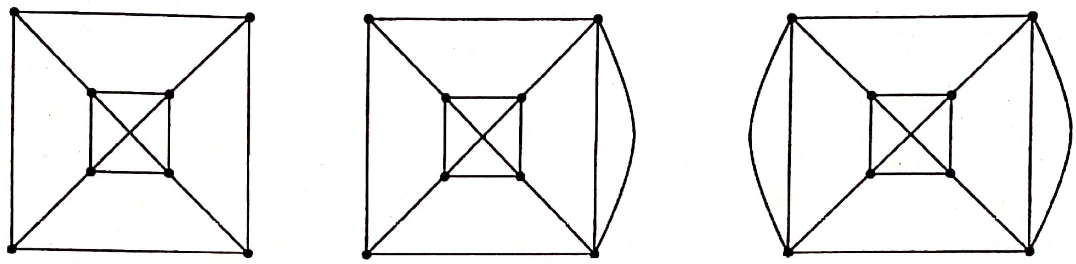
26 Graphs: initial concepts

(26.1) Discussion. Graph theory has found its application in many areas, as it provides us with a way to solve problems by use of 'dot' and 'lines'. We mention one such problem.

If we were to start from a dot, move through each line exactly once and complete the drawing, then which of the following pictures can be drawn? What if, we want the 'starting dot to be the finishing dot'?



Later, we shall see a theorem by Euler addressing this question.

(26.2) Discussion.

1. A **general graph** G is a pair (V, E) where V is a nonempty set and E is a multiset of unordered pairs of points of V.
2. The set V is called the **vertex set** and its elements are called **vertices**.
3. The set E is called the **edge set** and its elements are called **edges**.
4. An edge $\{u, v\}$ is also denoted by uv in some texts. For our convenience we will use $[u, v]$, which is relatively less common.
5. An edge of the form $[u, u]$ is called a **loop**.
6. If $G = (V, E)$ is a general graph, by $\boldsymbol{V(G)}$ we mean the vertex set V and by $\boldsymbol{E(G)}$ we mean the edge set E.
7. Let G be a general graph and $u, v \in V(G)$. If $[u, v] \in E(G)$, then we say that 'the vertices u and v are the **endvertices** of the edge $[u, v]$' or 'the vertices u and v are **adjacent**'.
8. We sometimes write $\boldsymbol{u \sim v}$ to mean that 'u is adjacent to v'.
9. Let e be an edge in a general graph G. We say 'e is **incident** on u' to mean that 'u is an endvertex of e'.

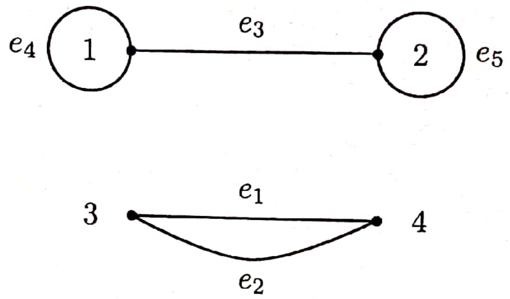
(26.3) Example. Take $G = \Big([4], \big\{[1, 1], [1, 2], [2, 2], [3, 4], [3, 4]\big\}\Big)$. It is a general graph.

(26.4) Discussion. (`Pictorial representation of a general graph.`) A general graph can be represented in picture in the following way.

1. Put different points on the paper for vertices and label them.
2. If $[u, v]$ appears in E some k times, draw k distinct lines joining the points u and v label the edges using subscripts. If $k = 1$, then we can address the edge by $[u, v]$. If you wish you can even label all the edges.

3. A loop at u is drawn if $[u, u] \in E$.

(26.5) Example. A picture for the general graph in (26.3) is given below.



Consider the general graph given in the above picture. [To be strict, it is not a general graph, it is just a pictorial representation. But we understand that we are referring to a graph whose pictorial representation is given here, namely the graph in (26.3).] Here 1 is adjacent to 1, whereas 3 is not adjacent to 3 and 3 is adjacent to 4. The vertices 1, 2 are endvertices of the edge e_3.

(26.6) Discussion. (`Multigraph, simple graph.`)

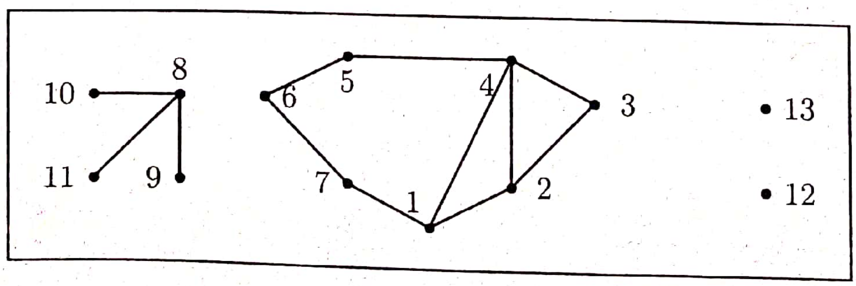
1. A **multigraph** is a general graph without loops.
2. A **simple graph** is a multigraph graph if no edge appears twice.
3. Henceforth, we shall assume that a graph is simple with a finite vertex set, unless otherwise stated. Hence, there is no need to label an edge $[u, v]$, as it can be referred as '$[u, v]$' itself.

(26.7) Discussion. (`Order, size, neighbor, isolated vertex, degree.`)

1. Let G be a graph. The number $|V(G)|$ is the **order** of the graph G. Sometimes it is denoted by $|G|$.
2. The number $|E(G)|$ is called the **size** of G, sometimes denoted by $||G||$.
3. If u is adjacent to v, then we say u is a **neighbor** of v.
4. By $\boldsymbol{N_G(v)}$ we denote the set of all neighbors of v.
5. At time we write $N(v)$ to denote $N_G(v)$, when there is only one graph under discussion.
6. A vertex v with $N(v) = \emptyset$ is called an **isolated vertex**.
7. The number $|N(v)|$ is called the **degree of v**. It is denoted by $\boldsymbol{d_G(v)}$ or simply $\boldsymbol{d(v)}$.

(26.8) Example. In the following graph G the vertex 12 is an isolated vertex, $N(1) = \{2, 4, 7\}$, and $d(1) = 3$.

G



(26.9) Facts.

1. In any graph G we have $\sum\limits_{v \in V} d(v) = 2|E|$.

   Proof. Each edge contributes 2 to the sum $\sum\limits_{v\in V} d(v)$. Hence $\sum\limits_{v \in V} d(v) = 2|E|$.

2. In any graph G, the number of vertices of odd degree is even.

   Proof. Note that

   $$\sum_{v\in V} d(v) = \sum_{d(v)\ \text{odd}} d(v) + \sum_{d(v)\ \text{even}} d(v) = 2|E|$$

   is even. So $\sum\limits_{d(v)\ \text{odd}} d(v)$ is even. So the number of vertices of odd degree is even.

3. If G is a graph of order n, then it can have at most $\binom{n}{2}$ edges.

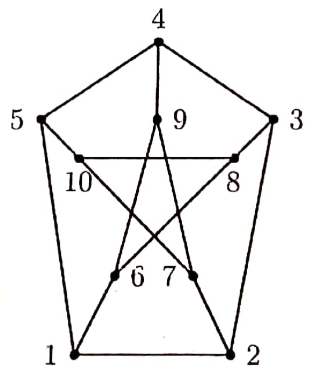
4. In a graph G with order $n \geq 2$, there are two vertices of equal degree.

   Proof. If there is an isolated vertex, then each vertex has degree less than $n - 1$. Apply PHP. If there a vertex of degree $n - 1$, then each vertex has degree at least 1. Apply PHP.

(26.10) Exercise. In a party of 27 persons, prove that someone must have an even number of friends (friendship is mutual).

(26.11) Example. (Petersen's graph.) The graph given below is called the **Petersen graph**. We shall use it as an example in many places.

Petersen graph



(26.12) Discussion.

1. A graph G is called **complete** if each pair of vertices in G are adjacent. We denote a complete graph of order n by K_n.

2. Let G be a graph. A **walk** is a sequence $[u_1, u_2, \ldots, u_k]$ of adjacent vertices, where vertices may repeat. At times we refer to this as an u_1-u_k-**walk**.

3. An u-v-walk is called **closed** if $u = v$.

4. The **length** of a walk is the number of edges on it.

5. An u-v-**path** is an u-v-walk $[u = u_1, u_2, \ldots, u_k = v]$ where the involved vertices and edges are distinct, except that we allow $u = v$. When $u = v$, we call an u-v-path a **closed path** or a **cycle**.

6. By P_n we mean a graph which is only a path on n vertices.
7. By C_n we mean a graph which is only a cycle on n vertices.

(26.13) Example.

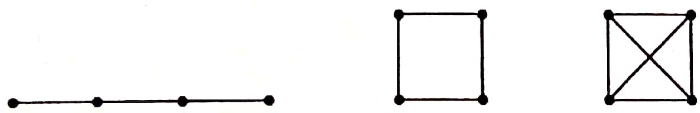
1. In the graph G in (26.11), we have $[1, 5, 1, 2, 7, 9]$ is a 1-9-walk of length 5 and $[1, 2, 1]$ is a closed walk of length 2.
2. Also, $[1, 2, 3, 4]$ is a 1-4-path of length 3 and $[1, 2, 1]$ is not a path as an edge is repeating.
3. We see that $[1, 2, 3, 4, 5, 1]$ is a cycle of length 5.

(26.14) Facts.

1. As our graphs are simple, the length of a cycle in any graph is at least 3.
2. In K_5, the maximum length of a cycle in G is 5 and the minimum length of a cycle in G is 3.
3. There are $10 = C(5, 3)$ many 3-cycles in K_5 (labeled).
4. What is the number of 4-cycles in a labeled K_5? **Ans:** $5 \times 3!/2$.
5. Let G be the Petersen graph, see (26.11). Then $[6, 8, 10, 5, 4, 3, 2, 7, 9, 6]$ is a 9-cycle in G. There are no 10-cycles in G. We shall see this when we discuss the Hamiltonian graphs.

(26.15) Remark. By definition all our graphs have vertices labeled. When the labels are not required for the intended discussion, we drop them in a picture, with the understanding that the discussion will remain the same even if we label the vertices.

(26.16) Example. The following are pictures of K_4, C_4 and P_4. Draw K_5, C_5 and P_5.



P_4 $\qquad$ C_4 $\qquad$ K_4

(26.17) Discussion.

1. The minimum degree of a vertex in a graph G is denoted by $\boldsymbol{\delta(G)}$.
2. The maximum degree of a vertex in G is denoted by $\boldsymbol{\Delta(G)}$.
3. A graph G is called **k-regular** if $d(v) = k$ for each $v \in V(G)$.
4. A 3-regular graph is called **cubic**.

(26.18) Facts.

1. For the Petersen graph, we have $\delta(G) = \Delta(G) = 3$. So it is cubic.
2. A graph has $\delta(G) = 0$ means that it has an isolated vertex.
3. The graph K_n is $n - 1$-regular.

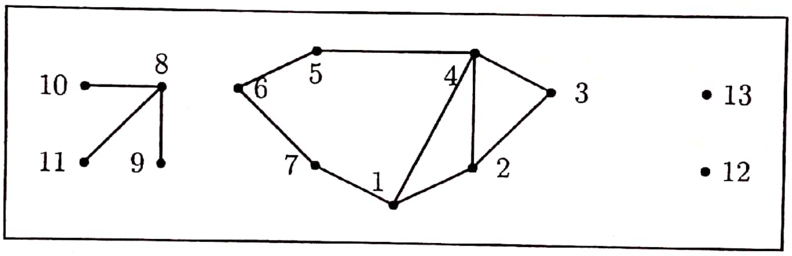
4. The graph K_4 is cubic.
5. The graph C_4 is 2-regular.
6. The graph P_4 is not regular.
7. Can we have a cubic graph on 5 vertices? **Ans:** No, as $\sum d(v) = 15$, not even.

(26.19) Discussion. (`Subgraph, induced subgraph, spanning subgraph,` k`-factor.`)

1. A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
2. If $U \subseteq V(G)$, then the subgraph **induced** by U is $\langle U \rangle := (U, E)$, where $E = \{[u, v] \in E(G) : u, v \in U\}$.
3. A subgraph H of G is a **spanning subgraph** if $V(G) = V(H)$.
4. A k-regular spanning subgraph is called a k**-factor**.

(26.20) Example. Consider the graph G below.

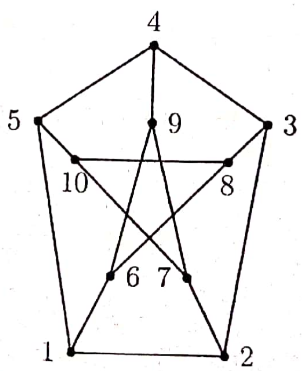
G



1. Let H_1 be the graph with $V(H_1) = \{6, 7, 8, 9, 10, 12\}$ and $E(H_1) = \{[6, 7], [9, 10]\}$. Then H_1 is not a subgraph of G, as H_1 has an edge $[9, 10]$ which is not in G.
2. Let H_2 be the graph with $V(H_2) = \{6, 7, 8, 9, 10, 12\}$ and $E(H_2) = \{[6, 7], [8, 10]\}$. Then H_2 is a subgraph of G but it is not an induced subgraph, because the induced subgraph should contain the edge $[8, 9]$ existing in G.
3. Let H_3 be the graph with $V(H_3) = \{6, 7, 8, 9, 10, 12\}$ and $E(H_3) = \{[6, 7], [8, 9], [8, 10]\}$. Then H_3 is an induced subgraph of G. That is, $H_3 = \langle \{6, 7, 8, 9, 10, 12\} \rangle$.
4. The graph G does not have a 1-factor, because if there is a 1-factor, the minimum degree cannot be 0.

(26.21) Facts.

1. A complete graph has a 1-factor iff it has an even order.
2. Consider the Petersen graph.



It has many 1-factors. Is $\{[1,6],[2,7],[3,8],[4,9],[5,10]\}$ a 1-factor? Yes. How many more are there?

Ans: There are 6 in total. Argue that a 1-factor either contains all the five edges $[1,6],[2,7],[3,8],[4,9],[5,10]$ in which case these constitute the 1-factor or contains exactly one of these edges in which case we have five more 1-factors.

One is $\{[1,6],[8,10],[7,9],[2,3],[4,5]\}$. The other four can be written similarly.

3. Remember that, by our definition $V(G)$ is always nonempty. Write an expression for the number of subgraphs of the complete graph on vertices $[10]$ (that is, $V(G)=[10]$).

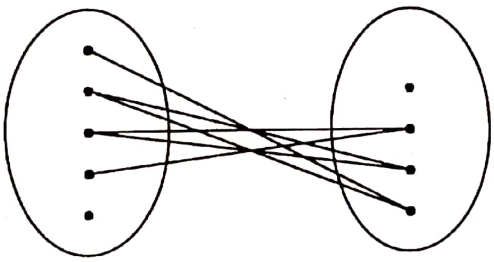
   Ans: $\sum_{k=1}^{10}\binom{10}{k}2^{\binom{k}{2}}$.

4. Consider K_8 on the vertex set $[8]$. How many 1-factors does it have? **Ans:** $8!/(2!)^4$.

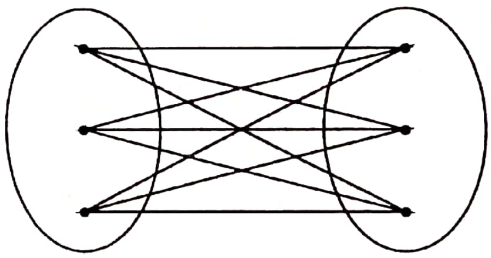
(26.22) Discussion. `(Bipartite graphs.)`

1. A graph $G=(V,E)$ is **bipartite** if $V=V_1\cup V_2$ such that $|V_1|,|V_2|\ge 1$, $V_1\cap V_2=\emptyset$, and $\|\langle V_1\rangle\|=0=\|\langle V_2\rangle\|$.
2. The complete bipartite graph with parts of size m and n is denoted by $\boldsymbol{K_{m,n}}$.

(26.23) Example. The following are some examples of bipartite graphs.



a bipartite graph



$K_{3,3}$

(26.24) Facts.

1. The number of edges in $K_{m,n}$ is mn.
2. The length of a cycle (if it exists) in any bipartite graph is always even.
3. The path P_n is bipartite.
4. The cycle C_n, $n\ge 3$ is bipartite iff n is even.
5. The graph $K_{n,n}$ is n-regular.
6. What is the maximum possible number of edges in a bipartite graph of order 11?

   Ans: 30.
7. If a bipartite graph has parts of size m and n and it is k-regular, then must $m=n$?

   Ans: Yes.

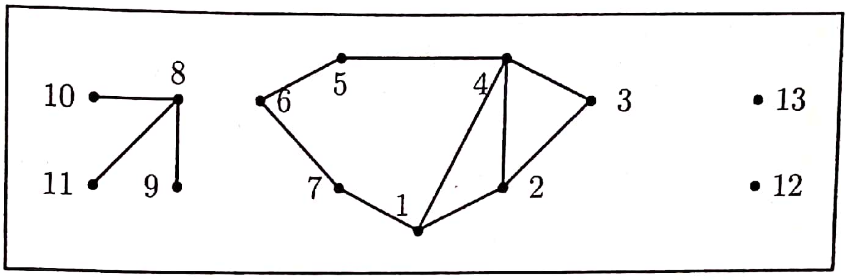
(26.25) Discussion. `(Deletion of a vertex, of an edge, addition of a non-existing edge.)`

1. Let G be a graph and v be a vertex. Then the graph $G-v$ is obtained by deleting v and all the edges that are incident with v. That is, $G-v=\langle V(G)\setminus\{v\}\rangle$.

2. If $e \in E(G)$, then then graph $G - e = (V, E(G) \setminus \{e\})$.
3. If $u, v \in V(G)$ such that $u \nsim v$, then $G + [u, v] = (V, E(G) \cup \{[u, v]\})$.

(26.26) Example. Consider the graph G below.

G



Let H be the graph with $V(H) = \{6, 7, 8, 9, 10, 12\}$ and $E(H) = \{[6, 7], [8, 10]\}$. Then $H + [8, 9]$ is the induced subgraph $\langle\{6, 7, 8, 9, 10, 12\}\rangle$ and $H - 8 = \langle\{6, 7, 9, 10, 12\}\rangle$.

(26.27) Discussion. `(Complement.)` The **complement** $\overline{G}$ of a graph G is defined as $(V(G), E)$, where $E = \{[u, v] : u \neq v, [u, v] \notin E(G)\}$.

(26.28) Example. The complement of K_3 contains 3 isolated points and the complement of P_3 consists of a P_2 and an isolated vertex.

(26.29) Facts.

1. For any graph G, we have $\|G\| + \|\overline{G}\| = C(|G|, 2)$.
2. In any graph G of order n, we have $d_G(v) + d_{\overline{G}}(v) = n - 1$. Thus $\Delta(G) + \Delta(\overline{G}) \geq n - 1$.
3. Characterize graphs G such that $\Delta(G) + \Delta(\overline{G}) = n - 1$.

   Ans: If possible, let $d_G(u) < d_G(v)$ for some two vertices u and v. Note that $d_{\overline{G}}(u) = n - 1 - d_G(u) > n - 1 - d_G(v)$. Hence,

   $$\Delta(G) + \Delta(\overline{G}) \geq d_G(v) + n - 1 - d_G(u) > d_G(v) + n - 1 - d_G(v) \geq n,$$

   a contradiction. Thus, it has to be a regular graph. Conversely, for any regular graph of order n, we see that the identity holds.

   So our answer is 'regular graphs'.
4. Can we have a family of graphs G such that $\Delta(G) + \Delta(\overline{G}) = n$?

   Ans: Yes. Consider a graph in which $\delta(G) + 1 = \Delta(G)$. Then in $\overline{G}$, we have

   $$\Delta(\overline{G}) = n - 1 - \delta(G) = n - 1 - \Delta(G) + 1.$$

   Hence $\Delta(G) + \Delta(\overline{G}) = n$.

   Such a graph G can be found by taking a disjoint collection of paths (meaning a collection paths with disjoint vertex sets).

(26.30) Discussion. `(Intersection, union.)` Let G and H be two graphs. Then the **intersection** $G \cap H$ is defined as $(V(G) \cap V(H), E(G) \cap E(H))$ and the **union** $G \cup H$ is defined as $(V(G) \cup V(H), E(G) \cup E(H))$.

(26.31) Example. Consider the graphs G and H below. Then their union and intersection are shown in the picture.

G $\quad$ H $\quad$ $G \cup H$ $\quad$ $G \cap H$

(26.32) Discussion. (Join.)

1. Let G and G' be two graphs on disjoint vertex sets. Then the **join** $G \vee G'$ is defined as
$$G \vee G' := G \cup G' + \{[v, v'] : v \in V,\, v' \in V'\}.$$
Some texts denote join by '$G + H$'.
2. When we talk about the join, we understand that the underlying vertex sets are disjoint.

(26.33) Example. We have, $K_2 \vee K_3 = K_5$ and $\overline{K_2} \vee \overline{K_2} = C_4$.

(26.34) Facts.

1. What is the complement of the disjoint union of $\overline{G}$ and $\overline{H}$? **Ans:** $G \vee H$.
2. If G and H are k-regular graphs of order m and n, respectively, determine the number of edges in $G \vee H$.

   Ans: It is $\|G\| + \|H\| + mn = \frac{mk}{2} + \frac{nk}{2} + mn$.

   When is it regular?

   Ans: Try to draw. The degree of a vertex on the left side must be the same as that of the right side. So m and n must be equal. Conversely, if $m = n$, then it is easy to see that the degree of each vertex v in $G \vee H$ is $d(v) = m + k$.
3. Is $(G_1 \cap G_2) \cup H = (G_1 \cup H) \cap (G_2 \cup H)$? **Ans:** Yes, by definition.
4. Is $(G_1 \cup G_2) \cap H = (G_1 \cap H) \cup (G_2 \cap H)$?
5. Suppose that $V(H)$ is disjoint from $V(G_1) \cup V(G_2)$. (We want to talk about join.)

   Is $(G_1 \cap G_2) \vee H = (G_1 \vee H) \cap (G_2 \vee H)$? **Ans:** Yes.

   Is $(G_1 \cup G_2) \vee H = (G_1 \vee H) \cup (G_2 \vee H)$? **Ans:** Yes.
6. If the underlying vertex sets are all disjoint, is $(G_1 \cup G_2) \vee (H_1 \cup H_2)$ a proper subgraph of $(G_1 \vee G_2) \vee (H_1 \vee H_2)$? **Ans:** Yes.

(26.35) Exercise. Show that a k-regular simple graph on n vertices exists iff kn is even and $n \geq k + 1$.

(26.36) Exercise. (Edge density.) The **edge density** $\varepsilon(G)$ is defined to be the number $\frac{|E(G)|}{|V(G)|}$.