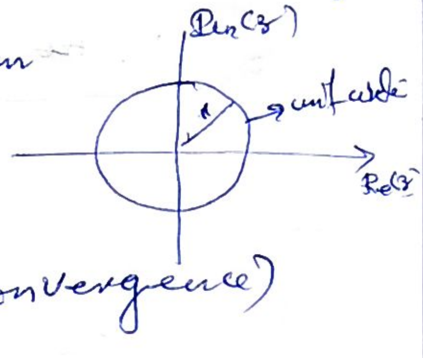
Relationship of DFT to other transforms:

— DFT is an important computational tool for performing frequency analysis of signals on digital signal processors.

① Relationship of DFT to z-transform $\quad z = re^{j\omega}$

Let $x(n)$ has a z-transform

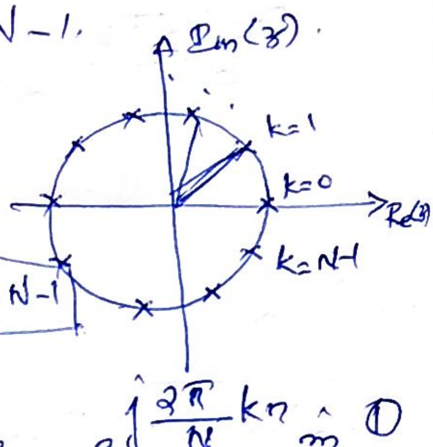


$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad \text{—— ①}$$

with an ROC (Region of convergence) that include unit circle.

— If $X(z)$ is sampled at the N equally spaced points on the unit circle.

$$z_k = e^{j\frac{2\pi k}{N}} \quad ; \; k = 0, 1, \ldots N-1.$$



ie
$$\boxed{X(k) \equiv X(z)\Big|_{z = e^{j\frac{2\pi}{N}kn}} \quad k=0,1\ldots N-1} \quad \text{—— ②}$$

substituting ② in ① ie put $z = e^{j\frac{2\pi}{N}kn}$ in ①

$$X(k) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}nk} \quad \text{—— ③}$$

If $X(k)$ is DFT of $x(n)$

then we have.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} \quad \text{---} (4)$$

~~substituting (4) in (1)~~ To express $X(z)$ as a function of $X(k)$

~~$X(z) = \sum_{n=0}$~~

If we are considering finite duration sequence $x(n)$ of length N

$$\text{then} \quad X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n} \quad \text{---} (5)$$

substituting (4) in (5).

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn} \right] z^{-n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1} \left(e^{j2\pi k/N} z^{-1} \right)^{n} \quad \text{---} (6)$$

Consider term $\sum_{n=0}^{N-1} \left(e^{j\frac{2\pi k}{N}} z^{-1} \right)^{n}$

- this represents sum of finite GP with N terms and first term 1,
  common ratio $e^{j\frac{2\pi k}{N}} z^{-1}$.

- sum of finite GP $= \frac{a(1-r^{n})}{1-r}$

$a \rightarrow$ first term

$r \rightarrow$ common ratio

$n \rightarrow$ no. of terms

$\therefore$ in this cas

$$\sum_{n=0}^{N-1}\left(e^{j\frac{2\pi k}{N}}z^{-1}\right)^n = 1\left[\frac{1-\left(e^{j\frac{2\pi k}{N}}z^{-1}\right)^N}{1-\left(e^{j\frac{2\pi k}{N}}z^{-1}\right)}\right]$$

$$= \left[\frac{1-e^{j2\pi k}z^{-N}}{1-\left(e^{j\frac{2\pi}{N}k}\cdot z^{-1}\right)}\right] \quad \left(\because e^{j2\pi k}=1\right)$$

$$= \frac{1-z^{-N}}{\left(1-e^{j\frac{2\pi}{N}k}z^{-1}\right)} \quad \text{—— (7)}$$

substituting (7) in (6)

$$X(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left[\frac{1-z^{-N}}{1-e^{j\frac{2\pi}{N}k}z^{-1}}\right]$$

$$\boxed{X(z) = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{\left[1-e^{j\frac{2\pi k}{N}}z^{-1}\right]}} \quad \text{—— (8)}$$

(2) Relationship of DFT to Fourier transform

$X(z)$ when evaluated on unit circle

i.e., we get fourier transform $X(\omega)$

$$\cancel{X(\omega)} \quad X(\omega) = X(z)\Big|_{z=e^{j\omega}}$$

substituting $z = e^{j\omega}$ in eqn (8).

$$X(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{1 - e^{j\frac{2\pi k}{N}} e^{-j\omega}}$$

$$\therefore \boxed{X(\omega) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{X(k)}{\left[1 - e^{-j(\omega - 2\pi k/N)}\right]}}$$