MPO 762 – Problem Set 2
Due Sept 26 2016

1 BTCS scheme

The following PDE is discretized according to the following finite difference approximation

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{c}{2}\left[\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x} + \frac{u_{j+1}^{n} - u_{j-1}^{n}}{2\Delta x}\right] \tag{1}$$

1. Derive the truncation error for this scheme and show that it is second order in space and time. Hint: make your space-time Taylor series expansion about the space-time point $(x_j, t_{n+\frac{1}{2}})$, where $t_{n+\frac{1}{2}} = \frac{t_n + t_{n+1}}{2}$.
2. Use the Von-Neumann stability analysis to derive the amplification factor and to show that the scheme is unconditionally stable.
3. How would the scheme need to be modified if for $c > 0$ the boundary condition imposed on the upstream end is changed to $u(x = 0, t) = \sin \pi t$.
4. Discuss what to do with the downstream boundary point which becomes an open boundary.

2 Fourier Analysis

We are solving the advection equation $u_t + u_x = 0$ with periodic boundary conditions on the unit interval $0 \le x \le 1$. The initial conditions are given by the superposition of 3 waves:

$$u_j^0 = \cos 2\pi x_j \;+\; 0.2 \cos \frac{2\pi M x_j}{8} \;+\; 0.1 \cos \frac{2\pi M x_j}{2} \tag{2}$$

where M is the number of cells in an eventual discrete computational grid (there should be M+1 points).

1. Identify the amplitudes a_m, wavenumbers k_m, and wavelengths λ_m (m=1,2,3) of the 3 waves.
2. Identify the number of cells per wavelength in each wave, and identify where they fall on the $k\Delta x$ scale.
3. Take a single time step using the FDA $u_j^1 = u_j^0 - \mu(u_j^0 - u_{j-1}^0)$, where μ is the Courant which is set at $\frac{1}{2}$ (assume the BC are periodic in space).
4. Plot the original and update solution u_j^0 and u_j^1, note the major differences and try to explain what you see using the amplification factor study.

5. Take the FFT of the new signal and confirm that the amplitude of the highest wavenumber wave is indeed zero.
6. Show that the number of time step for the initial condition to complete a full cycle is $n = M/\mu$.
7. Integrate until a full cycle has been completed, identify the Fourier spectrum of the end result and confirm whether the amplitude of the individual waves confirms the theoretical findings of the amplification factor.
8. The function used in homework 1 contains a number of Fourier modes. Use the discretization with 3200 cells and fft to identify the wave amplitude spectrum.

3 Background information on using FFT

The matlab function `fft` implements the following transform and its inverse:

$$\tilde{u}_n = \sum_{j=1}^{N} u_j e^{-\frac{2\pi i(j-1)(n-1)}{N}}, \text{ where } n = 1, 2, \ldots, N \tag{3}$$

$$u_j = \frac{1}{N} \sum_{n=1}^{N} \tilde{u}_n e^{\frac{2\pi i(j-1)(n-1)}{N}} \text{ where } j = 1, 2, \ldots, N \tag{4}$$

where the indeces are the appropriate ones for a vector components in matlab.

Remarks on FFT

1. Note that if you use N cells, you will have $N + 1$ edges. The last edge is a duplicate of the first edge if the signal is periodic. The FFT's are usually implemented by omitting the last edge from the computations. So if your matlab script duplicates this edge for ease of programming an advection equation, or if you use halo points for that matter, please do not forget to omit the extra portions of the array from the FFT calculation. If you are using H halo points then you should use the call `fft( u(H+1:H+N) )`.
2. Remember that the shortest wavelength in the signal occurs for a wavelength of $2\Delta x$ and hence the highest mode number is $n = N/2 + 1$
3. The FFT does not do the "proper" scaling for the amplitudes (it does it only on the ifft), so you would have to scale them by the number of samples provided:
`amps(1:N) = fft( u(1:N) ) / N`
4. Check out `http://www.mathworks.com/help/matlab/ref/fft.html` for more information.