MPO 762 – Problem Set 4

1 The SWE with a background current

The non-linear shallow water equations are linearized about a steady and uniform current of magnitude U to yield the following evolution equations for the perturbation velocity and pressure:

$$
\begin{aligned}
u_t + U u_x + g\eta_x &= 0 \qquad (1)\\
\eta_t + H u_x + U\eta_x &= 0 \qquad (2)
\end{aligned}
$$

where U is the current, g is gravity, and H the depth of the channel, all assumed constant.

- Determine the eigenvalues of the system and decide if it is still hyperbolic.
- What are the characteristic lines and the characteristic variables?
- What happens to the characteristic directions when the flow is supercritical ($U > \sqrt{gH}$)? How does the supercritical flow impact boundary conditions?

2 Numerical Experimentation

Write a code to solve the linearized 2D shallow water equation on a staggered C-grid:

$$
\begin{aligned}
\frac{\partial u_{i+\frac{1}{2},j}}{\partial t} + \delta_x \Phi_{i+\frac{1}{2},j} &= 0 \qquad (3)\\
\frac{\partial v_{i,j+\frac{1}{2}}}{\partial t} + \delta_y \Phi_{i,j+\frac{1}{2}} &= 0 \qquad (4)\\
\frac{\partial \eta_{i,j}}{\partial t} + \delta_x U_{i,j} + \delta_y V_{i,j} &= 0 \qquad (5)
\end{aligned}
$$

with the following definition for the variables Φ, U and V:

- $U_{i+\frac{1}{2},j} = \overline{H}^x_{i+\frac{1}{2},j} u_{i+\frac{1}{2},j}$ is the mass fluxing through the x-face of the cell.
- $V_{i,j+\frac{1}{2}} = \overline{H}^y_{i,j+\frac{1}{2}} v_{i,j+\frac{1}{2}}$ is the mass fluxing through the y-face of the cell.
- $\Phi_{i,j} = g\eta_{i,j}$ is the (pressure) potential

Use the RK3 or an RK4 scheme to integrate the equations in time. Use the code to solve the following problems.

2.1 Standing wave

The wave is sloshing in a **closed** rectangular basin. The initial conditions are

$$u(x,y,0) = 0,\ v(x,y,0) = 0,\ \eta(x,y,0) = \alpha \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \qquad (6)$$

Use the following parameters: Constant depth $H = 10m$; a channel size of $a = 1280m$ and b=640m; a wave amplitude of α=10cm; wave modes $m - 2$ and $n = 1$; and for simplicity set $g = 10m/s^2$. Integrate the equations until time $t = 256$ second which would allow the wave to travel twice the distance of the channel and reconstitute itself. Study the impact of the resolution by varying the grid spacing, setting Δx to 20, 10, 5 and 2 m. Make contour plots of the solution at times t=32, 64, 128, and 256 for the 2 m. resolution case. Convince yourself that the scheme is second order accurate in space by monitoring the rms errors of the velocity and pressure variables at time $t = 256$. The exact solution is

$$\begin{aligned}
u &= U \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \sin\omega t,\ U = \alpha\frac{g}{\omega}\frac{m\pi}{a} \\
v &= V \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\omega t,\ V = \alpha\frac{g}{\omega}\frac{n\pi}{a} \\
\eta &= \alpha \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} \cos\omega t \\
\omega &= c\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}, c = \sqrt{gH}.
\end{aligned}$$

Monitor the total volume of displaced water

$$V = \Delta x \Delta y \sum_i \sum_j \eta_{i,j}$$

and convince yourself that it stays constant throughout the calculations.

Calculate the following energy quantity defined on the p-points:

$$E = \Delta x \Delta y \sum_i \sum_j g\frac{\eta_{i,j}^2}{2} + H\frac{\overline{u^2}^x}{2} + H\frac{\overline{v^2}^y}{2}$$

and confirm that it also remains substantially constant throughout the calculation. Notice that the kinetic energy term involve squaring the velocity components *before* averaging in x or y.

2.2 Wave propagation in a channel of constant depth

The basin is the same as before but the initial condition is modified to be:

$$u(x, y, 0) = v(x, y, 0) = 0,\ \eta = \alpha e^{-\frac{r^2}{l^2}} \tag{7}$$

where r is the radial distance from the center of the basin and l =25m. This initial condition does not have an easy analytical solution but attempt the solution with Δx=8, 4 and 2 m. Again, report on the growth or decay of the water volume and the energy.

Repeat the above problem using a variable bathymetry of the form:

$$H = 10 + 5\tanh\frac{x - 100}{16}$$

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