6CS012-ARTIFICIAL INTELLIGENCE AND MACHINE LEARNING
COHORT-08;HERALD COLLEGE
UNIVERSITY OF WOLVERHAMPTON

Problem Set-I
Linear Algebra and Calculus: A Review.

Siman Giri

March 7, 2024

Instructions:

- Solve all the Problems.

1 Preliminary Concepts-Basic set Theory and Function

1. Suppose that:

$$A = \{x : x \in \mathbb{N} \text{ and } x \text{ is even}\},$$
$$B = \{x : x \in \mathbb{N} \text{ and } x \text{ is prime}\}, \text{ and}$$
$$C = \{x : x \in \mathbb{N} \text{ and } x \text{ is a multiple of } 5\}.$$

Describe Following Sets:
(a) $A \cap B$ (The intersection of sets A and B):

(b) $B \cap C$ (The intersection of sets B and C):

(c) $A \cup B$ (The union of sets A and B):

(d) $A \cap (B \cup C)$ (The intersection of set A and the union of sets B and C):

Solution:

2. Prove:

$$i) A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$ii) A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Solution:

3. Define a function:

$$f : \mathbb{N} \to \mathbb{N}$$

a) that is one-to-one but not onto. b) that is onto but not one-to-one.

Solution:

---

2 Matrix Operations.

1. The diagonal of a matrix A are the entries a_{ij}; where $i = j$.
   a) Write down the three-by-three matrix with ones on diagonal and zeros else where.
   b) Write down the three-by-four matrix with ones on diagonal and zeros elsewhere.
   c) Write down the four-by-three matrix with ones pn the diagonal and zeros elsewhere.

---

2. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ Compute AD and DA.
{observe and Explain the Results}.

---

3. Find the inverse of the matrices $A = \begin{bmatrix} 5 & 6 \\ 4 & 5 \end{bmatrix}; B = \begin{bmatrix} 6 & 4 \\ 3 & 3 \end{bmatrix}$.

   Hint:

   - We know:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det(A)} . \begin{bmatrix} C_{11} & C_{21} \\ c_{12} & C_{21} \end{bmatrix} = \frac{1}{ad - bc} . \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

---

4. Given matrices A and B below, verify that they are inverses.

$$A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}; B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \end{bmatrix}$$

   Hint:

   - The matrices are inverses if the product AB and BA both equal the identity matrix I_{2X2}.

2.1 SOLVE: SYSTEM OF LINEAR EQUATIONS

2.1.1 USING INVERSE METHODS

1. Express the system as AX = B ; then solve using matrix inverse.

   a)

$$\begin{aligned} x + 2y &= 4 \\ 3x - 5y &= 1 \end{aligned}$$

   Answers: x = 2 and y = 1.

   b)

$$\begin{aligned} 5x + y &= 13 \\ 3x + 2y &= 5 \end{aligned}$$

   Answers: $x = 3; y = -2$

   c)

$$\begin{aligned} 3x + 2y &= -2 \\ x + 4y &= 6 \end{aligned}$$

   Answers: $x = -2; y = 2$

2. **Self Study Notes and Hints:**

   - The Method for Finding the inverse of a Matrix with Elementary Row Operations:
     a) Write the augmented matrix $[A|I_n]$
     b) Write the augmented matrix in step a in reduced row echelon form.
     c) If the reduced row echelon form in b is in $[I_n|B]$,then B is the inverse of A.
     d) If the left side of the row reduced echelon is not an identity matrix, the inverse does not exist.
   - The Method for Solving a System of Equations when a Unique Solution Exists:
     a) Express the system in the matrix equation $AX = B$.
     b) To solve the equation $AX = B$, we multiply both sides by A^{-1}:

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= A^{-1}B \\ \{\textit{Where I is the identity matrix}\}. \end{aligned}$$

---

3. Solve the following system:

$$\begin{aligned} x - y + z &= 6 \\ 2x + 3y &= 1 \\ -2y + z &= 5 \end{aligned}$$

Solutions:
Write the system in matrix form AX = B as follows:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

To solve this system, we need inverse of A.
Let's first find A^{-1} with row echelon methods:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Write in a Augmented Matrix Form:

$$\begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 2 & 3 & 0 & | & 0 & 1 & 0 \\ 0 & -2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

Reduce to Row Echelon Form using Gauss-Jordan Method.
Performing Elementary Row operations:

$R2 : -2 * R1 + R2 \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 2 & 5 & -2 & | & -2 & 1 & 0 \\ 0 & -2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$

Swap $R2 <--> R3 \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & -2 & 1 & | & 0 & 0 & 1 \\ 0 & 5 & -2 & | & -2 & 1 & 0 \end{bmatrix}$

$R2 : \frac{R2}{-2} \begin{bmatrix} 1 & -1 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & -1/2 & | & 0 & 0 & -1/2 \\ 0 & 5 & -2 & | & -2 & 1 & 0 \end{bmatrix}$

$R1 : R1 + R2$ and $R3 : -5 * R2 + R3 \begin{bmatrix} 1 & 0 & 1/2 & | & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & | & 0 & 0 & -1/2 \\ 0 & 0 & 1/2 & | & -2 & 1 & 5/2 \end{bmatrix}$

$R3 : 2 * R3 \begin{bmatrix} 1 & 0 & 1/2 & | & 1 & 0 & -1/2 \\ 0 & 1 & -1/2 & | & 0 & 0 & -1/2 \\ 0 & 0 & 1 & | & -4 & 2 & 5 \end{bmatrix}$

$R2 : \frac{1}{1} * R3 + R2$ and $R1 : \frac{-1}{2} * R3 + R1 \begin{bmatrix} 1 & 0 & 0 & | & 3 & -1 & -3 \\ 0 & 1 & 0 & | & -2 & 1 & 2 \\ 0 & 0 & 1 & | & -4 & 2 & 5 \end{bmatrix}$

Thus the inverse of A is:

$A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$

Multiplying both sides of the matrix equation $AX = B$ on the left by A^{-1}, we get

$$\begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \\ 5 \end{bmatrix}$$

After multiplying the matrix, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

4. Solve the following system{Gauss Elimination or Gauss-jordan Method}:

$$\begin{aligned} x + y - z &= 2 \\ x + z &= 7 \\ 2x + y + z &= 13 \end{aligned} \qquad \begin{aligned} x + y + z &= 2 \\ 3x + y &= 7 \\ x + y + 2z &= 3 \end{aligned}$$

Solutions (a):
Writing in the Matrix Form AX = B:

$$\begin{bmatrix}1 & 1 & -1\\ 1 & 0 & 1\\ 2 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2\\ 7\\ 13\end{bmatrix}$$

Gauss Elimination Method:
Write in the Augmented Form:

$$\begin{bmatrix}1 & 1 & -1 & | & 2\\ 1 & 0 & 1 & | & 7\\ 2 & 1 & 1 & | & 13\end{bmatrix}$$

Performing Elementary Row Operations:

$$R2 : R2 - 1.R1 \begin{bmatrix}1 & 1 & -1 & | & 2\\ 0 & -1 & 2 & | & 5\\ 2 & 1 & 1 & | & 13\end{bmatrix}$$

$$R3 : R3 - 2.R1 \begin{bmatrix}1 & 1 & -1 & | & 2\\ 0 & -1 & 2 & | & 5\\ 0 & -1 & 3 & | & 9\end{bmatrix}$$

$$R3 : R3 - 1.R2 \begin{bmatrix}1 & 1 & -1 & | & 2\\ 0 & -1 & 2 & | & 5\\ 0 & 0 & 1 & | & 4\end{bmatrix}$$

We have reached the Row echelon Form, Thus rewriting the matrix equation:

$$\begin{bmatrix}1 & 1 & -1\\ 0 & -1 & 2\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}2\\ 5\\ 4\end{bmatrix}$$

This can also be written as linear combinations of columns as:

$$x\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} + y\begin{bmatrix}1\\ -1\\ 0\end{bmatrix} + z\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix} = \begin{bmatrix}2\\ 5\\ 4\end{bmatrix}$$

So the equation now becomes:

$$\begin{aligned} x + y - z &= 2\\ -y + 2z &= 5\\ z &= 4 \end{aligned}$$

Solving the above equation:

$$\begin{aligned} x &= 3\\ y &= 3\\ z &= 4 \end{aligned}$$

Solution(b):{Using Gauss-Jordan Method}
Writing the Matrix in the form AX = B:

$$\begin{bmatrix}1 & 1 & 1\\3 & 1 & 0\\1 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2\\7\\3\end{bmatrix}$$

Convert to augmented matrix and perform elementary row operation to achieve reduced row echelon form:

$$\begin{bmatrix}1 & 1 & 1 & | & 2\\3 & 1 & 0 & | & 7\\1 & 1 & 2 & | & 3\end{bmatrix}$$

$$R2 : R2-3R1 \begin{bmatrix}1 & 1 & 1 & | & 2\\0 & -2 & -3 & | & 1\\1 & 1 & 2 & | & 3\end{bmatrix}$$

$$R3 : R3-1.R1 \begin{bmatrix}1 & 1 & 1 & | & 2\\0 & -2 & -3 & | & 1\\0 & 0 & 1 & | & 1\end{bmatrix}$$

$$R2 : \tfrac{R2}{-2} \begin{bmatrix}1 & 1 & 1 & | & 2\\0 & 1 & 3/2 & | & -1/2\\0 & 0 & 1 & | & 1\end{bmatrix}$$

$$R2 : R2-3/2R3 \begin{bmatrix}1 & 1 & 1 & | & 2\\0 & 1 & 0 & | & -2\\0 & 0 & 1 & | & 1\end{bmatrix}$$

$$R1 : R1-1.R3 \begin{bmatrix}1 & 1 & 0 & | & 1\\0 & 1 & 0 & | & -2\\0 & 0 & 1 & | & 1\end{bmatrix}$$

$$R1 : R1-1.R2 \begin{bmatrix}1 & 0 & 0 & | & 3\\0 & 1 & 0 & | & -2\\0 & 0 & 1 & | & 1\end{bmatrix}$$

We have reached the reduced row echelon form, thus the solution is:

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-2\\1\end{bmatrix}$$

---

5. For Practise:

   - Solve the following:

$$\begin{aligned}-x-2y+z&=1\\2x+3y&=2\\y-2z&=0\end{aligned}\qquad\begin{aligned}x+4y-z&=4\\2x+5y+8z&=15\\x+3y-3z&=1\end{aligned}\qquad\begin{aligned}5x+3y+9z&=-1\\-2x+3y-z&=-2\\-x-4y+5z&=1\end{aligned}$$

3 Calculus: Derivative Review

3.1 Scalar Derivative.

1. Things to remember from derivative:

   a) The derivative of a function at a point represents the slope of the tangent line at that point.i.e. $f^{'}(a)$ = "the slope of the tangent line to the graph of $f(x)$ at $x = a$..

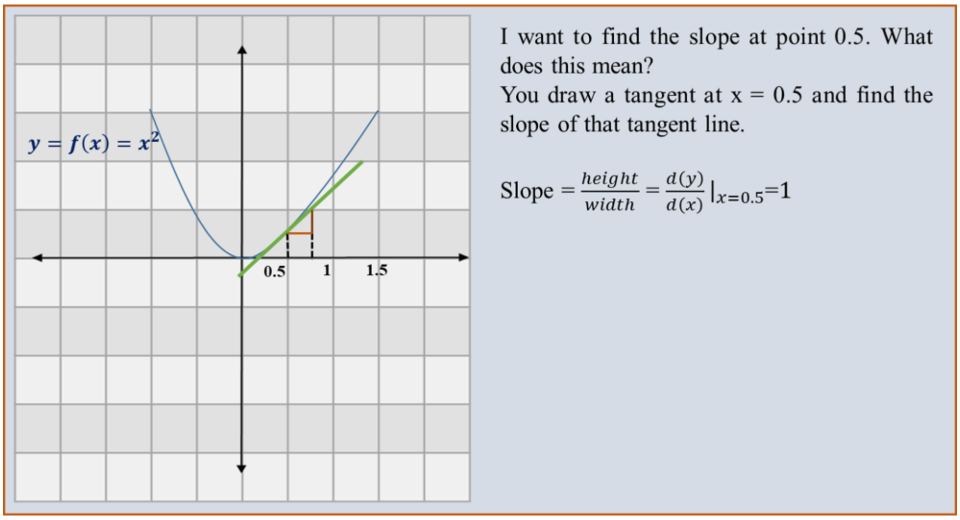

Figure 3.1: Derivative of a Non-linear Function{Slide No-38}

   b) Derivative can be used to understand the graph of any function such as $f(x)$:
      - if $\mathbf{f^{'}(x) > 0}$, then the graph of **f(x)** is increasing.
      - if $\mathbf{f^{'}(x) < 0}$, then the graph of **f(x)** is decreasing.
      - if $\mathbf{f^{'}(x) = 0}$, then the graph of **f(x)** has a horizontal tangent.{Neither Decreasing nor Increasing}.
      - if derivative of any function $f(x)$ i.e. $f^{'}(x)$ changes
        - from positive to negative at some point $x = c$, then f has local maximum at $x = c$.
        - from negative to positive at some point $x = c$, then f has local minimum at $x = c$.

c) Derivative of specific kind of functions:

Function - Type	Function - Notation	Derivative
Constant function	$f(x) = c$; where c is real constant.	$f'(x) = (c)' = 0$.
Identity function	$f(x) = x$	$f'(x) = (x)' = 1$.
Linear function	$f(x) = mx$	$f'(x) = (mx)' = m$.
Function of the form	$f(x) = x^n$	$f'(x) = (x^n)' = nx^{n-1}$.
Exponential function of the form	$f(x) = a^x$; where $a > 0$	$f'(x) = (a^x)' = a^x \ln(a)$.
Exponential function	$f(x) = e^x$	$f'(x) = (e^x)' = e^x$.
Logarithmic function	$f(x) = \ln(x)$	$f'(x) = (\ln(x))' = \frac{1}{x}$.
Sinusoidal function	$f(x) = \sin(x)$	$f'(x) = (\sin(x))' = \cos(x)$.
Cosine function	$f(x) = \cos(x)$	$f'(x) = (\cos(x))' = -\sin(x)$.
Tangent function	$f(x) = \tan(x)$	$f'(x) = (\tan(x))' = \sec^2(x)$.

Figure 3.2: Derivative of some specific functions.{Slide No-41}

d) Derivative Rules:

Rule	Function	Derivative
Sum – Difference Rule	$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
Multiplication by Constant	$c.f(x)$	$c.f'(x)$
Product Rule	$f(x).g(x)$	$f'(x).g(x) + f(x).g'(x)$
Quotient Rule	$f(x)/g(x)$	$\frac{f'(x).g(x) - f(x).g'(x)}{(g(x))^2}$
Chain Rule	$f(g(x))$	$f'(g(x)).g'(x)$

Figure 3.3: Derivative of some specific functions.{Slide No-41}

2. Let's take some derivatives! Find $\mathbf{f}'$; also write down which methods you use to find the derivative.

   a) $\mathbf{f(x) = x + \sqrt{x}}$
   Solution: Divide and Conquer.

      i. Derivative of First part **x**:
      The derivative of x w.r.t x is simply 1 i.e. {Derivative of an identity function.}

$$\frac{d}{dx}(x) = 1$$

      ii. Derivative of Second part $\mathbf{\sqrt{x} = x^{1/2}}$ {Applying power rule.}

$$\frac{d}{dx}(x^{1/2} = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$$

   Now combining the derivatives of both parts using the **sum rule**.

$$f'(x) = \frac{d}{dx}(x) + \frac{d}{dx}(\sqrt{x})$$

$$f'(x) = 1 + \frac{1}{2}x^{-1/2}$$

$$f'(x) = 1 + \frac{1}{2\sqrt{x}} \quad \square$$

---

   b) $\mathbf{f(x) = 2 + 1/x + 1/x^2}$ –> {$\mathbf{f'(x) = -\frac{1}{x^2} - \frac{2}{x^3}}$}
   Hint: Derivative of Constant; Product Rule and Sum Rule.

   c) $f(x) = \sqrt{x}cos(x)$ —> {$\mathbf{f'(x) = \frac{cos(x)}{2\sqrt{x}} - \sqrt{x}sin(x)}$}

---

   d) $f(x) = [x^2 + sin(x)]^4$
   Hint: For this problem we will apply chain rule and power rule to find the derivative.

   - **Chain Rule:** To apply chain rule we divide a composite function of our question into two function.
   {Composite function is a function that contains another function i.e. function with the form f(g(x))}
     - outer function: u^4 where inner function: $u = x^2 + sin(x)$
     Our Function now becomes: $f(x) = [u]^4$. Finding the derivative of $f(x) = [u]^4$ i.e.

$$f'(x) = \frac{d}{dx}(u^4)$$

$$f'(x) = 4u^3.\frac{du}{dx}$$

Putting back the $u = x^2 + sin(x)$

$$f'(x) = 4[x^2 + \sin(x)]^3 \cdot \frac{d}{dx}(x^2 + \sin(x))$$

Finding the derivative of $\mathbf{\frac{d}{dx}(x^2 + \sin(x))}$:

$$\frac{d}{dx}(x^2 + sin(x)) = 2x + cos(x)$$

Putting everything together:

$$f'(x) = 4[x^2 + \sin(x)]^3 \cdot [2x + \cos(x)$$

$$f'(x) = 4[x^2 + \sin(x)]^3 \cdot 2x + 4[x^2 + \sin(x)]^3 \cdot \cos(x)$$

$$f'(x) = 8x[x^2 + \sin(x)]^3 + 4\cos(x)[x^2 + \sin(x)]^3 \quad \square$$

e) $f(x) = x^3 e^x$
Hint: Apply the Product Rule:

3. Practise Problem: Find the derivative of:
   a) $\frac{x^3}{e^x}$ Answer -> { $\mathbf{\frac{3x^2 - x^3}{e^x}}$}
   Hint: Quotient Rule:
   b) $lnx - \frac{1}{x^2} + 8$ { **Ans:** $\mathbf{\frac{1}{x} + \frac{2}{x^3}}$}
   c) $3\sqrt{x} + 2x - \frac{8}{x}$ { **Ans:** $\mathbf{3/2 \cdot x^{-1/2} + 2 + 8x^{-}2}$}

4. Suppose that $\mathbf{g(x) = x^2 - 3x + 2}$ and $\mathbf{f(x)}$ is a differentiable function. All we know about $\mathbf{f(x)}$ is the following:
   - $\mathbf{f(0) = 3 \text{and} f'(0) = -1}$
   - $\mathbf{f(1) = 5 \text{and} f'(1) = 0}$
   - $\mathbf{f(2) = -2 \text{and} f'(2) = 3}$
   - $\mathbf{f(3) = 6 \text{and} f'(3) = 1}$

   If possible:

   a) Find the derivative of $\mathbf{f(x) \cdot g(x)}$ at $\mathbf{x = 1}$.
   Solution:

$$\begin{aligned}[f(x) \cdot g(x)]' &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \text{ \{Product Rule\}} \\ &= f'(1) \cdot g(1) + f(1) \cdot f(1) \cdot (2(1) - 3) \text{\{At } \mathbf{x = 1}\text{\} and } g'(x) = 2x - 3 \\ &= 0 \cdot g(1) + 5 \cdot -1 \\ &= -5 \quad \square\end{aligned}$$

b) Find the derivative of $\frac{f(x)}{g(x)}$ at $x = 0$:
Ans: $[\mathbf{7/4}]$

c) Find the derivative of $\mathbf{f}(\mathbf{g}(\mathbf{x}))at\mathbf{x} = \mathbf{0}$:
Ans: $[-\mathbf{9}]$ **Hint: Chain Rule.**

5. Find the maximum and minimum of $\mathbf{f(x) = x^3 - 3x + 1}$ on the interval $[\mathbf{0}, \mathbf{3}]$:
Solution:
Steps to find the maximum and minimum value:

a) Find the critical points by setting the derivative equal to zero.
Critical Points are: For any function $f(x)$ critical point occurs at $x = c$ if:

$$f^{'}(c) = 0$$
$$f^{'}(c) = \textbf{undefined}$$

These points are important because they can provide information about the behaviour of the function.Critical points are potential locations of maxima, minima, or points of inflection. Cautions!! However, not every critical point guarantees the presence of a maximum, minimum or inflection point. More further analysis is required for guarantees.
Endpoints: In the context of interval, the values at the "end" or boundaries of the interval are called endpoints.For example, if you have and interval$[a, b]$ and both a, b are real numbers then a is the left endpoint and b is the right endpoint.The interval includes all real numbers x such-that: $\mathbf{a \le x \le b}$.

b) Determine the values of $f(x)$ at the critical points and endpoints of the interval.

c) Compare the values obtained to identify the maximum and minimum.

Solution:

- **Find the critical points i.e. $\mathbf{f^{'}(x) = 0}$:**

$$f^{'}(x) = \left[x^3 - 3x + 1\right]^{'} = \frac{d}{dx}(x^3 - 3x + 1) = 0$$

$$\frac{d}{dx}(x^3 - 3x + 1) = 3x^2 - 3 = 0$$

$$3x^2 - 3 = 0$$

Solving for x:

$$3(x^2 - 1) = 0$$
$$(x^2 - 1) = 0$$
$$(x - 1) \cdot (x + 1) = 0$$

$$\textbf{so;} x = 1 \textbf{and} x = -1 \textbf{are the critical points.}$$

- Evaluate $\mathbf{f(x)}$ at critical points and end points.
  Endpoints are from the interval $[\mathbf{0,3}]$ from the question.

$$\begin{aligned} f(0) &= 1 \qquad \textbf{at left endpoint } x = 0 \\ f(1) &= 1 \qquad \textbf{at crtical point } x = 1 \\ f(-1) &= 3 \qquad \textbf{at crtical point } x = -1 \\ f(3) &= 19 \qquad \textbf{at right endpoint } x = 3 \end{aligned}$$

- Compare values to find maximum and minimum.
  So the maximum value of f(x) on the interval$[\mathbf{0,3}]$ is 19 and the minimum value is 3.

6. Find the maximum and minimum of following functions at given interval:
   a) $\mathbf{g(x) = x^2 - 4x + 5}$ at interval $[\mathbf{1,4}]$
   b) $\mathbf{h(x) = \frac{x^2}{2} - 2x}$ at interval $[\mathbf{-2,3}]$
   c) $\mathbf{p(x) = 3x^3 - 9x^2 + 7}$ at interval $[\mathbf{-1,2}]$
   d) $\mathbf{q(x) = e^x - x^2}$ at interval $[\mathbf{-1,2}]$

3.2 Find the Gradient:

Before we begin the exercise: please review the lecture slide: 46 - 49.

1. Evaluate the gradient of $\mathbf{f(x,y) = x^2 + y^2}$ at:

   (a) $(0,0)$ **Ans:**$[0,0]$ **(b)** $(1,3)$ **Ans:**$[2,6]$ **(c)** $(-1,-5)$ **Ans:** $[-2,-10]$

   Hint: To evaluate the magnitude of gradient we first need to find the derivative of $\mathbf{f(x,y)}$ it has two variable, Thus we need to apply partial derivative.
   The gradient of:

$$f(x,y) = \nabla f(x,y) = \left[\frac{\partial f(x,y)}{\partial x}, \frac{\partial f(x,y)}{\partial y}\right]$$

2. Evaluate the gradient of $\mathbf{f(x,y,z) = x^3z - 2y^2x + 5z}$ at:

   (a) $(1,1,-4)$ **Ans:**$[-10,-4,6]$ **(b)** $(0,1,0)$ **Ans:**$[-2,0,5]$ **(c)** $(-3,-2,1)$ **Ans:**$[33,24,-16]$

3. Suppose we are maximizing the function $\mathbf{f(x,y) = 4x + 2y - x^2 - 3y^2}$. Find where the gradient is 0. **Ans:** $[\mathbf{2,1/3}]$
   Hints:
   To find the where the gradient of $\mathbf{f(x,y)}$ is zero, we must:
   - **Find the partial derivatives w.r.to xandy**.

- **Then set them to zero i.e.**

$$\frac{\partial f}{\partial x} = 0$$
$$\frac{\partial f}{\partial y} = 0$$

4. **Thinking Question for the week.**

   Suppose we are minimizing the function $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{x^2} + \mathbf{2y} + \mathbf{2y^2}$.
   Along what vector should you travel from $[\mathbf{5}, \mathbf{12}]$.