Question: For any language L, let $HALF(L) = \{w : (\exists u)[wu \in L \text{ and } |w| = |u|]\}$.
Show that if L is regular, then $HALF(L)$ is regular.

Frist method:
Suppose L is regular language accepted by the DFA $D = (Q, \Sigma, \delta, q_0, F)$.
Define, NFA $N = (Q', \Sigma, \delta', S, F')$ as follows.
$Q' = \{(q, q', q'') : q, q', q'' \in Q\} \cup \{S\}$, where S is a new starting state.
$F' = \{(q, q'', q) : q'' \in F\}$
$\delta'(S, \epsilon) = \{(q_0, q, q) : q \in Q\}$.
For $a \in \Sigma$, $q', q'', q \in Q$, $\delta'((q', q'', q), a) = \{(\delta(q', a), \delta(q'', b), q) : b \in \Sigma\}$.

Intuitively, in the above construction, the machine N being in state (q', q'', q) represents that we have guessed the middle state to be q and starting from q_0, after reading the input seen so far, the DFA D would have reached state q' and some string of the same length as the input seen so far would have taken the DFA D from "middle" state q to q''. The start state S is used only to transfer control to (q_0, q, q), where q is the guessed middle state.

To show that above works, show by induction on length of w that, for all $w \in \Sigma^*$,

$$\hat{\delta}'(S, w) = \{(\hat{\delta}(q_0, w), q'', q) : q \in Q \text{ and } (\exists u \in \Sigma^*)[|u| = |w|, \hat{\delta}(q, u) = q'']\}$$

Thus, $\hat{\delta}'(S, w) \cap F' \neq \emptyset$, iff there exists a q such that $\hat{\delta}(q_0, w) = q$ and $(\exists u \in \Sigma^*)[|u| = |w|$ and $\hat{\delta}(q, u) \in F]$, that is $w \in HALF(L)$.

Second method:
Suppose L is regular language accepted by the DFA $D = (Q, \Sigma, \delta, q_0, F)$.
Define, NFA $N = (Q', \Sigma, \delta', S, F')$ as follows.
$Q' = \{(q, q') : q, q' \in Q\} \cup \{S\}$, where S is a new starting state.
$F' = \{(q, q) : q \in Q\}$
$\delta'(S, \epsilon) = \{(q_0, q') : q' \in F\}$.
For $a \in \Sigma$, $q, q' \in Q$, $\delta'((q, q'), a) = \{(\delta(q, a), q'') : \delta(q'', b) = q', b \in \Sigma\}$.

Intuitively, in the above construction, the machine N being in state (q, q') represents that starting from q_0, after reading the input seen so far the DFA D would have reached state q and some string of the same length as the input seen so far, would have taken the DFA D from state q' to a final state.

To show that above works, show by induction on length of w that, for all $w \in \Sigma^*$,

$$\hat{\delta}'(S, w) = \{(\hat{\delta}(q_0, w), q'') : (\exists u \in \Sigma^*)[|u| = |w|, \hat{\delta}(q'', u) \in F]\}$$

Thus, $\hat{\delta}'(S, w) \cap F' \neq \emptyset$, iff there exists a $q \in Q$ such that $\hat{\delta}(q_0, w) = q$ and $(\exists u \in \Sigma^*)[|u| = |w|$ and $\hat{\delta}(q, u) \in F]$, that is $w \in HALF(L)$.