ANALOG COMMUNICATION

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INTRODUCTION

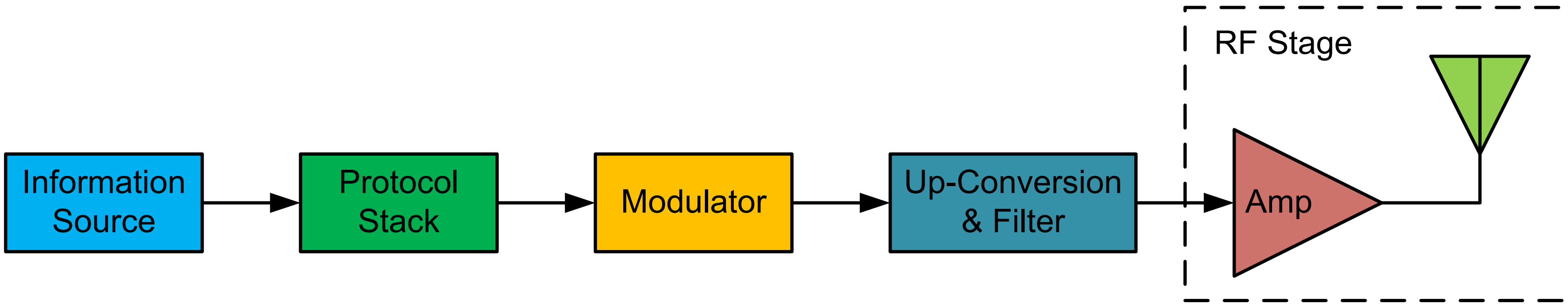
- Communication: Transmission of information from one point to another point.
- Information:
  - Audio: Speech, FM, AM, DAB, HDRadio, Phone, GSM, AMPS, etc.
  - Video: TV Broadcasting, Security cameras, Satellite images, etc.
  - Data: WiFi, Bluetooth, RS232, PCI, CD, DVD, etc.
- Important Parameters
  - Spectral Efficiency:
    - Amount of information per given bandwidth
  - Power Efficiency
    - Amount of information per used power
  - Multiplexing
    - Frequency
    - Time
    - Spatial

$$\boxed{a(t)\cos\left(2\pi f_c t + \phi(t)\right)}$$

How is Communication System Organized?

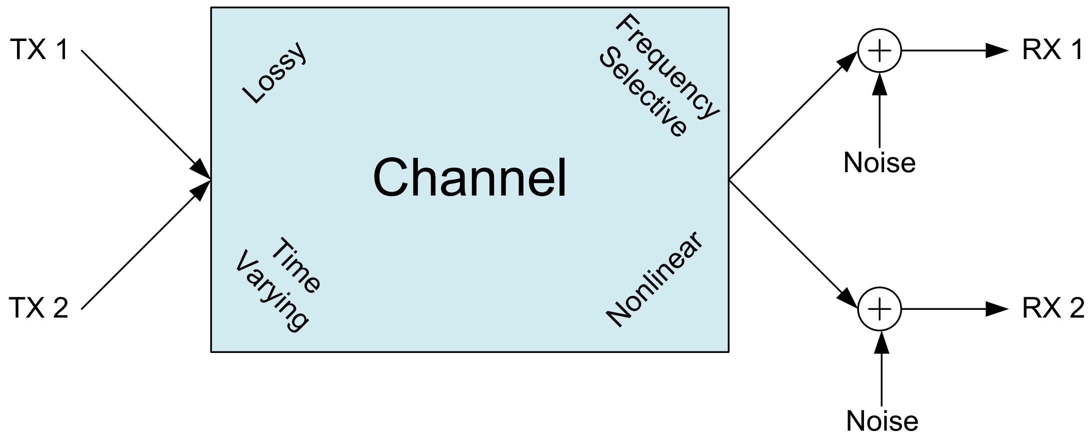
- **Source of information**: voice, music, pictures, videos, data
- **Transmitter**: Process the information in the form that is suitable for transmitting over the *channel*.
- **Channel (Transmission Medium)**: optical fiber, free space, twisted cable, copper cable, CD, cassette, etc.
- **Receiver**: Converts the signal transmitted over the channel back to a form that may be **understood** (it may not be exactly the same transmitted data) at the intended destination. The receiver may also compensate the distortions introduced by the channel and perform other functions such as synchronization of the receiver to the transmitter.
- **Sink of information**: user, computer, etc.

Wireless Communication Transmitter



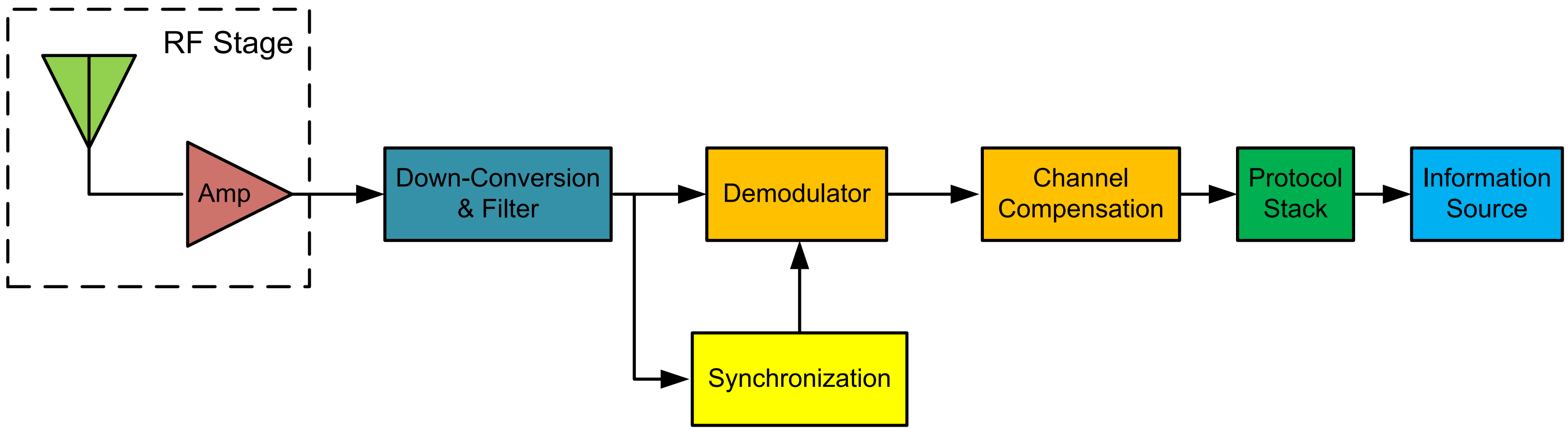
- **Protocol Stack**: It packages the data so that it can reliably get to the desired destination once it crosses the radio link.
- **Modulator**: It transform the information upon a carrier frequency in order to be recovered at the receiver.
- **Up-Conversion & Filter**: The modulated signal is converted to the final RF frequency at which it will be transmitted.
- **RF Stage**: The RF signal is amplified to an appropriate power level and then emitted via an antenna. In other words, the modulated signal is converted to an electromagnetic wave.

Wireless Communication Channel



- **Propagation Loss:** Loss of a signal strength with increasing distance.
- **Frequency Selectivity:** Many transmission medium conducts well over a relatively small range of frequencies.
- **Time Varying:** Some channel's characteristics vary with time. For example, mobile channels.
- **Nonlinear:** Nonlinear elements in the channel creates distortions.
- **Shared Usage:** For efficiency the communication channel is shared among different users. This creates interference between different users.
- **Noise:** The random motion of electrons creates uncertainty of the received signal. This usually is the reason of fundamental performance limitation.

Wireless Communication Receiver

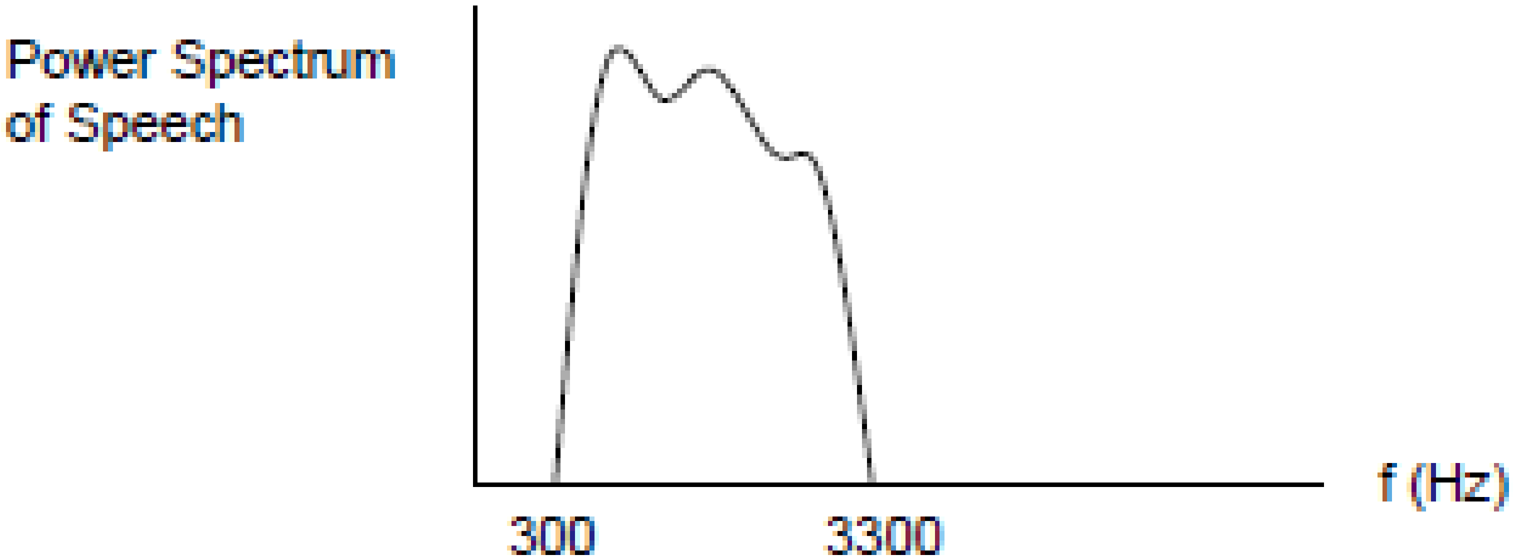


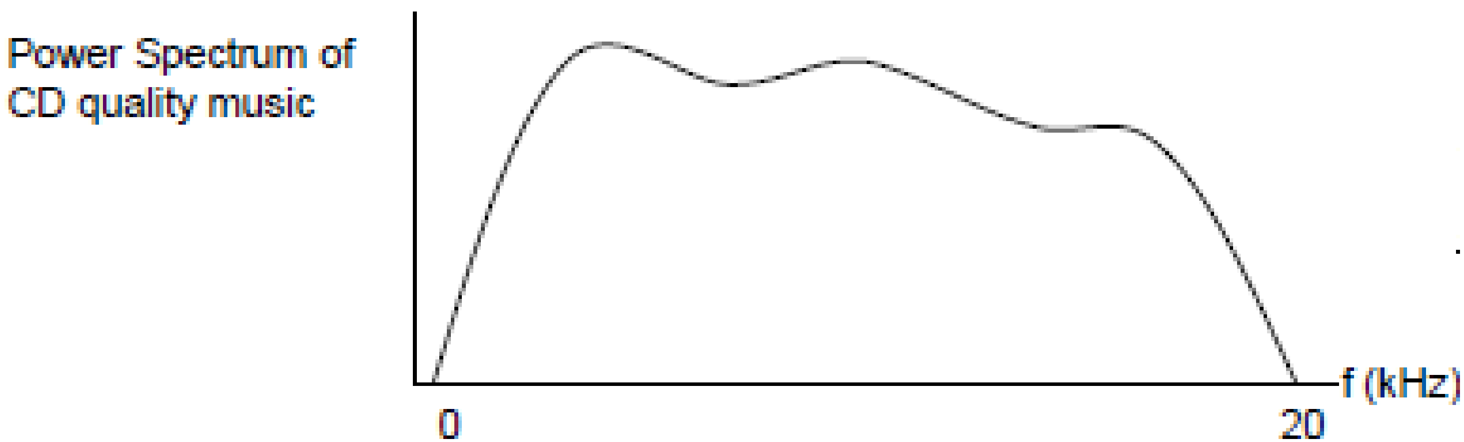
- **RF Stage**: The antenna collets RF energy in the desired band (may collect energy from unwanted as well). The first amplifier, called low-noise amplifier, boost the signal power.
- **Down Conversion**: Translate the RF signal to a frequency where the signal more easily demodulated.
- **Demodulation**: Transmitted signal is recovered.
- **Synchronization**: Compensates the time and frequency difference between transmitter and receiver.
- **Channel Compensation**: Counteract some of the impairments that the signal encountered in the channel. For example, equalization for frequency-selective channels, error correction for noisy channels.
- **Protocol Stack:** The receiver determines whether the detected message was intended for it or not.

Baseband and Passband Signals

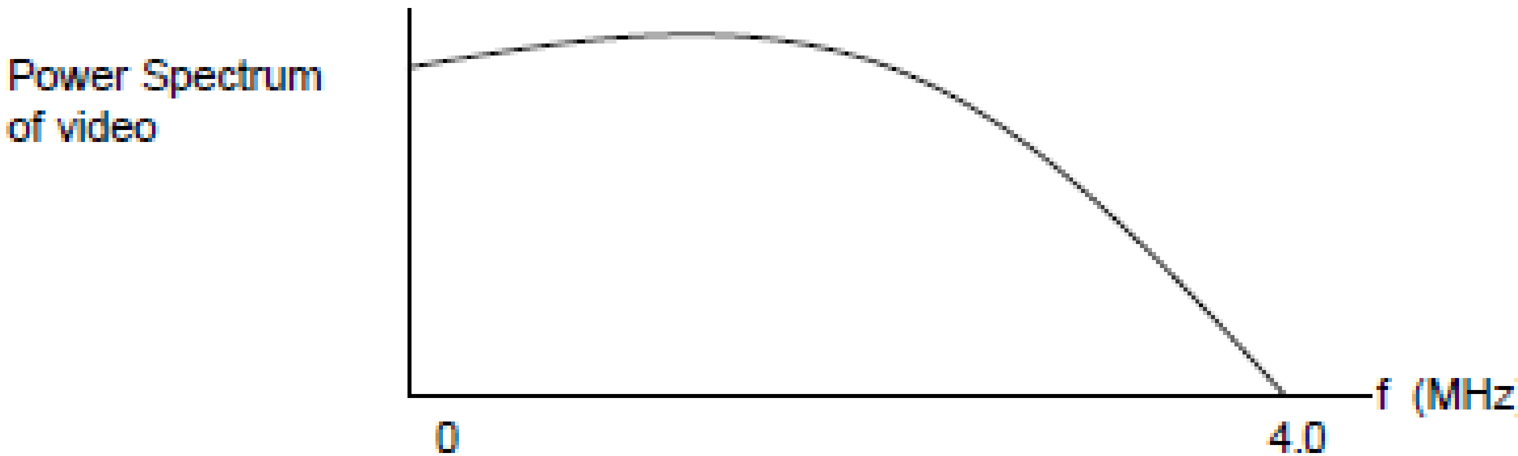
- Baseband signal:
  - uses the band of frequencies representing the message signal
  - The message signal band and transmission band matches.
  - analog or digital signal
  - Real or Complex signal
  - For example: Sound is an analog signal, computer signals are digital
- Passband signal:
  - Transmitted signal. Its characteristics is determined by the channel type.
  - The transmission band of the channel is centered at a frequency much higher than the highest frequency component of the message signal.

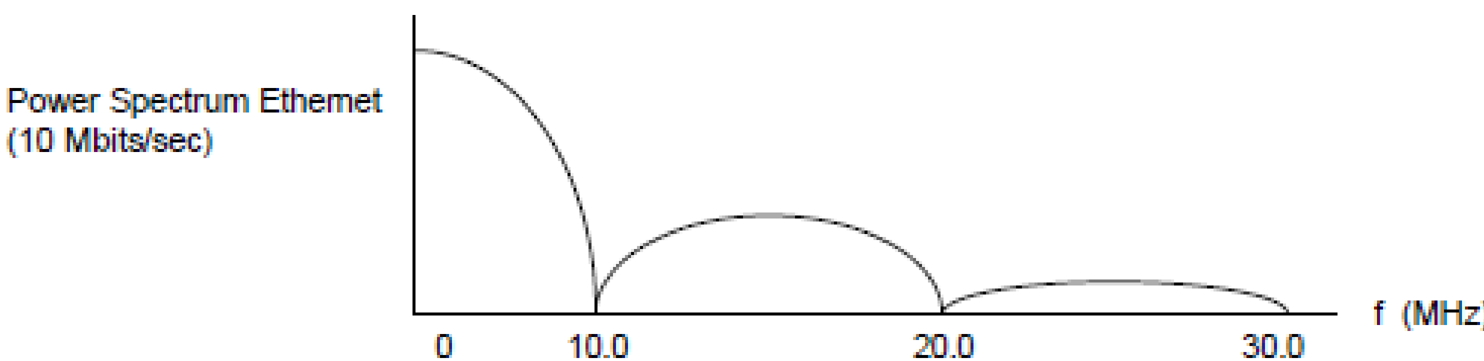
Signal Spectrum (Baseband) Examples





Signal Spectrum (Baseband) Examples





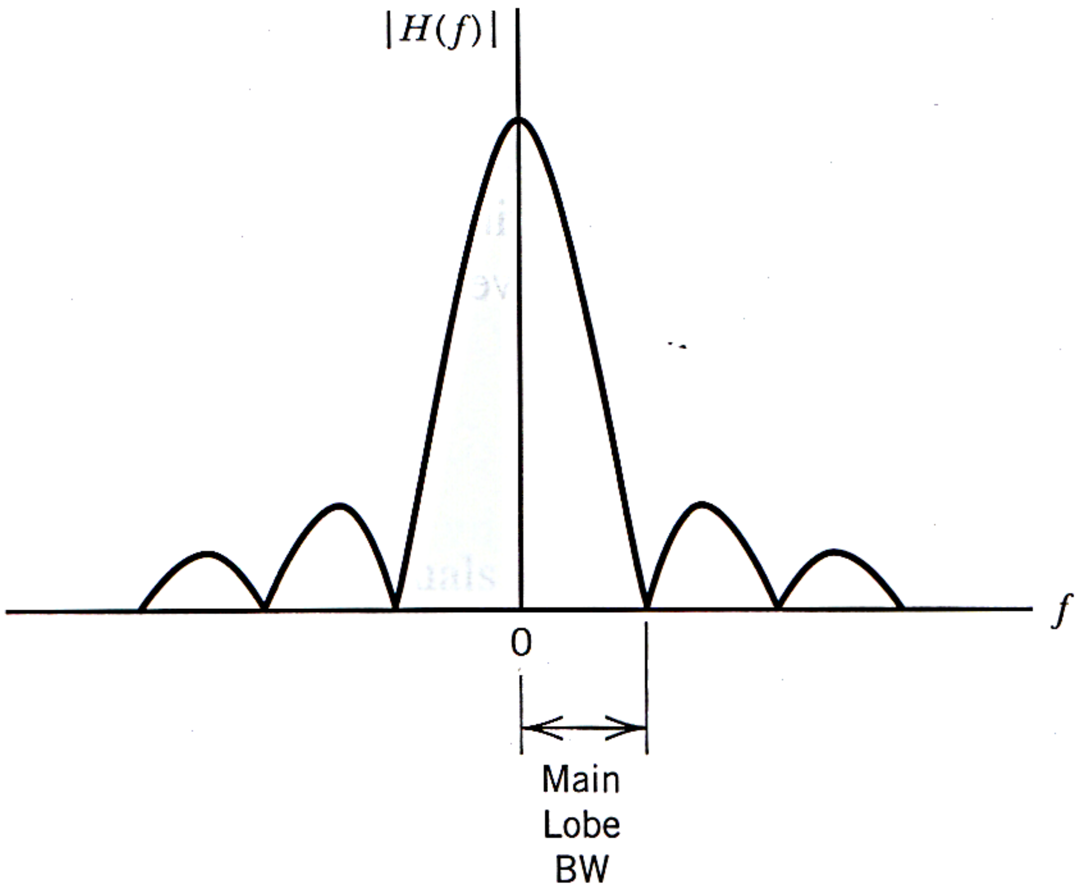
The Inverse Relationship between Time and Frequency

- If the time-domain description of a signal is changed, then the frequency-domain description of the signal is changed in inverse manner, and vice versa. This inverse relationship prevents arbitrary specifications of a signal in both domains.
- If a signal is strictly limited in frequency, then the time-domain description of the signal will trail on indefinitely.
- Example:
  - sinc pulse is strictly limited in frequency, but it is asymptotically limited in time
  - rectangular pulse is strictly limited in time, but it is asymptotically limited in frequency.

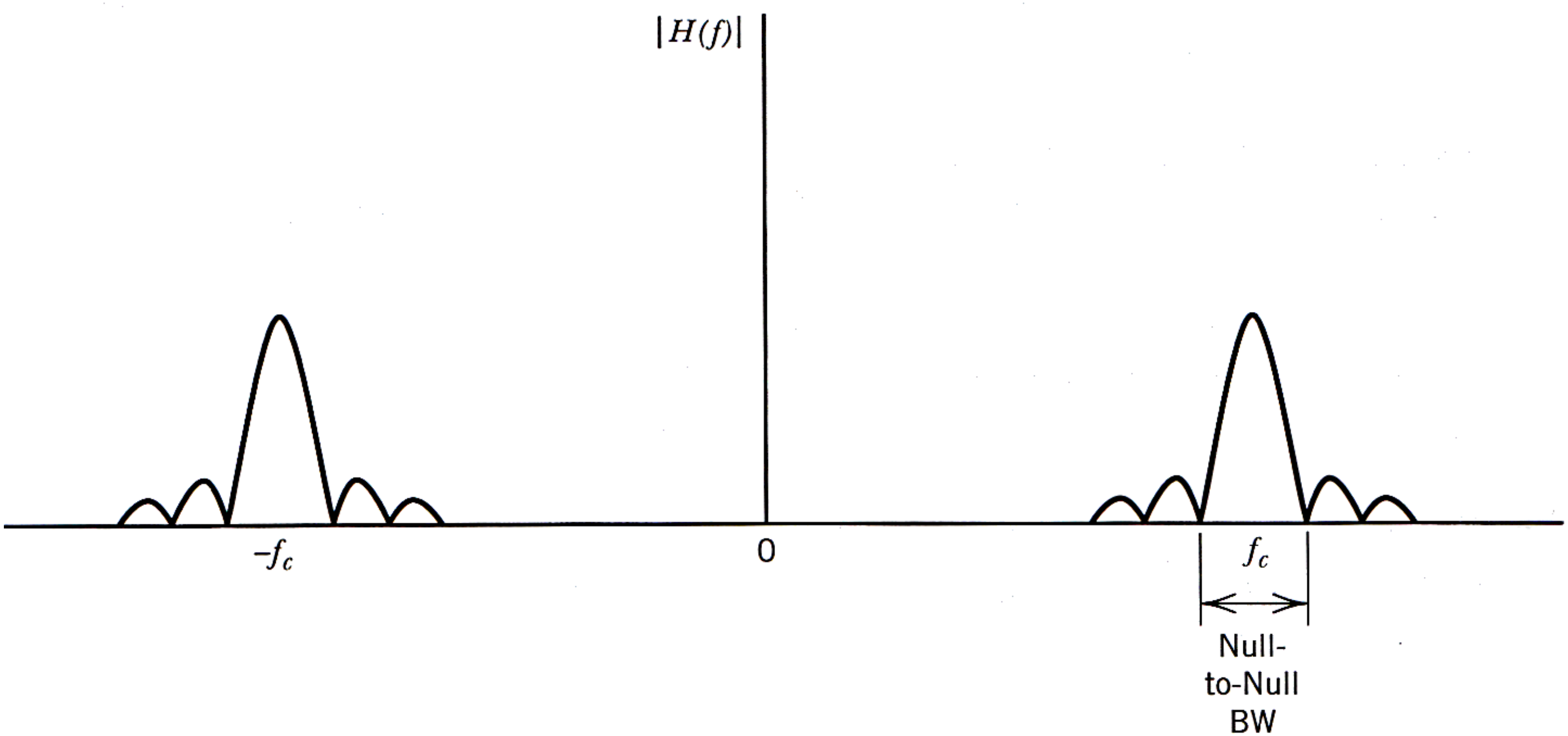
Bandwidth

- The bandwidth of a signal provides a measure of the extent of significant spectral content of the signal for positive frequencies.
- When the signal is strictly band-limited, the bandwidth is well defined.
- However, when the signal is not strictly band-limited, which is generally the case, it is difficult to define the bandwidth of the signal.
- Because the meaning of significant spectral content is mathematically imprecise.
- A signal is a low-pass signal if its significant spectral content is centered around the origin (zero frequency)
- A signal is a band-pass signal if its significant spectral content is centered around $\pm f_c$ where f_c is nonzero frequency.

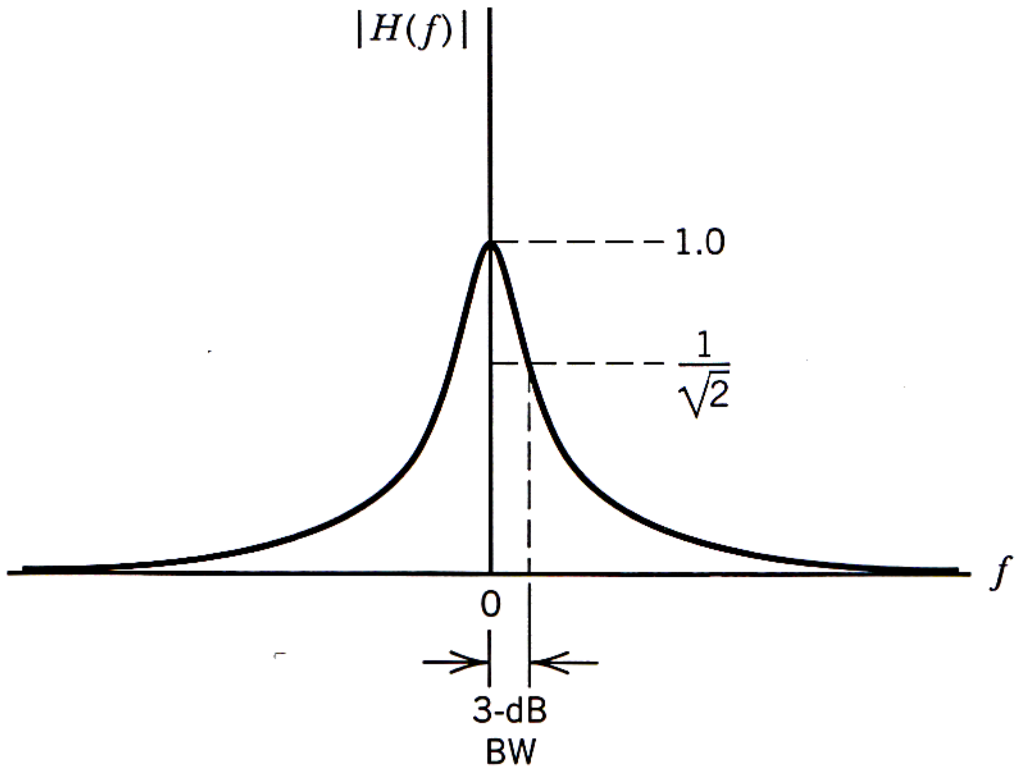
Null to Null Bandwidth (Low-Pass Signal)



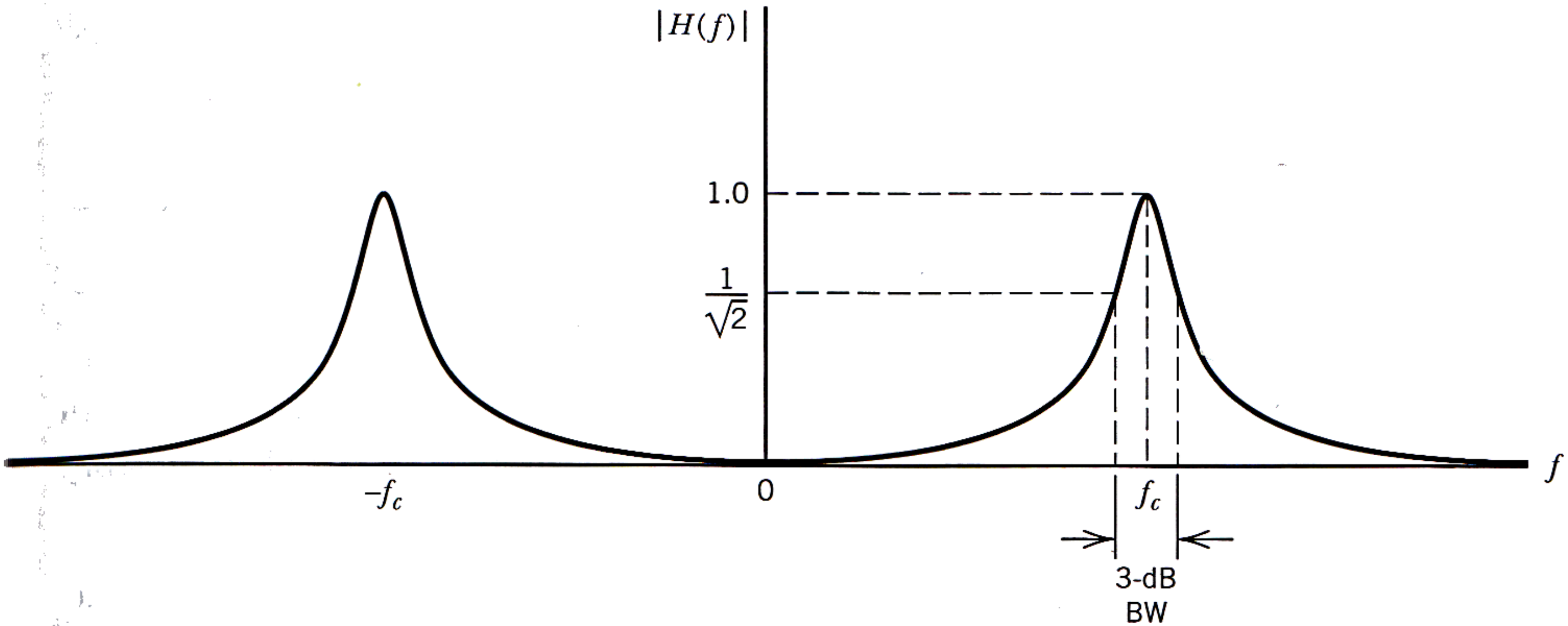
Null to Null Bandwidth (Band-Pass Signal)



3-DB BANDWIDTH (LOW-PASS SIGNAL)



3-DB BANDWIDTH (BAND-PASS SIGNAL)



Time-Bandwidth Product

- For any family of pulse signals that differ in time scale, the product of the signal's duration and its bandwidth is always constant.

  (duration) • (bandwidth) = constant
- This product is called the time-bandwidth product.
- This constancy is another manifestation of the inverse relationship between time-domain and frequency-domain.

Dirac Delta Function

- The Dirac delta function is defined as having zero amplitude everywhere except t=0, where it is infinitely large in such a way that it contains unit area under its curve.

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(d)dt = 1$$

$$\int_{-\infty}^{\infty} e^{j2\pi ft} df = \delta(t) \qquad \lim_{\tau \to 0} \frac{1}{\tau} e^{-\frac{\pi t^2}{\tau^2}} = \delta(t)$$

DIRAC DELTA FUNCTION

- Consider the product of g(t) and time-shifted delta function $\delta(t-t_0)$. The integral of this product

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)dt = g(t_0)$$

since $\delta(t)$ is an even function

$$\int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)dt = g(t)$$

$$g(t)*\delta(t) = g(t)$$

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}dt = 1$$

$$\delta(t)\overset{F}{\leftrightarrow}1$$

using the duality

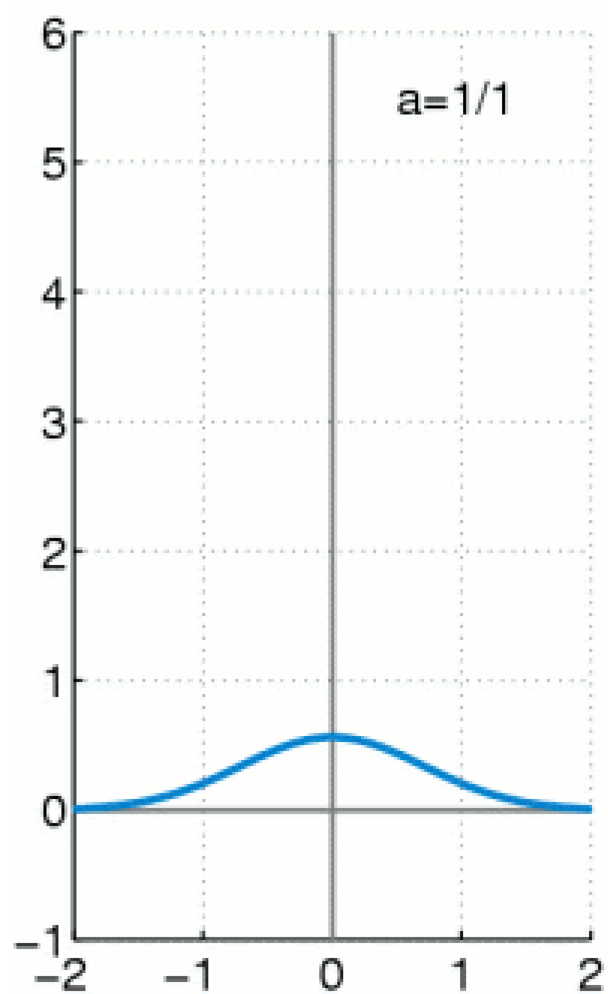
$$1\overset{F}{\leftrightarrow}\delta(f)$$

DELTA FUNCTION AS A LIMITING FORM OF GAUSSIAN PULSE

- Consider a Gaussian pulse of unit area

$$g(t) = \frac{1}{\tau} e^{-\frac{\pi t^2}{\tau^2}}$$

$$G(f) = e^{-\pi t^2 f^2}$$



Applications of Dirac Delta Function

- DC Signal: A dc signal is transformed in the frequency domain into a delta function.

$$1 \overset{F}{\leftrightarrow} \delta(f)$$

Using the Fourier Transform

$$\int_{-\infty}^{\infty} e^{-j2\pi ft} dt = \delta(f)$$

since $\delta(f)$ is real valued, and using Euler relationship

$$\int_{-\infty}^{\infty} \cos(2\pi ft) dt = \delta(f)$$

$$e^{-jx} = \cos x - j \sin x$$

APPLICATIONS OF DIRAC DELTA FUNCTION

- Complex Exponential Function: Let's apply the frequency shifting property to DC signal

$$\boxed{e^{j2\pi f_c t} \overset{F}{\leftrightarrow} \delta(f - f_c)}$$

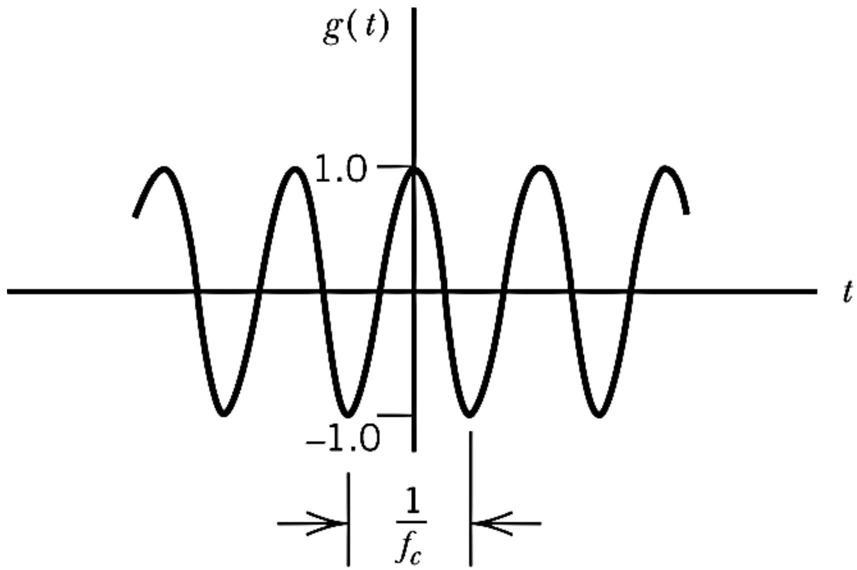
Complex exponential function of frequency f_c is transformed in the frequency domain into a delta function at $f=f_c$

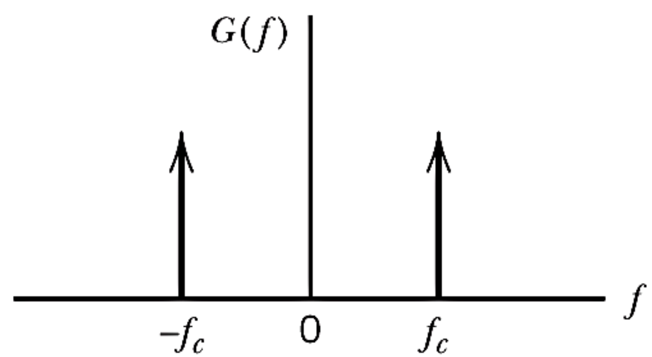
- Sinusoidal Function: Let's use Euler property

$$\cos(2\pi f_c t) = \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right] \Rightarrow \cos(2\pi f_c t) \overset{F}{\leftrightarrow} \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

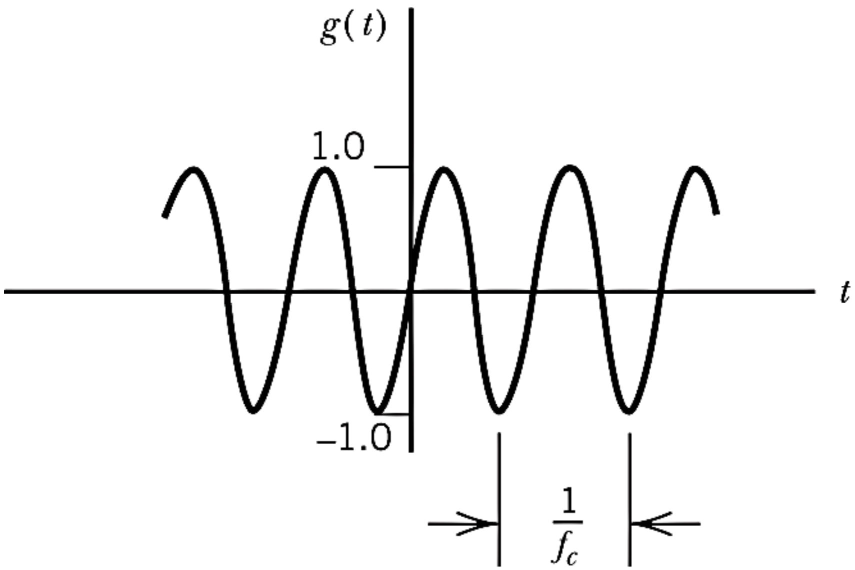
$$\sin(2\pi f_c t) = \frac{1}{2j}\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right] \Rightarrow \sin(2\pi f_c t) \overset{F}{\leftrightarrow} \frac{1}{2j}\left[\delta(f - f_c) - \delta(f + f_c)\right]$$

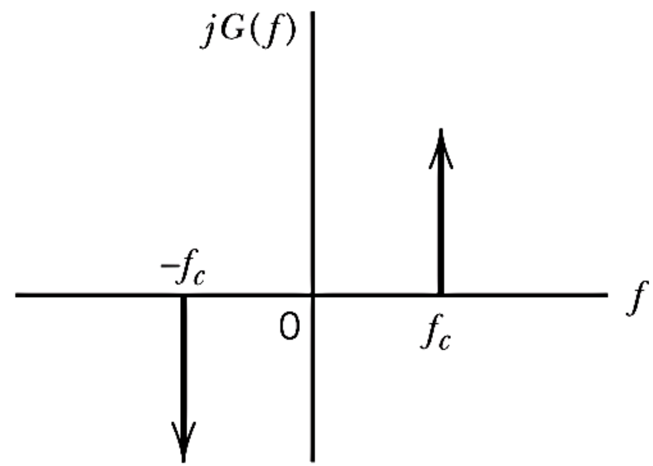
Spectrum of Cosine





Spectrum of Sine





Applications of Dirac Delta Function

- Signum Function: Signum function does not satisfy Diriclet conditions, since its energy is not finite.
- We may define its Fourier transform by viewing it as the limiting form if the antisymmetric double exponential pulse

$$\operatorname{sgn}(t)=\begin{cases}1 & t>0\\ 0 & t=0\\ -1 & t<0\end{cases}$$

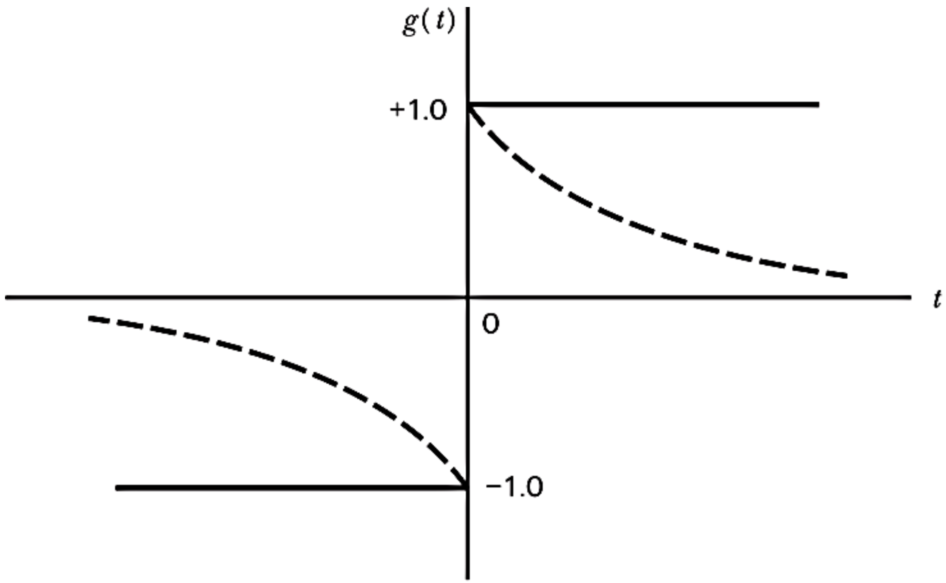
$$\operatorname{sgn}(t)=\lim_{a\to 0} g(t)=\lim_{a\to 0}\begin{cases}e^{-at} & t>0\\ 0 & t=0\\ -e^{at} & t<0\end{cases}$$

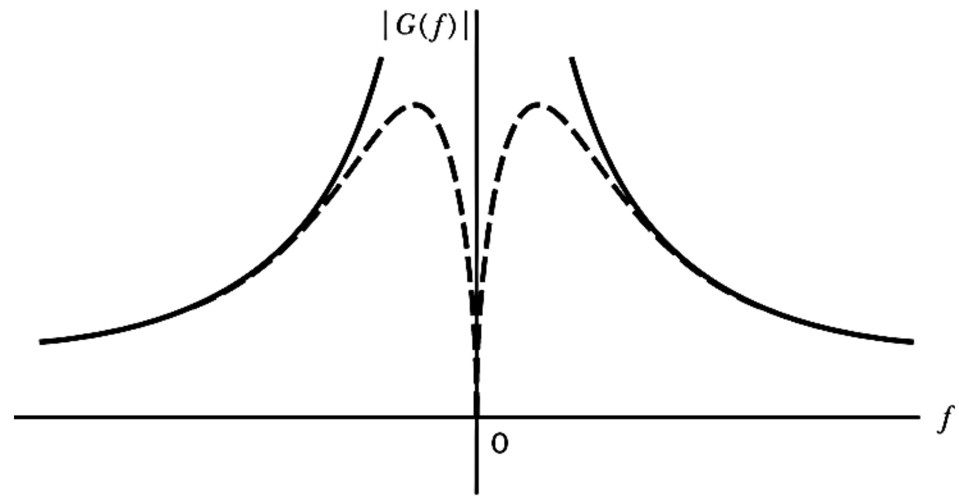
$$G(f)=\frac{-j4\pi f}{a^2+(2\pi f)^2}$$

$$F[\operatorname{sgn}(t)]=\lim_{a\to 0}\frac{-j4\pi f}{a^2+(2\pi f)^2}=\frac{1}{j\pi f}$$

$$\operatorname{sgn}(t)\overset{F}{\leftrightarrow}\frac{1}{j\pi f}$$

Spectrum of Signum Function





Applications of Dirac Delta Function

- Unit Step Function: The Fourier transform of unit step can be derived using the Fourier transform of signum function and linearity property.

$$u(t)=\begin{cases}1 & t>0\\ \frac{1}{2} & t=0\\ 0 & t<0\end{cases}$$

$$u(t)=\frac{1}{2}\left[\operatorname{sgn}(t)+1\right]$$

$$u(t)\overset{F}{\leftrightarrow}\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)$$

Transmission of Signals Through Linear Systems

- A system refers to any physical device that produces an output signal in response to an input signal.
- In a linear system, the superposition holds.
- Time-invariant means that system characteristics do not change in time.
- Time Response or Impulse Response:
  - the response of the system with zero initial conditions to a unit impulse applied to the input of the system.

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

Transmission of Signals Through Linear Systems

Causality and Stability:

- A system is causal if it does not respond before the input is applied.
- For a linear time-invariant system, the necessary and sufficient condition for causality is

$$\boxed{h(t) = 0 \quad t < 0}$$

- A system is stable if the output signal is bounded for all bounded input signals. (bounded input-bounded output (BIBO))
- The necessary and sufficient condition for BIBO stability is

$$\boxed{\int_{-\infty}^{\infty} |h(t)| dt < \infty}$$

Transmission of Signals Through Linear Systems

- Frequency Response: Consider a linear time-invariant system of impulse response h(t) driven by a complex exponential input of unit amplitude and frequency f

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{j2\pi f(t-\tau)} d\tau = e^{j2\pi ft} \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f\tau} d\tau$$

$$H(f) = \int_{-\infty}^{\infty} h(\tau) e^{-j2\pi f\tau} d\tau$$

- The Fourier Transform of the output signal y(t) is

$$\boxed{Y(f) = H(f)X(f)}$$

Transmission of Signals Through Linear Systems

- The transfer function H(f) is , in general, a complex quantity

$$\boxed{H(f)=|H(f)|e^{j\angle H(f)}}$$

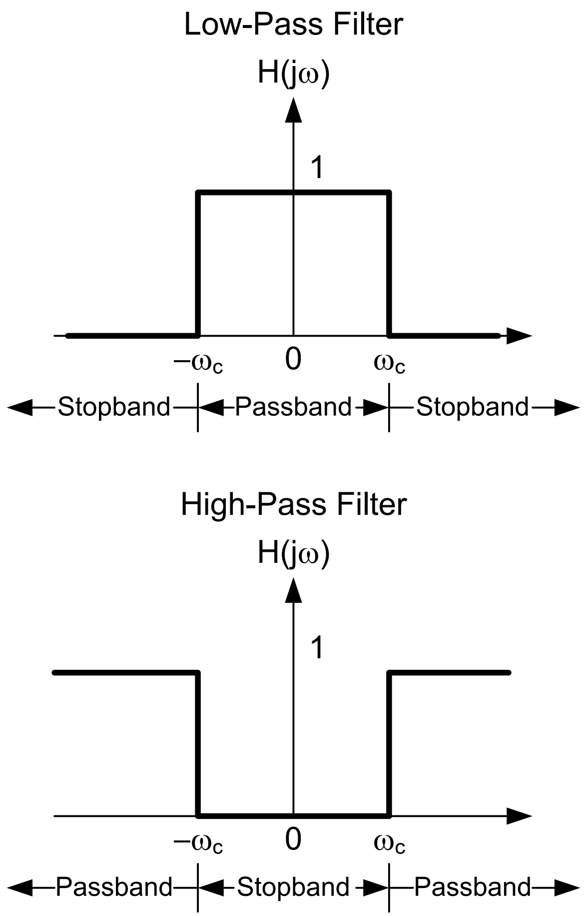
- |H(f)| is called the amplitude response and ∠H(f) is the phase response.
- The gain may also expressed in decibels (dB)

$$\boxed{\alpha(f)=20\log_{10}|H(f)|}$$

Filters

- A filter is a frequency-selective device that limits the spectrum of a signal to some specified band of frequencies.
- Filter types:
  - Low-pass: Transmits low frequencies and rejects high frequencies
  - High-pass: Transmits high frequencies and rejects low frequencies
  - Band-pass: Transmits a band of frequencies and rejects the rest
  - Band-reject: Rejects a band of frequencies and transmits the rest

Filters



Low-Pass Filter
H(jω)
1
−ωc
0
ωc
Stopband
Passband
Stopband
Band-Pass Filter
H(jω)
1
−ωcu
−ωcl
0
ωcl
ωcu
Stopband
Passband
Stopband
Passband
Stopband
High-Pass Filter
H(jω)
1
−ωc
0
ωc
Passband
Stopband
Passband
Band-Stop Filter
H(jω)
1
−ωcu
−ωcl
0
ωcl
ωcu
Passband
Stopband
Passband
Stopband
Passband

FILTERS

Design of Filters:

- A filter is characterized by specifying its impulse response h(t) or frequency response H(f).
- Most of the time the filters are used to separate signals on the basis of their frequency content. Therefore, the filters are usually designed in frequency domain.
- Using the Laplace, the filter transfer function is written as poles and zeros

$$\boxed{H(s)=K\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}}$$

- For stability, the poles must be inside the left half of the s-plane.

Different Types of Filters

- Butterworth: The poles of the transfer function H(s) lie on a circle with origin centered and $2\pi B$ as the radius, where B is the 3-dB bandwidth.
- Chebyshev: The poles lie on an ellipse. The Chebyshev filters has faster roll-off than Butterworth filter at the expense of higher phase distortion.
- Elliptic: It has faster roll-off than Chebyshev at the expense of ripple in the passband and stopband.

Hilbert Transform

- Shifting phase angles of all component of a given signal ±90 degrees is known as the Hilbert Transform of the signal.
- The Hilbert Transform of the signal g(t)

$$\boxed{\hat{g}(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(\tau)}{t-\tau}d\tau}$$

- The Hilbert Transform is a linear operation. The inverse Hilbert Transform

$$\boxed{g(t)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\hat{g}(\tau)}{t-\tau}d\tau}$$

HILBERT TRANSFORM

- Note that the Hilbert transform may be interpreted as the convolution of g(t) with the time function 1/(πt)
- The Fourier transform 1/(πt)

$$\frac{1}{\pi t} \overset{F}{\leftrightarrow} -j\,\mathrm{sgn}(f)$$

$$\hat{G}(f) = -j\,\mathrm{sgn}(f)G(f)$$

- Hilbert transform of a signal is basically passing it through a filter whose transfer function is equal to –jsgn(f)
- This filter produces a phase shift of -90 degrees for all positive frequencies and +90 degrees for all negative frequencies.

HILBERT TRANSFORM

- Example: Consider the cosine function

$$g(t) = \cos(2\pi f_c t)$$

$$G(f) = \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

$$\hat{G}(f) = -j\,\mathrm{sgn}(f)G(f)$$

$$= -\frac{j}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]\mathrm{sgn}(f)$$

$$= \frac{1}{2j}\left[\delta(f - f_c) - \delta(f + f_c)\right]$$

$$\hat{g}(t) = \sin(2\pi f_c t)$$

Properties of Hilbert Transform

- A signal g(t) and its Hilbert transform ĝ(t) have the same amplitude spectrum.
  - the magnitude of −jsgn(f) is equal to one
- If ĝ(t) is the Hilbert transform of g(t), then the Hilbert transform of ĝ(t) is −g(t)

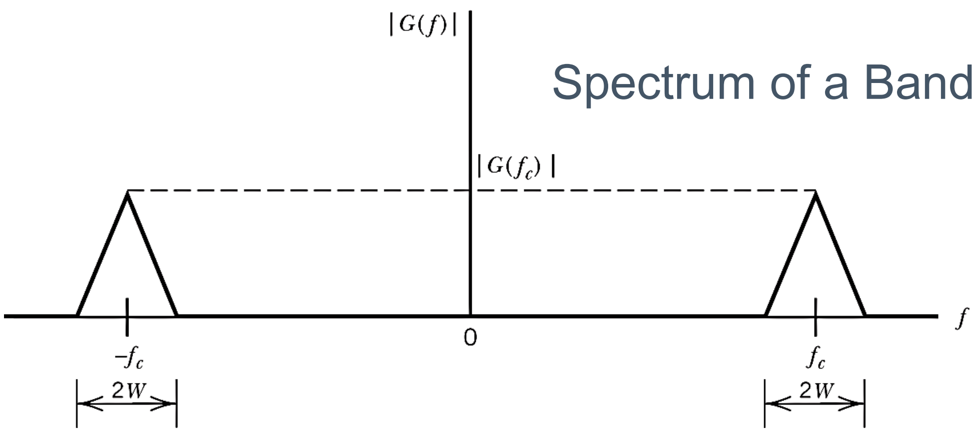
$$\boxed{\left[-j\,\mathrm{sgn}(f)\right]^2 = 1 \quad \text{for all } f}$$

- A signal g(t) and its Hilbert transform ĝ(t) are orthogonal.

$$\boxed{\int_{-\infty}^{\infty} g(t)\,\hat{g}(t)\,dt = 0}$$

Low-Pass and Band-Pass Signals

- Low-Pass Signals: Frequency content is centered at the origin and limited by |f|<W
  - Communication using low-pass signals is referred to as baseband communication
  - Limited to wired or cabled communication systems
- Band-Pass Signals: Frequency content is centered at a frequency (carrier frequency) much higher than the bandwidth.



Complex Baseband Representation

- The spectrum of complex signals is not complex conjugate like the spectrum of real signals. Therefore, more information can be send using complex signaling.
- Consider a general signal

$$\boxed{g(t) = a(t)\cos[2\pi f_c t + \phi(t)]}$$

- a(t) is the envelope and f(t) is the phase of the signal.
- We may rewrite this equation in terms of cosine and sine

$$\boxed{\begin{array}{l} g(t) = g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \\ g_I(t) = a(t)\cos(\phi(t)) \quad g_Q(t) = a(t)\sin(\phi(t)) \end{array}}$$

- are called in-phase and quadrature components of g(t)

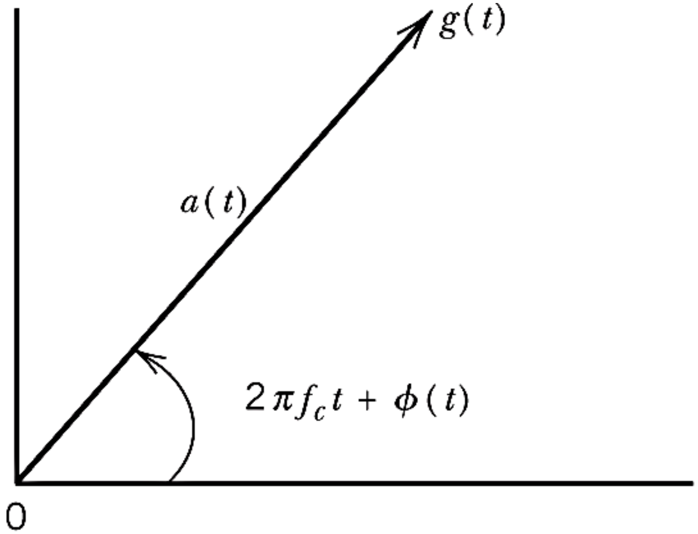
Complex Baseband Representation

- The envelope and phase of the complex signal is derived using the following relationship

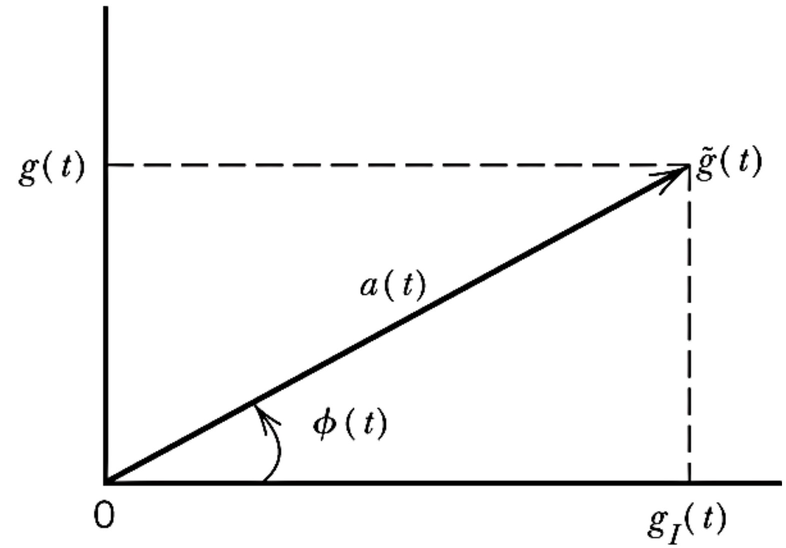
$$a(t) = \sqrt{g_I^2(t) + g_Q^2(t)}$$

$$\phi(t) = \tan^{-1}\left(\frac{g_Q(t)}{g_I(t)}\right)$$

Phasor Representation of a Band-Pass Signal g(t)



Phasor representation of a band-pass signal g(t)



Phasor representation of the corresponding complex envelope g(t)

Complex Baseband Representation

- g(t) may be written in complex form

$$
\begin{aligned}
g(t) &= a(t)\cos[2\pi f_c t + \phi(t)] \\
&= g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \\
&= \mathrm{Re}\left[g_I(t)e^{j2\pi f_c t} - g_Q(t)(-j)e^{j2\pi f_c t}\right] \\
&= \mathrm{Re}\left[g_I(t)e^{j2\pi f_c t} + jg_Q(t)e^{j2\pi f_c t}\right] \\
&= \mathrm{Re}\left[\tilde{g}(t)e^{j2\pi f_c t}\right] \\
\tilde{g}(t) &= g_I(t) + jg_Q(t)
\end{aligned}
$$

$\tilde{g}(t)$ is the complex envelope of the band-pass signal

The complex envelope corresponds to a phasor that has the constant phase rotation is suppressed.

$$e^{jx} = \cos x + j\sin x$$

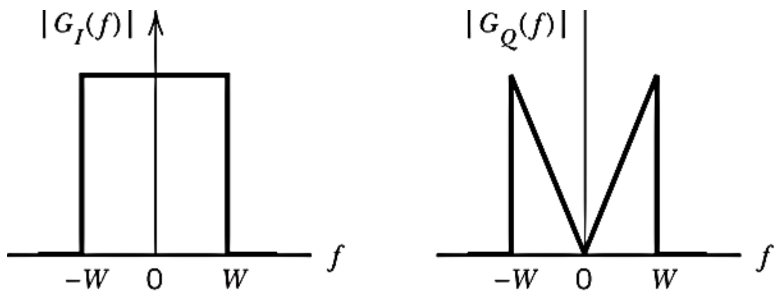
Complex Baseband Representation

- Fourier Transform of g(t) can be obtained shifting the frequency of the Fourier transform of complex envelope

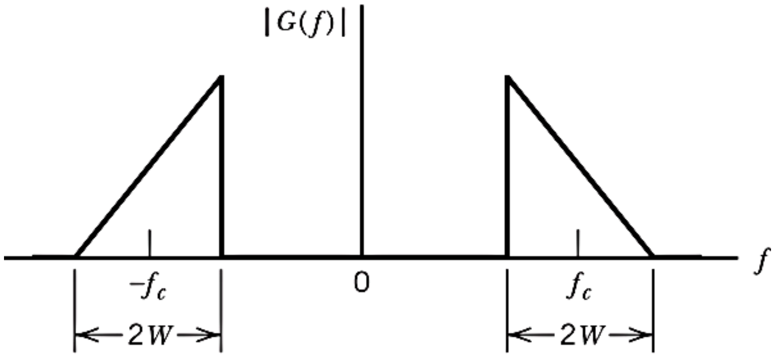
$$\boxed{\begin{aligned} g(t) &= \frac{1}{2}\left[\tilde{g}(t)e^{j2\pi f_c t} + \tilde{g}^*(t)e^{-j2\pi f_c t}\right] \\ G(f) &= \frac{1}{2}\left[\tilde{G}(f-f_c) + \tilde{G}^*(-f-f_c)\right] \end{aligned}}$$

note that Re[x]=x+x*

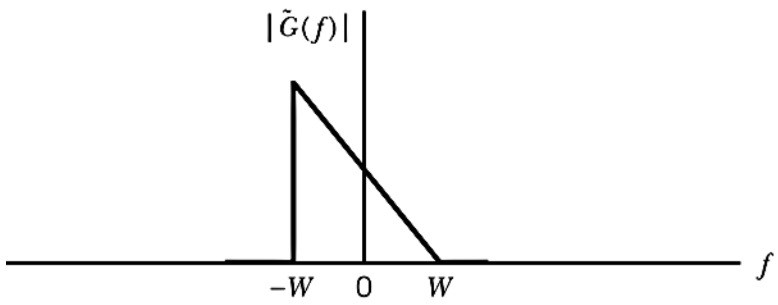
Complex Baseband Representation



In-Phase and Quadrature components Fourier transforms of $g_i(t)$ and $g_q(t)$ are symmetric about the origin

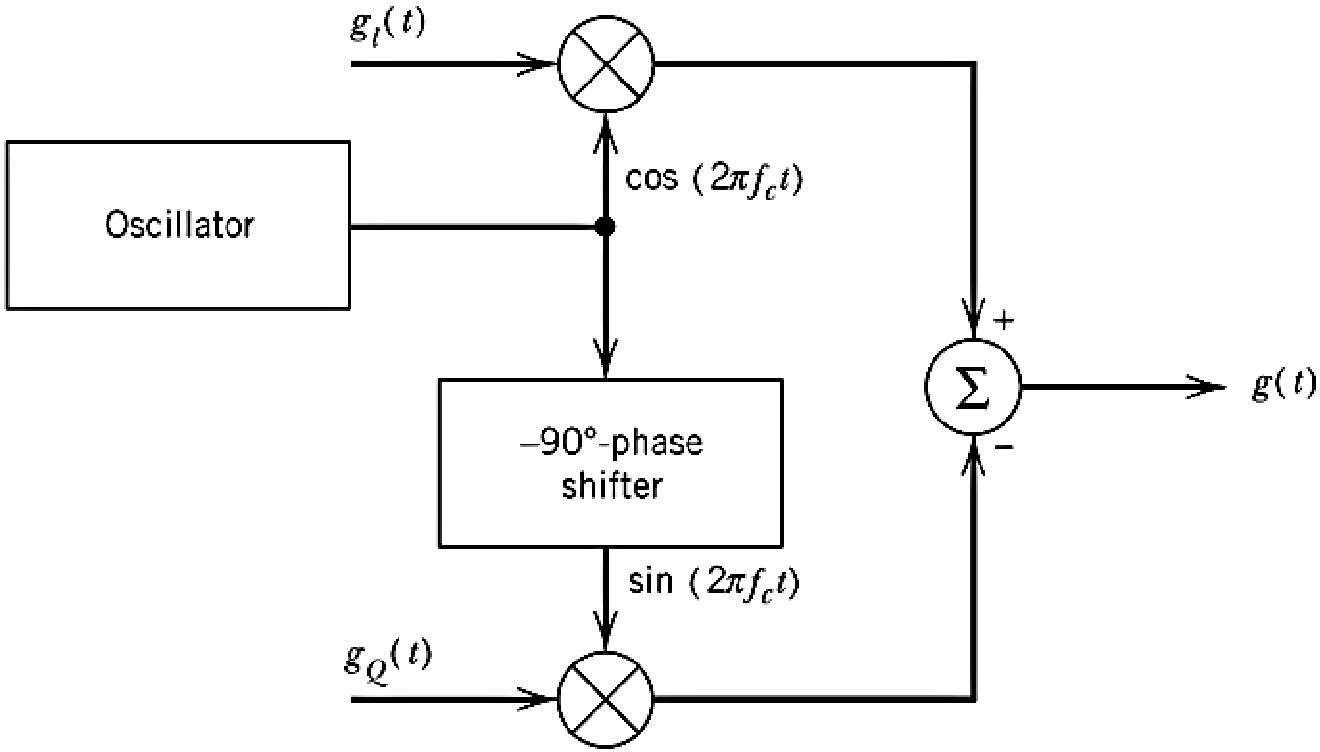


Corresponding band-pass spectrum Fourier transform of band-pass signal is symmetric about the origin but it is not guaranteed to be symmetric about fc

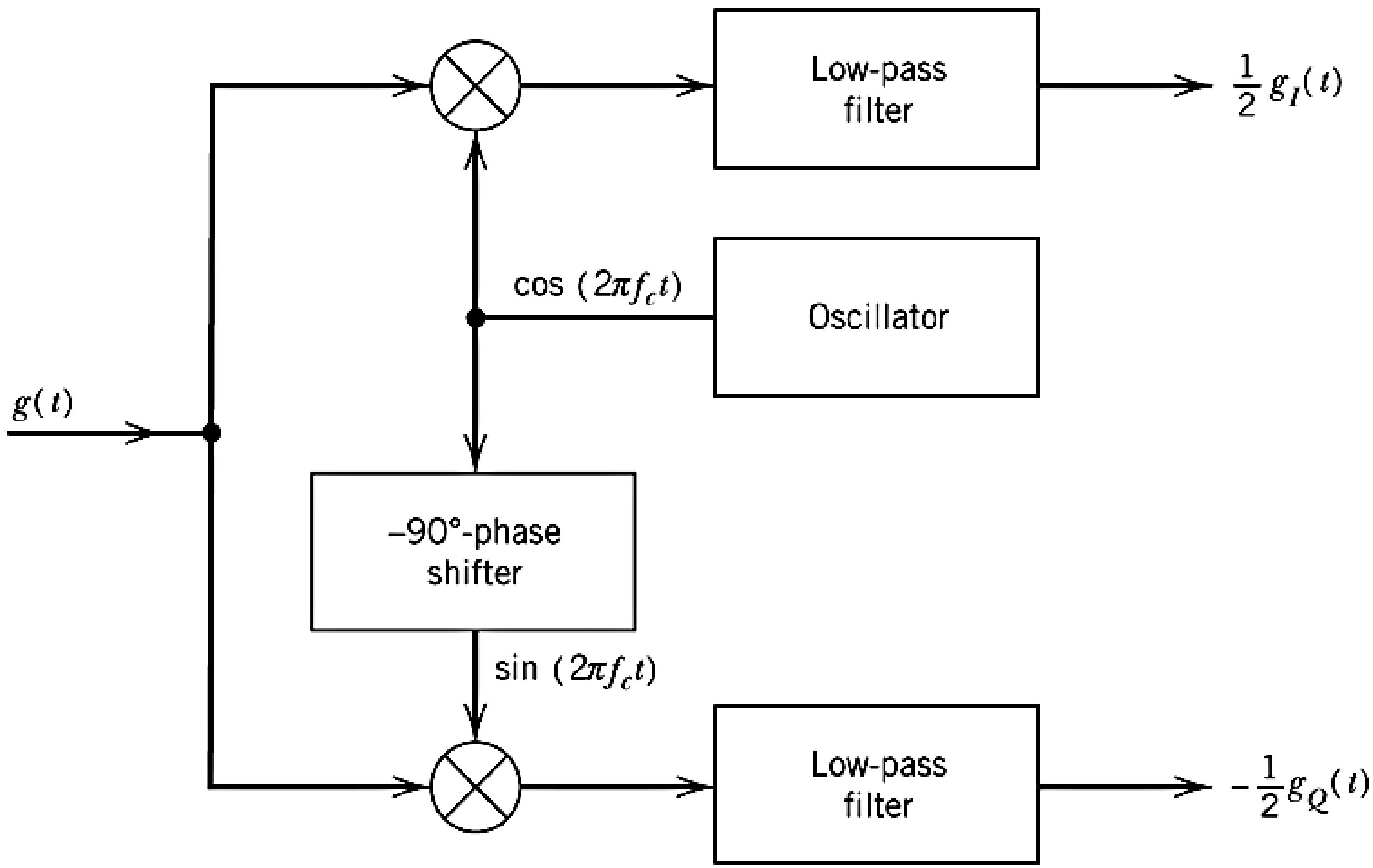


Corresponding complex envelope

GENERATING BAND-PASS SIGNAL FROM COMPLEX ENVELOPE



Demodulating Complex Envelope from Band-Pass Signal



PHASE AND GROUP DELAY

- When a signal is transmitted through frequency selective system, some delay is introduced into the output signal in relation to the input signal.
- Phase delay: It is the delay of the pure sinusoidal signal. The delay is equal to $\angle H(f_c)/2\pi f_c$ seconds.
- Group delay: The delay between the envelope of the input signal and output signal.
- Phase and Group delays can be derived by expanding the phase response to Taylor series.

$$\angle H(f) \approx \angle H(f_c) + (f - f_c)\left.\frac{\partial \angle H(f)}{\partial f}\right|_{f=f_c}$$

$$\angle H(f) \approx -2\pi f_c \tau_p - 2\pi (f - f_c)\tau_g$$

$$\tau_p = -\frac{\angle H(f_c)}{2\pi f_c}$$

$$\tau_g = -\frac{1}{2\pi}\left.\frac{\partial \angle H(f)}{\partial f}\right|_{f=f_c}$$

Amplitude Modulation

- Consider a sinusoidal carrier wave c(t)

$$\boxed{c(t) = A_c \cos(2\pi f_c t)}$$

  A_c is carrier amplitude and f_c is the carrier frequency
- Let m(t) is the baseband signal that carries the message.
- Amplitude modulation (AM) is defined as a process in which the amplitude of the carrier wave c(t) is varied about a mean value, linearly with the baseband signal m(t)

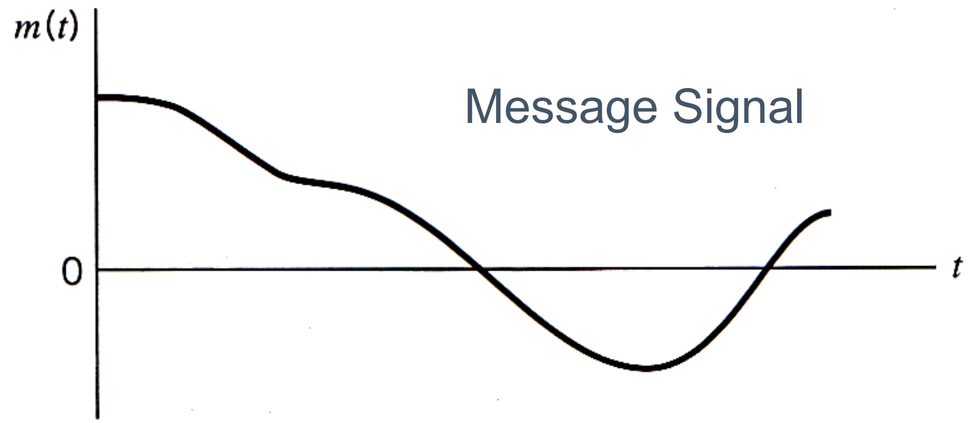
Amplitude Modulation

- In general, an Amplitude Modulated wave

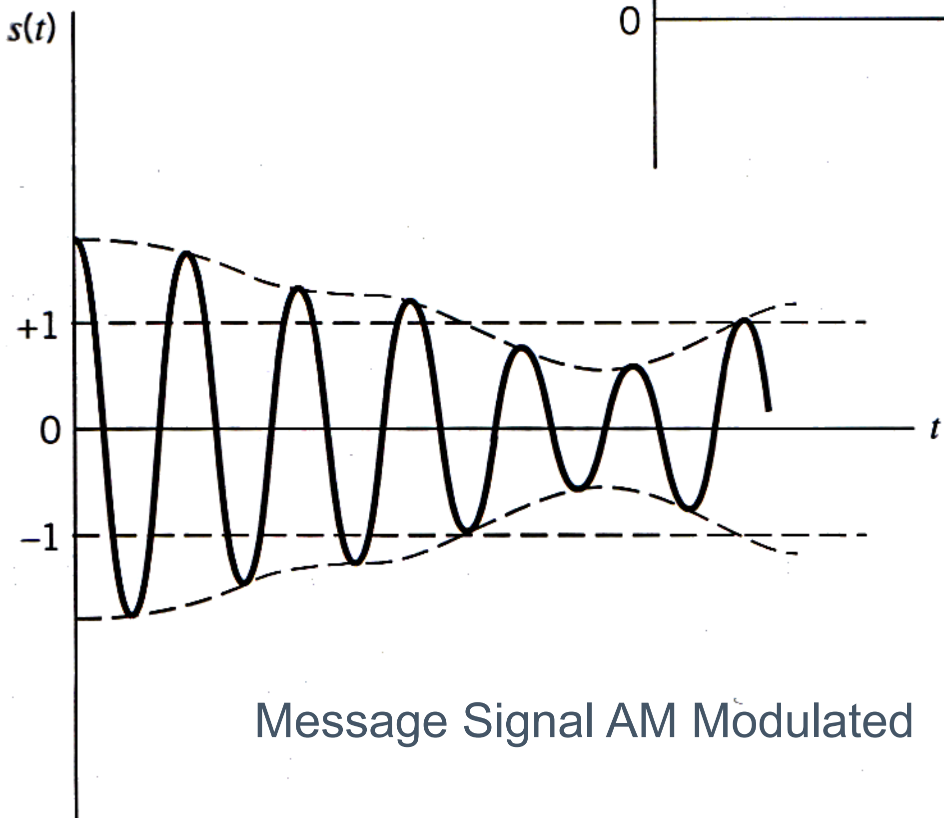
$$\boxed{s(t) = A_c\left[1 + k_a m(t)\right]\cos\left(2\pi f_c t\right)}$$

- k_a: amplitude sensitivity in $volt^{-1}$
  - The envelope of s(t) has the same shape as the baseband signal m(t), if these two requirements are met
    - $|k_a m(t)| < 1$ for all t
    - $f_c >> W$ in other words carrier frequency is greater than message bandwidth, otherwise the envelope can not be visualized

Amplitude Modulation



Message Signal

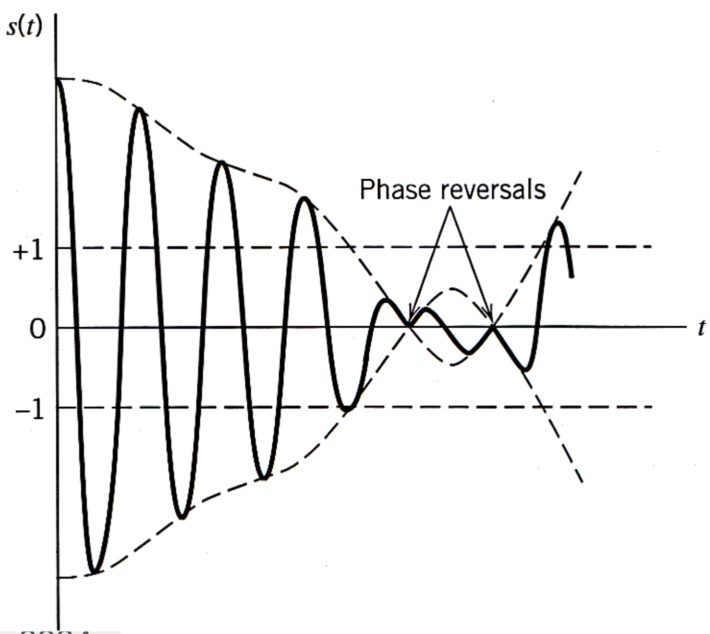


Message Signal AM Modulated

The absolute maximum value of $k_a m(t)$ multiplied by 100 is referred to as the percentage modulation

Amplitude Modulation

- When the amplitude sensitivity k_a is large enough to make $|k_a m(t)|>1$ for any t, the carrier wave becomes overmodulated, resulting in carrier phase reversals whenever the factor $1+k_a m(t)$ crosses zero.



Envelope of the modulated wave envelope does not match the message signal anymore.
The envelope has envelope distortion

Fourier Transform of AM Wave

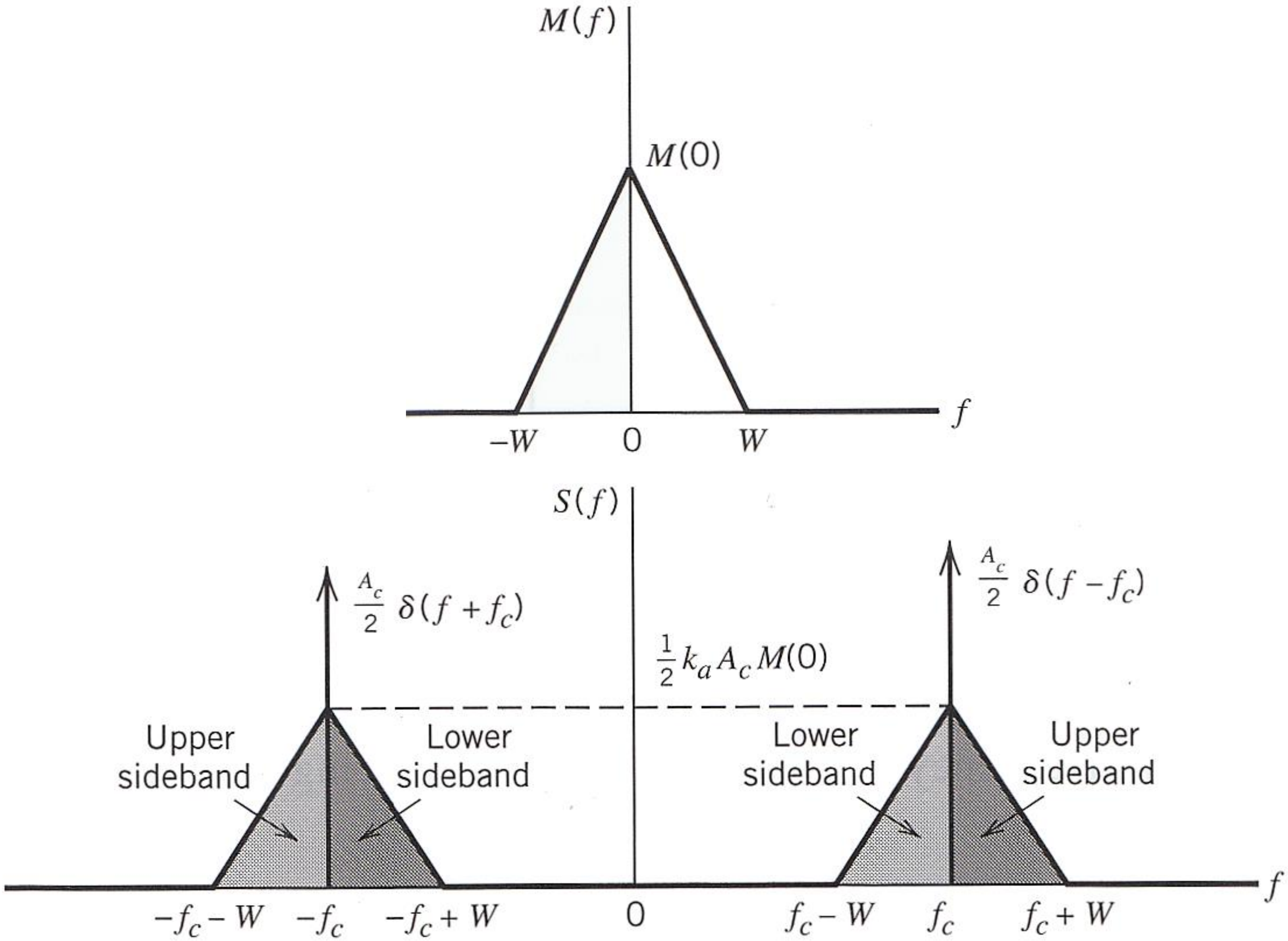
- Suppose that the message signal m(t) is band-limited to the interval –W ≤ f ≤ W
- The Fourier Transform of m(t) is M(f)
- Using the linearity and multiplication property

$$s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$S(f) = A_c[\delta(f) + k_a M(f)] * \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

$$= \frac{A_c}{2}[\delta(f - f_c) + \delta(f + f_c)] + \frac{k_a A_c}{2}[M(f - f_c) + M(f + f_c)]$$

Fourier Transform of AM Wave



Fourier Transform of AM Wave

- The spectrum of m(t) for negative frequencies becomes visible for positive frequencies.
- The portion of the spectrum lying above the carrier frequency f_c is referred as **upper sideband**, whereas the symmetric portion below f_c is referred as **lower sideband**.
- The transmission bandwidth B_T for an AM wave is exactly twice the message bandwidth W.

$$\boxed{B_T = 2W}$$

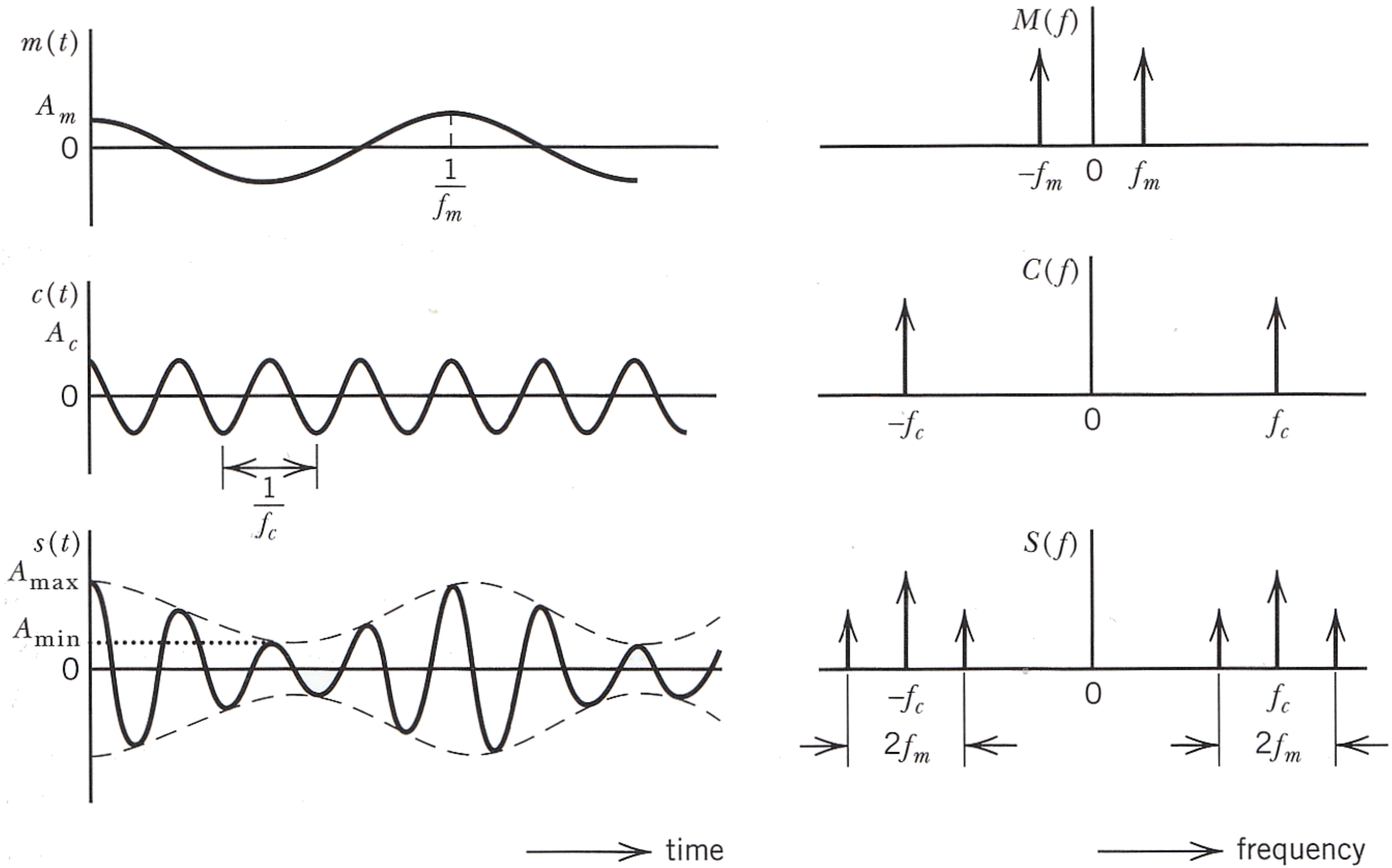
Single-Tone Modulation

- m(t) is a single tone sinusoidal signal

$$\boxed{\begin{aligned} m(t) &= A_m \cos(2\pi f_m t) \\ s(t) &= A_c[1+\mu\cos(2\pi f_m t)]\cos(2\pi f_c t) \\ \mu &= k_a A_m \end{aligned}}$$

- μ is a dimensionless constant and called the modulation factor or modulation index or the percentage modulation when it is expressed numerically as a percentage.
- To avoid envelope distortion, the modulation factor μ must be kept below unity.

Single-Tone Modulation



Single-Tone Modulation

- The ratio of maximum and minimum values of the envelope of the modulated wave

$$\boxed{\frac{A_{\max}}{A_{\min}}=\frac{A_c(1+\mu)}{A_c(1-\mu)}\Rightarrow \mu=\frac{A_{\max}-A_{\min}}{A_{\max}+A_{\min}}}$$

- The Fourier transform of s(t) is

$$\begin{aligned}
s(t) &= A_c[1+\mu\cos(2\pi f_m t)]\cos(2\pi f_c t)\\
&= A_c\cos(2\pi f_c t)+\frac{1}{2}\mu A_c\cos(2\pi(f_c+f_m)t)+\frac{1}{2}\mu A_c\cos(2\pi(f_c-f_m)t)\\
S(f) &= \frac{1}{2}A_c[\delta(f-f_c)+\delta(f+f_c)]\\
&\quad+\frac{1}{4}\mu A_c[[\delta(f-f_c-f_m)+\delta(f+f_c+f_m)]]\\
&\quad+\frac{1}{4}\mu A_c[[\delta(f-f_c+f_m)+\delta(f+f_c-f_m)]]
\end{aligned}$$

Single-Tone Modulation

- In practice, AM wave s(t) is a voltage or current wave. The average power delivered to 1-ohm resistor

$$\boxed{\begin{aligned} \text{Carrier Power} &= \frac{1}{2}A_c^2 \\ \text{Upper side-band Power} &= \frac{1}{8}\mu^2 A_c^2 \\ \text{Lower side-band Power} &= \frac{1}{8}\mu^2 A_c^2 \end{aligned}}$$

- The Carrier power does not carry any information and is wasted.
- Upper side-band is complex conjugate of lower side-band since m(t) is real signal.

EXAMPLE – AM POWER

- A zero mean sinusoidal message is applied to AM transmitter with 10kW power. Compute carrier power if modulation index is 0.6. What percentage of the total power is in the carrier? Calculate the power in each sideband.

$$
\begin{aligned}
\text{Total Transmit Power} &= \text{Carrier Power} \\
&\quad + \text{Upperside-band} + \text{Lowerside-band} \\
&= \frac{1}{2}A_c^2 + \frac{1}{8}\mu^2 A_c^2 + \frac{1}{8}\mu^2 A_c^2 \\
&= \frac{1}{2}A_c^2\left[1 + \frac{1}{4}\mu^2 + \frac{1}{4}\mu^2\right] \\
&= \frac{1}{2}A_c^2\left[1 + \frac{1}{2}\mu^2\right]
\end{aligned}
$$

Example – AM Power

$$P_C = \frac{P_{AM}}{1+\frac{1}{2}\mu^2} = \frac{10}{1+\frac{1}{2}0.6^2} = 8.47kW$$

Percentage power in the carrier

$$\frac{P_C}{P_{AM}} \times 100 = \frac{8.47}{10} \times 100 = 84.7\%$$

Power in each sideband

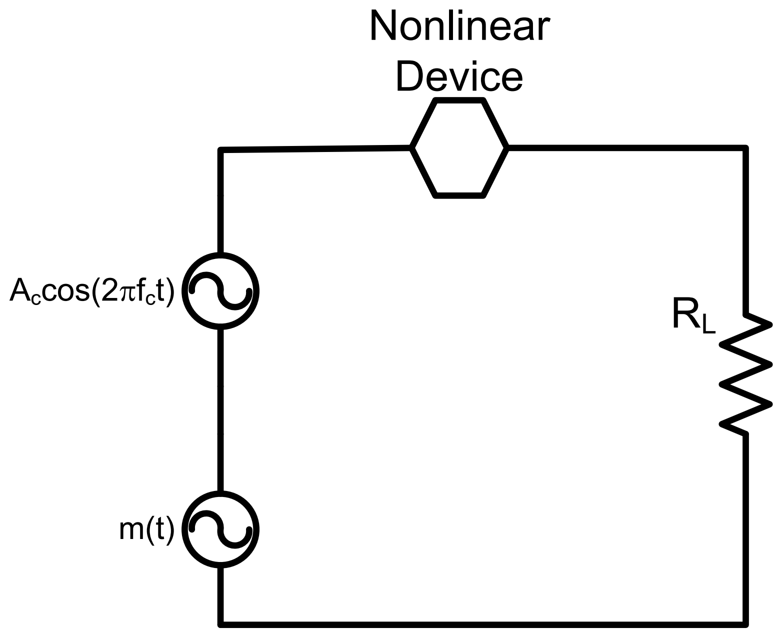
$$\frac{1}{2}\left(P_{AM} - P_C\right) = \frac{1}{2}\left(10 - 8.47\right) = 0.765kW$$

Percentage power in the single sideband

$$\frac{P_S}{P_{AM}} \times 100 = \frac{0.765}{10} \times 100 = 7.65\%$$

AM Generation

- Since the mathematical expression for AM appears as a product function, an obvious way producing such a signal is to develop a device in which the output is proportional to the product of the two input functions.



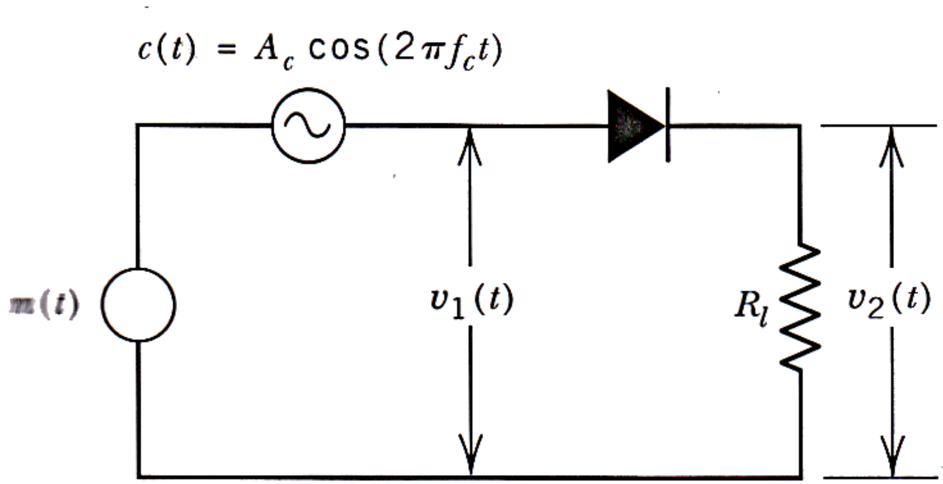
AM GENERATION
SQUARE-LAW MODULATOR

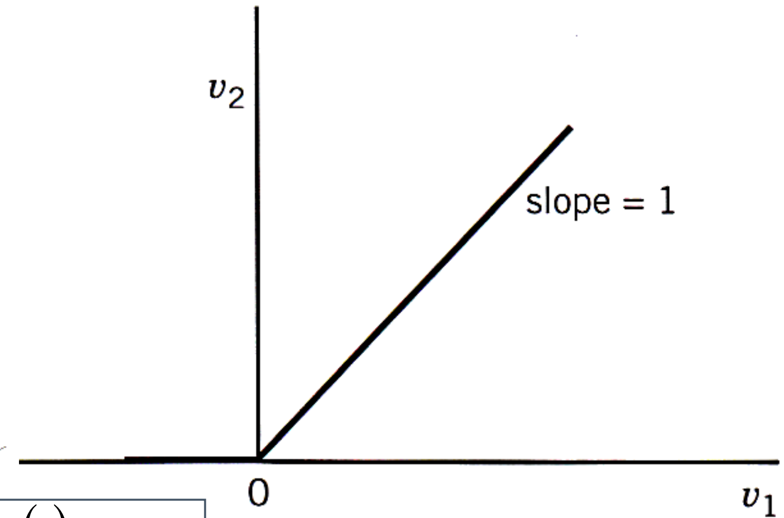
- As a nonlinear device a square wave modulator in which the output is proportional to the square of the input.

$$s(t) = a_1 x(t) + a_2 x^2(t)$$

$$\begin{aligned} s(t) &= a_1\left(A_c \cos(2\pi f_c t) + m(t)\right) + a_2\left(A_c \cos(2\pi f_c t) + m(t)\right)^2 \\ &= a_1 m(t) + a_2 A_c^2 \cos^2(2\pi f_c t) + a_2 m^2(t) + a_1 A_c \cos(2\pi f_c t) + 2a_2 A_c m(t)\cos(2\pi f_c t) \\ &= \underbrace{a_1 m(t) + a_2 A_c^2 \cos^2(2\pi f_c t) + a_2 m^2(t)}_{\text{Unwanted Terms}} + \underbrace{a_1 A_c \cos(2\pi f_c t)\left[1 + \frac{2a_2}{a_1} m(t)\right]}_{\text{AM Terms}} \end{aligned}$$

AM Generation
Switching Modulator

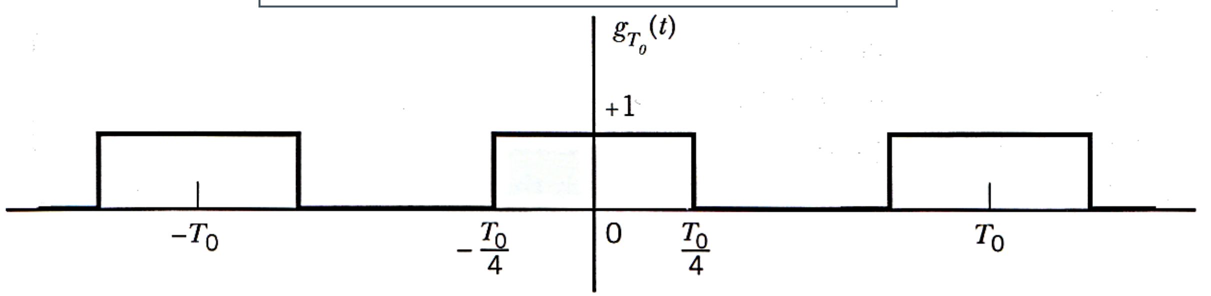




$$v_1(t) = A_c \cos(2\pi f_c t) + m(t)$$

$$v_2(t) \approx \begin{cases} v_1(t) & c(t) > 0 \\ 0 & c(t) < 0 \end{cases}$$

$$v_2(t) \approx \left[A_c \cos(2\pi f_c t) + m(t)\right] g_{T_0}(t)$$



AM GENERATION
SWITCHING MODULATOR

- Mathematically the operation is multiplying the sum of message and carrier with periodic pulse train.

$$g_{T_0}(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos[2\pi f_c t(2n-1)]$$

- The output has the following wanted component

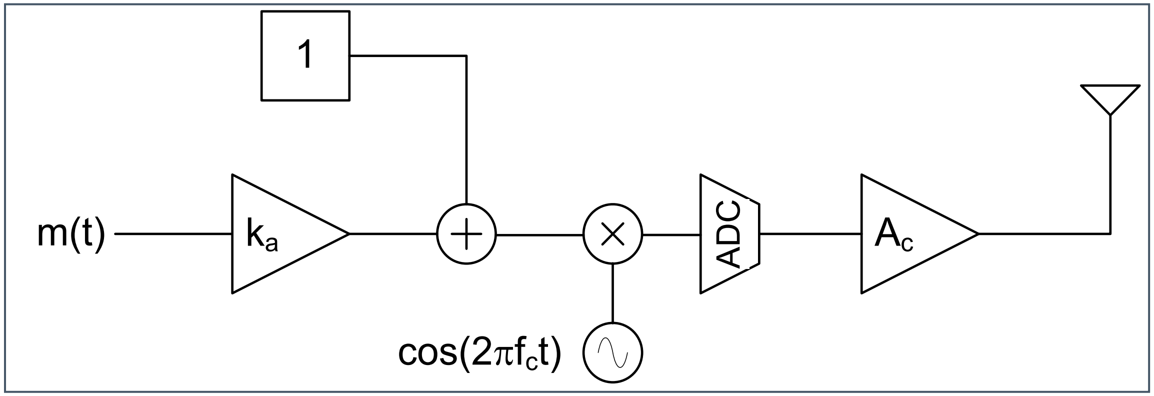
$$\frac{A_c}{2}\left[1+\frac{4}{\pi A_c}m(t)\right]\cos(2\pi f_c t)$$

The unwanted components at multiples of f_c are removed by band-pass filter

- The amplitude sensitivity of this modulator depends on carrier amplitude. Reducing it increases the sensitivity. However, it must be large enough to make the diode act like a switch

AM Generation
Product Modulator

- Square-Law modulator requires a device which has perfect relationship between its input and output. In practice this is usually not the case.
- Switching modulator requires an ideal switch which does not exist in analog world.
- Since current digital circuit speeds are much higher than AM requirements, a digital solution does also exist.



AM Demodulation
Envelope Detection

- A common AM demodulator is the envelope detector.
- Its main advantage is simple to construct.
- An envelope detector consists of a diode and a RC filter.
- The diode acts as a switch which is mathematically equivalent to multiply the received AM signal with pulse train.
- The RC filter is used to suppress the unwanted components.

AM Demodulation
Envelope Detection

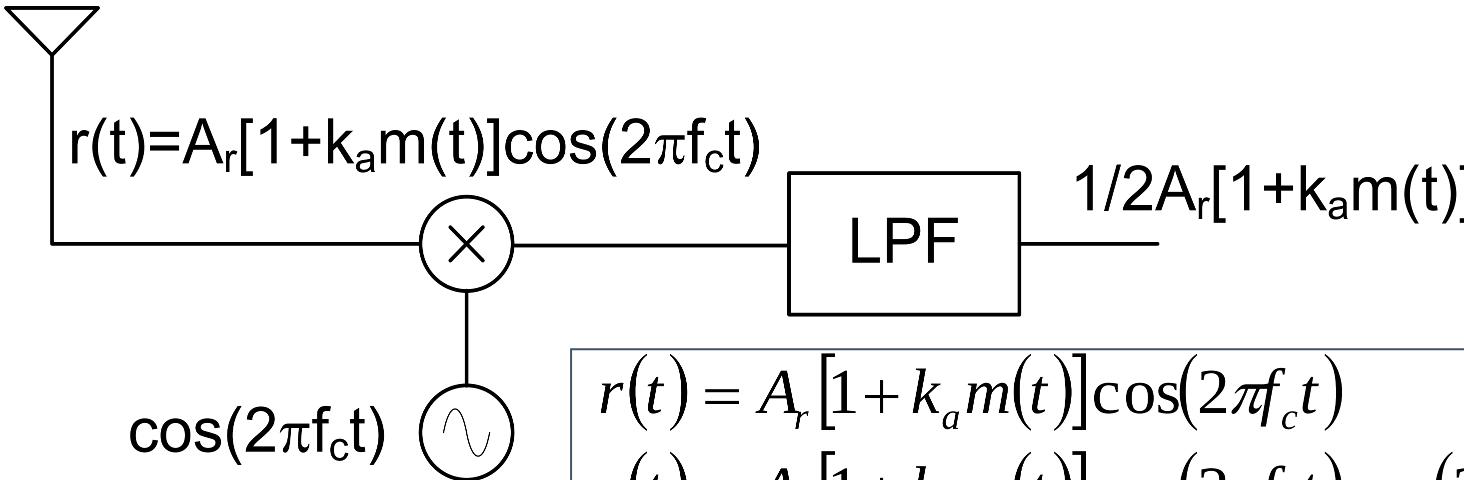
$$r(t) = A_r\left[1 + k_a m(t)\right]\cos(2\pi f_c t)$$

$$g_{T_0}(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\left[2\pi f_c t(2n-1)\right]$$

$$r(t)g_{T_0}(t) = \underbrace{\frac{A_r}{\pi}\left[1 + k_a m(t)\right]}_{\text{Wanted Term + DC}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\left[2\pi f_c tn\right]}_{\substack{\text{Unwanted Term} \\ \text{Removed by RC Filter}}}$$

- If the received signal level is equal to transmitted signal level, the wanted term is the same as the message signal multiplied by a constant term.
- Since this is not case, the signal level changes with the received signal power level

AM Demodulation
Coherent Demodulation



$$r(t) = A_r\left[1 + k_a m(t)\right]\cos(2\pi f_c t)$$

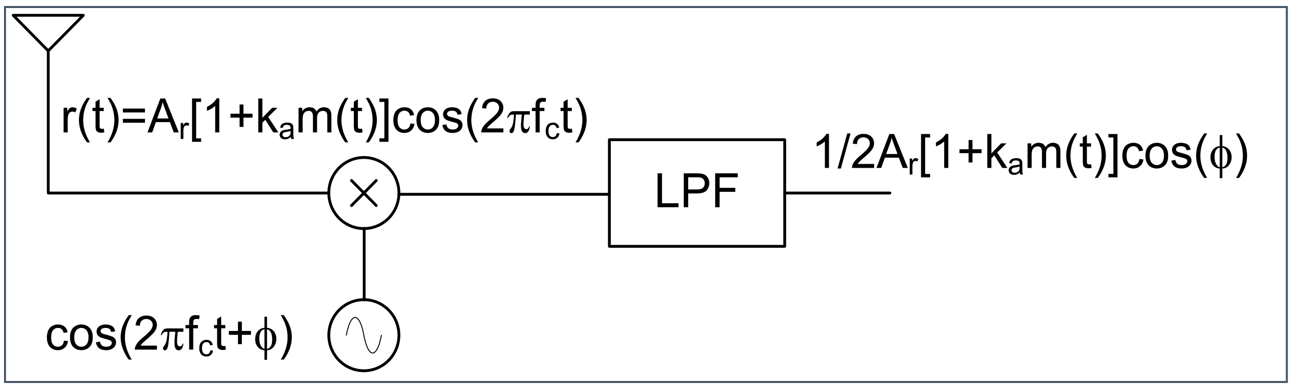
$$r_b(t) = A_r\left[1 + k_a m(t)\right]\cos(2\pi f_c t)\cos(2\pi f_c t)$$

$$= \frac{A_r}{2}\left[1 + k_a m(t)\right]\left[\underbrace{\cos(0)}_{\text{wanted term}} + \underbrace{\cos(4\pi f_c t)}_{\text{unwanted term}}\right]$$

- Coherent modulator uses the phase information and gives better SNR performance than envelope detector in the expense of increased complexity.

AM Demodulation
Coherent Demodulation

- What happens if there is a phase difference between receiver and transmitter?



$$r(t)=A_r[1+k_a m(t)]\cos(2\pi f_c t)$$

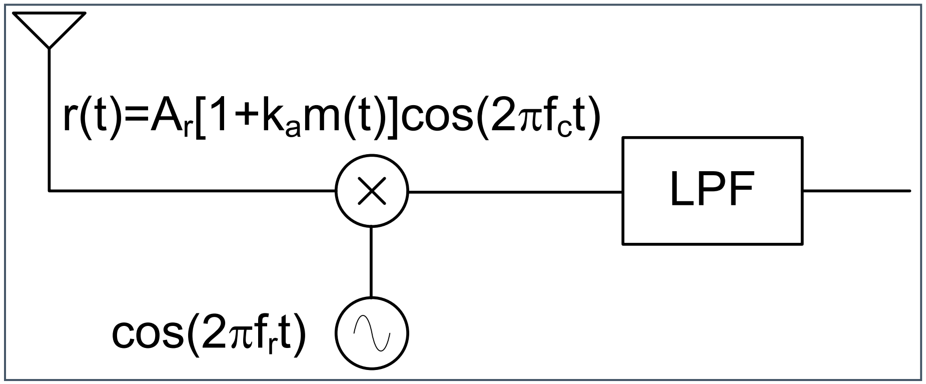
$$r_b(t)=A_r[1+k_a m(t)]\cos(2\pi f_c t)\cos(2\pi f_c t+\phi)$$

$$=\frac{A_r}{2}[1+k_a m(t)]\left[\underbrace{\cos(\phi)}_{\text{wanted term}}+\underbrace{\cos(4\pi f_c t+\phi)}_{\text{unwanted term}}\right]$$

if $\phi=\pi/2$ the wanted term will be suppressed.

AM Demodulation
Coherent Demodulation

- What happens if there is a frequency difference between receiver and transmitter?



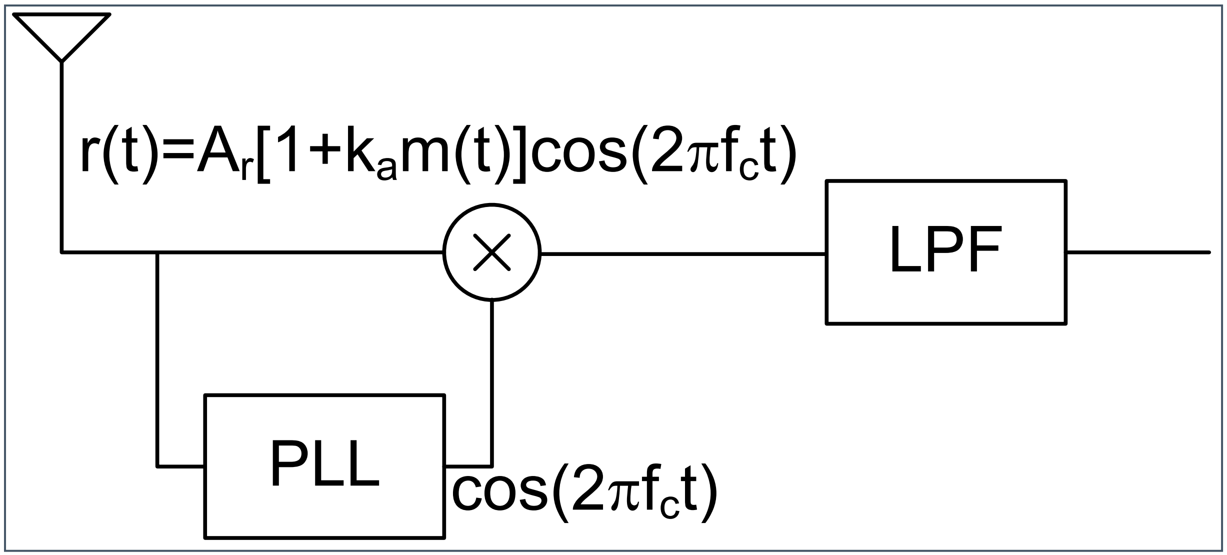
if f_c-f_r=20Hz there will be audible effect

$$r(t) = A_r\left[1 + k_a m(t)\right]\cos(2\pi f_c t)$$

$$r_b(t) = A_r\left[1 + k_a m(t)\right]\cos(2\pi f_c t)\cos(2\pi f_r t)$$

$$= \frac{A_r}{2}\left[1 + k_a m(t)\right]\left[\underbrace{\cos(f_c - f_r)}_{\text{wanted term}} + \underbrace{\cos(4\pi(f_c + f_r)t)}_{\text{unwanted term}}\right]$$

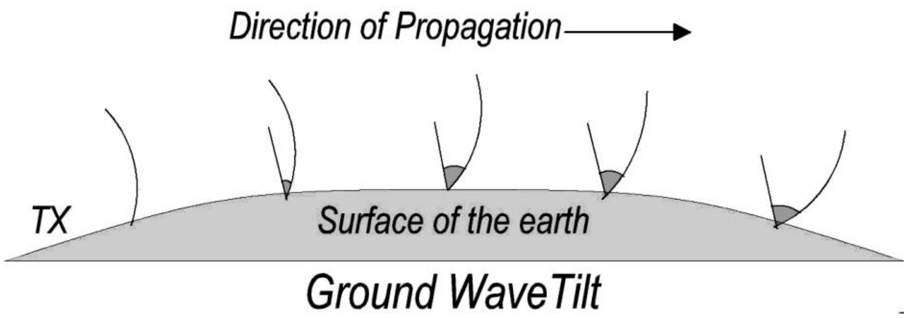
AM Demodulation
Coherent Demodulation

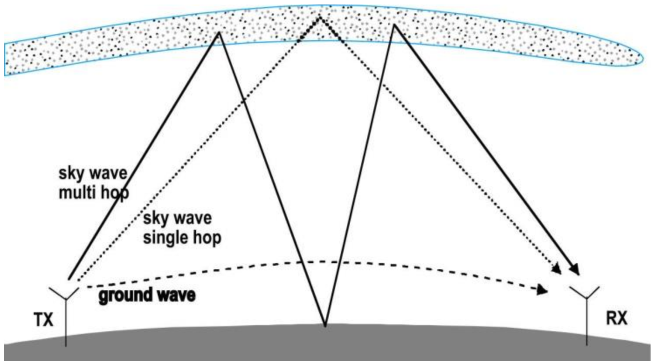


- The receiver oscillator is generated using a PLL (Phase Locked Loop) which is locked to the received signal carrier.
- This guarantees perfect match of the transmit carrier frequency to receiver oscillator frequency.

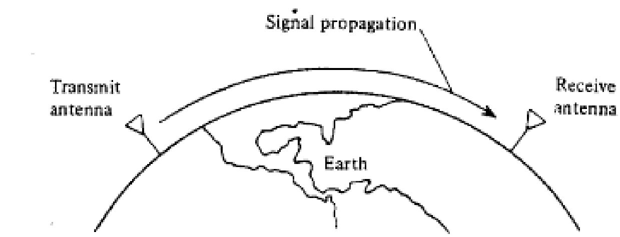
AM BROADCASTING

- Long Wave: 148.5KHz – 283.5KHz with 9KHz channel spacing
- Medium Wave: 520KHz – 1610KHz with 10KHz channel spacing in Americas, 9KHz elsewhere.
  - This is the “AM radio” that most people are familiar with
  - Medium wave signals follows the curvature of the earth (groundwave) at all the times and also refracts off the ionosphere at night (skywave)
- Short Wave: 1.711MHz – 30MHz divided 15 broadcast bands with 5KHz channel spacing. Short wave is intended for audio services at great distances.
- Time Signals: 40KHz – 80KHz band is used to transmit time signals to radio clocks.

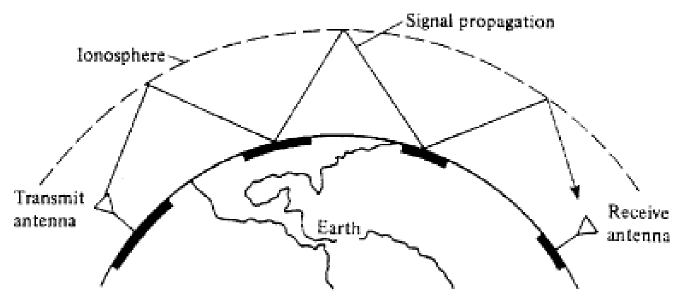




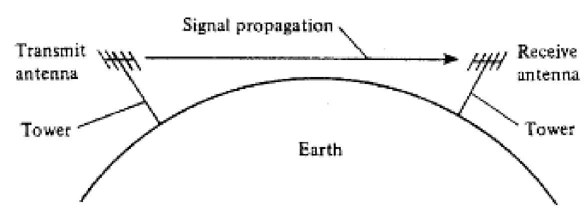
Propagation of Radio Frequencies



(a) Ground-Wave Propagation (Below 2 MHz)



(b) Sky-Wave Propagation (2 to 30 MHz)



(c) Line-of-Sight (LOS) Propagation (Above 30 MHz)

Limitations of Amplitude Modulation

- AM is the oldest modulation method
- Biggest advantage
  - easy to generate using a switching modulator or square-wave modulator
  - easy and cheap to demodulate using an envelope detector
- Disadvantages
  - Waste of power
  - Waste of bandwidth

Limitations of AM – Waste of Power

- The carrier wave is completely independent of the information
- The power of the carrier wave is wasted.
- For example, for single-tone AM modulation

$$
\begin{aligned}
\text{Total Transmit Power} &= \text{Carrier Power} \\
&\quad + \text{Upper side - band} + \text{Lower side - band} \\
&= \frac{1}{2}A_c^2 + \frac{1}{8}\mu^2 A_c^2 + \frac{1}{8}\mu^2 A_c^2 \\
&= \frac{1}{2}A_c^2\left[1 + \frac{1}{4}\mu^2 + \frac{1}{4}\mu^2\right] \\
&= \frac{1}{2}A_c^2\left[1 + \frac{1}{2}\mu^2\right]
\end{aligned}
$$

$$
\text{Wasted Power Ratio on Carrier} = \frac{\frac{1}{2}A_c^2}{\frac{1}{2}A_c^2\left[1 + \frac{1}{2}\mu^2\right]} = \frac{1}{1 + \frac{1}{2}\mu^2}
$$

EXAMPLE 1

- An AM transmitter's output power is 1KW. The modulation index is 0.4. What is the power wasted on the carrier?

$$\boxed{\text{Wasted Power on Carrier} = \frac{1}{1+\frac{1}{2}\mu^2} P_T = \frac{1}{1+\frac{1}{2}0.4^2} 1KW \approx 0.93KW}$$

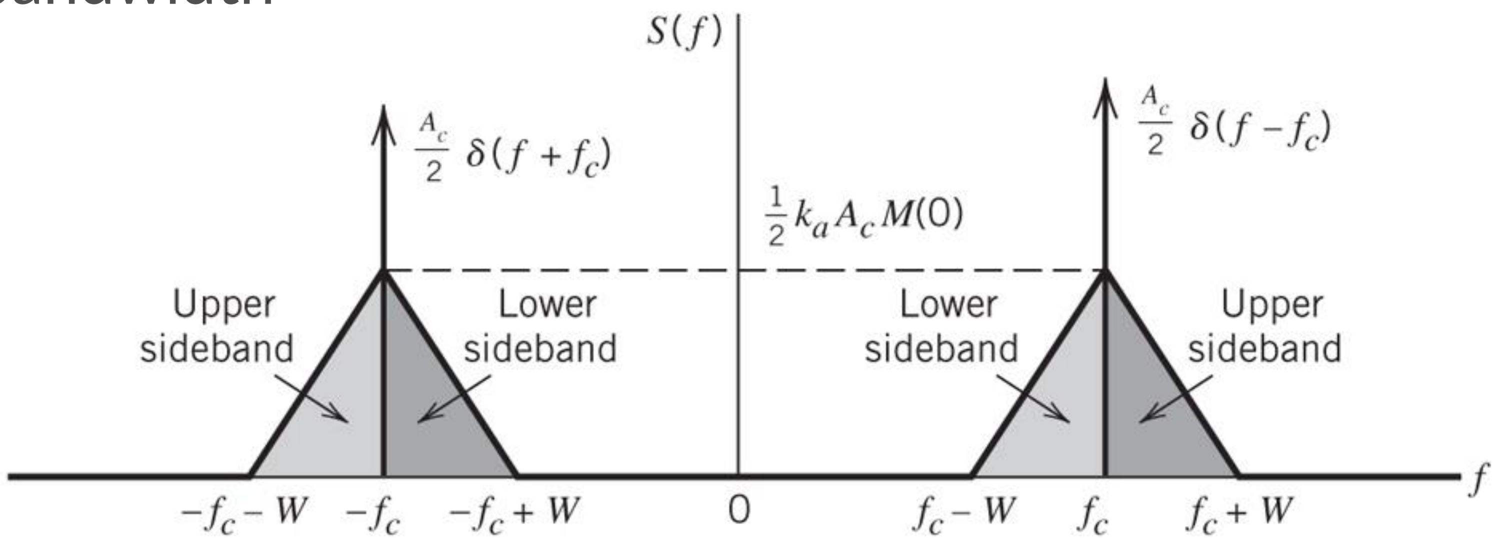
EXAMPLE 2

- A sinusoidal signal is applied to an AM system. The modulation percentage is 50% and the minimum envelope level is 1V. What is the maximum level of the envelope?

$$\boxed{\begin{aligned} &\mu = \frac{A_{\max} - A_{\min}}{A_{\max} + A_{\min}} \\ &0.5 = \frac{A_{\max} - 1V}{A_{\max} + 1V} \Rightarrow A_{\max} = 3V \end{aligned}}$$

LIMITATIONS OF AM – WASTE OF BANDWIDTH

- The upper and lower sidebands of and AM wave are uniquely related to each other.
- Only one sideband will be adequate to transmit the information.
- Waste of Bandwidth = twice of the message bandwidth



Modifications to AM

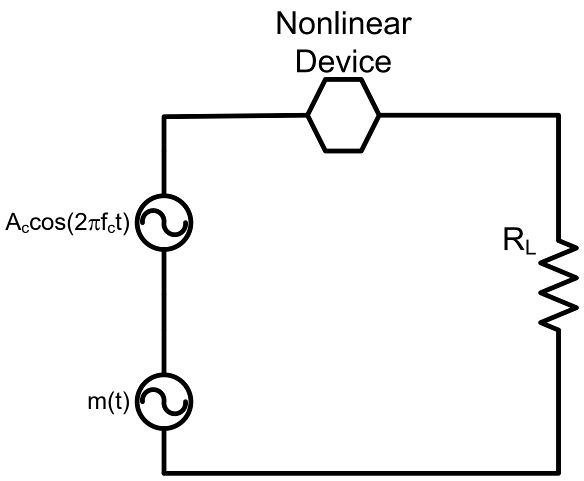
- Double Sideband Suppressed Carrier (DSB-SC) Modulation
  - The carrier is not transmitted
    - carrier power is saved.
  - The upper and lower sidebands are transmitted
    - bandwidth requirement is the same
  - Simple envelope detector can not be used as detector. Demodulation is more complex than AM with carrier.
- Single Sideband (SSB) Modulation
  - Only upper or lower sideband is transmitted.
    - Carrier and one of the sideband power is saved
    - Bandwidth requirement is the same as the message bandwidth
  - Simple envelope detector can not be used as detector. Demodulation is more complex than AM with carrier.
  - It is not possible to use signals down to DC due to the filter requirements.

Modifications to AM

- Vestigial Sideband (VSB) Modulation
  - one sideband is almost passed completely and just a trace, or vestige, of the other sideband is retained.
  - The required bandwidth is slightly higher than the message bandwidth.
  - It is better suited for transmitting signals that contain components at low frequencies.
  - This modulation scheme is used in commercial television broadcasting to send the picture information.
  - In TV broadcasting a carrier is also transmitted in order to use envelope detector to demodulate the incoming signal.

EXERCISE 1

- A device which has the following input-output relationship is wanted to be used for AM modulation.
  - Prove that it can be used as an DSB-SC modulator
  - Can this device be used as an AM modulator
  - If it is used as an AM modulator, determine its amplitude sensitivity



$$\boxed{s(t) = a_1 x(t) + a_3 x^3(t)}$$

Exercise 1 – Solution

$$s(t) = a_1 x(t) + a_3 x^3(t)$$

$$s(t) = a_1\left(A_c \cos(2\pi f_c t) + m(t)\right) + a_3\left(A_c \cos(2\pi f_c t) + m(t)\right)^3$$

$$= a_1 m(t) + a_1 A_c \cos(2\pi f_c t) + a_3 A_c^3 \cos^3(2\pi f_c t) + 3a_3 A_c^2 m(t)\cos^2(2\pi f_c t)$$

$$+ 3a_3 A_c m^2(t)\cos(2\pi f_c t) + a_3 m^3(t)$$

$$= a_1 m(t) + a_1 A_c \cos(2\pi f_c t) + \frac{3a_3 A_c^3}{4}\cos(2\pi f_c t) + \frac{a_3 A_c^3}{4}\cos(2\pi 3 f_c t)$$

$$+ \frac{3a_3 A_c^2}{2} m(t) + \frac{3a_3 A_c^2}{2} m(t)\cos(2\pi 2 f_c t) + 3a_3 A_c m^2(t)\cos(2\pi f_c t) + a_3 m^3(t)$$

$$= \underbrace{\left[a_1 A_c + \frac{3a_3 A_c^3}{4} + 3a_3 A_c m^2(t)\right]\cos(2\pi f_c t)}_{\text{AM Terms}} + \underbrace{\left[a_1 + \frac{3a_3 A_c^2}{2}\right] m(t) + a_3 m^3(t) + \frac{3a_3 A_c^2}{2} m(t)\cos(2\pi 2 f_c t) + \frac{a_3 A_c^3}{4}\cos(2\pi 3 f_c t)}_{\text{Unwanted Terms}}$$

$$= \underbrace{\left[a_1 A_c + \frac{3a_3 A_c^3}{4}\right]\left[1 + \frac{3a_3 A_c}{a_1 A_c + \frac{3a_3 A_c^3}{4}} m^2(t)\right]\cos(2\pi f_c t)}_{\text{AM Terms}} + \underbrace{\left[a_1 + \frac{3a_3 A_c^2}{2}\right] m(t) + a_3 m^3(t) + \frac{3a_3 A_c^2}{2} m(t)\cos(2\pi 2 f_c t) + \frac{a_3 A_c^3}{4}\cos(2\pi 3 f_c t)}_{\text{Unwanted Terms}}$$

$$k_a = \frac{3a_3}{a_1 + \frac{3a_3}{4}}$$

This device modulate the square of the message signal, therefore, the square root of message signal should be applied to this modulator

EXERCISE 1 - SOLUTION

$$s(t) = a_1 x(t) + a_3 x^3(t)$$

$$s(t) = a_1\left(A_c \cos(2\pi f_c t) + m(t)\right) + a_3\left(A_c \cos(2\pi f_c t) + m(t)\right)^3$$

$$= a_1 m(t) + a_1 A_c \cos(2\pi f_c t) + a_3 A_c^3 \cos^3(2\pi f_c t) + 3a_3 A_c^2 m(t)\cos^2(2\pi f_c t)$$

$$+ 3a_3 A_c m^2(t)\cos(2\pi f_c t) + a_3 m^3(t)$$

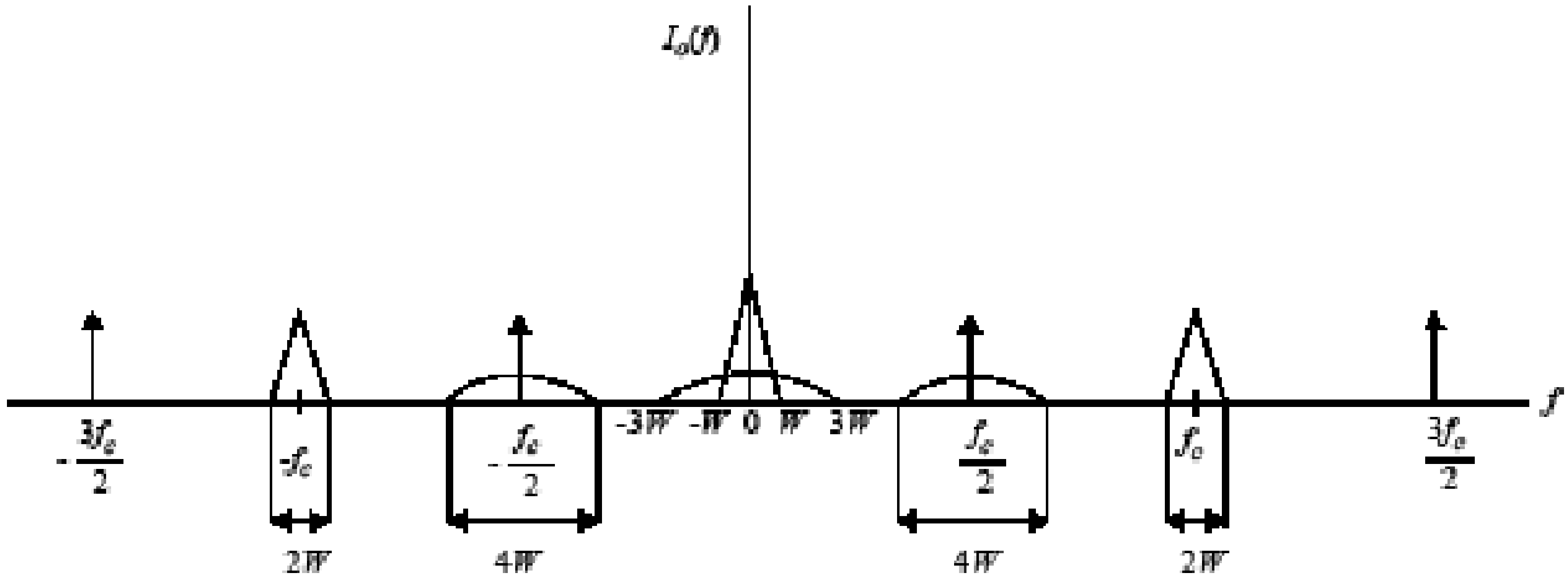
$$= a_1 m(t) + a_1 A_c \cos(2\pi f_c t) + \frac{3a_3 A_c^3}{4}\cos(2\pi f_c t) + \frac{a_3 A_c^3}{4}\cos(2\pi 3 f_c t)$$

$$+ \frac{3a_3 A_c^2}{2} m(t) + \frac{3a_3 A_c^2}{2} m(t)\cos(2\pi 2 f_c t) + 3a_3 A_c m^2(t)\cos(2\pi f_c t) + a_3 m^3(t)$$

$$= \underbrace{\frac{3a_3 A_c^2}{2} m(t)\cos(2\pi 2 f_c t)}_{\text{DSB-SC Terms}} + \underbrace{\left[a_1 + \frac{3a_3 A_c^2}{2}\right] m(t) + a_3 m^3(t) + \frac{a_3 A_c^3}{4}\cos(2\pi 3 f_c t) + \left[a_1 A_c + \frac{3a_3 A_c^3}{4} + 3a_3 A_c m^2(t)\right]\cos(2\pi f_c t)}_{\text{Unwanted Terms}}$$

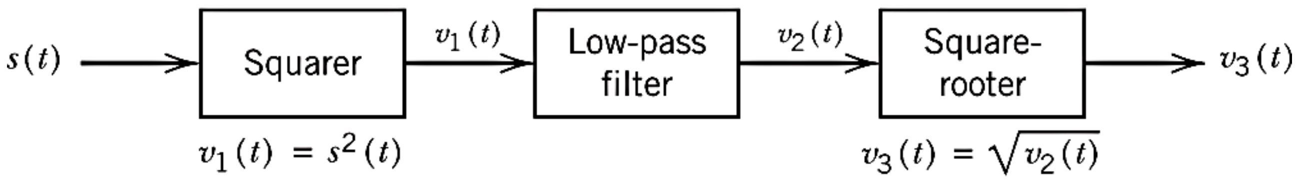
The DSB-SC signal is generated at the twice of the carrier frequency. Therefore, half of the desired carrier frequency should be applied to this modulator.

Exercise 1 – Solution



EXERCISE 2

- An AM signal is applied to the following system. The message signal is limited to the interval $-W<f<W$ and the carrier frequency $f_c>>W$. The low pass filter bandwidth is W. Show that m(t) can be obtained for the square-rooter output.



$$\boxed{s(t) = A_c\left[1+k_a m(t)\right]\cos(2\pi f_c t)}$$

Exercise 2 – Solution

$$s(t) = A_c\left[1 + k_a m(t)\right]\cos(2\pi f_c t)$$

$$s^2(t) = \left(A_c\left[1 + k_a m(t)\right]\right)^2 \cos^2(2\pi f_c t)$$

$$= \left(A_c\left[1 + k_a m(t)\right]\right)^2 \left(\frac{1}{2} + \frac{1}{2}\cos(2\pi 2 f_c t)\right)$$

$$s_{LPF}^2(t) = \frac{1}{2}\left(A_c\left[1 + k_a m(t)\right]\right)^2$$

$$\sqrt{s_{LPF}^2(t)} = \frac{A_c}{\sqrt{2}}\left[1 + k_a m(t)\right]$$

Double Sideband Suppressed Carrier Modulation

- Double Sideband Suppressed Carrier Modulation (DSB-SC) is product of message signal and carrier wave

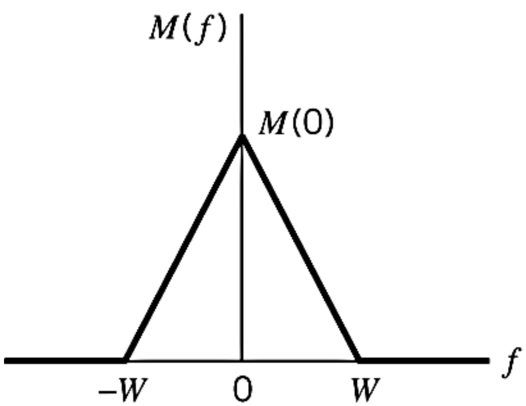
$$\boxed{\begin{aligned} s(t) &= c(t)m(t) \\ &= A_c \cos(2\pi f_c t)m(t) \end{aligned}}$$

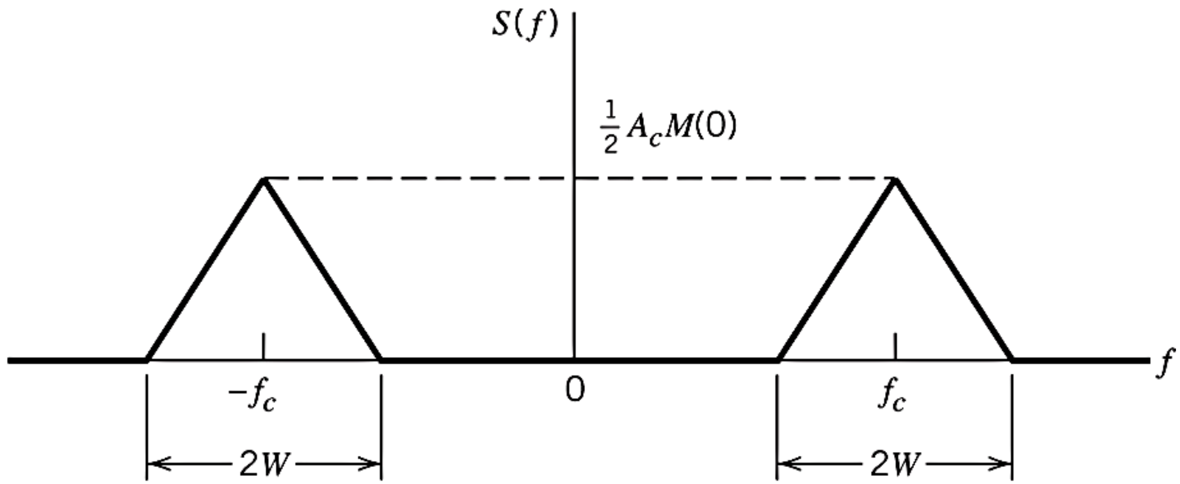
- The Fourier Transform of s(t) which is limited to the interval -W≤f≤W where W<<f_c is

$$\boxed{S(f) = \frac{1}{2}A_c\left[M(f - f_c) + M(f + f_c)\right]}$$

Double Sideband Suppressed Carrier Modulation

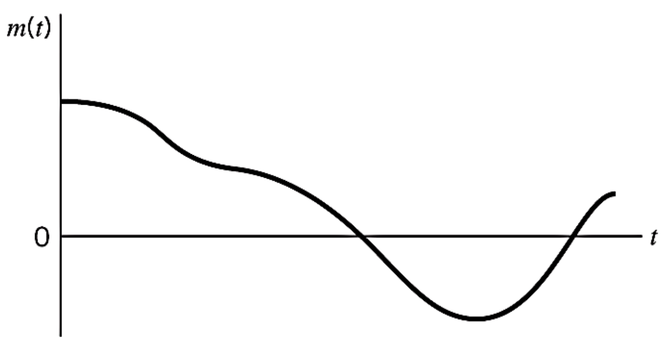
- The modulation process simply translates the baseband spectrum by $\pm f_c$
- The transmission bandwidth is twice of the bandwidth of the baseband signal.

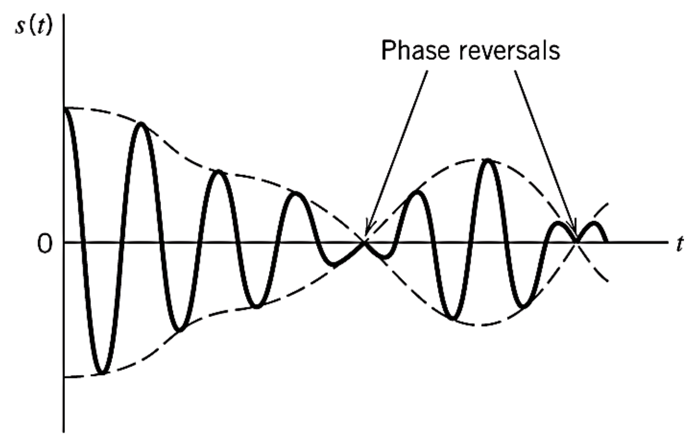




Double Sideband Suppressed Carrier Modulation

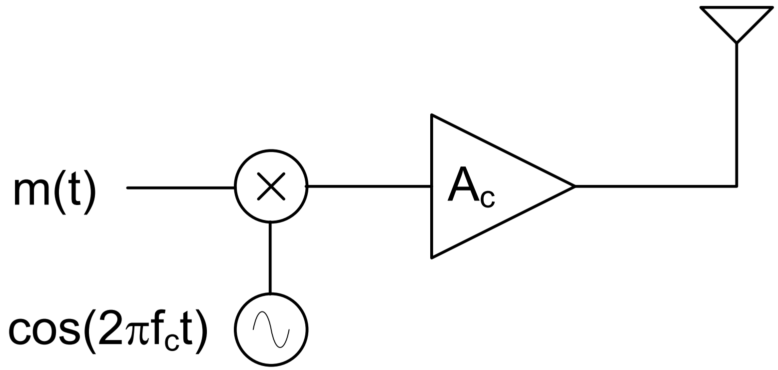
- The phase of s(t) reverses whenever the message signal m(t) crosses zero.
- The envelope of DSB-SC is different than m(t) ➔ simple envelope detector can not be used





DSB-SC Generation

- DSB-SC wave is easily generated using a mixer or product multiplier circuit.



- In some cases instead of using a sinusoidal oscillator a square wave oscillator is used.
- For example, ring modulator multiplies the message signal with square wave.

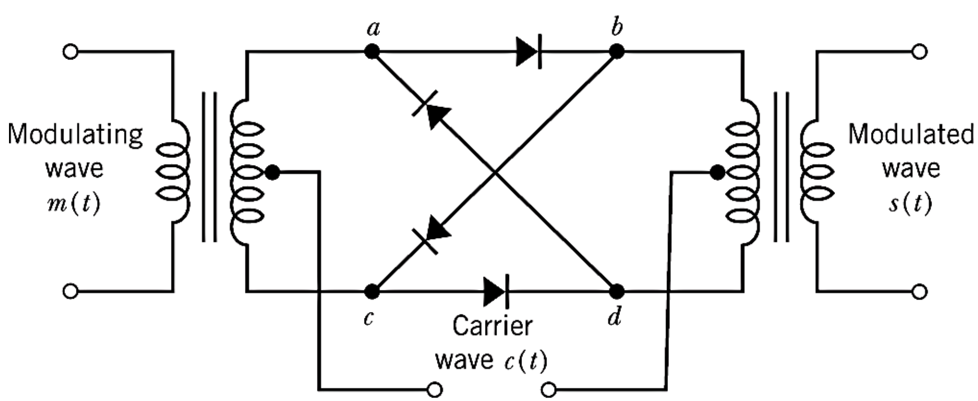
DSB-SC Generation

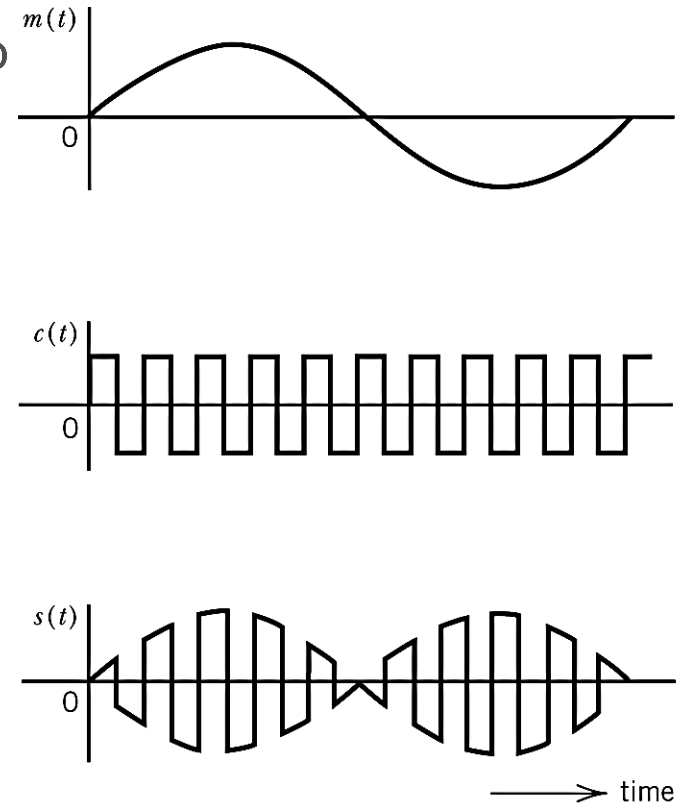
- Ring modulator needs a band pass filter to suppress the unwanted harmonics

$$c(t) = \frac{4}{\pi}\sum^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos[2\pi f_c t(2n-1)]$$

$$s(t) = c(t)m(t)$$

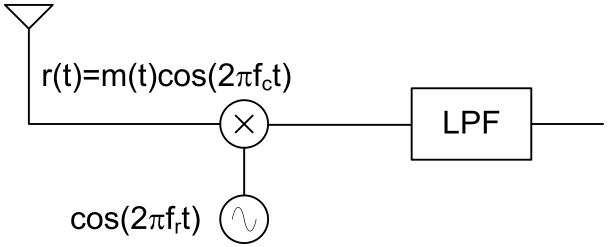
$$= \frac{4}{\pi}\sum^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos[2\pi f_c t(2n-1)]m(t)$$





DSB-SC DEMODULATION

- The baseband signal m(t) can be recovered by first multiplying the received signal with locally generated sinusoidal wave and then low-pass filtering the product to suppress unwanted terms.
- The locally generated sinusoidal wave must be synchronized with the transmitter local oscillator in frequency and phase.
- This method is also called coherent detection.



DSB-SC Demodulation

- If there is a phase difference between the receiver and transmitter the message signal level will reduce
- If there is a frequency difference between the receiver and transmitter the message signal will not translate exactly to baseband.

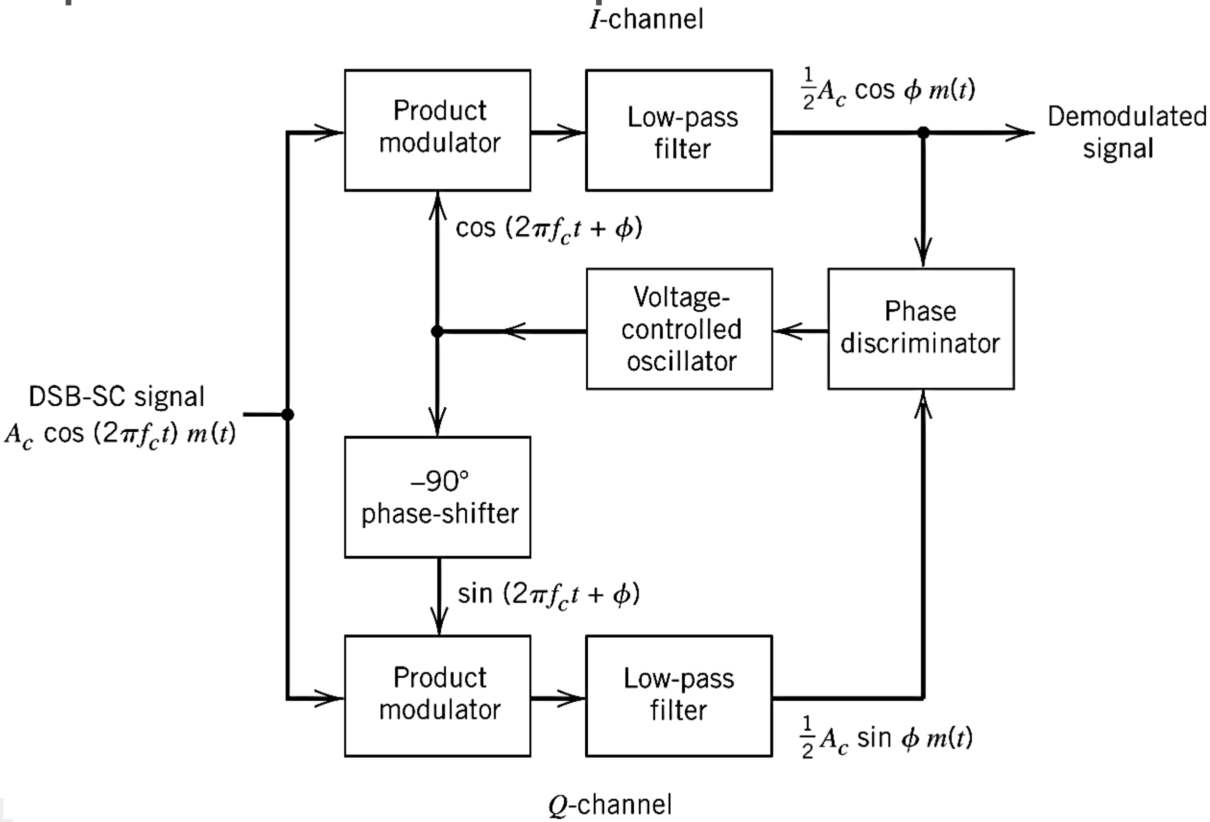
$$
\begin{aligned}
m_r(t) &= A_r\cos(2\pi(f_c+\Delta f)t+\phi)s(t) \\
&= A_r A_c\cos(2\pi f_c t)\cos(2\pi(f_c+\Delta f)t+\phi)m(t) \\
&= \frac{A_r A_c}{2}\cos(2\pi\Delta f t+\phi)m(t)+\frac{A_r A_c}{2}\cos(2\pi(2f_c+\Delta f)t+\phi)m(t) \\
&\quad \text{after low-pass filter} \\
m_r(t) &= \frac{A_r A_c}{2}\cos(2\pi\Delta f t+\phi)m(t)
\end{aligned}
$$

DSB-SC Demodulation

- The demodulated signal is proportional to m(t) if the phase and frequency errors are zero
- If the frequency is matched in other words Δf is zero, the demodulated signal is maximum when $\phi=0$ and it is minimum (zero) when $\phi=\pm\pi/2$. This effect is called quadrature null effect.
- In practice the phase error varies randomly in time. This may result in signal loss from time to time.
- Obviously a system which tracks the transmitter oscillator is required.

Costas Receiver

- This receiver consists of two coherent detectors which have the same input, but have local oscillators that are inphase quadrature with respect to each other



Costas Receiver

- In-phase coherent detector is also called I-channel
- Out-of-phase coherent detector is called Q-channel
- The operation is as follows
  - The output of the phase discriminator, which is basically a divider followed by atan operator.

$$\boxed{v_{err}(t) = \operatorname{atan}\left(\frac{\frac{1}{2}A_c\sin(\phi)m(t)}{\frac{1}{2}A_c\cos(\phi)m(t)}\right) = \operatorname{atan}\left(\frac{\sin(\phi)}{\cos(\phi)}\right) = \phi}$$

  - The DC signal which is feed to voltage controlled oscillator is used to correct the phase difference

Costas Receiver

- The loop in the Costas receiver requires the modulated signal in order to stay in lock.
- The reestablishment of the phase lock is not a problem for voice transmission, because the lock-up process occurs so rapidly that no distortion is perceptible.

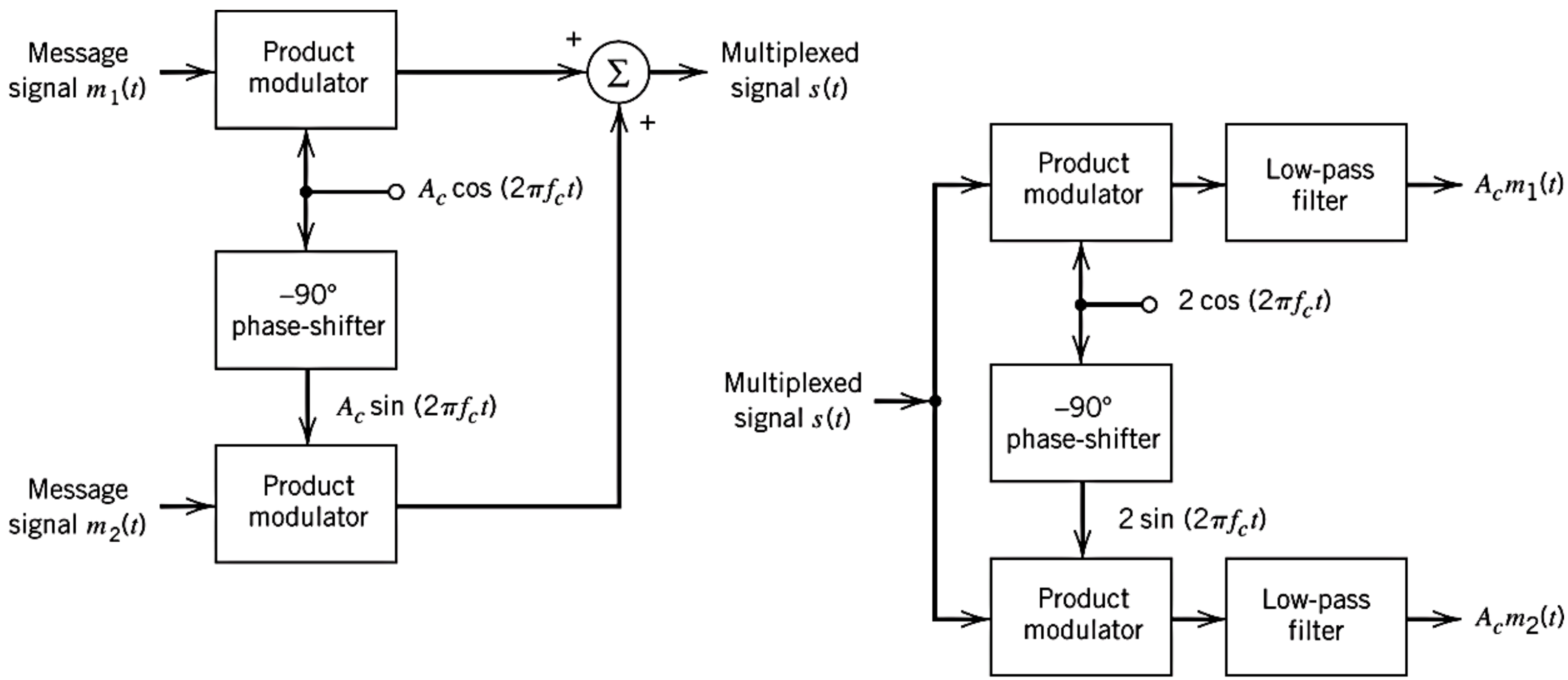
Quadrature Carrier Multiplexing

- The quadrature null effect of the coherent detector may also be used to increase the amount of information for a given bandwidth.
- Two DSB-SC modulated waves of two independent message signals occupy the same bandwidth at the same time, yet they can be recovered at the receiver.

$$\boxed{s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)}$$

- m_1 and m_2 are two independent message signals with the bandwidth of W.
- The success of the system highly depends on the synchronization of the oscillators of receiver and transmitter

Quadrature Carrier Multiplexing

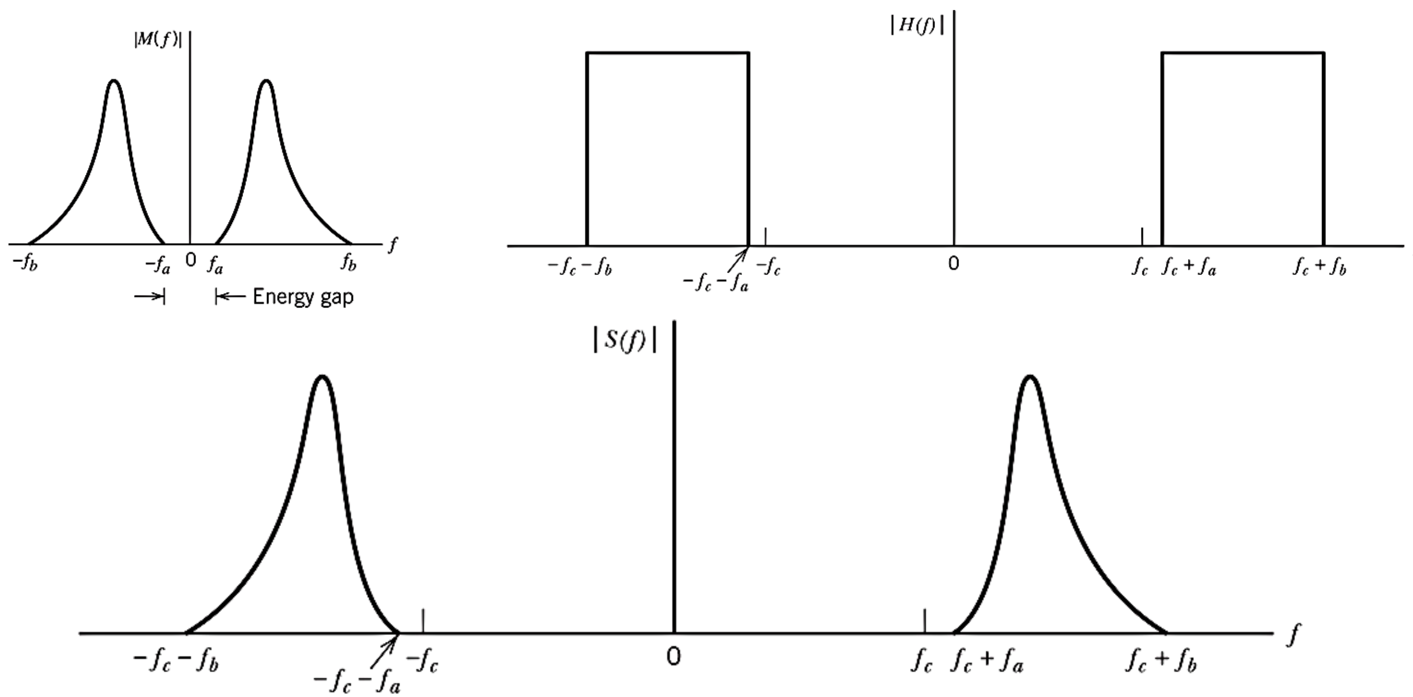


Single-Sideband Modulation

- Double Sideband Modulation schemes (AM and DSB-SC) transmit the same information at the lower and upper sidebands.
- The conserve the bandwidth one of the sideband can be transmitted.
- The generation of Single-Sideband (SSB) signal is straightforward.
  - First generate a DSB-SC signal
  - Then apply a band-pass filter which passes only the frequencies from f_c to f_c+W
- Although ideally the generation seems easy, it is not possible to design an ideal filter.

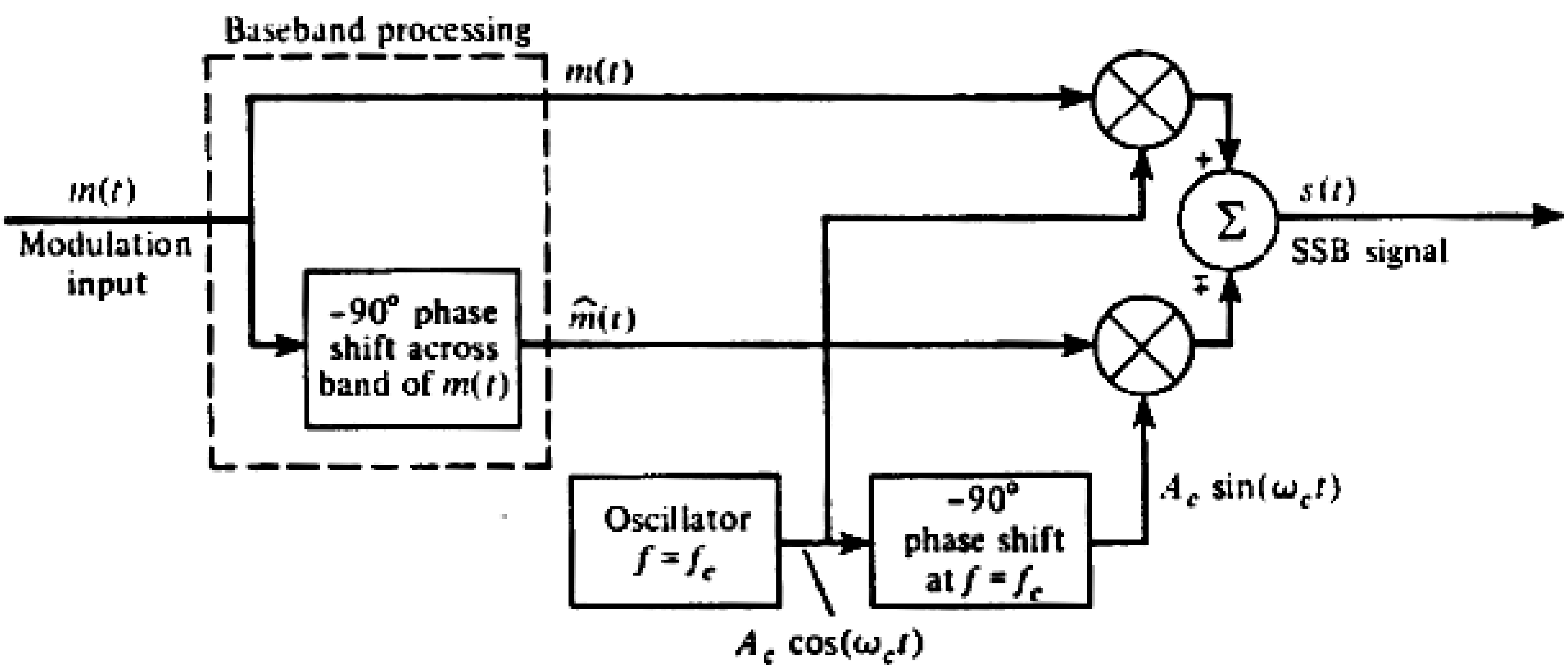
Single-Sideband Modulation

- SSB is mainly used for voice applications where almost no energy below 300Hz is present.



SSB Modulation
Phase Shift Method

- Depending on whether we add or subtract the outputs of balanced modulators, the output will have upper sideband or lower sideband.
- 90° phase shift operation is basically applying Hilbert transform



SSB Modulation

- The SSB modulation is analyzed using Hilbert Transform.

$$
\begin{aligned}
\hat{M}(f) &= -j\,\mathrm{sgn}(f)\left[M_U(f) + M_L(f)\right] \\
&= j\left[M_L(f) - M_U(f)\right] \\
S_1(f) &= \frac{1}{2}A_c\left[M_U(f-f_c) + M_L(f-f_c) + M_U(f+f_c) + M_L(f+f_c)\right] \\
S_2(f) &= \frac{1}{2j}A_c\,j\left[M_L(f-f_c) - M_U(f-f_c) + M_L(f+f_c) - M_U(f+f_c)\right] \\
S_L(f) &= S_1(f) + S_2(f) \\
&= A_c\left[M_L(f-f_c) + M_L(f+f_c)\right] \\
S_U(f) &= S_1(f) - S_2(f) \\
&= A_c\left[M_U(f-f_c) + M_U(f+f_c)\right]
\end{aligned}
$$

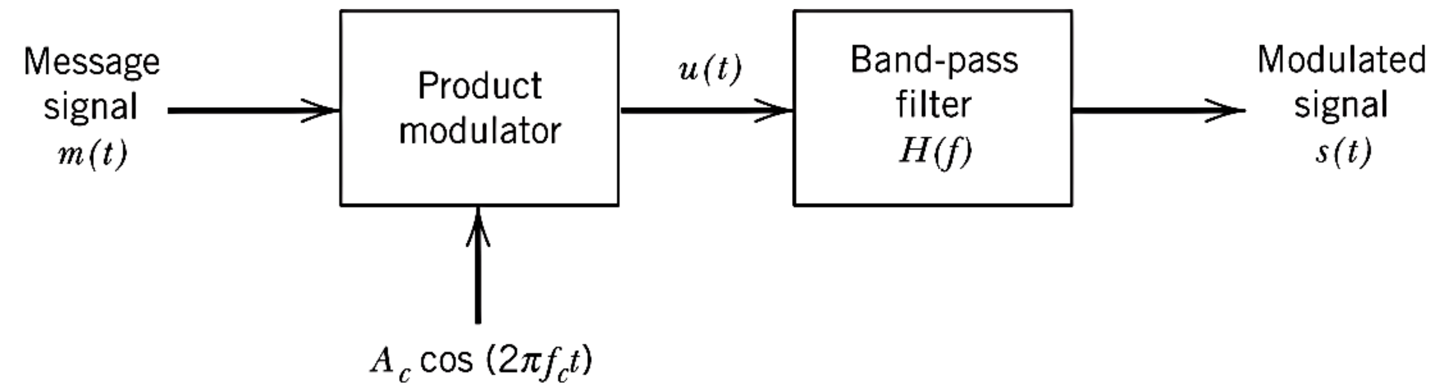
SSB Demodulation

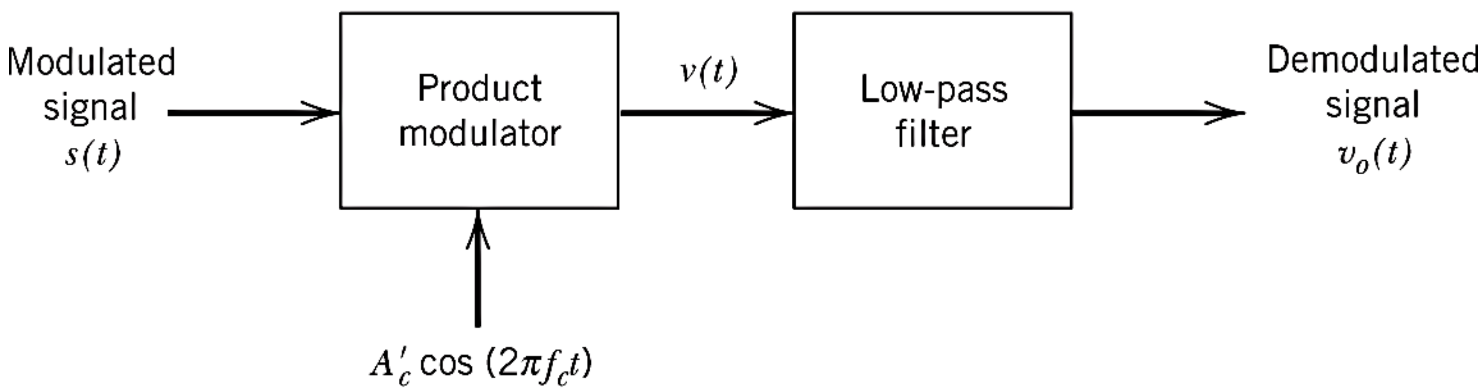
- SSB requires coherent demodulation.
- The synchronization is obtained often by one of the following methods
  - Transmitting a low power pilot carrier in addition to selected sideband. This pilot is used to generate local oscillator signal which is in sync with the transmitter oscillator
  - Using highly stable oscillators in both the transmitter and receiver for generating SSB signal
    - If there is a phase error between transmitter and receiver, the phase of the message signal is distorted.

Vestigial Sideband Modulation

- SSB modulation requires an energy gap at the origin.
- If the message signal goes down to the DC, all of the one sideband is transmitted and a small amount (vestige) of the other sideband is transmitted as well.
- In VSB, the filter is allowed to have a nonzero transition band.
- Obviously, there should be some restrictions on the filter in order to recover the message signal.

Vestigial Sideband Modulation





Vestigial Sideband Modulation

$$S(f) = U(f)H(f)$$

$$= \frac{A_c}{2}\left[M(f - f_c) + M(f + f_c)\right]H(f)$$

$$v(t) = A_c^{'} \cos(2\pi f_c t)s(t)$$

$$V(f) = \frac{A_c^{'}}{2}\left[S(f - f_c) + S(f + f_c)\right]$$

$$= \frac{A_c^{'} A_c}{4} M(f)\left[H(f - f_c) + H(f + f_c)\right]$$

$$+ \frac{A_c^{'} A_c}{4}\left[M(f - 2f_c)H(f - f_c) + M(f + 2f_c)H(f + f_c)\right]$$

$$V_o(f) = \frac{A_c^{'} A_c}{4} M(f)\left[H(f - f_c) + H(f + f_c)\right]$$

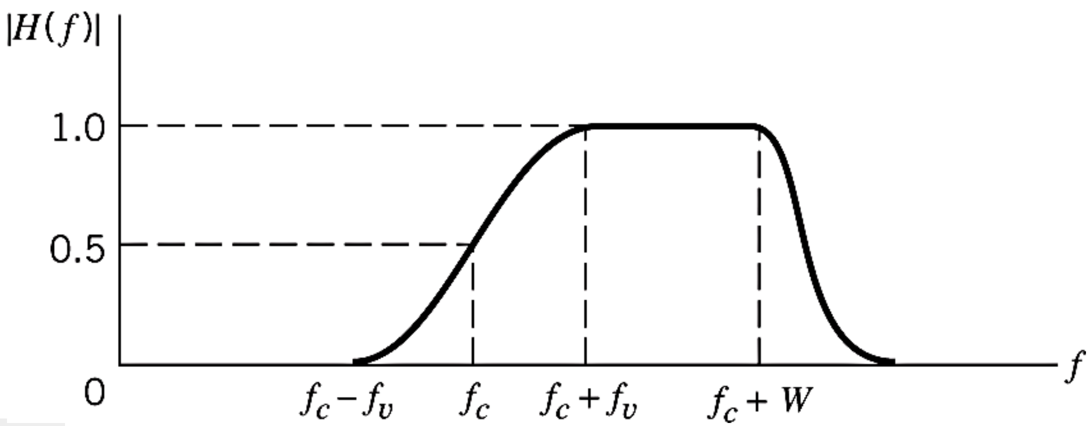
Vestigial Sideband Modulation

- For distortionless reproduction of m(t) at the detector output, $V_o(f)$ must be scaled version of M(f)
- The filter's transfer function must satisfy the condition

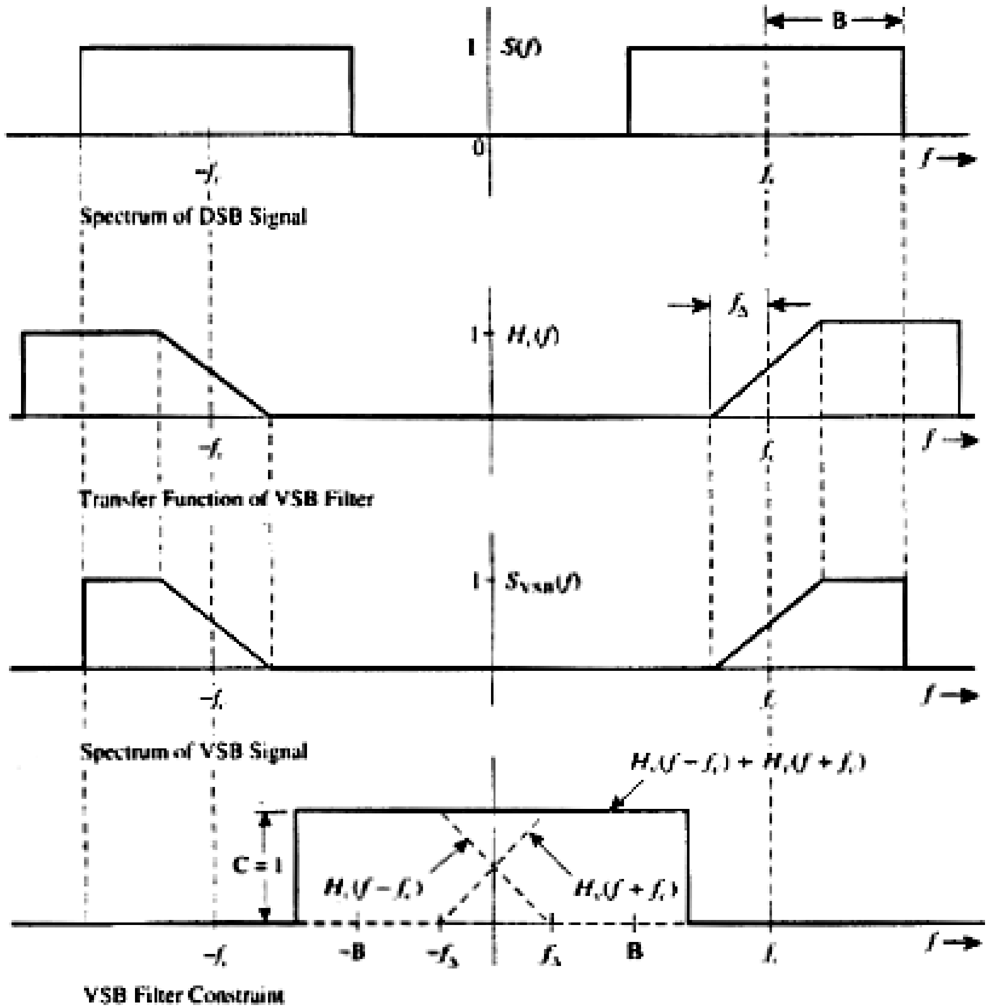
$$\boxed{H(f-f_c)+H(f+f_c)=2H(f_c)}$$

- If M(f) is limited to $-W \le f \le W$ interval, this condition needs to be satisfied only in this interval.
- Moreover, if we set $H(f_c)=1/2$, H(f) needs to satisfy

$$\boxed{H(f-f_c)+H(f+f_c)=1 \qquad -W \le f \le W}$$

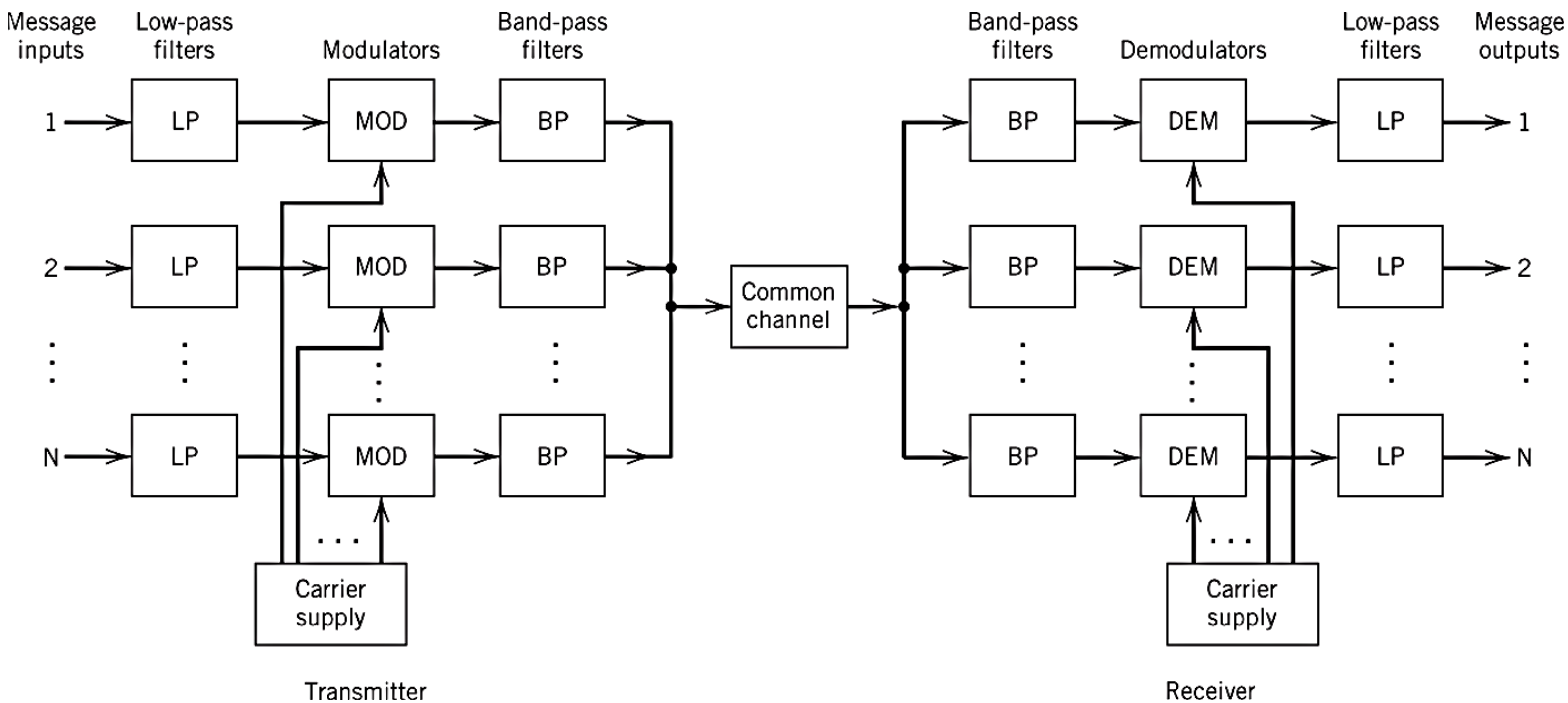


Vestigial Sideband Modulation



FREQUENCY DIVISION MULTIPLEXING

- In multiplexing operation, several independent signals are combined into a composite signal suitable for transmission over a common channel.
- If the signals are separated in frequency, it is called frequency-division multiplexing (FDM)



Introduction

- Angle Modulation:
  - The angle of the carrier wave is varied according to the baseband (message) signal.
  - The amplitude of the carrier wave is maintained constant.
  - There are two forms of angle modulation
    - Phase modulation
    - Frequency modulation
  - Angle modulation gives better performance in the presence of noise and interference at the expense of increased bandwidth.

$$\boxed{a(t)\cos\left(2\pi f_c t + \phi(t)\right)}$$

Basic Definitions

- $\theta_i(t)$ is the angle of a modulated sinusoidal carrier at time t.
- The angle modulated wave

$$\boxed{s(t) = A_c \cos[\theta_i(t)]}$$

- where A_c is the carrier amplitude.
- $\theta_i(t)$ changes 2π ➔ complete oscillation occurs
- $\theta_i(t)$ increases monotonically with time ➔ average frequency from t to $t+\Delta t$

$$\boxed{f_{\Delta t}(t) = \frac{\theta_i(t+\Delta t) - \theta_i(t)}{2\pi\Delta t}}$$

BASIC DEFINITIONS

- The instantaneous frequency of the angle-modulated signal

$$\boxed{\begin{aligned} f_i(t) &= \lim_{\Delta t \to 0} f_{\Delta t}(t) \\ &= \lim_{\Delta t \to 0}\left[\frac{\theta_i(t+\Delta t)-\theta_i(t)}{2\pi\Delta t}\right] \\ &= \frac{1}{2\pi}\frac{d\theta_i(t)}{dt} \end{aligned}}$$

- Frequency is the derivative of the angle with respect to time t

Basic Definitions

- Angle modulated signal s(t) may be interpreted as a rotating phasor of length Ac and angle $\theta_i(t)$.
- The angular velocity of this phasor is $d\theta_i(t)/dt$.
- As a simple case the angle of an unmodulated carrier

$$\boxed{\theta_i(t) = 2\pi f_c t + \phi_c}$$

- The phasor velocity is $2\pi f_c$
- The constant ϕ_c is the value of $\theta_i(t)$ at t=0

Basic Definitions

Phase Modulation

- Instantaneous angle $\theta_i(t)$ is varied linearly with the message signal

$$\boxed{\theta_i(t) = 2\pi f_c t + k_p m(t)}$$

- $2\pi f_c t$ represents the angle of unmodulated carrier
- k_p is the phase sensitivity of the modulator in radians per volt (m(t) is assumed a voltage waveform)
- For convenience the angle of unmodulated carrier at t=0 is assumed 0

$$\boxed{s(t) = A_c \cos\left[2\pi f_c t + k_p m(t)\right]}$$

Basic Definitions

- Frequency Modulation
  - Instantaneous frequency $f_i(t)$ is varied linearly with the message signal m(t)

$$\boxed{f_i(t) = f_c + k_f m(t)}$$

  - f_c is the frequency of the unmodulated carrier
  - k_f is frequency sensitivity in Hertz per volt
  - The instantaneous phase is

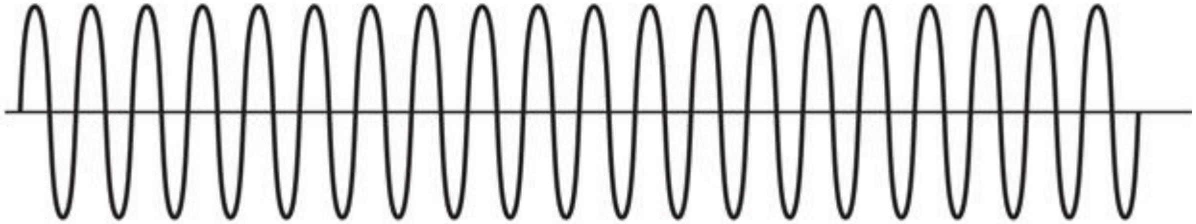
$$\boxed{\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau}$$

  - Assuming the angle at t=0 is 0

$$\boxed{s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau\right]}$$

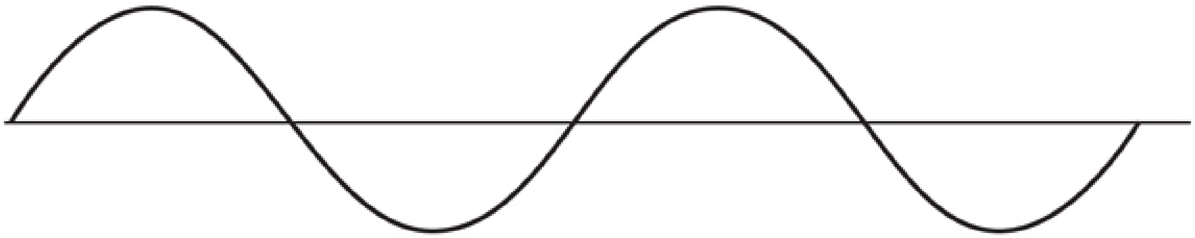
Basic Definitions

- Carrier Wave



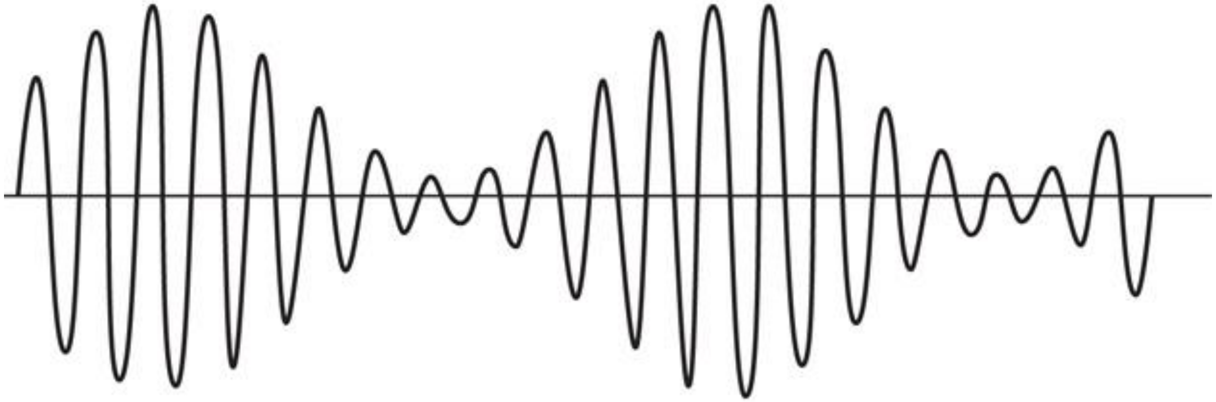
BASIC DEFINITIONS

- Message Signal



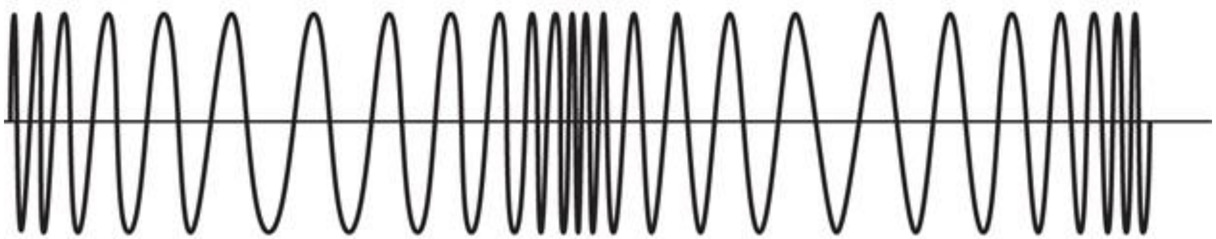
Basic Definitions

- AM Wave



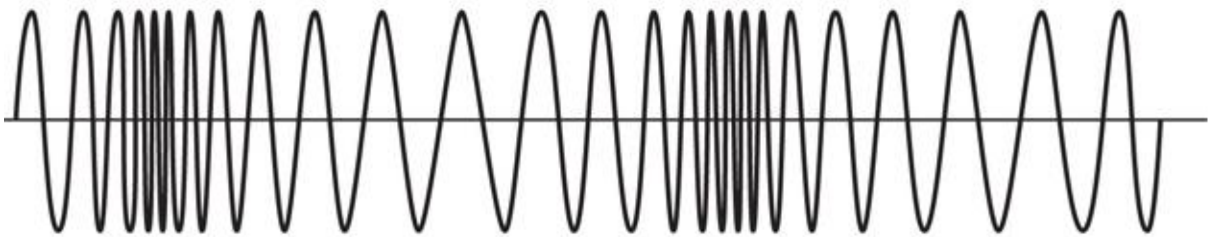
Basic Definitions

- Phase Modulated Wave



Basic Definitions

- Frequency Modulated Wave



Properties of Angle-Modulated Waves

- Property 1: Constancy of Transmitted Power
  - The amplitude of PM and FM waves equals to A_c which is a constant value.
  - The average transmitted power of angle modulated waves is constant.

$$\boxed{P_{av} = \frac{1}{2} A_c^2}$$

Properties of Angle-Modulated Waves

- Property 2: Nonlinearity of the modulation process
  - Angle modulation is a nonlinear process ➔ violates the principle of superposition

$$
\boxed{
\begin{aligned}
&m(t) = m_1(t) + m_2(t) \\
&s(t) = A_c \cos\left[2\pi f_c t + k_p\left(m_1(t) + m_2(t)\right)\right] \\
&s_1(t) = A_c \cos\left[2\pi f_c t + k_p m_1(t)\right] \\
&s_2(t) = A_c \cos\left[2\pi f_c t + k_p m_2(t)\right] \\
&s(t) \neq s_1(t) + s_2(t)
\end{aligned}
}
$$

  - This nonlinearity property complicates the spectral analysis of PM and FM waves.

Properties of Angle-Modulated Waves

- Property 3: Irregularity of Zero-Crossings
  - Since instantaneous angle depends on the message signal or integral of message signal, the zero-crossings of PM and FM wave have no perfect regularity in their spacing across time-scale.
  - Zero-crossings: the instant time at which a waveform changes its amplitude from positive to negative or the other way around.
  - The information content of the message signal m(t) resides in the zero-crossings of the modulated wave.
  - Due to the this property PM and FM waves shows better performance in the presence of interferers.

Properties of Angle-Modulated Waves

- Property 4: Visualization Difficulty of Message Waveform
  - In AM the message signal is seen as the envelope of the modulated wave. This is not so in angle modulated waves.
- Property 5: Trade-Off of Increased Transmission Bandwidth for Improved Noise Performance
  - The improvement in noise performance is at the expense of increasing the transmission bandwidth.
  - This trade-off is not possible with amplitude modulation.

Phase and Frequency Modulation

- Comparing PM and FM waveforms reveals that FM signal can be considered as a PM signal in which the message signal is integrated version of m(t).
- The properties of PM signals can be deduced from the properties of FM signal
- Therefore. we will concentrate on FM signals

$$\boxed{s(t) = A_c \cos\left[2\pi f_c t + k_p m(t)\right]}$$

$$\boxed{s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau) d\tau\right]}$$

Frequency Modulation

- Since the FM signal is a nonlinear function of the message signal m(t), it is not easy to analyze its spectrum.
- Let's start the simplest case of the message signal, which is a single sinusoidal tone.

$$\boxed{m(t) = A_m \cos(2\pi f_m t)}$$

Frequency Modulation

$$\begin{aligned}
s(t) &= A_C \cos\left[2\pi f_c t + 2\pi k_f \int_0^t A_m \cos(2\pi f_m \tau) d\tau\right] \\
&= A_C \cos\left[2\pi f_c t + A_m \frac{k_f}{f_m} \sin(2\pi f_m t)\right] \\
&= A_C \cos[2\pi f_c t + \beta \sin(2\pi f_m t)] \\
&= \Re\left\{A_C e^{j(2\pi f_c t + \beta \sin 2\pi f_m t)}\right\} \\
&= \Re\left\{A_C e^{j2\pi f_c t} e^{j\beta \sin 2\pi f_m t}\right\}
\end{aligned}$$

- The ratio of peak frequency deviation to the max modulation frequency is called modulation index

$$\beta = \frac{A_m k_f}{f_m} = \frac{\Delta_f}{f_m} = \frac{\text{peak frequency deviation}}{\text{max frequency of } m(t)}$$

Frequency Modulation

$$\boxed{e^{j\beta\sin 2\pi f_m t} \text{ is periodic with } T_0 = 1/f_m}$$

- Expanding this signal to Fourier Series

$$\boxed{e^{j\beta\sin 2\pi f_m t} = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_m t}}$$

- The Fourier Series Coefficients

$$\boxed{C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j\beta\sin 2\pi f_m t} e^{-j2\pi n f_m t}\, dt}$$

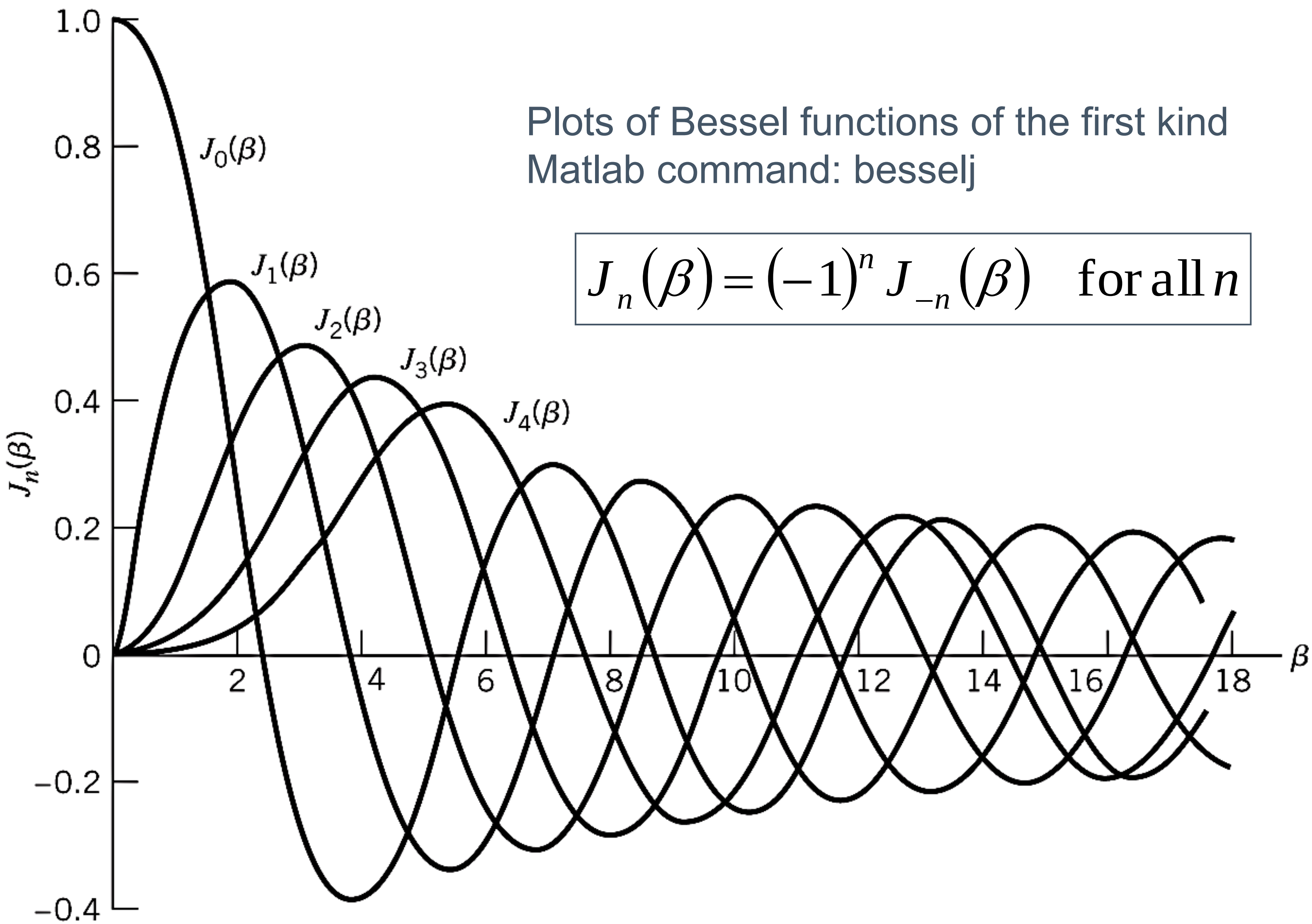
FREQUENCY MODULATION

$$\begin{aligned}
C_n &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{j\beta \sin 2\pi f_m t} e^{-j2\pi n f_m t} dt \\
&= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} e^{-j\left(2\pi n f_m t - \beta \sin 2\pi f_m t\right)} dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\left(n x - \beta \sin x\right)} dx \\
&= J_n(\beta)
\end{aligned}$$

- $J_n(\beta)$ nth order Bessel function of the first kind

$$e^{j\beta \sin 2\pi f_m t} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}$$

Frequency Modulation



Plots of Bessel functions of the first kind
Matlab command: besselj

$$\boxed{J_n(\beta) = (-1)^n J_{-n}(\beta) \quad \text{for all } n}$$

FREQUENCY MODULATION

$$s(t) = \Re\left\{A_C e^{j2\pi f_C t} \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi n f_m t}\right\}$$

$$= \Re\left\{A_C \sum_{n=-\infty}^{\infty} J_n(\beta) e^{j2\pi(f_C + n f_m)t}\right\}$$

$$= A_C \sum_{n=-\infty}^{\infty} J_n(\beta) \cos\left[2\pi\left(f_C + nf_m\right)t\right]$$

- The spectrum of FM signal with sinusoidal input is

$$S(f) = \frac{A_C}{2} \sum_{n=-\infty}^{\infty} J_n(\beta)\left[\delta\left(f + f_C + nf_m\right) + \delta\left(f - f_C - nf_m\right)\right]$$

Narrowband Frequency Modulation

- For small values of the modulation index β

$$\boxed{\begin{aligned} &J_0(\beta) \approx 1 \\ &J_1(\beta) \approx \frac{\beta}{2} \quad J_{-1}(\beta) \approx -\frac{\beta}{2} \\ &J_n(\beta) \approx 0 \quad |n| > 2 \end{aligned}}$$

$$\boxed{s(t) = A_C\left(\cos[2\pi f_C t] + \frac{\beta}{2}\cos[2\pi(f_C + f_m)t] - \frac{\beta}{2}\cos[2\pi(f_C - f_m)t]\right)}$$

$$\boxed{S(f) = \frac{A_C}{2}\left(\begin{aligned} &[\delta(f + f_C) + \delta(f - f_C)] + \frac{\beta}{2}[\delta(f + f_C + f_m) + \delta(f - f_C - f_m)] \\ &-\frac{\beta}{2}[\delta(f + f_C - f_m) + \delta(f - f_C + f_m)] \end{aligned}\right)}$$

Properties of FM Spectrum

- The spectrum of an FM signal contains carrier components and an infinite set of side frequencies located symmetrically on either side of the carrier at frequency separations of f_m, $2f_m$, $3f_m$, ...
- For the special case of β small compared with unity, only the Bessel coefficients $J_0(\beta)$ and $J_1(\beta)$ have significant values, so that the FM signal is effectively composed of a carrier and a single pair of side frequencies at $f_c \pm f_m$.

Properties of FM Spectrum

- The amplitude of the carrier component varies with β according to $J_0(\beta)$. The physical explanation for this property is that the envelope of an FM signal is constant, so that the average power across 1-ohm resistor is also constant

$$\boxed{P = \frac{1}{2}A_c^2}$$

When the carrier is modulated to generate FM signal, the power in the side frequencies appear only at the expense of the carrier power.

$$\boxed{P = \frac{1}{2}A_c^2 \sum_{n=-\infty}^{\infty} J_n^2(\beta) \Rightarrow 1 = \sum_{n=-\infty}^{\infty} J_n^2(\beta)}$$

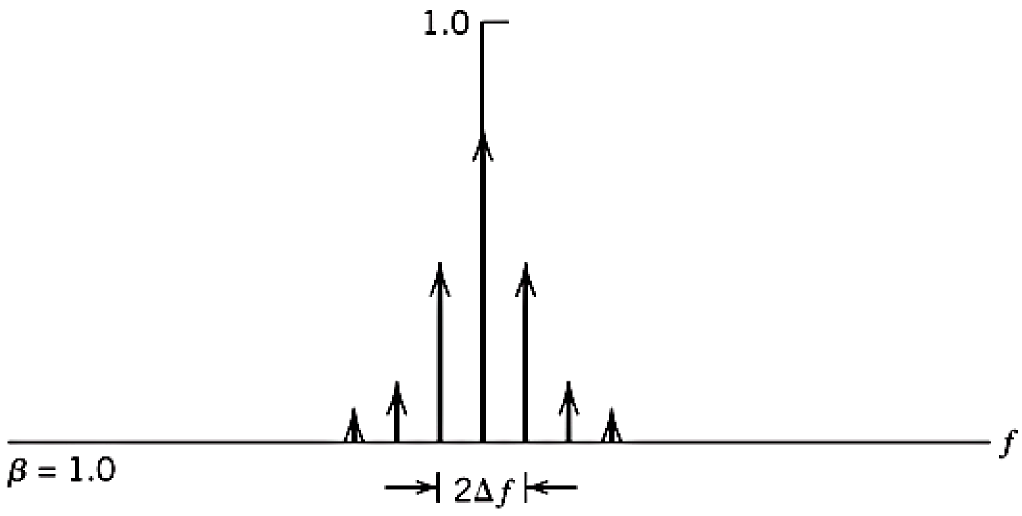
Properties of FM Spectrum

- The integral of the power spectral density of s(t) will give the total energy of the FM signal.

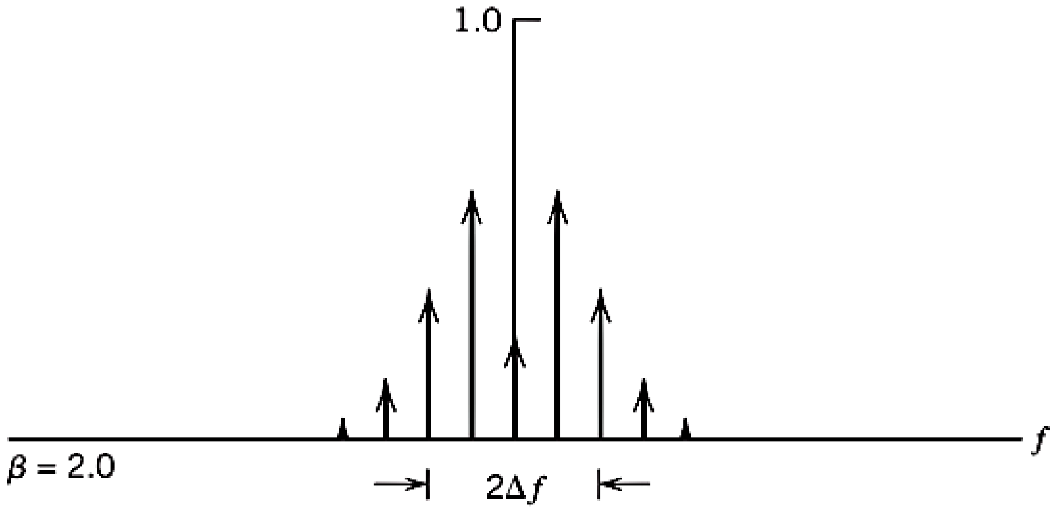
$$S(f)=\sum_{n=-\infty}^{\infty}\frac{A_C^2 J_n^2(\beta)}{4}\delta(f+f_C+nf_m)+\sum_{n=-\infty}^{\infty}\frac{A_C^2 J_n^2(\beta)}{4}\delta(f-f_C-nf_m)$$

$$\begin{aligned}
P &= \int_{-\infty}^{\infty} S(f)df \\
&= \sum_{n=-\infty}^{\infty}\frac{A_C^2 J_n^2(\beta)}{4}+\sum_{n=-\infty}^{\infty}\frac{A_C^2 J_n^2(\beta)}{4} \\
&= \sum_{n=-\infty}^{\infty}\frac{A_C^2 J_n^2(\beta)}{2} \\
&= \frac{A_C^2}{2}\sum_{n=-\infty}^{\infty}J_n^2(\beta) \\
&= \frac{A_C^2}{2}
\end{aligned}$$

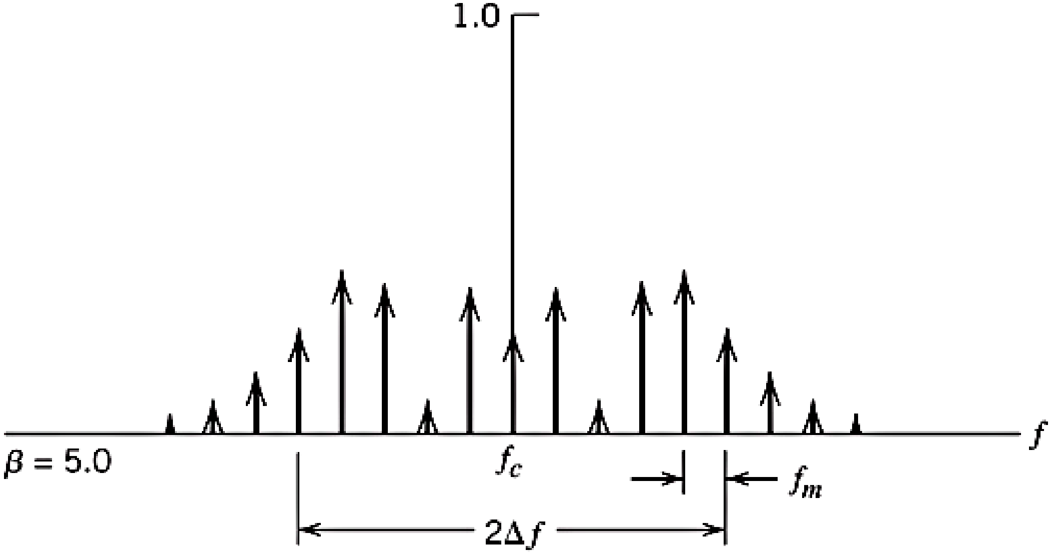
SPECTRA OF FM SIGNALS



Spectra of FM Signals



Spectra of FM Signals



Transmission Bandwidth of FM Signals

- In theory, FM signals has infinite number of side frequencies, so the required bandwidth is infinite.
- In practice, FM signals are limited to a finite number of significant side frequencies.
- For large values of modulation index β, the bandwidth approaches the total frequency excursion 2Δf.
- For small values of modulation index β, the bandwidth approaches $2f_m$.

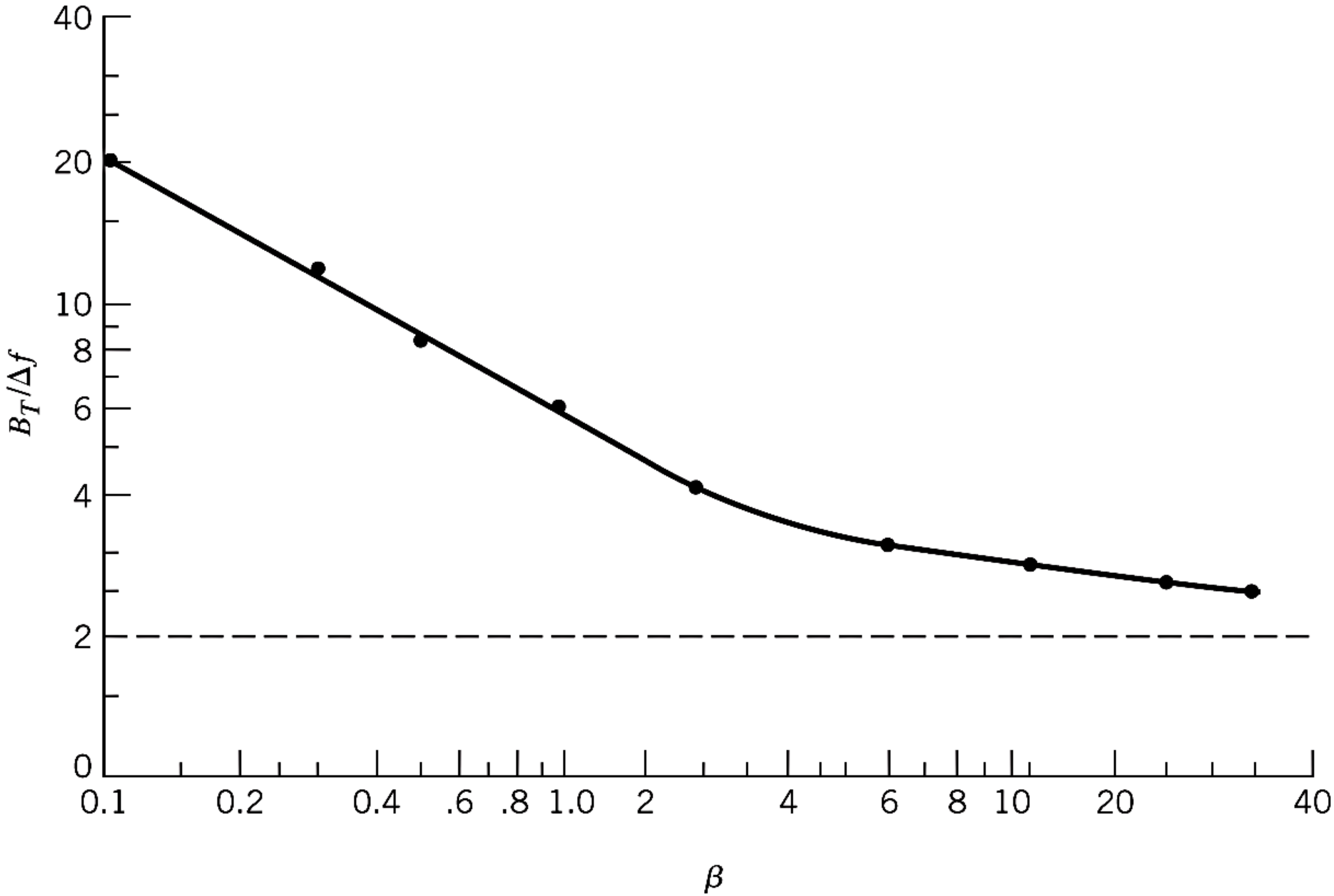
$$\boxed{B_T \approx 2\Delta f + 2 f_m = 2\Delta f\left(1+\frac{1}{\beta}\right)}$$

- This empirical relation is known as **Carson's rule**.

Transmission Bandwidth of FM Signals

- More accurate definition:
  - Transmission bandwidth of an FM wave as the separation between the two frequencies beyond which none of the side frequencies is greater than 1 percent of the carrier amplitude obtained when the modulation is removed.
  - The bandwidth is $2n_{max}f_m$, where fm is the frequency modulation and n_{max} is the largest value of the integer n that satisfies the requirement $|J_n(\beta)|>0.01$

Transmission Bandwidth of FM Signals



Transmission Bandwidth of FM Signals

- Let's consider that m(t) is an arbitrary waveform.
- The deviation ratio D, defined as the ratio of the frequency deviation Δf, which corresponds to the maximum possible amplitude of the modulation signal m(t), to the highest modulation frequency W.

$$\boxed{D = \frac{\Delta f}{W}}$$

- The deviation ratio D plays the same role for nonsinusoidal modulation that modulation index β plays for the case of sinusoidal modulation.

EXAMPLE

- The frequency deviation Δf of FM broadcasting is fixed at 75KHz. The maximum audio frequency of interest is 15KHz. What is deviation ratio? Calculate the bandwidth using Carson's rule and using the curve.

$$D = \frac{75KHz}{15KHz} = 5$$

$$B_T = 2\left(75KHz + 15KHz\right) = 180KHz$$

$$B_T = 3.2\Delta f = 3.2 * 75 = 240KHz$$

Carson's rule underestimates the bandwidth by 25% compared with the result of using the universal curve

Generation of FM Signals

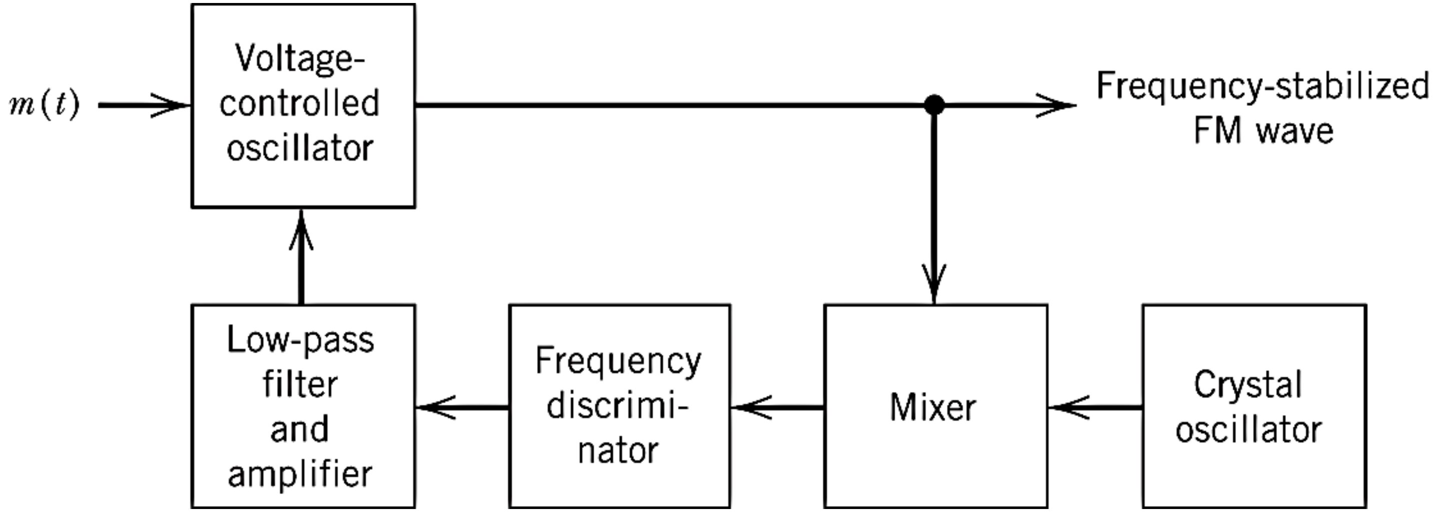
- The instantaneous frequency of the carrier wave is varied directly in accordance with the message signal using voltage-controlled oscillator.
- Voltage-Controlled Oscillator (VCO) has the following input-output relationship

$$\boxed{f_{VCO} = f_c + k_f V_{in}}$$

- In practice, VCO output frequency is not stable across process, temperature and voltage. Therefore, a Frequency-Locked Loop (FLL) can be used.

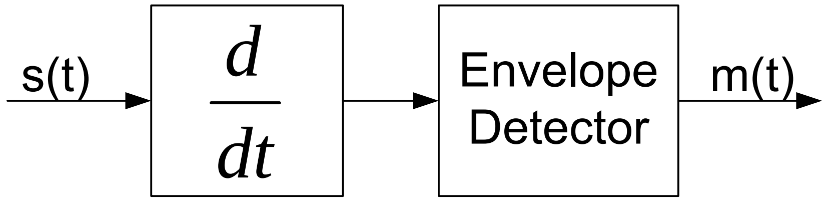
Generation of FM Signals

- In FLL the average output frequency is detected and compared with a reference frequency.
- The error between the actual average output frequency and desired output frequency is minimized using a feedback loop.



DEMODULATION OF FM SIGNALS

- Using a differentiator followed by envelope detector



$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right]$$

$$\frac{ds(t)}{dt} = -A_c \underbrace{\left(2\pi f_c + 2\pi k_f m(t)\right)}_{\text{AM Term}} \sin\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right]$$

Demodulation of FM Signals

- Consider FM signal as a complex signal

$$
\begin{aligned}
s(t) &= A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right] \\
&= A_c \cos(2\pi f_c t + \phi(t)) \\
&= s_I(t)\cos(2\pi f_c t) - s_Q(t)\sin(2\pi f_c t)
\end{aligned}
$$

$$s_I(t) = A_c \cos\left(2\pi k_f \int_0^t m(\tau)d\tau\right)$$

$$s_Q(t) = A_c \sin\left(2\pi k_f \int_0^t m(\tau)d\tau\right)$$

$$\tan^{-1}\left(\frac{s_Q(t)}{s_I(t)}\right) = 2\pi k_f \int_0^t m(\tau)d\tau$$

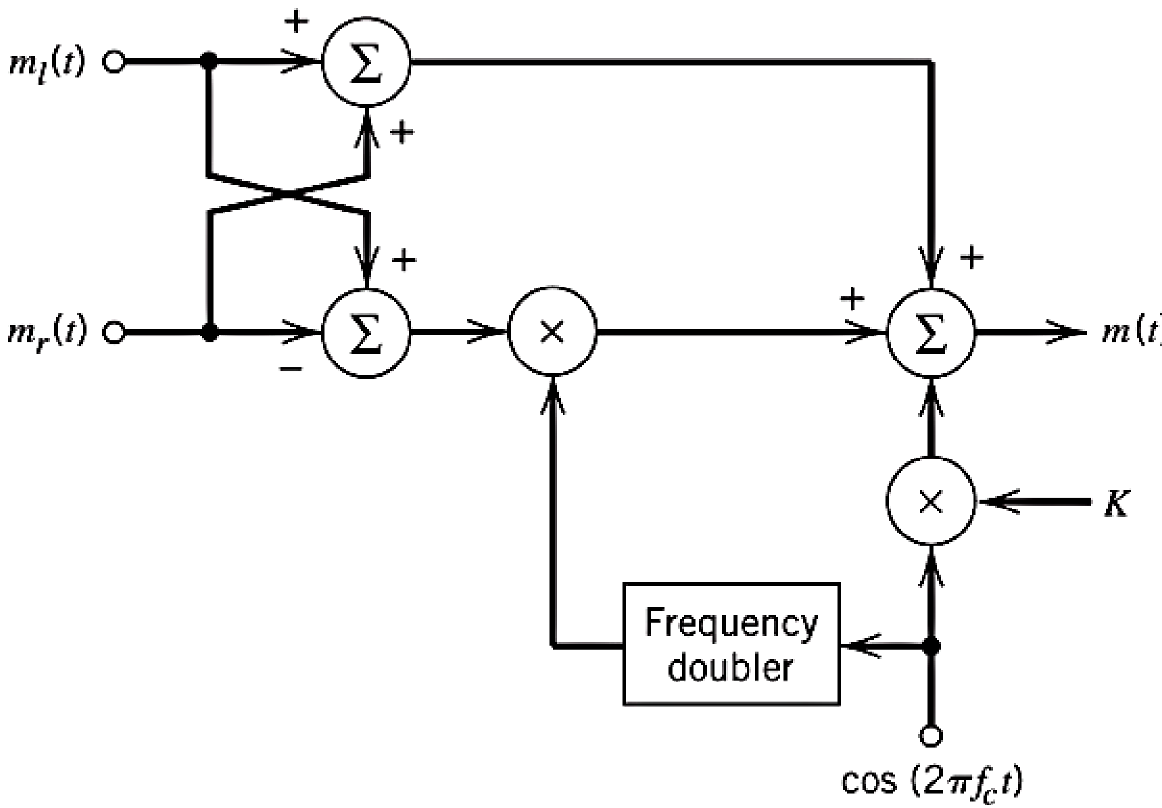
$$\frac{d}{dt}\tan^{-1}\left(\frac{s_Q(t)}{s_I(t)}\right) = 2\pi k_f m(t)$$

FM Stereo Multiplexing

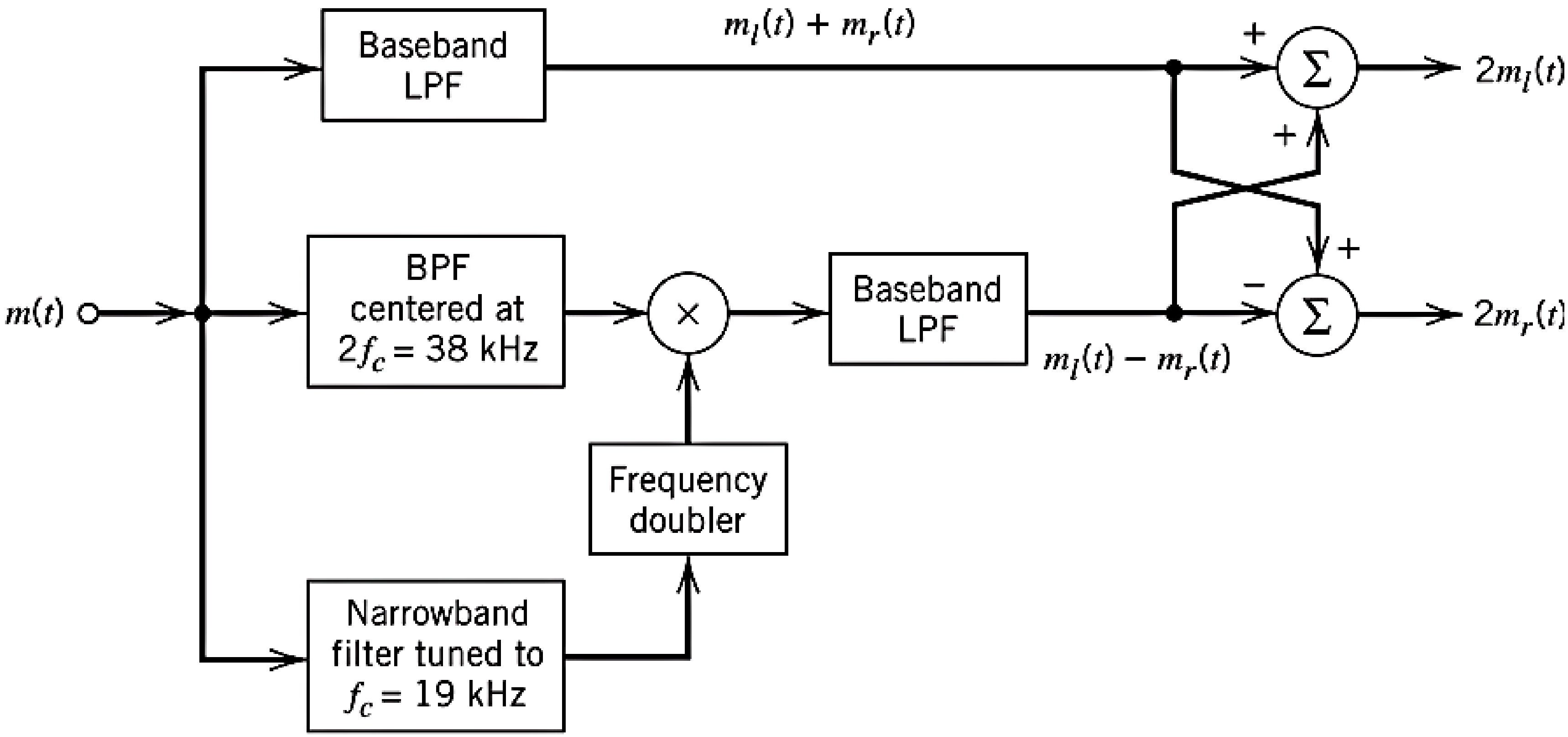
- Stereo multiplexing is a form of frequency-division multiplexing (FDM) designed to transmit two separate signals via the same carrier.
- It is widely used in FM broadcasting to transmit left, right audio and Digital Services such as RDS.
- The stereo broadcasting standard must
  - operate within the allocated FM broadcast channels
  - be compatible with monophonic radio receivers.
- Since DSB-SC modulation requires generators phase information for coherent demodulation, a pilot tone is also added at 19KHz

$$\boxed{m(t)=[m_L(t)+m_R(t)]+[m_L(t)-m_R(t)]\cos(2\pi t38KHz)+K\cos(2\pi t19KHz)}$$

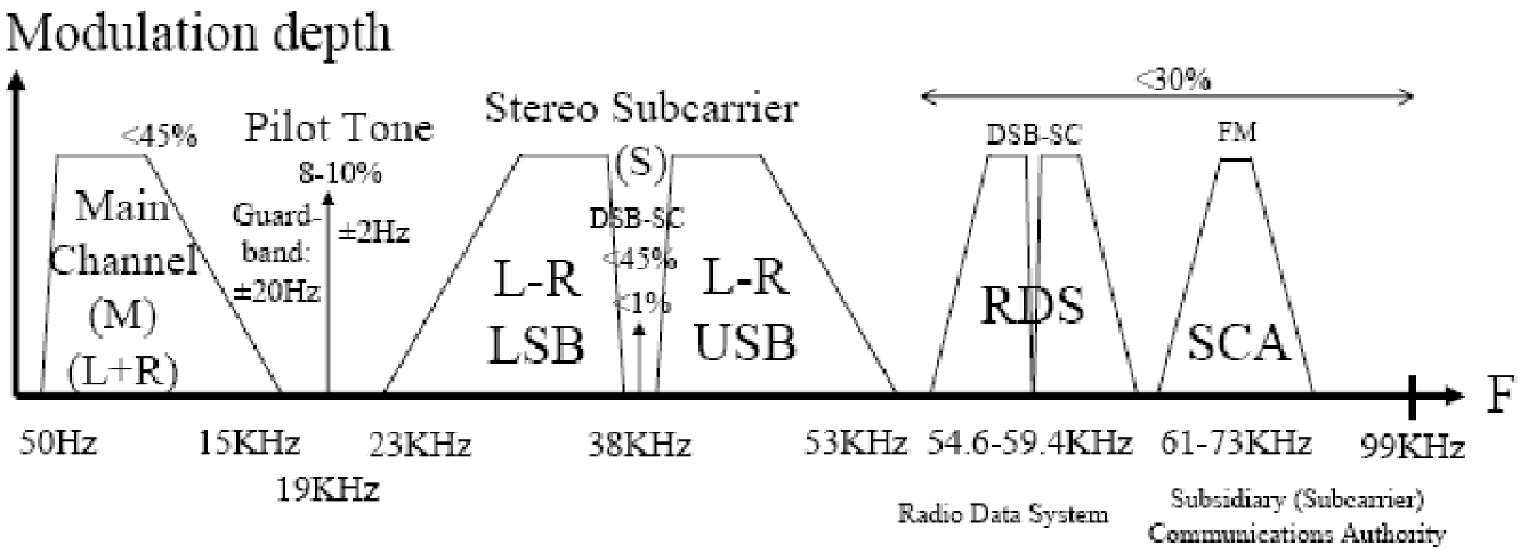
FM Stereo Multiplexer



FM Stereo Demultiplexer



FM Stereo Spectrum



RDS carry data at 1187.5 bits/second on a 57KHz carrier, so there are 48 cycles of subcarrier during every bit.

57KHz is third harmonic of 19KHz pilot. L-R signal finishes at 38KHz+15KHz=53KHz ➔ 4KHz for lower sideband of RDS signal

30% modulation (100%=75KHZ ➔ 30%=22.5KHz) is the maximum allowable level when mono transmission (L+R) is also off.

In normal operation max RDS modulation is 10%➔7.5KHZ

Nonlinear Effects in FM Systems

- Consider a communication channel with the following input-output relationship

$$\boxed{v_0(t) = a_1 v_i(t) + a_2 v_i^2(t) + a_3 v_i^3(t)}$$

- Let's apply the following FM signal to this channel.

$$v_i(t) = A_c \cos[2\pi f_c t + \phi(t)]$$

$$\phi(t) = 2\pi k_f \int_0^t m(\tau)d\tau$$

$$v_0(t) = a_1 A_c \cos[2\pi f_c t + \phi(t)] + a_2 A_c^2 \cos^2[2\pi f_c t + \phi(t)]$$
$$+ a_3 A_c^3 \cos^3[2\pi f_c t + \phi(t)]$$

Nonlinear Effects in FM Systems

- Let's expand the squared and cubed terms

$$\boxed{\begin{aligned} v_0(t) &= \frac{1}{2}a_2A_c^2 + \left(a_1A_c + \frac{3}{4}a_3A_c^3\right)\cos\left[2\pi f_c t + \phi(t)\right] \\ &+ \frac{1}{2}a_2A_c^2\cos\left[4\pi f_c t + 2\phi(t)\right] \\ &+ \frac{1}{4}a_3A_c^3\cos\left[6\pi f_c t + 3\phi(t)\right] \end{aligned}}$$

Nonlinear Effects in FM Systems

- Let Δf is the frequency deviation of the incoming FM signal and W is the highest frequency component of the message signal m(t)
- Let's apply Carson's rule to determine the bandwidth of the second and third harmonics
- Note that the frequency deviation of the second harmonic is doubled and third harmonic is tripled.

$$
\begin{array}{|ll|}
\hline
B_{T1} = 2\Delta f + 2f_m & \text{for large } \beta \\
B_{T2} = 2(2\Delta f) + 2f_m & B_{T2} \approx 2B_{T1} \\
B_{T3} = 3(2\Delta f) + 2f_m & B_{T3} \approx 3B_{T1} \\
\hline
\end{array}
$$

EXAMPLE 1

- A carrier wave of frequency 100MHz is frequency modulated by a sinusoidal wave of amplitude 20 volts and frequency 100KHz. The frequency sensitivity of the modulator is 25 KHz per volt.
  - Determine the bandwidth using Carson’s rule
  - Determine the bandwidth using the universal curve

Example 1 - Solutions

- The frequency deviation is

$$\boxed{\Delta f = k_f A_m = 25\times 10^3 \times 20 = 5\times 10^5\, Hz}$$

- The corresponding modulation index

$$\boxed{\beta = \frac{\Delta f}{f_m} = \frac{5\times 10^5}{10^5} = 5}$$

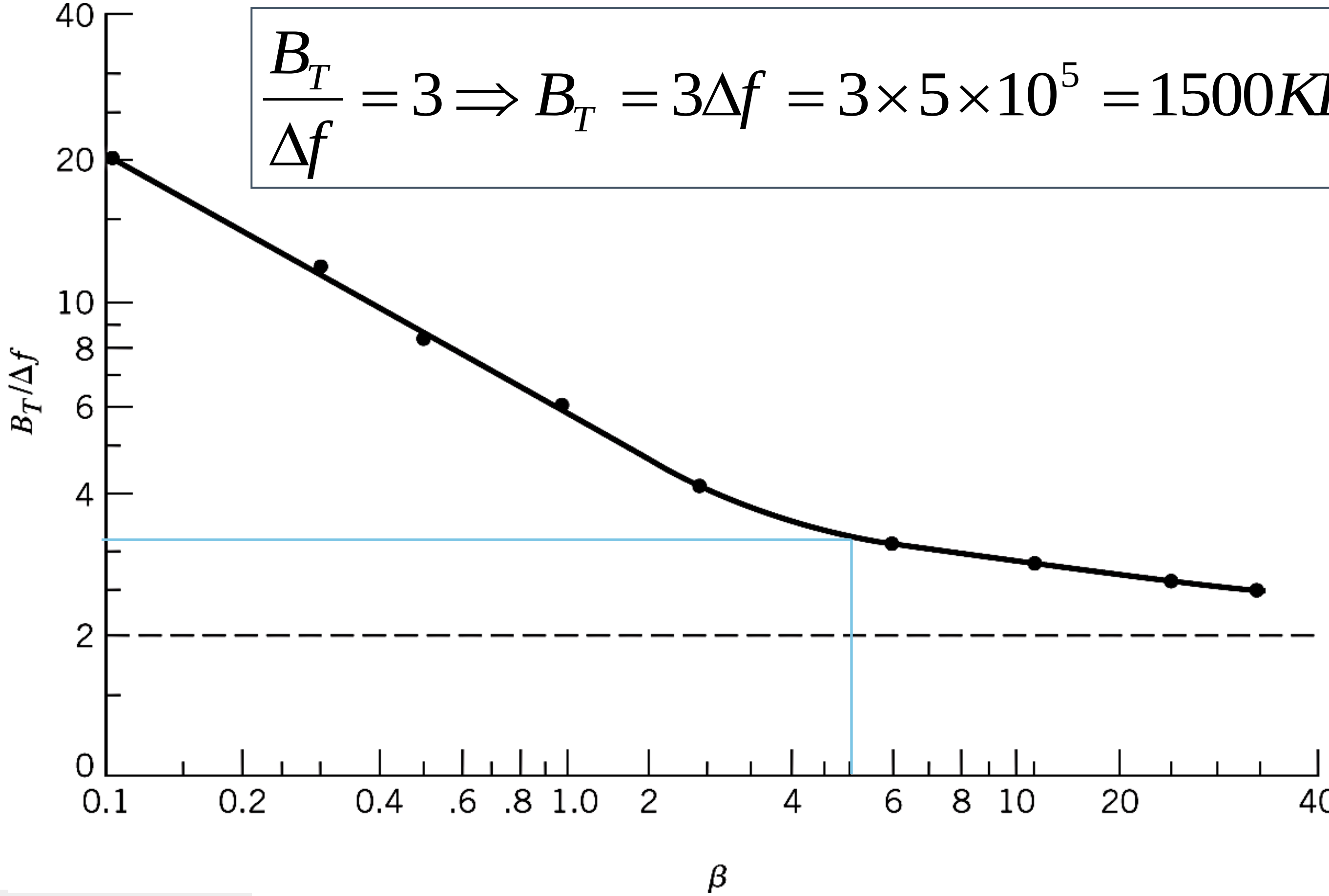
- The transmission bandwidth of the FM wave using Carson's rule

$$\boxed{B_T = 2 f_m (1+\beta) = 2\times 10^5 (1+5) = 1200 KHz = 1.2 MHz}$$

EXAMPLE 1 - SOLUTION

- Using the universal curve for β=5 $B_T/\Delta f=3$

$$\frac{B_T}{\Delta f}=3 \Rightarrow B_T=3\Delta f=3\times 5\times 10^5=1500KHz=1.5MHz$$



Example 2

- An FM signal with a frequency deviation of 10KHz at a modulation frequency of 5KHz is applied to a frequency multiplier, which triples the frequency.
- Determine the frequency deviation and the modulation index of the FM signal at the output of frequency multiplier.
- What is the frequency separation of the adjacent side frequencies of this FM signal?

Example 2 - Solution

- Assume that the instantaneous frequency of the FM wave at the input of frequency multiplier

$$\boxed{f_i(t) = f_c + \Delta f \cos(2\pi f_m t)}$$

- The instantaneous frequency at the output of frequency multiplier is

$$\boxed{f_o(t) = nf_c + n\Delta f \cos(2\pi f_m t)}$$

- Therefore the frequency deviation of this FM signal is equal to

$$\boxed{n\Delta f = 3 \times 10 KHz = 30 KHz}$$

EXAMPLE 2 - SOLUTION

- The modulation index

$$\boxed{\frac{n\Delta f}{f_m} = \frac{30KHz}{5} = 6}$$

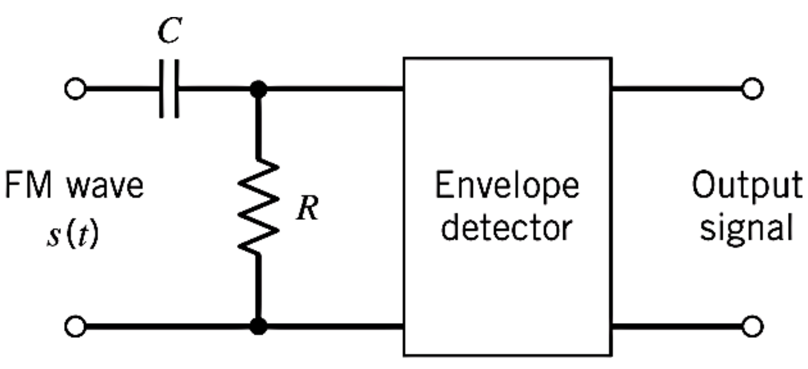
- The frequency separation of the adjacent side-frequencies of this FM signal is unchanged and is equal to f_m=5KHz

EXAMPLE 3

- The FM Signal

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)\,d\tau\right]$$

- is applied to the following system. Assume that the resistance R is small compared to the reactance of the capacitor C and envelope detector does not load the RC filter.
- Determine the signal at the envelope detector output, assuming $k_f|m(t)|<f_c$ for all t



Example 3 - Solution

- The transfer function of the RC filter

$$\boxed{H(f) = \frac{j2\pi fCR}{1 + j2\pi fCR}}$$

If $2\pi fCR \ll 1$ for all frequencies of interest, then H(f) is approximated as

$$\boxed{H(f) \approx j2\pi fCR}$$

- Multiplication by $j2\pi f$ in the frequency domain is equivalent to differentiation in the time domain.

EXAMPLE 3 - SOLUTION

- Therefore, at the RC filter output

$$\boxed{\begin{aligned} v_{RC} &\approx CR\frac{ds(t)}{dt} \\ &= CR\frac{d}{dt}\left\{A_c\cos\left[2\pi f_c t + 2\pi k_f\int_0^t m(\tau)d\tau\right]\right\} \\ &= -CRA_c\left[2\pi f_c + 2\pi k_f m(t)\right]\sin\left[2\pi f_c t + 2\pi k_f\int_0^t m(\tau)d\tau\right] \end{aligned}}$$

- The corresponding envelope detector output

$$\boxed{v_{ENV} \approx 2\pi f_c CRA_c\left[1+\frac{k_f}{f_c}m(t)\right]}$$

- except for a DC bias, the output is proportional to the modulating signal m(t)

Random Signals

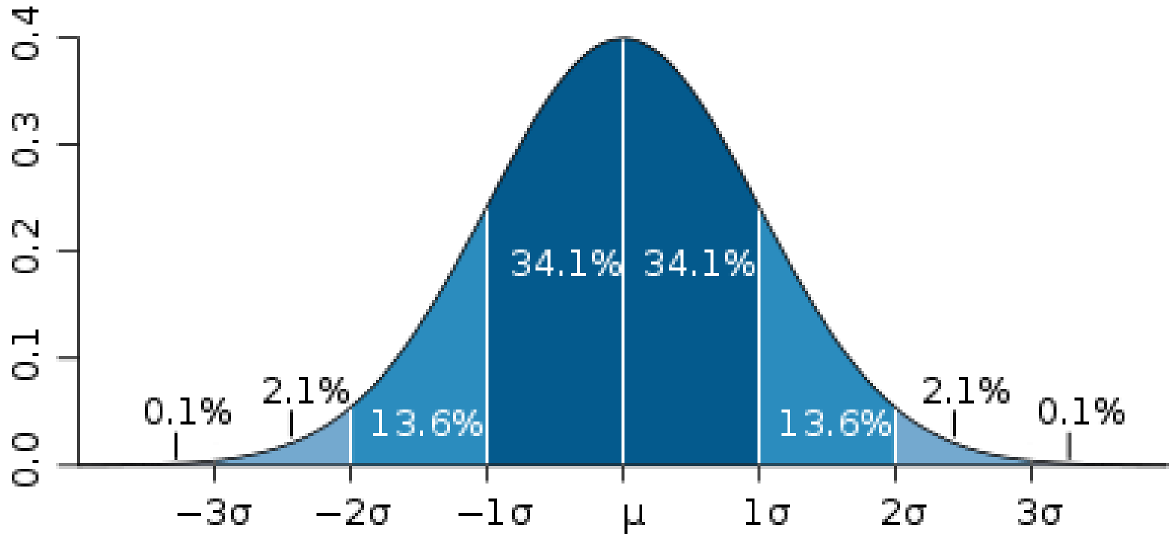
- Random signal: it is not possible to predict its precise value in advance.
- It may be possible to describe its statistical properties, such as average power, average spectral distribution of its power.
- The mathematical discipline that deals with the statistical characterization of random signals is probability theory.
- Probability theory is rooted in phenomena that can be modeled by an experiment with an outcome that is subject to chance.

Random Variables

- While the meaning of the outcome of a random experiment is clear, such outcomes are often not the most convenient representation for mathematical analysis.
- We use the expression random variable to describe this process of assigning a number to the outcome of a random experiment.
- Random variables may be discrete and take only a finite number of values, such as coin-tossing experiment. Alternatively, random variables may be continuous and take real values.
- Probability density function (pdf) is a function that describes the relative likelihood for this random variable to occur at a given point.

Probability Density Function

- The probability for the random variable to fall within a particular region is given by the integral of this variable's density over the region.
- The probability density function is nonnegative everywhere, and its integral over the entire space is equal to one.



Statistical Averages

- The expected value or mean of a random variable x is defined by

$$\boxed{\mu_x = E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx}$$

E donates the statistical expectation operator.

- Function of Random Variable:

$$Y = g(x)$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x)\,dx$$

Example

- Let Y=g(X)=cos(X) where X is a random variable uniformly distributed in the interval (-π, π)

$$f_X(x) = \begin{cases} \dfrac{1}{2\pi} & -\pi < x < \pi \\ 0 & otherwise \end{cases}$$

What is the expected value of Y

$$E[Y] = \int_{-\pi}^{\pi} \cos(x) \frac{1}{2\pi} dx = -\frac{1}{2\pi} \sin(x) \Big|_{x=-\pi}^{\pi} = 0$$

Moments

- For the special case of $g(X)=X^n$, we obtain nth moment of the probability distribution of the random variable X.

$$\boxed{E\left[X^n\right]=\int_{-\infty}^{\infty} x^n f_X(x)dx}$$

- n=1 gives the mean and n=2 gives the mean-square value of X

$$\boxed{E\left[X^2\right]=\int_{-\infty}^{\infty} x^2 f_X(x)dx}$$

Central Moments

- Central moments are simply the moments of the difference between a random variable X and its mean. Thus, the nth central moment is

$$\boxed{E\left[\left(X-\mu_x\right)^n\right]=\int_{-\infty}^{\infty}\left(x-\mu_x\right)^n f_X(x)dx}$$

- n=1 is zero, whereas n=2 is defined as the variance of the random variable.

$$\boxed{\operatorname{var}(X)=E\left[\left(X-\mu_x\right)^2\right]=\int_{-\infty}^{\infty}\left(x-\mu_x\right)^2 f_X(x)dx}$$

The square-root of the variance is called the standard deviation of the random variable X.

Example

- Let Y=g(X)=cos(X) where X is a random variable uniformly distributed in the interval (-π, π)

$$f_X(x)=\begin{cases}\frac{1}{2\pi} & -\pi<x<\pi\\ 0 & otherwise\end{cases}$$

What is the variance of Y

$$\mathrm{var}(X)=E\left[(X-\mu_x)^2\right]=\int_{-\infty}^{\infty}(x-\mu_x)^2 f_X(x)dx$$

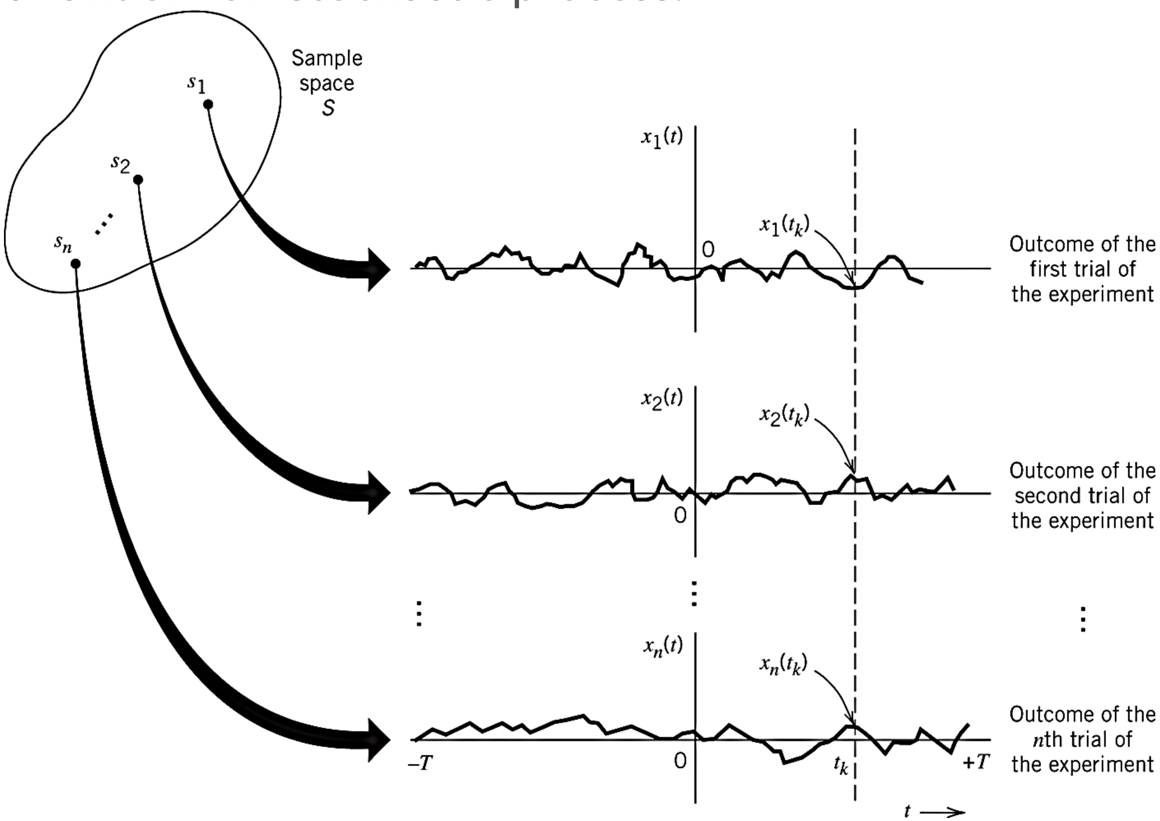
$$\mathrm{var}(X)=\int_{-\pi}^{\pi}(\cos(x)-0)^2\frac{1}{2\pi}dx$$

$$=\int_{-\pi}^{\pi}\left(\frac{1}{2}+\frac{1}{2}\cos(2x)\right)\frac{1}{2\pi}dx$$

$$=\frac{1}{2}\frac{1}{2\pi}x\Big|_{x=-\pi}^{\pi}=\frac{1}{2}$$

Random Process

- The random signals in communication are functions of time, defined on some observation interval.
- The sample space or ensemble comprised of functions of time is called a random or stochastic process.



Mean, Correlation

- The mean of the random process X(t) is obtained by observing the process at some time t,

$$\boxed{\mu_x = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\,dx}$$

- The autocorrelation function of the process X(t) as the expectation of the product of two random variables $X(t_1)$ and $X(t_2)$, obtained by observing X(t) at time t_1 and t_2.

$$\begin{aligned} R_X(t_1,t_2) &= E[X(t_1)X(t_2)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1)X(t_2)}(x_1,x_2)\,dx_1 dx_2 \end{aligned}$$

Properties of the Autocorrelation Function

- For convenience we redefine the autocorrelation function of X(t)

$$\boxed{R_X(\tau) = E[X(t+\tau)X(t)] \quad \text{for all } t}$$

- Mean square value of the process is

$$\boxed{R_X(0) = E[X^2(t)]}$$

- The autocorrelation function is an even function

$$\boxed{R_X(\tau) = R_X(-\tau)}$$

- The autocorrelation function has its maximum magnitude at τ=0

$$\boxed{|R_X(\tau)| \le R_X(0)}$$

Power Spectral Density

- The Fourier transform of the autocorrelation function is called the power spectral density $S_X(f)$ of the random process X(t)

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) e^{-j2\pi f\tau} d\tau$$

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{j2\pi f\tau} d\tau$$

Gaussian Process

- The random variable Y has Gaussian distribution if its probability density function has the form

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}}$$

- For the special case when the Gaussian random variable Y is normalized to have a mean of zero and a variance of one

$$\boxed{N(0,1) = f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}$$

- Most of the physical phenomena are modeled using Gaussian distribution.

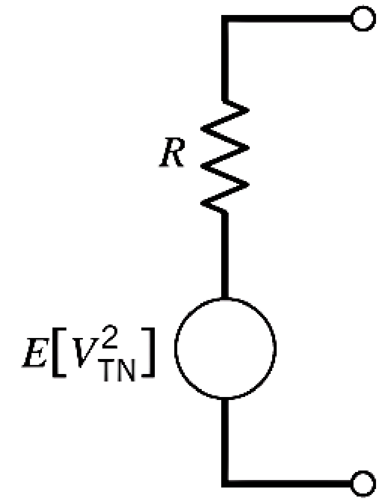
Noise

- Noise is the unwanted signals that disturb the transmission and processing of communication signals.
- The sources of noise may be external to the system, such as atmospheric noise, galactic noise, man-made noise.
- Internally generated noise arises from spontaneous fluctuations of current and voltage in electrical circuits. This type of noise limits the transmission and quality of the communication signals. Most common noise of this type is thermal noise.

THERMAL NOISE

- Thermal noise arises from the random motion of electrons in a conductor.
- The mean-square value of the thermal noise voltage V_{TN} appearing across the terminals of a resistor, measured in a bandwidth of Δf Hertz is

$$\boxed{E\left[V_{TN}^2\right] = 4kTR\Delta f}$$



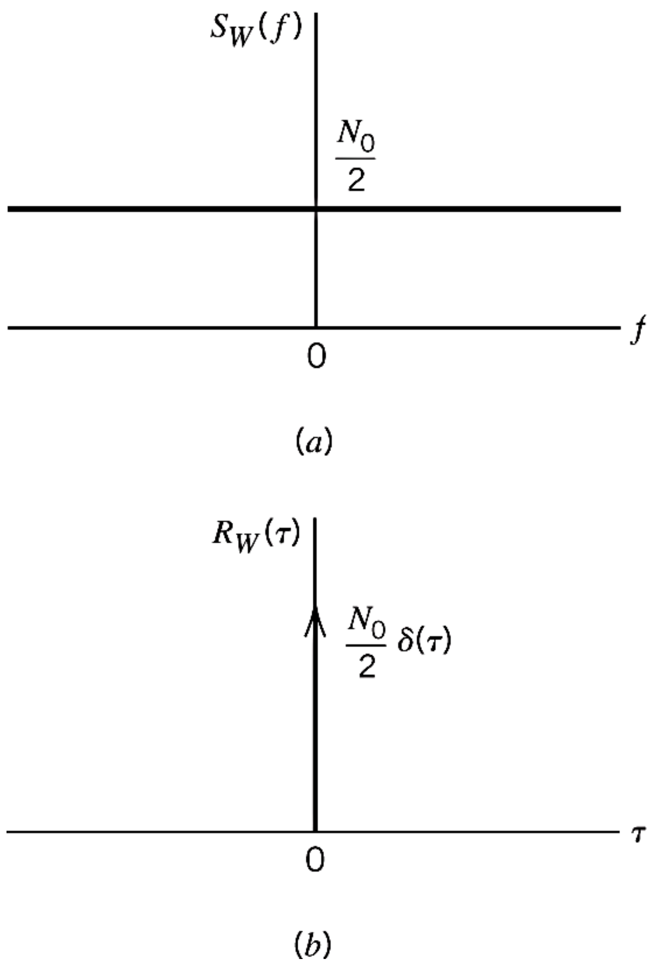
- where k is Boltzmann constant equal to 1.38×10^{-23} joules per degree Kelvin, T is the absolute temperature in degrees Kelvin and R is the resistance in ohms.

White Noise

- The noise analysis of communication systems is based on idealized form of noise called white noise, which has constant power spectral density across frequency.
- N_0 is in watts per Hertz and referenced to the input stage of the receiver.

$$\boxed{N_0 = kT_e}$$

- k is Boltzmann constant and Te is the equivalent noise temperature of the receiver.



Example - Ideal Low-Pass Filtered White Noise

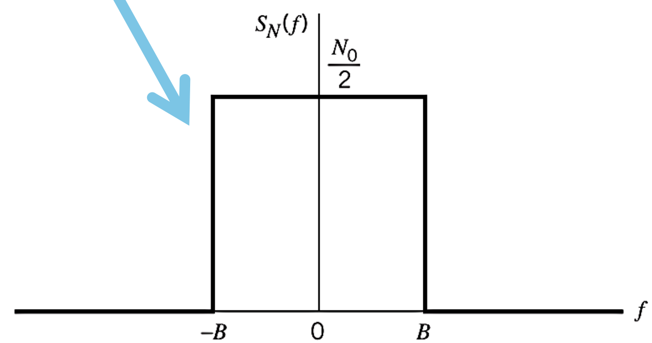
- Suppose that a white Gaussian noise w(t) of zero mean and power spectral density $N_0/2$ is applied to an ideal low-pass filter of Bandwidth B and passband response of one.
- Calculate the autocorrelation of the noise at the filter output.
- What is the noise power at the filter output?

$$H(f) = \begin{cases} 1 & -B < f < B \\ 0 & |f| > B \end{cases}$$

Example - Solution

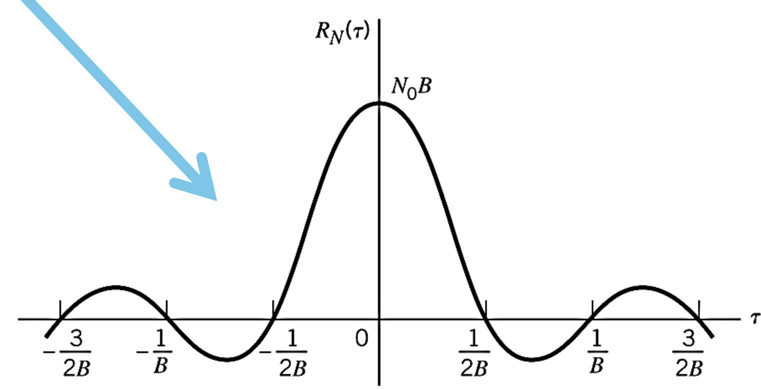
- The autocorrelation function of n(t) is the inverse Fourier transform of the power spectral density.

$$S_N(f) = \begin{cases} \dfrac{N_0}{2} & -B < f < B \\ 0 & |f| > B \end{cases}$$



(a)

$$\boxed{\begin{aligned} R_N(\tau) &= \int_{-B}^{B} \frac{N_0}{2} e^{j2\pi f\tau}\, df \\ &= N_0 B \,\mathrm{sinc}(2B\tau) \end{aligned}}$$



(b)

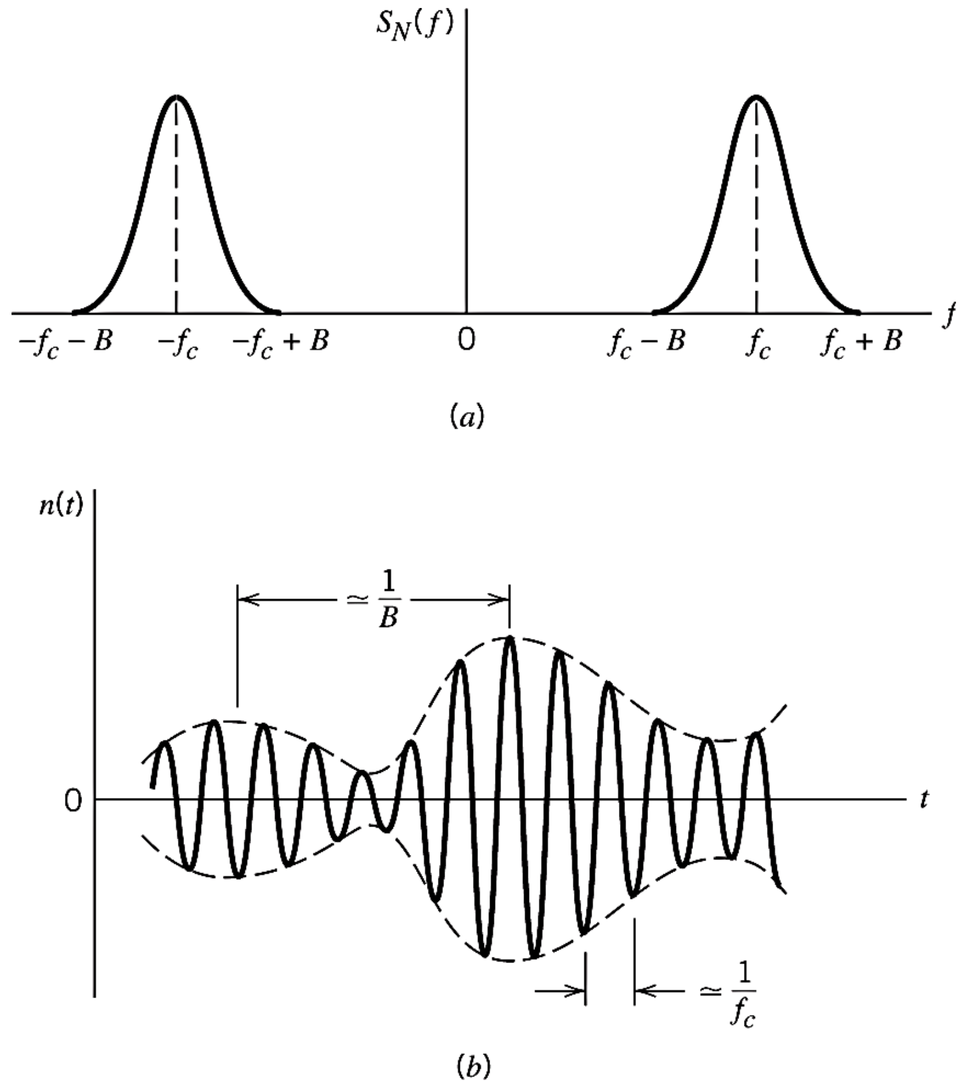
- The power at the filter output can be calculated
  - Integrate the power spectral density
  - $R_N(0)$ is also the noise power. ➔ N_0B

Narrowband Noise

- Usually there is a bandpass filter before the frontend.
- This bandpass filter is just large enough to pass the desired channel and suppress the rest of the band
- However, noise within the desired band also passes this filter. This noise has narrow bandwidth comparing white noise is called narrowband noise.

Narrowband Noise

- Depending on the application there are two different representation of this noise
  - In-phase and Quadrature
  - Envelope and phase



Narrowband Noise in term of In-Phase and Quadrature Components

- Consider a narrowband noise n(t) of bandwidth 2B centered on frequency f_c.
- Using complex notation, the narrowband noise

$$\boxed{n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)}$$

  where nI(t) is in-phase component of n(t) and nQ(t) is quadrature component of n(t).
- This type of representation is very helpful to analyze the noise on digital communication systems.

PROPERTIES OF THE IN-PHASE AND QUADRATURE COMPONENTS OF NARROWBAND NOISE

- The in-phase component $n_I(t)$ and quadrature component $n_Q(t)$ of narrowband noise has zero mean.
- $n(t)$ is Gaussian ➔ $n_I(t)$ and $n_Q(t)$ are jointly Gaussian
- $n(t)$ is stationary ➔ $n_I(t)$ and $n_Q(t)$ are jointly stationary
- They have the same power spectral density, which is related to the $S_N(f)$ of the narrowband noise $n(t)$

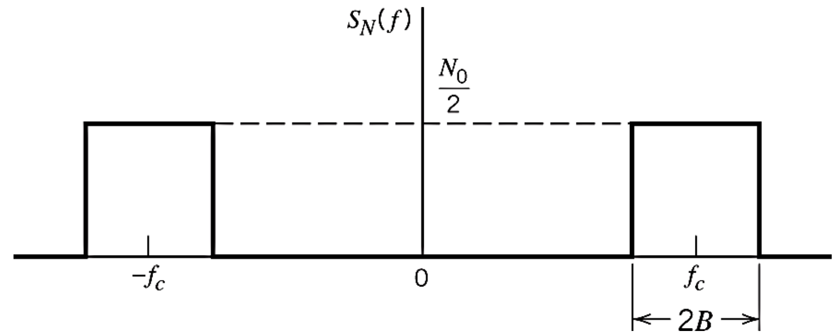
$$\boxed{S_{NI}(f) = S_{NQ}(f) = \begin{cases} S_N(f - f_c) + S_N(f + f_c) & -B \le f \le B \\ 0 & otherwise \end{cases}}$$

- They have the same variance as the narrowband noise $n(t)$

Example – Ideal Band-Pass Filtered White Noise

- Consider a white Gaussian noise of zero mean and power spectral density $N_0/2$ is passed through an ideal band-pass filter

$$\boxed{H(f)=\begin{cases}1 & f_c-B<|f|<f_c+B\\ 0 & otherwise\end{cases}}$$

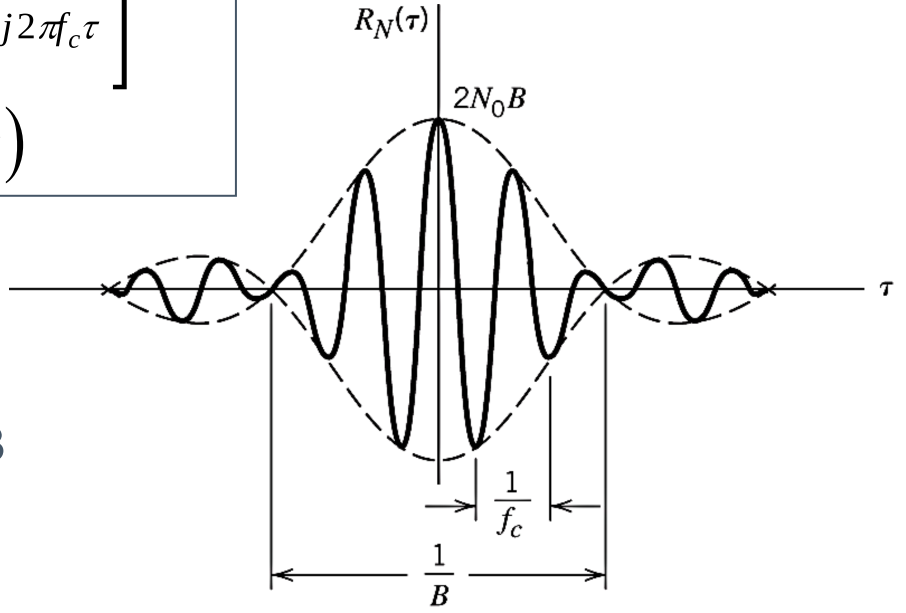


- Calculate the autocorrelation of the noise at the filter output
- Calculate the noise power at the filter output
- Calculate the in-phase and quadrature noise autocorrelations

EXAMPLE - SOLUTION

- The autocorrelation function of n(t) is the inverse Fourier transform of the power spectral density

$$
\begin{aligned}
R_N(\tau) &= \int_{-f_c-B}^{-f_c+B} \frac{N_0}{2} e^{j2\pi f\tau} df + \int_{f_c-B}^{f_c+B} \frac{N_0}{2} e^{j2\pi f\tau} df \\
&= N_0 B \operatorname{sinc}(2B\tau)\left[e^{-j2\pi f_c\tau} + e^{j2\pi f_c\tau}\right] \\
&= 2N_0 B \operatorname{sinc}(2B\tau)\cos(2\pi f_c\tau)
\end{aligned}
$$

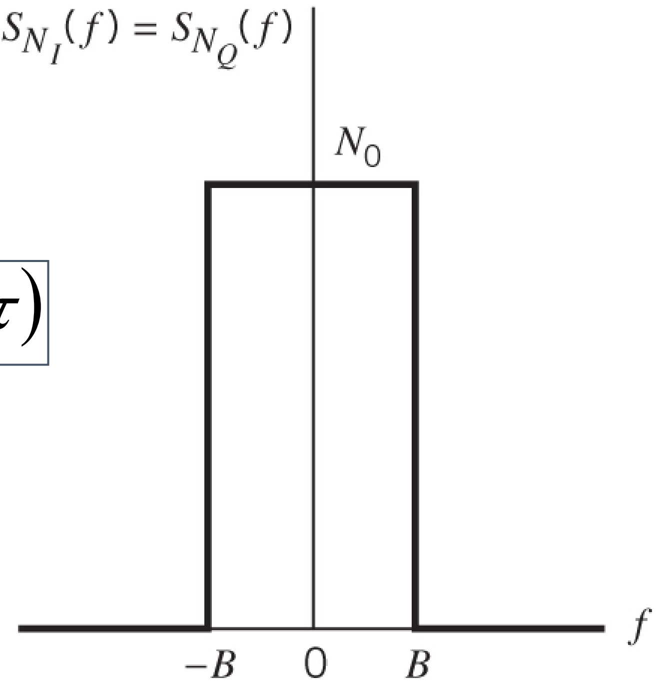


Total noise power at the output is $2N_0B$

EXAMPLE - SOLUTION

- The spectral density of the noise at the filter output is symmetric about $\pm f_c$. Therefore the spectral density of $n_I(t)$ and $n_Q(t)$ is

$$\boxed{R_{NI}(\tau) = R_{NQ}(\tau) = 2N_0 B \operatorname{sinc}(2B\tau)}$$



Narrowband Noise in term of Envelope and Phase Components

- We may represent the noise n(t) in terms of its envelope and phase components.

$$n(t) = r(t)\cos[2\pi f_c t + \Psi(t)]$$

$$r(t) = \sqrt{\left|n_I^2(t) + n_Q^2(t)\right|}$$

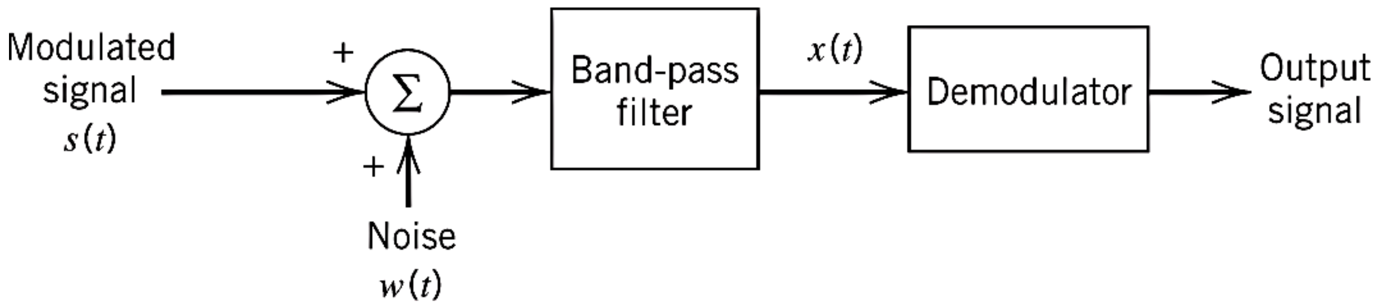
$$\Psi(t) = \tan^{-1}\left[\frac{n_Q(t)}{n_I(t)}\right]$$

- r(t) and Ψ(t) are statistically independent.

Receiver Model

- The noise at the input of receiver
  - additive, white and Gaussian
- This enables
  - a method to understand the effect of the noise
  - a framework for the comparison of the noise performance of different CW modulations
- The receiver model
  - must provide an adequate description of the receiver noise
  - must account for the inherent filtering and modulation characteristics of the system
  - must be simple enough for statistical analysis

Receiver Model



- The power spectral density of the noise w(t) is $N_0/2$ defined for both positive and negative frequencies.
- N_0 is the average noise power per unit bandwidth measured at the front-end of the receiver.
- The band-pass filter's bandwidth is equal to the transmission bandwidth of the modulated signal s(t) and mid-frequency is equal to the carrier frequency f_c

Receiver Model

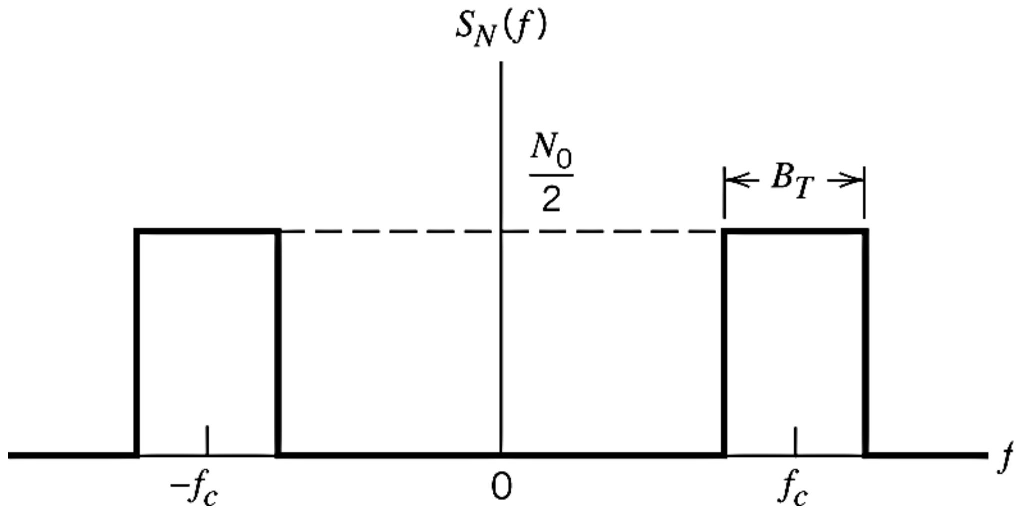
- The carrier frequency f_c is large compared to the transmission bandwidth B_T.
- The filtered noise n(t) as narrowband noise is represented as

$$\boxed{n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)}$$

- where $n_I(t)$ is the in-phase noise component and $n_Q(t)$ is the quadrature noise component.
- The filtered signal x(t)

$$\boxed{x(t) = s(t) + n(t)}$$

Receiver Model



- The average noise power at the demodulator input is equal to the total area under the curve of the power spectral density $S_N(f)$.
- The average noise power is equal to N_0B_T

Receiver Model

- Since the noise is additive, we can define the signal to noise ratio (SNR), which is defined as the ratio of average power of the modulated signal s(t) to the average power of the filtered noise.
- The SNR defined at the channel output or demodulator input is called as SNR_C
- The SNR defined at the demodulator or receiver output is called SNR_O

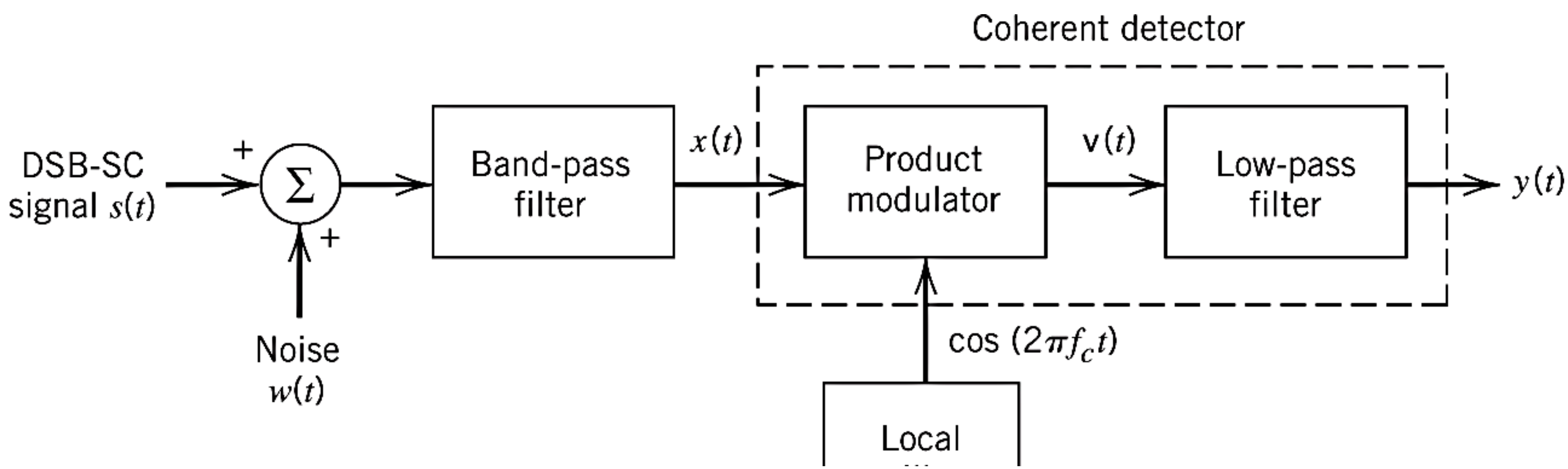
RECEIVER MODEL

- We can define a figure of merit to compare different CW modulation schemes.

$$\boxed{\text{Figure of Merit} = \frac{SNR_O}{SNR_C}}$$

- Depending on the modulation scheme the figure of merit can be
  - less than one
  - equal to one
  - greater than one

Noise in DSB-SC Receivers



- The DSB-SC signal is demodulated using coherent detection.
- The locally generated sinusoidal signals phase perfectly matches with the transmitter's local oscillators phase.
- This synchronization is usually performed using a phase locked loop (PLL)

Noise in DSB-SC Receivers

- The DSB-SC component of the filtered signal x(t) is

$$\boxed{s(t) = CA_c \cos(2\pi f_c t) m(t)}$$

where C is the system dependent scaling factor in order to ensure that the signal component s(t) is measured in the same units as the additive noise component n(t).

- The average power P of the message signal is

$$\boxed{P = \int_{-W}^{W} S_M(f)\, df}$$

Noise in DSB-SC Receivers

- The signal power in DSB-SC signal is

$$\boxed{P_S = \frac{C^2 A_c^2 P}{2}}$$

- With a noise spectral density of $N_0/2$, the average noise power in the message bandwidth W is equal to WN_0.
- The channel signal-to-noise ratio is

$$\boxed{SNR_{C,DSB} = \frac{C^2 A_c^2 P}{2WN_0}}$$

- where C ensures that this ratio is dimensionless.

Noise in DSB-SC Receivers

- Let's determine the output signal-to-noise ratio at the coherent demodulator output

$$\boxed{\begin{aligned} x(t) &= s(t) + n(t) \\ &= CA_c \cos(2\pi f_c t) m(t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}}$$

- The output of the product-modulator component of the coherent detector is

$$\boxed{\begin{aligned} v(t) &= x(t)\cos(2\pi f_c t) \\ &= \frac{1}{2} CA_c m(t) + \frac{1}{2} n_I(t) \\ &\quad + \frac{1}{2}\left[CA_c m(t) + n_I(t)\right]\cos(4\pi f_c t) - \frac{1}{2} n_Q(t)\sin(4\pi f_c t) \end{aligned}}$$

Noise in DSB-SC Receivers

- The low-pass filter removes the high frequency components of v(t) and the receiver output is

$$\boxed{y(t) = \frac{1}{2} C A_c m(t) + \frac{1}{2} n_I(t)}$$

- The message signal m(t) and in-phase noise component $n_I(t)$ appear additively at the receiver output.
- The quadrature component $n_Q(t)$ is completely rejected by the coherent detector.
- Coherent demodulator guarantees that the signal and noise component are additive irrespective of signal-to-noise ratio.

Noise in DSB-SC Receivers

- The signal power at the detector output is

$$\boxed{P_s = \frac{C^2 A_c^2 P}{4}}$$

- The noise power at the detector output is

$$\boxed{P_N = \left(\frac{1}{2}\right)^2 2WN_0 = \frac{1}{2} WN_0}$$

- The signal-to-noise ratio at the output is

$$\boxed{SNR_O = \frac{C^2 A_c^2 P / 4}{WN_0 / 2} = \frac{C^2 A_c^2 P}{2WN_0}}$$

NOISE IN DSB-SC RECEIVERS

- The figure of merit of DSB-SC system is

$$\boxed{\frac{SNR_O}{SNR_C} = 1}$$

- Note that C^2 factor is common and cancels out in evaluating the figure of merit.

Noise in AM Receivers

- In an AM signal the transmitted signal is

$$\boxed{s(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)}$$

  k_a is small enough not to create any phase reversal
- For receiver simplicity envelope detector is used to demodulate the AM signal.
- The average carrier power is

$$\boxed{\frac{A_c^2}{2}}$$

Noise in AM Receivers

- The average message signal component of AM signal is

$$\boxed{\frac{A_c^2 k_a^2 P}{2}}$$

  where P is the average power of the message signal.
- Total average power of the full AM signal is

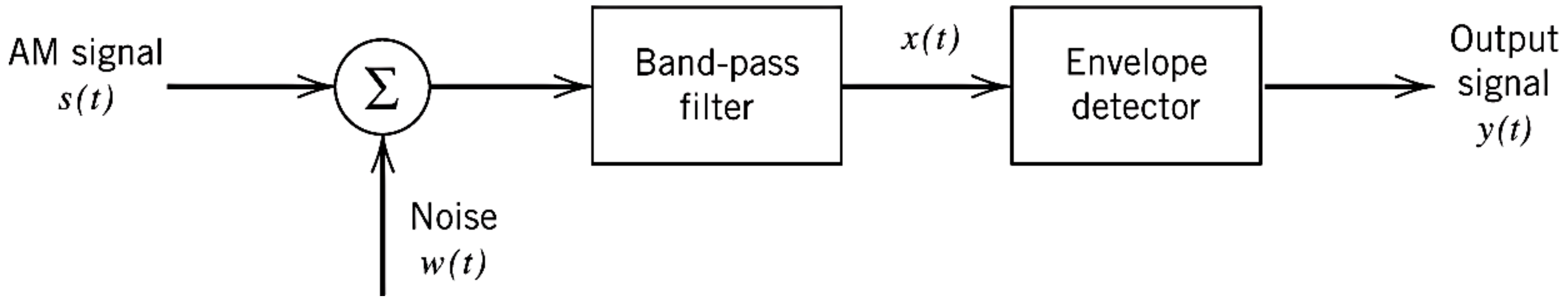
$$\boxed{\frac{A_c^2}{2}\left(1+k_a^2 P\right)}$$

NOISE IN AM RECEIVERS

- The average noise power in the message bandwidth is similar to DSB-SC and equal to WN_0
- The channel signal-to-noise ratio for AM is

$$\boxed{SNR_{C,AM} = \frac{A_c^2\left(1+k_a^2 P\right)}{2WN_0}}$$

- The AM receiver with noise model is



Noise in AM Receivers

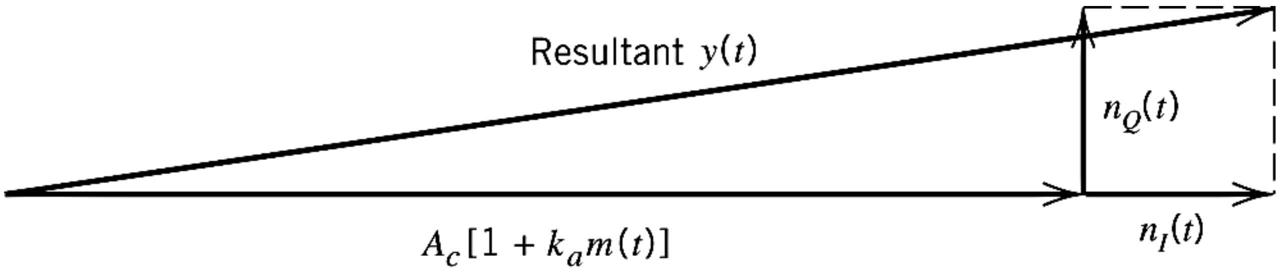
- To evaluate the output signal-to-noise ratio, we need to write the filtered noise in terms of its in-phase and quadrature components.

$$\boxed{\begin{aligned} x(t) &= s(t) + n(t) \\ &= \left[A_c + A_c k_a m(t) + n_I(t)\right]\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}}$$

- The envelope of the AM signal carries the message signal.
- Therefore, the envelope of x(t) needs to be determined from its phasor diagram.

NOISE IN AM RECEIVERS

- Phasor diagram for AM wave plus narrowband noise for the case of high carrier-to-noise ratio



- From this phasor diagram the receiver output is obtained as

$$\boxed{\begin{aligned} y(t) &= \text{envelope of } x(t) \\ &= \sqrt{\left[A_c + A_c k_a m(t) + n_I(t)\right]^2 + n_Q^2(t)} \end{aligned}}$$

Noise in AM Receivers

- The signal y(t) is the output of an ideal envelope detector.
- The envelope detector output is complex and hard to analyze. It needs to be simplified in order to achieve some meaningful results.
- The first assumption is the average carrier power is large compared to noise so that the signal term is larger then the noise term.
- y(t) can be approximated as

$$\boxed{y(t) \approx A_c + A_c k_a m(t) + n_I(t)}$$

Noise in AM Receivers

- The dc term A_c in the envelope detector output y(t) is due to the demodulation of the carrier wave.
- If we neglect this dc term, the output signal-to-noise ratio at the envelope detector output is

$$\boxed{SNR_{O,AM} \approx \frac{A_c^2 k_a^2 P}{2WN_0}}$$

- The Figure of Merit of the AM system is

$$\boxed{\frac{SNR_O}{SNR_C} \approx \frac{k_a^2 P}{1 + k_a^2 P}}$$

Noise in AM Receivers

- Although the noise performance of DSB-SC or SSB is one, the noise performance of an AM system is always less than unity.
- This is due to the waste of the transmitted carrier power, which does not carry any information.

EXAMPLE

- Compare the DSB-SC noise performance to the AM noise performance (100 percent modulation) of the following sinusoidal wave.

$$\boxed{m(t) = A_m \cos(2\pi f_m t)}$$

Example - Solution

- The average power of the message signal is

$$\boxed{P = \frac{1}{2} A_m^2}$$

Therefore, the figure of merit of AM system is

$$\boxed{\frac{SNR_O}{SNR_C} = \frac{\frac{1}{2} k_a^2 A_m^2}{1 + \frac{1}{2} k_a^2 A_m^2} = \frac{k_a^2 A_m^2}{2 + k_a^2 A_m^2}}$$

- For 100 percent modulation $k_a A_m = 1$.
- Therefore the AM systems figure of merit is 1/3, on the other hand DSB-SC FOM is 1.
- This means we need to transmit 3x power in AM to achieve the same noise performance as DSB-SC

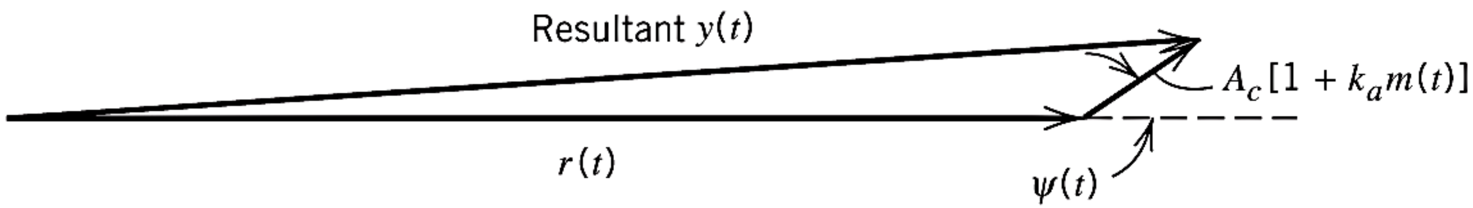
Threshold Effect

- When the carrier-to-noise ratio is small compared with unity, the noise term dominates and the performance of the envelope detector changes completely.
- In this case it is more convenient to represent the noise n(t) in terms of its envelope r(t) and phase Ψ(t)

$$\boxed{n(t) = r(t)\cos\left[2\pi f_c t + \Psi(t)\right]}$$

THRESHOLD EFFECT

- The corresponding phase diagram is



- Assuming that the carrier-to-noise ratio is so low that the carrier amplitude A_c is small compared with the noise envelope r(t) the envelope detector output is approximately

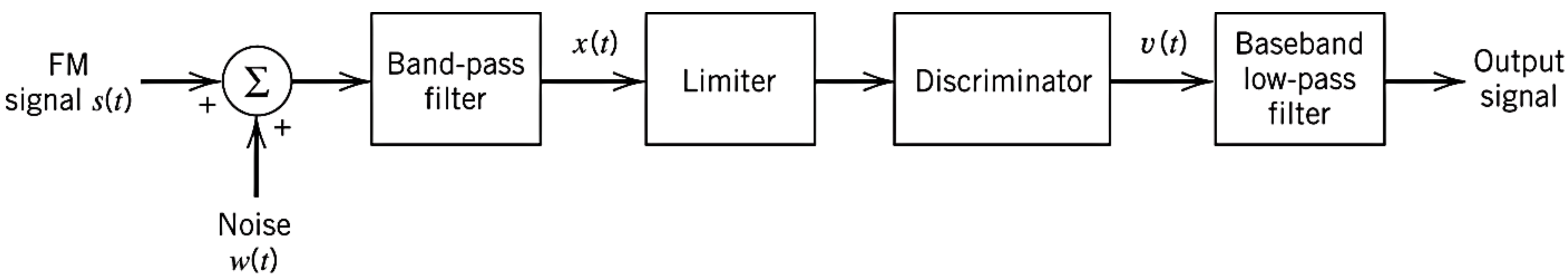
$$\boxed{y(t) \approx r(t) + A_c \cos[\Psi(t)] + A_c k_a m(t) \cos[\Psi(t)]}$$

Threshold Effect

- This relation shows that when carrier-to-noise ratio is low, the detector output has no component strictly proportional to message signal m(t).
- Note that the envelope detector is not sensitive to the phase $\Psi(t)$, which is multiplied by the message signal. This means total loss of the message signal and referred as the threshold effect.
- Every nonlinear detector shows this threshold effect. On the other hand, this effect does not arise in a coherent detector.

Noise in FM Receivers

- In an FM system, the message information is transmitted by variations of the instantaneous frequency of a sinusoidal carrier wave, and its amplitude is maintained constant.
- Therefore, any variation on its amplitude is removed by the limiter.



Noise in FM Receivers

- The discriminator consists of two components
  - A slope network or differentiator with a purely imaginary transfer function that varies linearly with frequency.
  - An envelope detector that recovers amplitude variation and thus reproduce the message signal.
- The post detection filter (baseband low-pass filter) has a bandwidth that is just large enough to accommodate the highest frequency component of the message signal. This filter basically removes any out-of-band components of the noise in order to minimize the amount of noise.

Noise in FM Receivers

- Let's rewrite the noise in terms of its amplitude and phase.

$$\boxed{n(t) = r(t)\cos[2\pi f_c t + \Psi(t)]}$$

- Let's write the FM signal in its simplest form

$$\boxed{s(t) = A_c \cos[2\pi f_c t + \phi(t)]}$$

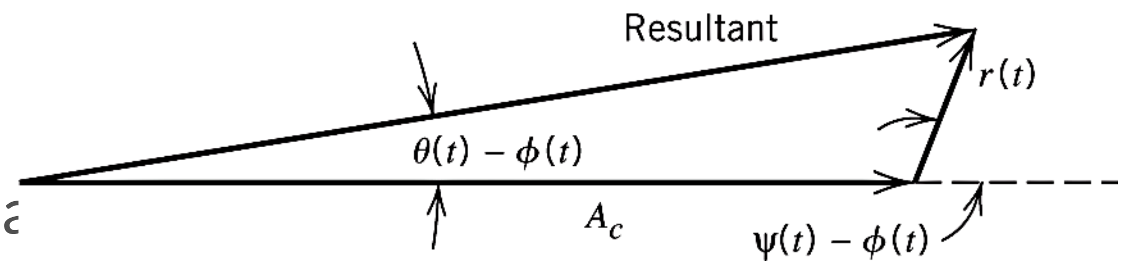
$$\boxed{\phi(t) = 2\pi k_f \int_0^t m(\tau)\,d\tau}$$

- The noisy signal at the band-pass filter output is

$$\boxed{\begin{aligned} x(t) &= s(t) + n(t) \\ &A_c \cos[2\pi f_c t + \phi(t)] + r(t)\cos[2\pi f_c t + \Psi(t)] \end{aligned}}$$

NOISE IN FM RECEIVERS

- Let's represent x(t) by means of a phasor diagram.



- The pha

$$\boxed{\theta(t) = \phi(t) + \tan^{-1}\left\{\frac{r(t)\sin[\Psi(t) - \phi(t)]}{A_c + r(t)\cos[\Psi(t) - \phi(t)]}\right\}}$$

- Assuming the discriminator ideal, its output is proportional to $\theta'(t)/2\pi$ where $\theta'(t)$ is the derivative $\theta(t)$

Noise in FM Receivers

- We need to make some assumption in order to achieve some meaningful results.
- Let's assume that the carrier-to-noise ratio at the discriminator input is large compared with unity.
- Another assumption is the noise amplitude is smaller than the carrier amplitude.
- So the phase simplifies as follows.

$$\boxed{\theta(t) \approx \phi(t) + \frac{r(t)}{A_c}\sin[\Psi(t) - \phi(t)]}$$

Noise in FM Receivers

- Using the expression for $\phi(t)$

$$\theta(t) \approx 2\pi k_f \int_0^t m(\tau)d\tau + \frac{r(t)}{A_c}\sin[\Psi(t) - \phi(t)]$$

- The discriminator output is

$$\begin{aligned} v(t) &= \frac{1}{2\pi}\frac{d\theta(t)}{dt} \\ &\approx k_f m(t) + n_d(t) \end{aligned}$$

- where the noise term is defined by

$$n_d(t) = \frac{1}{2\pi A_c}\frac{d}{dt}\{r(t)\sin[\Psi(t) - \phi(t)]\}$$

Noise in FM Receivers

- Since we assumed that the carrier-to-noise ratio is high, the discriminator output consists of the original message signal multiplied by the constant factor k_f plus an additive noise component $n_d(t)$.
- According to the definition of $n_d(t)$, the noise term has a signal component. On the other hand $\Psi(t)$ is uniformly distributed over 2π radians.
- We can assume the phase difference $\Psi(t)-\phi(t)$ is also uniformly distributed over 2π radians.
- This assumption results in that the discriminator output is independent from the modulating signal.

NOISE IN FM RECEIVERS

- Under these assumption the noise term is

$$\boxed{n_d(t) \approx \frac{1}{2\pi A_c} \frac{d}{dt} \{r(t) \sin[\Psi(t)]\}}$$

- Note that the quadrature component of the filtered noise n(t) is

$$\boxed{n_Q(t) = r(t) \sin[\Psi(t)]}$$

- Therefore, we may rewrite

$$\boxed{n_d(t) \approx \frac{1}{2\pi A_c} \frac{dn_Q(t)}{dt}}$$

Noise in FM Receivers

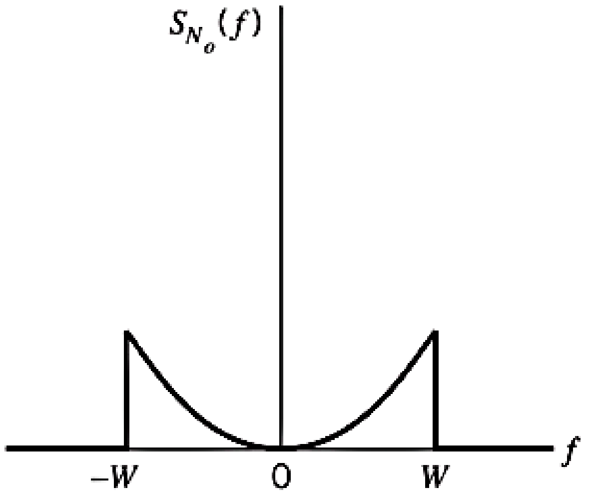
- The average output noise power is proportional to the time derivative of the quadrature noise component.
- Since the derivation of a function with respect to time corresponds multiplication of its Fourier transform by j2πf, the power spectral density of $S_{Nd}(f)$ is

$$\boxed{S_{N_d}(f)=\frac{f^2}{A_c^2}S_{N_Q}(f)}$$

Noise in FM Receivers

- The low-pass filter after the discriminator removes the excessive noise and the spectrum is as follows

$$\boxed{S_{N_d}(f) = \begin{cases} \dfrac{N_0 f^2}{A_c^2} & |f| \le W \\ 0 & \text{otherwise} \end{cases}}$$



Noise in FM Receivers

- The average output noise is determined by integrating the power spectral density from –W to W

$$\boxed{\begin{aligned}\text{Average output noise power} &= \frac{N_0}{A_c^2}\int_{-W}^{W} f^2\,df \\ &= \frac{2N_0W^3}{3A_c^2}\end{aligned}}$$

- Note that the average output noise power is inversely proportional to the average carrier power $A_c^2/2$. This means increasing the carrier power has a noise-quieting effect.

Noise in FM Receivers

- The output signal-to-noise ratio is

$$\boxed{SNR_{O,FM} = \frac{3A_c^2 k_f^2 P}{2N_0 W^3}}$$

- The average carrier power is

$$\boxed{SNR_{O,FM} = \frac{A_c^2}{2N_0 W}}$$

- The figure of merit is

$$\boxed{\frac{SNR_{O,FM}}{SNR_{C,FM}} = \frac{3k_f^2 P}{W^2}}$$

Noise in FM Receivers

- Recall that the deviation ratio D is equal to the frequency deviation Δf, which is proportional to k_f, divided by the message bandwidth.
- In other words, the deviation ratio D is proportional to the ratio $k_f P^{1/2}/W$.
- Note that the transmission bandwidth B_T is approximately proportional to the deviation ratio D.
- When the carrier-to-noise ratio is high, an increase in the transmission bandwidth B_T provides a corresponding quadratic increase in the output signal-to-noise ratio or figure of merit of the FM system.

EXAMPLE

- Consider the case of a sinusoidal wave of frequency f_m and peak frequency deviation Δf.
- Calculate the Figure of Merit in terms of the modulation index β.

Example - Solution

- The message signal is

$$\boxed{m(t) = \frac{\Delta f}{k_f}\cos(2\pi f_m t)}$$

- The power of the message signal is

$$\boxed{P = \frac{(\Delta f)^2}{2k_f^2}}$$

- The Figure of Merit is

$$\boxed{\frac{SNR_{O,FM}}{SNR_{C,FM}} = \frac{3}{2}\left(\frac{\Delta f}{W}\right)^2 = \frac{3}{2}\beta^2}$$

Example - Solution

- The figure of merit of an AM system operating at 100% modulation is

$$\boxed{\left.\frac{SNR_O}{SNR_C}\right|_{AM} = \frac{1}{3}}$$

- Comparing the figure of merit with the FOM of FM system, FM system shows better noise performance than AM when

$$\boxed{\begin{gathered}\frac{3}{2}\beta^2 > \frac{1}{3} \\ \beta > \frac{\sqrt{2}}{3} = 0.471\end{gathered}}$$

Roughly the transition between narrowband FM and wideband FM is considered when β is 0.5.
This statement is based on the noise considerations.

Capture Effect

- Since in FM the information is coded at the zero-crossings, it has also ability to minimize the effects of unwanted interference.
- The FM receiver locks the stronger signal and suppress the unwanted interference.
- If the desired signal and interference strengths are of nearly equal strength, the receiver fluctuates back and forth between them.
- This phenomenon is called as the capture effect.

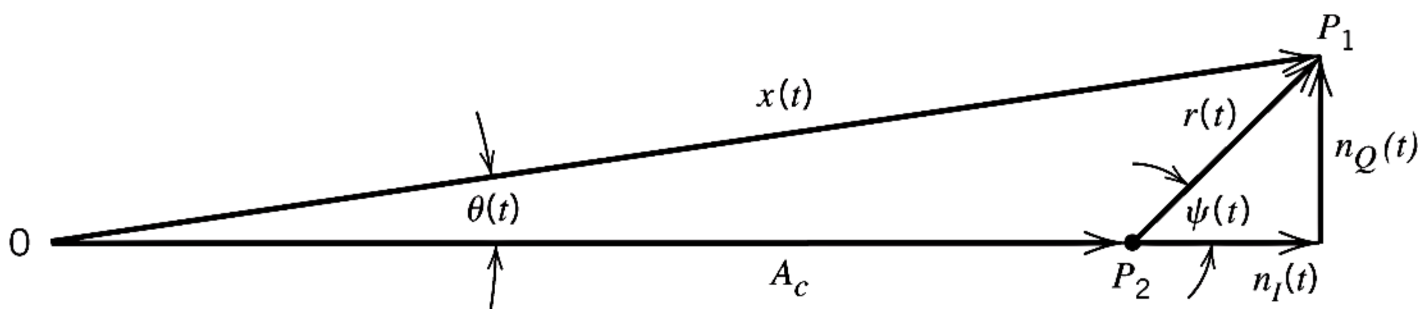
FM Threshold Effect

- As the input noise power is increased so that the carrier-to-noise ratio is decreased, the FM receiver breaks.
- At first individual clicks are heard.
- The clicks rapidly merge into a crackling sound.
- Near the breaking point, the input/output SNR relationship deviates from the estimated value calculated for large input SNR.
- This phenomenon is called as the threshold effect.

FM Threshold Effect

- Let's consider the case where there is no message signal is present.

$$x(t)=[A_c+n_I(t)]\cos(2\pi f_c t)-n_Q(t)\sin(2\pi f_c t)$$

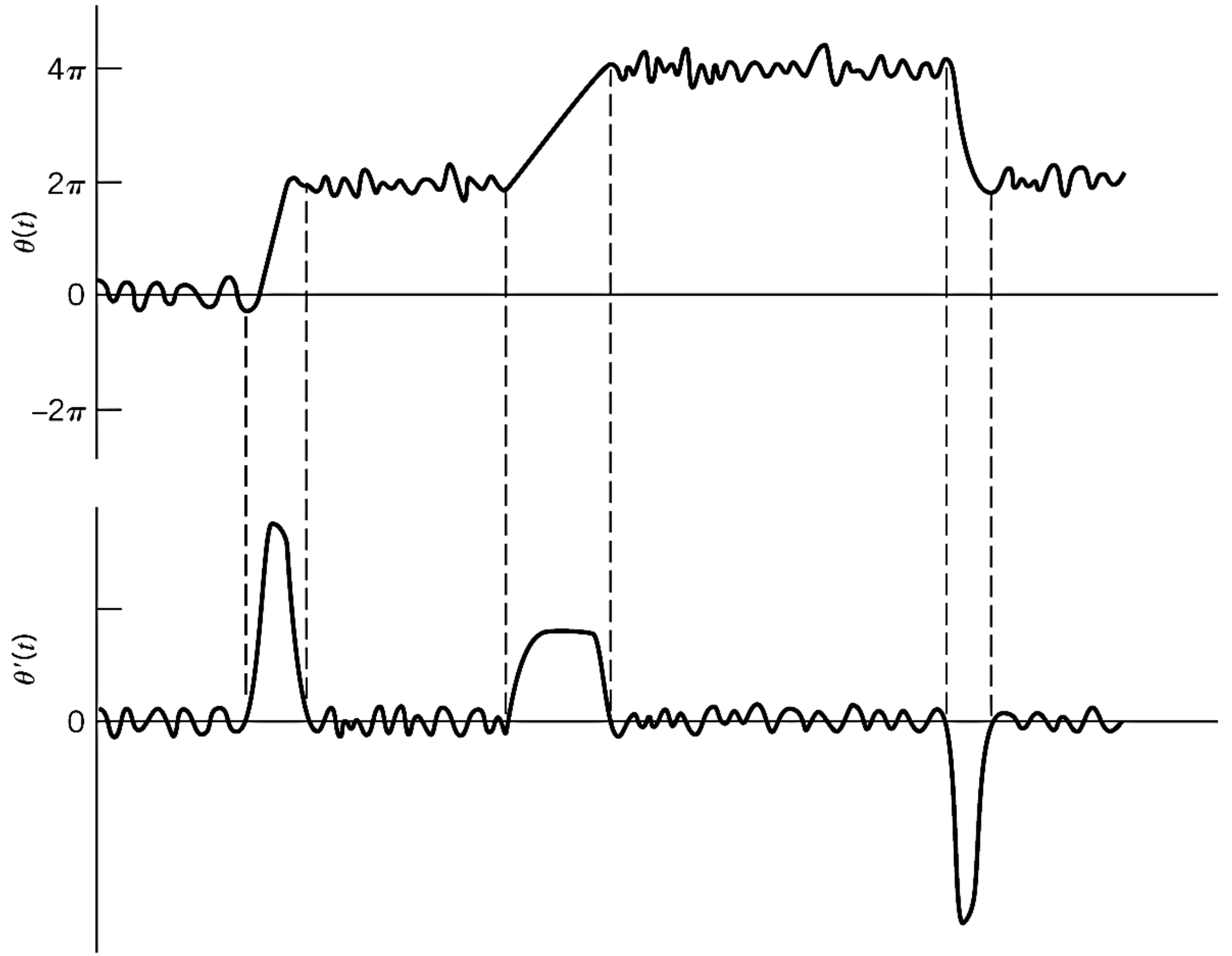


- As $n_I(t)$ and $n_Q(t)$ changes with time in a random manned, the point P_1 wanders around P_2.
- When the CNR is large, P_1 spends most of its time near P_2, so the angle is approximately $n_Q(t)/A_c$

FM Threshold Effect

- When CNR is low, P_1 occasionally sweeps around origin and $\theta(t)$ increases or decreases by 2π radian.
- The discriminator output is equal to $\theta'(t)/ 2\pi$, so 2π changes creates impulselike components.
- This impulses are heard as clicks.

FM Threshold Effect



FM Threshold Effect

- According to the phasor diagram, the requirement for a positive click happens when the envelope r(t) and phase Ψ(t) and n(t) is

$$\boxed{\begin{gathered} r(t) > A_c \\ \Psi(t) < \pi < \Psi(t) + d\Psi(t) \\ \frac{d\Psi(t)}{dt} > 0 \end{gathered}}$$

for negative going click

$$\boxed{\begin{gathered} r(t) > A_c \\ \Psi(t) > -\pi > \Psi(t) + d\Psi(t) \\ \frac{d\Psi(t)}{dt} > 0 \end{gathered}}$$

FM Threshold Effect

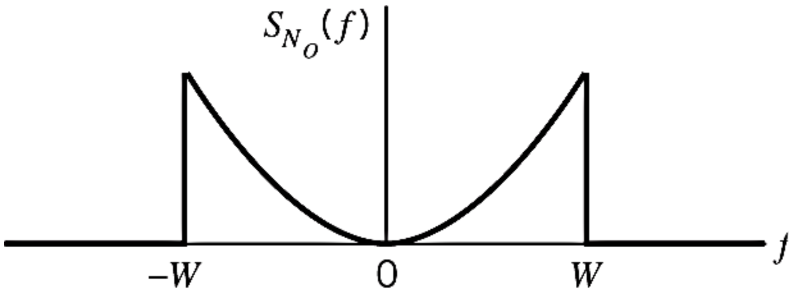
- The CNR (Carrier-to-Noise Ratio) is defined as

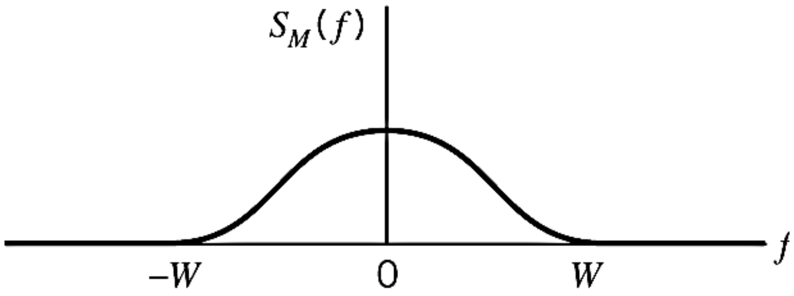
$$\boxed{CNR = \frac{A_c^2}{2B_T N_0}}$$

- In most practical cases the threshold effect may be avoided if the CNR is greater than 20 (13dB)
- Since the transmission bandwidth is a function of peak frequency deviation, increasing the peak frequency deviation increases the threshold point.

Pre-Emphasis and De-Emphasis in FM

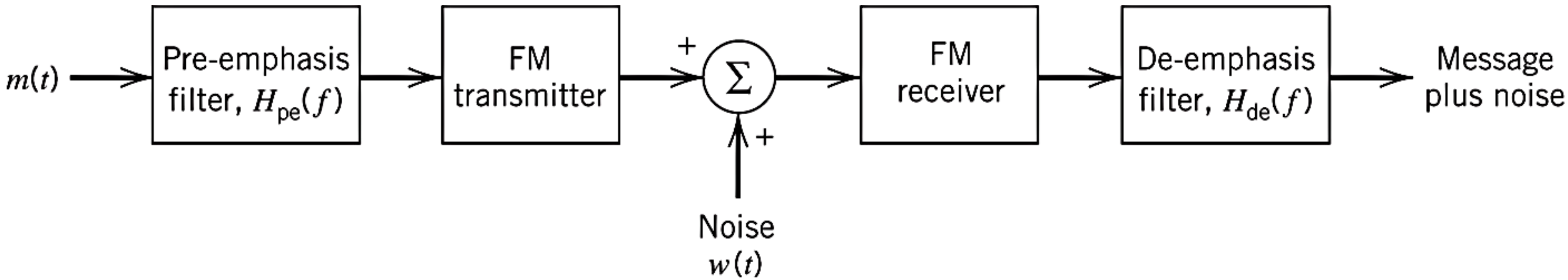
- The power spectral density of the noise at the output of FM receiver has a square-law dependence on the operating frequency.
- Most of the message signals utilizes low-frequency spectrum.
- Therefore, the high frequency components of the message signal has lower SNR than low-frequency components.





Pre-Emphasis and De-Emphasis in FM

- To improve the SNR at the receiver output, pre-emphasis filter is applied at the transmitter and de-emphasis filter is applied at the receiver.



$$H_{de}(f) = \frac{1}{H_{pe}(f)} \quad -W \le f \le W$$

Pre-Emphasis and De-Emphasis in FM

- The noise at the FM receiver without de-emphasis output for high CNR

$$S_{N_d}(f) = \begin{cases} \dfrac{N_0 f^2}{A_c^2} & |f| \le W \\ 0 & \text{otherwise} \end{cases}$$

- The noise at the de-emphasis filter output is

$$P_{N,out} = \frac{N_0}{A_c^2} \int_{-W}^{W} f^2 \left|H_{de}(f)\right|^2 df$$

Pre-Emphasis and De-Emphasis in FM

- We can calculate the SNR improvement due to the de-emphasis is

$$\boxed{I = \frac{2W^3}{3\int_{-W}^{W} f^2 \left|H_{de}(f)\right|^2 df}}$$

- A simple pre-emphasis and de-emphasis filter is constructed using first order RC filter.

EXAMPLE

- A simple pre-emphasis filter in practice is defined by the transfer function

$$\boxed{H_{pe}(f) = 1 + \frac{jf}{f_0}}$$

- The corresponding de-emphasis filter is

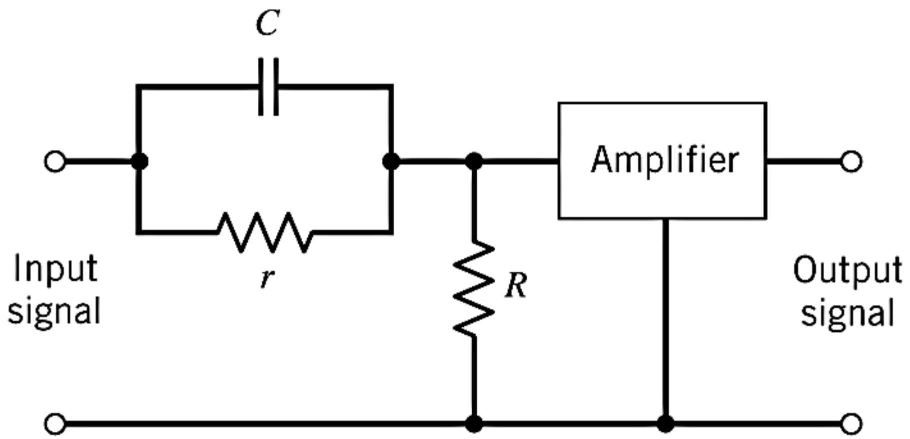
$$\boxed{H_{de}(f) = \frac{1}{1 + \frac{jf}{f_0}}}$$

EXAMPLE (CONT.)

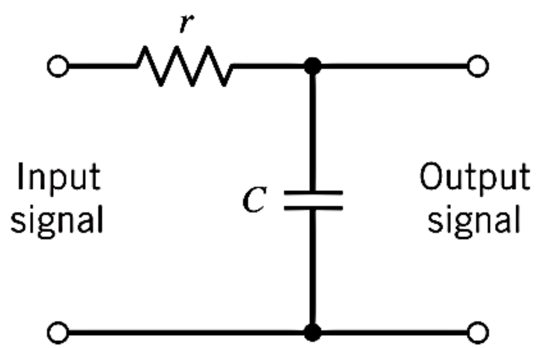
- These transfer functions can be realized for the band of interest using the simple RC network.
- The improvement factor is

$$I = \frac{2W^3}{3\int_{-W}^{W} \frac{f^2}{1+(f/f_0)^2}\,df} = \frac{(W/f_0)^3}{3\left[(W/f_0) - \tan^{-1}(W/f_0)\right]}$$

In commercial FM broadcasting, f0=2.1KHz and W=15KHz. This gives the improvement factor of 22 (13dB)

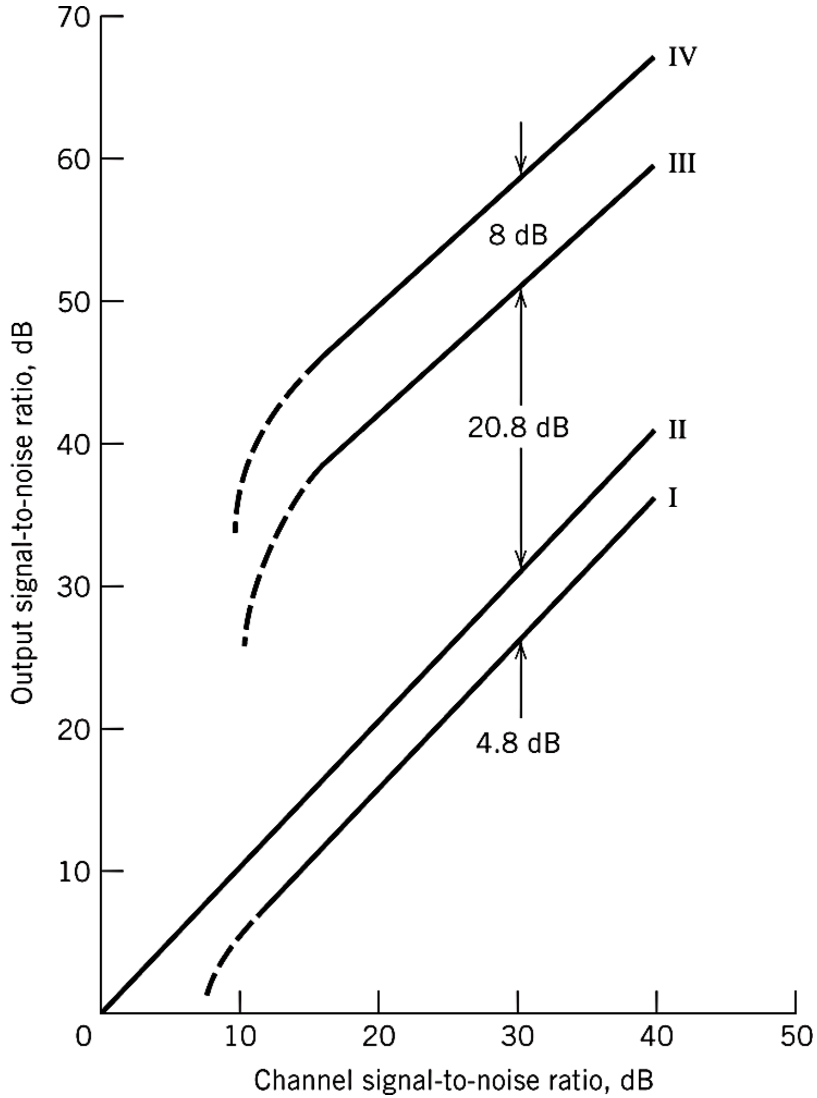


Pre-Emphasis Filter



De-Emphasis Filter

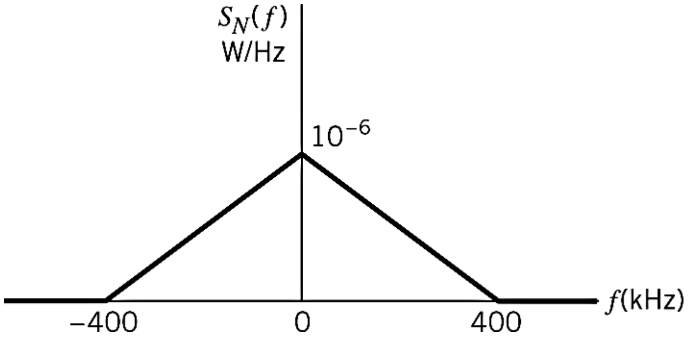
Comparison of the Noise Performance of various CW Modulation Systems



I. Standard AM $\mu=1$
II. DSB-SC
III. FM $\beta=2$ with pre-emphasis and de-emphasis
IV. FM $\beta=5$ with pre-emphasis and de-emphasis

EXAMPLE

- A DSB-SC modulated signal is transmitted over a noisy channel with the power spectral density shown below.
- The message signal is 4KHz and the carrier frequency is 200KHz.
- Assuming that the average power of the modulated wave is 10 Watts, determine the output signal-to-noise ratio of the receiver.



Example – Solution

- After passing the received signal through narrow-band filter of bandwidth 8KHz centered on f_c=200KHz,

$$\boxed{\begin{aligned} x(t) &= A_c m(t)\cos(2\pi f_c t) + n(t) \\ &= A_c m(t)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\ &= [A_c m(t) + n_I(t)]\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \end{aligned}}$$

- After coherent demodulation the output term become

$$\boxed{y(t) = A_c m(t) + n_I(t)}$$

Example - Solution

- Since the message power is 10Watt, we need to calculate the noise power in order to calculate SNR.
- We can write the noise power spectral density as

$$\boxed{S_N(f) = -\frac{10^{-6}}{400KHz} f + 10^{-6}}$$

- Since the signal bandwidth is much smaller compared to carrier frequency, we can assume the noise power spectral density is flat around carrier frequency and is equal to 0.5×10^{-6} W/Hz

EXAMPLE - SOLUTION

- The total noise power is

$$\boxed{0.5\times10^{-6}\cdot 8KHz = 0.008\,Watts}$$

- The SNR is

$$\boxed{SNR = \frac{10}{0.008} = 1250(31\,dB)}$$