PROBABILITY REFRESHER: +2 LEVEL CONCEPTS

Arun P R

CONTENTS

- Ref : NCERT Mathematics Textbooks for Std XI and XII
  - Latest versions can be found here : NCERT
- Sets (XI, Chapter 1)
- Permutations and Combinations (XI, Chapter 7)
- Probability (XI, Chapter 16)
- Probability Distributions (XII, Chapter 13 + additional concepts)

Sets

- Foundation of relations and functions, sequences, geometry, probability theory, etc.
- A set is a well defined collection of objects.
  - Sets are usually denoted by upper case letters A, B, C, etc.
  - Elements of a set are represented by lower case letters a, b, c, etc.
  - If a is an element of a set A, we say that " a belongs to A" the Greek symbol $\in$ (epsilon) is used to denote the phrase 'belongs to'. Thus, we write $a \in A$.
  - If 'b' is not an element of a set A, we write $b \notin A$ and read "b does not belong to A".
- A few examples of sets used particularly in mathematics, viz.
  - **N** : the set of all natural numbers
  - **Z** : the set of all integers , $\mathbf{Z}^+$: the set of positive integers
  - **Q** : the set of all rational numbers , $\mathbf{Q}^+$: the set of positive rational numbers
  - **T** : the set of all irrational numbers
  - **R** : the set of real numbers , $\mathbf{R}^+$: the set of positive real numbers
  - **C** : the set of all complex numbers

Sets

- Empty Set /Null Set/Void Set
  - A set which does not contain any elements.
  - The empty set is denoted by the symbol φ or { }.
- Finite and Infinite sets
  - A set which is empty or consists of a definite number of elements is called finite
  - Otherwise, the set is called infinite.
- Equal Sets
  - Two sets A and B are said to be equal if they have exactly the same elements, $A = B$.
  - Otherwise, the sets are said to be unequal and we write $A \neq B$
- Equivalent Sets
  - Two finite sets A and B are said to be equivalent if they have the same number of elements, $A \leftrightarrow B$
  - For Example $A=\{1, 2, 3, 4, 5\}$, $B=\{a, b, c, d, e\}$

Sets

- Subsets
  - A set A is said to be a subset of a set B if every element of A is also an element of B.
    - $A \subset B$ if whenever $a \in A$, then $a \in B$
    - $A \subset B$, $a \in A \Rightarrow a \in B$, for all $a \in A$
  - Every set A is a subset of itself, i.e., $A \subset A$ (A is called improper subset of A).
  - Since the empty set φ has no elements, φ is a subset of every set
  - If $A \subset B$ and $A \neq B$, then A is called a proper subset of B and B is called superset of A
  - Proper subsets of A include all subsets of A including φ except the improper subset
  - If a set has "n" elements, then the number of subset of the given set is 2^n
  - **$\mathbf{N} \subset \mathbf{Z} \subset \mathbf{Q}$, $\mathbf{Q} \subset \mathbf{R}$, $\mathbf{T} \subset \mathbf{R}$, $\mathbf{N} \not\subset \mathbf{T}$**
- Power Set
  - The set of all subsets of set A is called the power set.
  - This includes the improper subset and all proper subsets.

Sets

- Universal Set
  - Usually, in a particular context, we have to deal with the elements and subsets of a basic set which is relevant to that particular context. This basic set is called the "Universal Set".
  - The universal set is usually denoted by U, and all its subsets by the letters A, B, C, etc.
  - For example, in human population studies, the universal set consists of all the people in the world
- Operations on sets
  - Union of Sets : $A \cup B$, elements present in both sets
  - Intersection of Sets : $A \cap B$, elements common to both sets
  - Set Difference : $A – B$, remove $A \cap B$ from A
  - Symmetric Difference : $A \Delta B = (A - B) \cup (B - A)$

Combinatorics

- Fundamental Principle of Counting (Multiplication Principle)
  - If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is $m \times n$.
  - The above principle can be generalized for any finite number of events.
  - For example, for 3 events, the principle is as follows: If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to the events in the given order is $m \times n \times p$.
  - How many 3 letter code words are possible using the first 10 letter of English Alphabet, if
    - No letter can be repeated – $10 \times 9 \times 8 = 720$
    - Letters are repeated – $10 \times 10 \times 10 = 1000$
- Factorial notation
  - n! represents the product of first n natural numbers
  - $n! = n \times (n - 1) \times (n - 2) \times \ldots. \times 2 \times 1 = n \times (n - 1)!$
  - $0! = 1$

PERMUTATIONS

- A permutation is an arrangement in a definite order of a number of objects taken some or all at a time.
- Permutations when all objects are distinct
  - The number of permutations of n different objects, taken all at a time is given by $P(n, n) = n!$
  - The number of permutations of n different objects taken r at a time, where $0 < r \leq n$ and the objects do not repeat is $n(n-1)(n-2)\ldots(n-r+1)$, which is denoted by $P(n, r)$.
  - $P(n, r) = n! / (n-r)!$
- Permutations when all objects are not distinct
  - The number of permutations of n objects, where p1 objects are of one kind, p2 are of second kind, ..., pk are of k th kind and the rest, if any, are of different kind is $n! / (p1! \times p2! \times \ldots \times pk!)$
- Circular permutations
  - No of distinguishable circular permutations for n distinct objects arranged in a circle is $(n-1)!$

COMBINATIONS

- A combination is a selection of some of all of a number of different objects. The order of selection of the objects is immaterial.
- The number of combinations of n distinct objects, taken r at a time, denoted by $C(n, r)$
  - $C(n, r) = n! / (r! \times (n-r)!) = P(n, r) / r!$
  - $C(n, n) = C(n, 0) = 1$
  - $C(n, r) = C(n, n - r)$
- Pascal's rule
  - $C(n+1, r) = C(n, r) + C(n, r-1)$

COMBINATORIAL PROBLEMS

- No of options to choose from : n
- Ordered samples of size r, without replacement
  - Permutation, No of possible Outcomes = P(n, r)
  - Example : No of 3 digit numbers that can be formed with 1,2,….,9 without repetition
  - Example : Choose 2 balls from a bag containing 5 numbered balls, order of numbers matters
- Unordered samples of size r, without replacement
  - Combination, No of possible Outcomes = C(n, r)
  - Example : Choose 2 balls from a bag containing 5 numbered balls, order of numbers do not matter

COMBINATORIAL PROBLEMS

- No of options to choose from : n
- Ordered samples of size r, with replacement
  - No of possible Outcomes = n^r
  - Example : Three dice are rolled together or a dice is rolled three times, order of numbers matters
  - Example : Choose 1 ball from a bag containing 5 numbered balls 2 times, order of numbers matters
- Unordered samples of size r, with replacement
  - Combination, No of possible Outcomes = C(n + r –1 , r)
  - Example : Three dice are rolled together or a dice is rolled three times, order of numbers do not matter
  - Example : Choose 1 ball from a bag containing 5 numbered balls 2 times, order of numbers do not matter

Basic Probability theory

- Probability is the measure of uncertainty of random experiments.
- Classical Definition : The probability of an event is the number of outcomes favorable to the event , divided by the total number of outcomes, where all outcomes are equally likely
  - It only considers experiments with a finite number of outcomes (and hence restrictive)
  - All outcomes are considered to be equally likely (again this is a circular definition since we are using the concept of probability to define probability itself)
  - Both these conditions are valid if we consider classical probability problems like tossing of an (**unbiased**) coin, rolling of (**fair**) die, or picking a card from a (**perfectly shuffled**) deck of cards
  - But this classical definition cannot be used in the construction of a mathematical theory of probability

Basic Probability theory

- Frequentist / Statistical approach of probability :
  - Find the probability on the basis of observations and collected data
  - Frequentist statistics uses rigid frameworks, the type of frameworks that you learn in basic statistics, like p-values and confidence intervals.
  - Frequentist probability has more applicability than the classical model, but is still very limited
- Axiomatic Approach of Probability
  - It sets down a set of axioms (rules) that apply to all of types of probability, including frequentist probability and classical probability.
  - These rules, based on Kolmogorov's Three Axioms, set starting points for mathematical probability.
  - Kolmogorov's Three Axioms
    1. For any event A, $P(A) \geq 0$
    2. $P(S) = 1$, where S is the sample space of an experiment; i.e., the set of all possible outcomes
    3. If A and B are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$.

Basic Probability theory

- Random Experiment : An experiment is called random experiment if it satisfies the following two conditions:
  1. It has more than one possible outcome.
  2. It is not possible to predict the outcome in advance.
- Outcomes and Sample Space (S)
  - A possible result of a random experiment is called its outcome. Each outcome of the random experiment is also called sample point.
  - The set of all possible outcomes of a random experiment is called the sample space associated with the experiment
  - A given performance of the experiment must produce a result corresponding to exactly one of the points of S.
- Event : An event is a subset of the sample space, I.e. a collection of points of the sample space.
- Occurrence of an event
  - The event E of a sample space S is said to have occurred if the outcome ω of the experiment is such that $\omega \in E$.
  - If the outcome ω is such that $\omega \notin E$, we say that the event E has not occurred

Types of events

- Simple event: A simple event or an elementary event is an event containing only a single sample point.
- Compound events: Compound events or decomposable events are those events that are obtained by combining together two or more elementary events.
  - For instance, the event of drawing a heart from a deck of cards is the subset A = {heart} of the sample space S = {heart, spade, club, diamond}. Therefore, A is a simple event.
  - The event B of drawing a red card is a compound event since B = {heart U diamond} = {heart or diamond}.
- Mutually exclusive or disjoint events: Events are said to be mutually exclusive or disjoint or incompatible if the occurrence of any one of them prevents the occurrence of all the other events.
  - Getting a head and getting a tail in toss of a coin are mutually exclusive
- Mutually non-exclusive events: The events which are not mutually exclusive are known as compatible events or mutually non-exclusive events.
  - Getting a card of heart suite and getting a red card are not mutually exclusive.

Types of events

- Independent events: Events are said to be independent, if the happening (or non-happening) of one event is not affected by the happening (or non-happening) of other events.
  - If I toss two coins simultaneously, the outcomes of both trials are independent of each other
- Dependent events: Two or more events are said to be dependent, if the happening of one event affects (partially or totally) the other event.
- Exhaustive Events: A Set of events is said to be exhaustive if the performance of random experiments always results in the occurrence of at least one of them.
  - For instance, consider an ordinary pack of cards. The events 'drawn card is heart', drawn card is diamond', 'drawn card is club' and 'drawn card is spade' is a set of events that is exhaustive.
  - In other words all sample points put together (i.e. sample space itself) would give us an exhaustive event.
  - If 'E' is an exhaustive event/sure event then P (E) = 1.

Example

- Random Experiment : Toss a coin
- Sample Space (Ω) : In tossing of a coin, the sample space for the number that shows up on the top face would be : Ω = {H, T}
- For the coin tossing case, (2^2) = 4 subsets can be formed out of that sample space.
  - These subsets are: Φ, {H}, {T} and {H, T}.
- Here getting a head or tail are equally likely events.
- Event :
  - Getting a head in a single toss of a coin, E1 = {H} ⊂ Ω
  - Getting a tail in a single toss of a coin, E2 = {T} ⊂ Ω
  - Getting a head or a tail in a single toss of a coin, E3 ⊆ Ω
  - Not Getting either a head nor a tail in a single toss of a coin, E4 = ∅ ⊂ Ω

Example

- Random Experiment : Picking a card of a particular suite from a deck of cards
- Sample Space (Ω) : In picking a card of a particular suite from a deck of cards, the sample space will be : Ω = {heart, spade, club, diamond}
- For the suite picking case, (2^4) = 16 subsets can be formed out of that sample space.
- Here picking any particular suite of card is equally likely
- Event :
  - Getting a Spade card from a deck of cards, E2 = {Spade} ⊂ Ω
  - Getting a Diamond card from a deck of cards, E2 = {Diamond } ⊂ Ω
  - Getting a red or black card from a deck of cards, E3 ⊆ Ω
  - Getting a card with symbol circle from a deck of cards, E4 = ∅ ⊂ Ω

Example

- Random Experiment : Rolling a die
- Sample Space (Ω) : In rolling a die, the sample space for the number that shows up on the top face would be : $\Omega = \{1, 2, 3, 4, 5, 6\}$
- For the die rolling case, $(2^6) = 64$ subsets can be formed out of that sample space.
- Here getting any number from 1 to 6 is equally likely
- Event :
  - Getting an odd no in a single throw of a die, $E1 = \{1,3,5\} \subset \Omega$
  - Getting a prime no in a single throw of a die, $E2 = \{2,3,5\} \subset \Omega$
  - Getting a no less than 7 in a single throw of a die, $E3 \subseteq \Omega$
  - Getting a no greater than or equal to 7 in a single throw of a die, $E4 = \emptyset \subset \Omega$

Example

- Random Experiment : Picking a card from a deck of cards
- Sample Space (Ω) : In picking a card from a deck of cards, the sample space will consist of 52 entries corresponding to each card of the deck.
- For the card picking case, (2^52) subsets can be formed out of that sample space.
- Here picking any particular card is equally likely
- Event :
  - Getting a King of Spades from a deck of cards, E1 = {King of Spades} $\subset \Omega$
  - Getting a Diamond card from a deck of cards, E2 = {K,Q,J,A,10,9,8,7,6,5,4,3,2 of Diamond } $\subset \Omega$
  - Getting a red or black card from a deck of cards, E3 $\subseteq \Omega$
  - Getting a card with number 1 from a deck of cards, E4 = $\emptyset \subset \Omega$

CONDITIONAL PROBABILITY

- The probability of occurrence of an event A, given that B has already occurred is called the conditional probability of occurrence of A.
  - It is denoted by $P(A \mid B)$.
  - If the event B has already occurred, then the sample space reduces to B.
  - Now the outcome favorable to the occurrence of A (given that B has already occurred) are those that are common to both A and B, that is, those which belong to $A \cap B$.
  - $P(A|B) = P(A \cap B) / P(B)$, $P(B) \neq 0$
- Similarly, $P(B|A) = P(A \cap B) / P(A)$, $P(A) \neq 0$
- If A and B are two independent events, then $P(B|A) = P(B)$, $P(A|B) = P(A)$

A FEW THEOREMS ON PROBABILITY

- If A and B are two events,
  - Addition Rule of Probability
    - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
    - If A and B are two mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$, since $P(A \cap B) = 0$
  - Multiplication Rule of Probability
    - $P(A \cap B) = P(B|A) * P(A) = P(A|B) * P(B)$
    - If A and B are two independent events, then $P(A \cap B) = P(A) * P(B$
- Probability of Complement of an Event, $P(A') = 1 – P(A)$
- If A, B and C are three events, then
  - $P(A \cup B \cup C) = P(A) + P(B) + P(C) – P(B \cap C) – P(C \cap A) – P(A \cap B) + P(A \cap B \cap C)$
  - $P(E \cap F \cap G) = P(E) * P(F|E) * P(G|(E \cap F)) = P(E) * P(F|E) * P(G|EF)$

Baye’s Theorem

- Theorem of total probability
  - Let {E1 , E2 ,...,En } be a partition of the sample space S, and suppose that E1, E2 ,..., En are mutually exclusive and exhaustive events
  - Let A be any event associated with S that occurs with E1, E2 ,..., En , then
  - $P(A) = P(E_1) P(A|E_1) + P(E_2) P(A|E_2) + \ldots + P(E_n) P(A|E_n) = \Sigma(P(E_i) P(A|E_i))$, $i=1,\ldots,n$
- Bayes’ Theorem
  - Let {E1 , E2 ,...,En } be a partition of the sample space S, and suppose that E1, E2 ,..., En are mutually exclusive and exhaustive events
  - Let A be any event associated with S that occurs with E1, E2 ,..., En , then
  - $P(E_i | A) = P(E_i) P(A|E_i) / P(A)$

Application

	P(A is true)	**P(A is not true)**	
P(Test for A is false)	P(A ∩ TA') – Type I Error (α)	P(A' ∩ TA')	P(TA')
P(Test for A is true)	P(A ∩ TA)	P(A' ∩ TA) – Type II Error (β)	P(TA)
	P(A)	P(A')	1

Example: Assume A is the Event that you are Covid +ve
Is a Type I error or Type II error more problematic in this case?

Thank You !!!