Probabilistic perspective of ML

set theoretic approach for probability

Let A be a non-empty set representing an event & S be the sample space

$P[A] = \frac{n(A)}{n(S)}$ (n: no. of elements in set)

$= \frac{\text{no. of favourable cases to A}}{\text{total no. of cases}}$

TN: $0 \le P[A] \le 1$

① 2 events A & B are independent of each other if occurence of A does not depend on occurence of B. Eg) Tossing 2 dies

② 2 events are said to be mutually exclusive if occ. of A avoids occ of B. Eg) Tossing a coin

∴ if 2 events A & B are mutually exclusive, $P[A \cup B] = P[A] + P[B]$

Addition theorem of probability

proof: use ★ venn diagram.

- if A & B are 2 diff. events then

$$P[A \cup B] = P(A) + P(B) - P[A \cap B]$$

Proof:

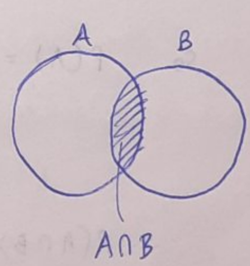
$A \cup B = A \cup (\bar{A} \cap B)$ ∵ [A & $\bar{A} \cap B$ are mutually exclusive]

LHS $= P(A \cup B) = P[A] + P[\bar{A} \cap B]$

$= P[A] + P[B - (A \cap B)]$

$= P(A) + P(B) - P(A \cap B)$ = RHS.

H.P.

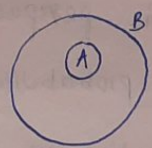


TN: mutually exclusive ≠ independent

PTO

* If $A \subseteq B$, $P(A) \leq P(B)$.

$B = A \cup (\bar{A} \cap B)$



To prove just make every thing mutually exclusive

$P(B) = P(A) + \underbrace{P(\bar{A} \cap B)}_{\text{hide}}$

$5 \geq 4 + \cancel{x}$

$\therefore P(B) \geqq P(A)$ #

TN: $P(A \cup B) = P(A) + P(B) - \underline{P(A \cap B)} \leq P(A) + P(B)$.

Conditional probability

If A, B : events and A is yet to occur when B has already occurred:

$$P[A/B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B \cap A]}{P[B]} \quad ①$$

without condition prob will increase

Note: If A and B are independent then

$$P[A/B] = P[A] \quad ②$$

Q. State & prove multiplication theorem of prob:

If A & B are independent of each other,

$$P(A \cap B) = P(A) \cdot P(B). \quad ③$$

Proof

By defn $P[A/B] = \dfrac{P(A \cap B)}{P(B)}$

A&B ind $\Rightarrow$ $P[A/B] = P(A)$

So $P(A) = \dfrac{P(A \cap B)}{P(B)}$

$\therefore P(A \cap B) = P(A) \cdot P(B)$.

H.P.

IMP

$$\star \; \underbrace{P[A \cap B]}_{\text{posterior}} = \underbrace{P[A]}_{\text{prior}} \cdot \underbrace{P[B/A]}_{\text{max. likelihood}} = P[B] \cdot P[A/B].$$

▸ State & prove

If $A_1, A_2 \cdots A_n$

occuring along

then P[A

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Here

B = (A, ∩ B

P(B) =

=

∴ P[

Proof:

B can be

where (

∴ By

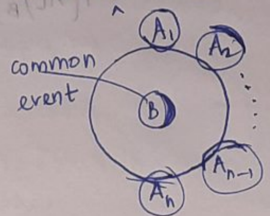
① → P

i.e. P(

- State & prove Bayes' theorem ☆IMP

If $A_1, A_2 \cdots A_n$ are mutually exclusive events & B is a common event occuring along with ~~some of~~ $A_1, A_2 \cdots A_n$

then $P[A_i/B] = \dfrac{P[A_i] \cdot P[B/A_i]}{\sum_{i=1}^{n} P[A_i] \cdot P[B/A_i]}$



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Here max likelihood is a conditional prob. entity

SVD
PCA/1
CE
Eigen values &vectors

$B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B)$

$P(B) = \sum_{i=1}^{n} P[A_i \cap B]$

$= \sum_{i=1}^{n} P[A_i] \cdot P[B/A_i]$

$\therefore P[A_i/B] = \dfrac{P[A_i \cap B]}{P[B]}$

$= \dfrac{P[A_i \cap B]}{\sum_{i=1}^{n} P[A_i]\, P[B/A_i]}$

$= \dfrac{P[A_i] \cdot P[B/A_i]}{\sum_{i=1}^{n} P[A_i] \cdot P[B/A_i]}$

Proof:

B can be written as : $B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots (A_n \cap B)$

where (A_i's $\cap$ B) are mutually exclusive

$\therefore$ By addition of prob.;

① → $P(B) = P(A_1 \cap B) + P(A_2 \cap B) \cdots P(A_n \cap B)$

i.e. $P(B) = \sum_{i=1}^{n} P(A_i \cap B)$

$= \sum_{i=1}^{n} P(A_i) \cdot P(B/A_i)$ — ② $\because$ by using the result

$P[A \cap B] = P(A) \cdot P(B/A)$ ∀ A, B.

By defⁿ,

$$P[A_i/B] = \frac{P(A_i \cap B)}{P(B)}$$

$$= \frac{P[A_i] \cdot P[B/A_i]}{\sum_{i=1}^{n} P(A_i) \cdot P(B/A_i)} \quad \because \text{ by using } ① \,\&\, ②.$$

▸ <u>Property</u>

If ϕ is an empty event, P.T. $P(\phi) = 0$.

Let S be the sample space

$S = S \cup \phi$

$P(S) = P(S) + P[\phi] \quad \because S \,\&\, \phi$ are mutually exclusive.

$\therefore P[\phi] = 0.$

★ If $\bar{A}$ is the complementary event of A

P.T. $P(\bar{A}) = 1 - P(A) \leq 1.$

Let S be the SS; A be event $\subset S$.

$S = A \cup \bar{A}$

$P(S) = P(A) + P(\bar{A}) \quad (\because A \,\&\, \bar{A}$ are mutually exclusive)

$1 = P(A) + P(\bar{A})$

$\therefore P(\bar{A}) = 1 - P(A) \leq 1 \quad \because$ by axiom of probability.

▸ If A & B are two independent events

i) P.T. $(\bar{A} \,\&\, B)$ $(A \,\&\, \bar{B})$ ~~$\& (\bar{A} \,\&\, \bar{B})$~~ are independent of each other.

$(\bar{A} \cap B) \cup (A \cap B) = B$

$P[B] = P[\bar{A} \cap B] + P[A \cap B]$

$(\because \bar{A} \cap B \,\&\, A \cap B$ are mutually exclusive) A B.

$P(B) = P(\bar{A} \cap B) + P(A) \cdot P(B)$

$P(\bar{A} \cap B) = P(B) - P(A) \cdot P(B)$

$= P[B] \cdot [1 - P[A]]$

$= P[B] \cdot P[\bar{A}]$

since $P(\bar{A} \cap B) = P(B) \cdot P(\bar{A})$

$\therefore$ $\bar{A}$ & B are independent of each other.

ii) $(A \cap \bar{B}) \cup (A \cap B) = A$

$P(A) = P(A \cap \bar{B}) + P(A \cap B)$

$P(A) = P(A \cap \bar{B}) + P(A) \cdot P(B)$

$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A) \cdot P(B)$

$P(A \cap \bar{B}) = P(A)\, [1 - P[B]]$

$P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$

$\therefore$ A & $\bar{B}$ are independent of each other.

iii) T.P. $\bar{A}$ & $\bar{B}$ are independent of each other i.e.

$P[\bar{A} \cap \bar{B}] = P[\bar{A}] \cdot P[\bar{B}]$

De Morgan's : $\bar{A} \cap \bar{B} = \overline{A \cup B}$

LHS :

$P[\bar{A} \cap \bar{B}] = P[\overline{A \cup B}]$

$= 1 - P[A \cup B]$

$= 1 - [P[A) + P(B) - P(A \cap B)]$ $\therefore$ by AT

$= 1 - P(A) - P(B) + P(A \cap B)$

$= P(\bar{A}) - P(B) + P(A) \cdot P(B)$

$= P(\bar{A}) - P(B)\,(1 - P(A))$

$= P(\bar{A}) - P(B) \cdot P(\bar{A})$

$= (1 - P(B)) \cdot P(\bar{A})$

$= P(\bar{B}) \cdot P(\bar{A})$.

$\therefore$ $\bar{A}$ & $\bar{B}$ $\Rightarrow$ indep.

Corollary to add^n theorem:

$P[A \cup B \cup C] = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Proof: Let $B \cup C = D$.

$\therefore P[A \cup B \cup C] = P[A \cup D]$

$= P(A) + P(D) - P(A \cap D)$ $\because$ By AT

~~$= P(A) + P(B \cap C$~~

$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$

$= P(A) + P(B) + P(C) - P(B \cap C) - P[(A \cap B) \cup P(A \cap C)] - P(A \cap B \cap C)]$ $\because$ by distributive property

$= \underline{RHS}$.

H.P.

Q. If A, B, C are any 3 events S.T. $P(A) = P(B) = P(C) = 0.25$

$P(A \cap B) = P(B \cap C) = 0$

$P(C \cap A) = (1/8)$

Find the prob. that atleast one of the events (U) A, B, C occurs. max

A) $P[A \cup B \cup C] = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

$0.75 - 0 - 0 - 0.125 + 0$.

(as $P(A \cap B \cap C) = 0$ if $P(A \cap B) = P(B \cap C) = 0$.

$= \underline{0.625}$

or $\frac{5}{8}$

Q. In a shooting test prob of hitting the target is $\frac{A}{2}$, $\frac{2B}{3}$ & $\frac{3C}{4}$. If all of them fire at the target, find the prob. that (a) none of them hit the target
(b) atleast one of them hits the target.

(a) $P(\bar{A}) = \frac{1}{2}$, and $P(\bar{B}) = \frac{1}{3}$, and $P(\bar{C}) = \frac{1}{4}$.

(or-add^n -intsn)

$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} = \frac{1}{24}$

($\because$ we find $P(\bar{A} \cap \bar{B} \cap \bar{C})$)

(b) $P(A \cup B \cup C) = \cancel{\frac{1}{2} + \frac{2}{3} + \frac{3}{4}} -$

$= 1 - [P[\bar{A} \cap \bar{B} \cap \bar{C}]]$

$= 1 - 1/24$

$= 23/24$.

(means getting sum of no.s on both dies as 6)

Q. A & B alternatively throw a pair of dies. A throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, s.t his chance of winning is 30/61.

A) P(Throwing 6 with 2 dice) = 5/36 (1,5), (2,4), (3,3) (4,2), (5,1)

P(throwing 7 with ") = 1/6. (6/36).

Let A = event of A throwing 6.

B = event of B throwing 7.

A plays in the first, third, ~~fifth~~.. trials.

$\therefore$ A will win, if he throws 6 in the first trial or third trial or in the subsequent (odd) trials.

$$P(A \text{ wins}) = P(A \text{ or } \bar{A}\bar{B}A \text{ or } \bar{A}\bar{B}\bar{A}\bar{B}A \text{ or } \cdots)$$

$$= P(A) + P(\bar{A}\bar{B}A) + P(\bar{A}\bar{B}\bar{A}\bar{B}A) + \cdots$$

$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)\frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \times \frac{5}{36} + \cdots \infty$$

$$= \frac{5/36}{1-(155/216)} = \underline{\underline{30/61}}.$$

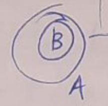
(∞ geometric series)

A ⊂ B ; P.T $P(B/A) = 1$

$$\frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} \; (\because A \subset B)$$

$$= \underline{\underline{1}}.$$

Q. If $B \subset A$, P.T. $P(B/A) \geq P(B)$



$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{P(B)}{P(A)} \leq 1$$

$$\Rightarrow P(B) \leq P(A) \leq P[B/A]$$

Q. If A & B are independent events, P.T.

$$P(A \cup B) = 1 - P(\bar{A}) \cdot P(\bar{B})$$

$$P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$$

$$P(A \cup B) = 1 - P(\bar{A} \cap \bar{B})$$

$$= 1 - P(\bar{A}) \cdot P(\bar{B}) \quad \text{(by multiplication theorem)}$$

H.P.

Q. - A, B, C - machines. (produce bolts).

- A produces twice of B & C produces same as that of B
- 2% of bolts produced by A & B are defective,
- 4% of bolts produced by C are defective.
- All bolts → 1 stock pile & 1 is chosen from the pile.

Q. Prob that it is defective?

Sol

A : event in which item has been produced by A, etc

D : event of item being defective

$P(A) = 1/2$, $P(B) = P(C) = 1/4$.

$P(D/A)$ = P(an item is defective, given A has produced it)

$P(D/A) = P(D/B) = \frac{2}{100}$. $P\left(\frac{D}{C}\right) = \frac{4}{100}$.

$$P(D) = P[(D\cap A) \cup (D\cap B) \cup (D \cap C)]$$

$$= P[D\cap A] + P[D\cap B] + P[D\cap C]$$

$$= P[A] \cdot P\left(\frac{D}{A}\right) + P(B) \cdot P\left(\frac{D}{B}\right) + P(C) \cdot P\left(\frac{D}{C}\right)$$

$$= \frac{1}{2}\left(\frac{2}{100}\right) + \frac{1}{4}\left(\frac{2}{100}\right) + \frac{1}{4}\left(\frac{4}{100}\right)$$

$$= \frac{1\cancel{0}}{4 \cdot 10\cancel{0}} = \frac{1}{40}.$$

Q. An urn contains 10 white & 3 black balls. Another urn has 3 W + 5 B balls. Two balls are drawn at random from the first urn and placed in the second urn & then 1 ball is taken randomly from the latter. What is the probability that it is a white ball?

(1) 10W 3B (2) 3W 5B

A] The two balls transferred may be both white or both black or 1 white 1 black.

Let B_1 = event of drawing 2 white balls from the first urn,

B_2 = event of drawing 2 black balls from it.

B_3 = event of drawing 1 white & 1 black ball from it.

Clearly B_1, B_2, B_3 : mutually exclusive.

- Let A be event of drawing a white ball from the second urn ~~from~~ after transfer.

$$P(B_1) = \frac{10_{C_2}}{13_{C_2}} = \frac{15}{26} \; ; \; P(B_2) = \frac{3_{C_2}}{13_{C_2}} = \frac{1}{26} \; ; \; P(B_3) = \frac{10\times 3}{13_{C_2}} = \frac{10}{26}.$$

$P(A/B_1)$ = P(drawing a white ball / 2 white balls have been transferred)

= P (drawing a white ball / urn 2 contains 5 W 5B balls)

$= 5/10$.

Similarly, $P(A/B_2) = \frac{5}{10}$. and $P(A/B_3) = \frac{4}{10}$.

$\therefore$ By theorem of probability:

$$P(A) = P(B_1)\cdot P\left(\frac{A}{B_1}\right) + P(B_2)\cdot P\left(\frac{A}{B_2}\right) + P(B_3)\cdot P\left(\frac{A}{B_3}\right)$$

$$= \frac{15}{26} \times \frac{5}{10} + \frac{1}{26} \times \frac{3}{10} + \frac{10}{26} \times \frac{4}{10}$$

$$= \frac{59}{130}.$$

Q. Find the prob. of getting at least 60 heads, when 100 fair coins are tossed.

A) Since the coins are fair $p = q = \frac{1}{2}$

$n = 100.$

$np = 50$ and $\sqrt{npq} = 5.$

Reqd. prob. $P = \sum_{r=60}^{100} {}^{100}C_r \left(\frac{1}{2}\right)^{100}$

$$= \int_{1.9}^{10.1} \phi(t)\, dt \cong \int_{1.9}^{\infty} \phi(t)\, dt$$ (By De-Moivre - Laplace approx.)

$$= 0.5 - \int_0^{1.9} \phi(t)\, dt$$

$= 0.5 - 0.4719$ (from the normal table)

$= 0.0281.$

Q. 3 true coins and 1 false coin with head on both sides. A coin is chosen at random & tossed 4 times. What is the prob. (b) that a false coin has been chosen and used?

A) $P(\text{true coin}) = 3/4$ $\quad$ $P(\text{false coin}) = 1/4$

A = event of getting 4 heads (all).

(can be true coin / false coin.

$$P(A/T) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}.$$

$P(A/F) = 1$ (sure head $\therefore$ 1).

$$P(F/A) = \frac{P(F) \cdot P(A/F)}{P(F) \cdot P(A/F) + P(T) \cdot P(A/T)} \qquad \left(\frac{P(F \cap A)}{P(A)}\right)$$

$\therefore$ by Bayes' theorem

Denom: Total probability theorem (calc for events whose prob. is obvious).

$$= \frac{\frac{1}{4} \times 1}{\frac{1}{4} \times 1 + \frac{3}{4} \times \frac{1}{16}} = \frac{1/4}{1/4 + \frac{3}{64}} = \frac{16/64}{19/64}$$

$$= \underline{\underline{\frac{16}{19}}}.$$

(a) prob. of true coin to be chosen?

$$P(T/A) = \frac{P(T) \cdot P(A/T)}{P(F) \cdot P(A/F) + P(T) P(A/T)}$$

$$= \frac{\frac{3}{4} \times \frac{1}{16}}{\frac{1}{4} \times 1 + \frac{3}{4} \times \frac{1}{16}} = \frac{\frac{3}{64}}{\frac{16}{64} + \frac{3}{64}} = \underline{\underline{\frac{3}{19}}}.$$

* denominator $\neq 1$ but it doesn't rep. prob but add^n of probs.

Reason: if denom. has an improper fraction there may be a case where LHS > 1 which is not possible.

$\therefore$ denom < 1.

Q. For a certain binary, comm. channel the prob. that a transmitted '0' is received as a '0' is 0.95 and the prob. that a transmitted '1' is received as '1' is 0.90. If prob. that a '0' is transmitted is 0.4 find the prob. that i) a '1' is recd. and ii) a '1' was transmitted given that a '1' was received.

A) Let A be the event of transmitting '1', $\bar{A}$ = event of transmitting '0'.

Let B be the event of receiving '1'. $\therefore \bar{B}$ = event of receiving '0'.

given: $P(\bar{A}) = 0.4$. $P(B/A) = 0.9$ and $P(\bar{B}/\bar{A}) = 0.95$.

$P(A) = 1-0.4 = 0.6$. $P(B/\bar{A}) = 0.05$ $\quad 1 - P\left(\frac{\bar{B}}{\bar{A}}\right)$ $\quad (1-0.95)$.

By theorem of total prob;

$$P(B) = P(A) \cdot P\left(\frac{B}{A}\right) + P(\bar{A}) \cdot P\left(\frac{B}{\bar{A}}\right)$$
$$= (0.6)(0.9) + (0.4)(0.05)$$
$$= 0.56.$$

By Bayes' theorem,

$$P(A/B) = \frac{P(A) \cdot P(B/A)}{P(B)} = \frac{0.6 \times 0.9}{0.56} = \frac{27}{28}$$

Type I conditional prob. & total prob. ~~theorem~~ data given in qn.

Type II conditional prob with p & c of independent events.

Type III an event in which complement plays a role. (well defined)

HW| Q 19