Poles of a Transfer function $H(s)$.

Consider the transfer function

$$H(s) = \frac{s-a}{(s-b)(s-c)}.$$

Poles of the transfer function $H(s)$, is the values of 's' for which the function $H(s)$ approaches infinity [or denominator of $H(s)$ becomes zero).

at poles: $(s-b)(s-c) = 0$

ie either $(s-b) = 0$ or $(s-c) = 0$

so the poles are $s = b$, $s = c$.

at zeros: The function $H(s)$ becomes zero.

ie $H(s) = 0$

for that numerator of $H(s) = 0$

ie $(s-a) = 0$

$s = a$.

For example.

$$H(s) = \frac{s-1}{(s+1)(s+2)}.$$

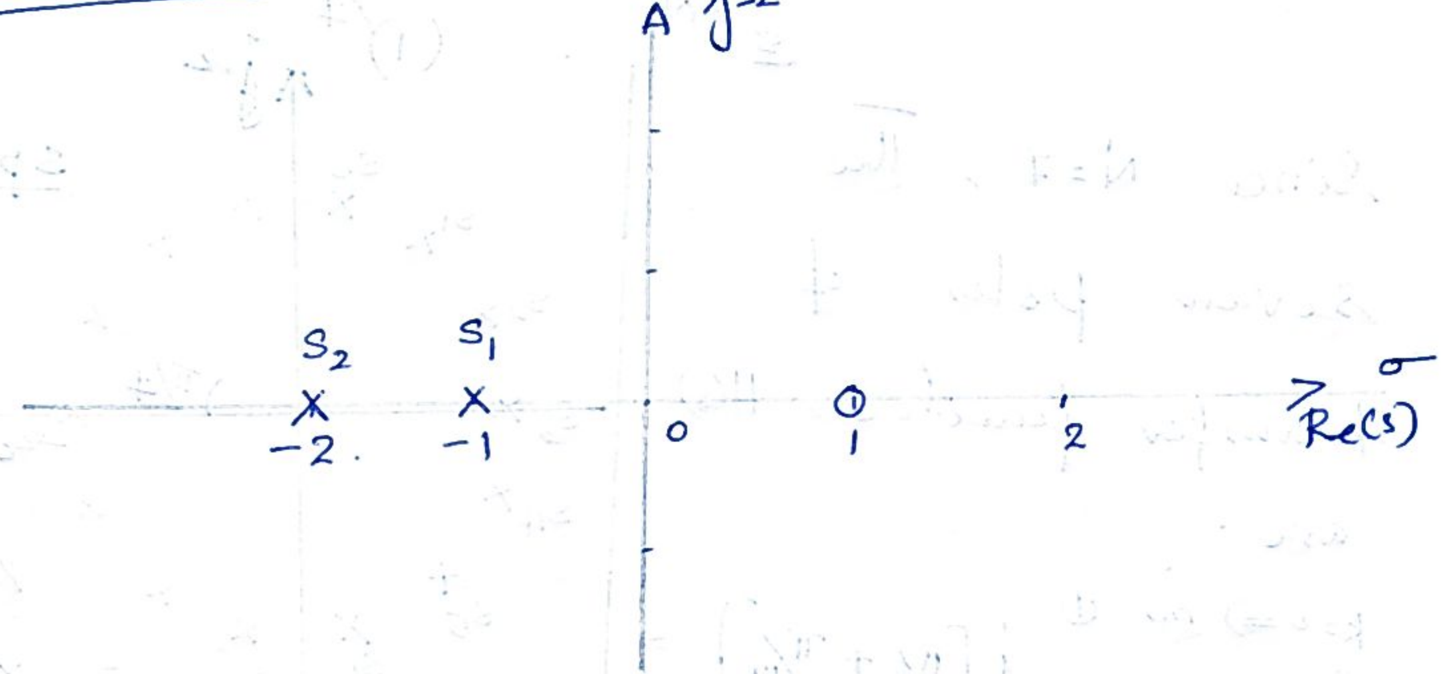
Zeros : $(s-1) = 0 \implies s = 1$. zeros denoted 'o' poles : $(s+1) = 0 \Rightarrow s = -1$

$(s+2) = 0 \Rightarrow s = -2.$

$\therefore$ $H(s)$ has two poles at $s=-1$ and $s=-2$.

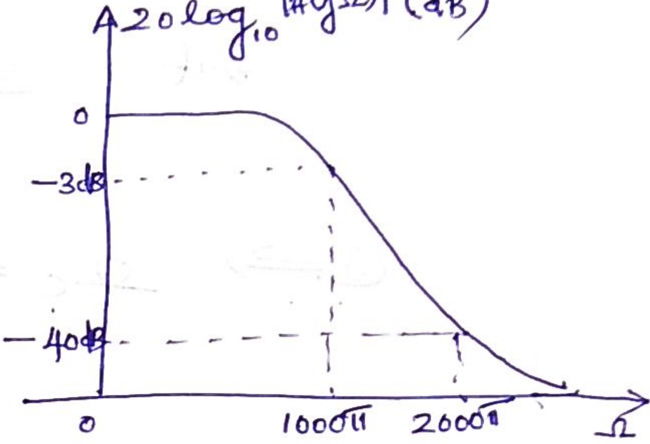
Poles are denoted as 'x'

<u>Pole zero plot of $H(s)$.</u>



Q) Determine the order and the poles of lowpass Butterworth filter that has a 3 dB attenuation at 500 Hz and an attenuation of 40 dB at 1000 Hz.

Ans.



Given $\alpha_p = 3\,dB$.

$f_p = 500\,Hz$.

$\therefore \Omega_p = 2\pi f_p$

$= 2\pi \times 500$

$= 1000\pi$ rad/sec

$\boxed{\Omega = 2\pi f}$

and $\alpha_s = 40\,dB$

$f_s = 1000\,Hz$.

$\therefore \Omega_s = 2\pi f_s$

$= 2\pi \times 1000 = 2000\pi$ rad/sec.

$\lambda = \sqrt{10^{0.1\alpha_s} - 1} =$

$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} =$

$$N \geq \frac{\log_{10}(\lambda/\varepsilon)}{\log_{10}(\Omega_s/\Omega_p)} = 6.6$$

Rounding 'N' to nearest higher value we get $N = 7$

Poles of Butterworth function is given by

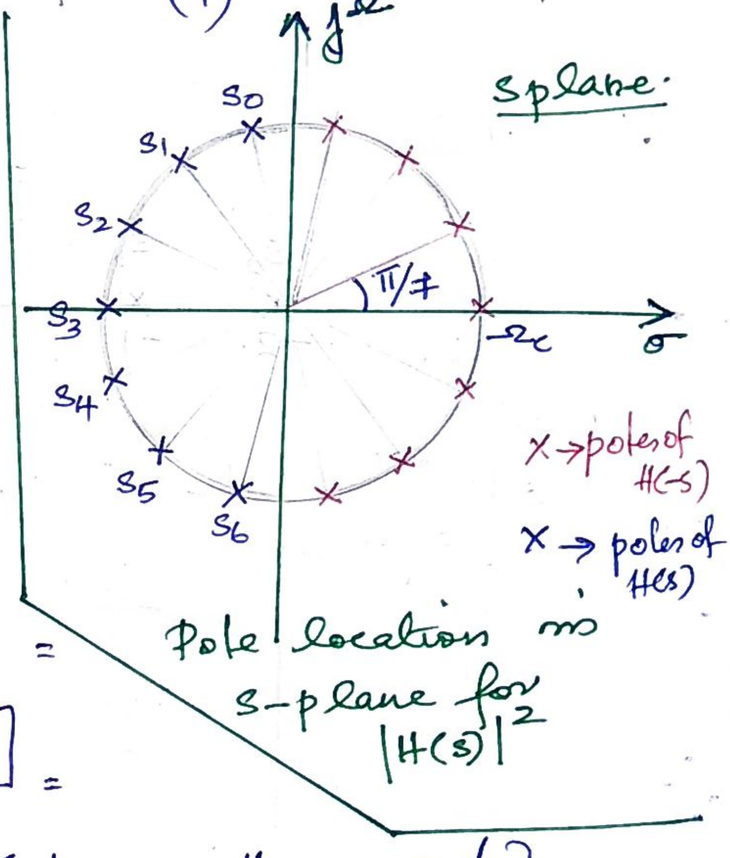
$$s_k = \Omega_c e^{j\left[\pi/2 + (2k+1)\pi/2N\right]} \quad \text{--- (1)}$$

$$k = 0, 1 \ldots\ldots N-1$$

Cutoff frequency

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{1000\pi}{(1)^{1/7}} = 1000\pi \text{ rad/sec.}$$

Since $N = 7$, the seven poles of transfer function $H(s)$ are:



$k=0 \Rightarrow$ in (1)

$$s_0 = \Omega_c e^{j\left[\pi/2 + \pi/14\right]} =$$

$$s_1 = \Omega_c e^{j\left[\pi/2 + 3\pi/14\right]} =$$

$$s_2 = \Omega_c e^{j\left[\pi/2 + 5\pi/14\right]} =$$

s_3, s_4, s_5, s_6 (do as Homework).

Then the transfer function

$$H(s) = \frac{1}{(s-s_0)(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)(s-s_6)}$$

[∵ Butterworth polynomial is all pole transfer function]

[For a stable transfer function $H(s)$ poles are on left half of s-plane]

The following table 5.1 gives Butterworth polynomials for various values of N for $\Omega_c = 1$ rad/sec.

Table 5.1 List of Butterworth Polynomials

N	Denominator of $H_N(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.8477s+1)$
5	$(s+1)(s^2+0.61803s+1)(s^2+1.61803s+1)$
6	$(s^2+1.931855s+1)(s^2+\sqrt{2}s+1)(s^2+0.51764s+1)$
7	$(s+1)(s^2+1.80194s+1)(s^2+1.247s+1)(s^2+0.445s+1)$

The Eq. (5.12) gives us the pole locations of Butterworth filter for $\Omega_c = 1$ rad/sec and are known as normalized poles. In general, the unnormalized poles are given by

$$s_k{}' = \Omega_c s_k \tag{5.14}$$

The transfer function of such type of Butterworth filter can be obtained by substituting $s \rightarrow s/\Omega_c$ in the transfer function of Butterworth filter.

5.6 Steps to design an analog Butterworth lowpass filter

1. From the given specifications find the order of the filter N.
2. Round off it to the next higher integer.
3. Find the transfer function $H(s)$ for $\Omega_c = 1$ rad/sec for the value of N.
4. Calculate the value of cutoff frequency Ω_c.
5. Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting $s \to \frac{s}{\Omega_c}$ in $H(s)$.

Example 5.4 Design an analog Butterworth filter that has a $-$ 2 dB passband attenuation at a frequency of 20 rad/sec and atleast $-$ 10 dB stopband attenuation at 30 rad/sec.

Solution

Given
$$\alpha_p = 2\,\text{dB}; \quad \Omega_p = 20\,\text{rad/sec}$$
$$\alpha_s = 10\,\text{dB}; \quad \Omega_s = 30\,\text{rad/sec}$$

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\dfrac{10-1}{10^{0.2}-1}}}{\log \dfrac{30}{20}}$$

$$\geq 3.37$$

Rounding off N to the next highest integer we get

$$N = 4$$

The normalized lowpass Butterworth filter for $N = 4$ can be found from table 5.1 as

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)}$$

From Eq. (5.31) we have

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{20}{(10^{0.2} - 1)^{1/8}} = 21.3868$$

Note:

To find the cutoff frequency Ω_c either (5.31) or (5.32a) can be used. The Eq. (5.31) satisfies passband specification at Ω_p, while the stopband specification at Ω_s is exceeded. The Eq. (5.32a) satisfies the stopband specification Ω_s, while the passband specification at Ω_p is exceeded. All the examples in this chapter are solved using Eq. (5.31). Students are advised to solve the exercise problems using Eq. (5.32a).

The transfer function for $\Omega_c = 21.3868$ can be obtained by substituting

$$s \rightarrow \frac{s}{21.3868} \text{ in } H(s)$$

i.e.,
$$H(s) = \frac{1}{\left(\dfrac{s}{21.3868}\right)^2 + 0.76537\left(\dfrac{s}{21.3868}\right) + 1} \cdot \frac{1}{\left(\dfrac{s}{21.3868}\right)^2 + 1.8477\left(\dfrac{s}{21.3868}\right) + 1}$$

$$= \frac{0.20921 \times 10^6}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)}$$

T.F