Spectrum of FM

In order to explain the spectrum we have to know about Bessel function. Bessel function is a standard mathematical equation which is related to many propagations in reallife.

Bessel function is given by

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(x\sin\theta - n\theta)}\, d\theta.$$

Properties of Bessel function

the value of Bessel function $J_n(x)$ decreases as the value of n increases (except some exceptional cases)

$$J_0(x) > J_1(x) > J_2(x) \ldots\ldots$$

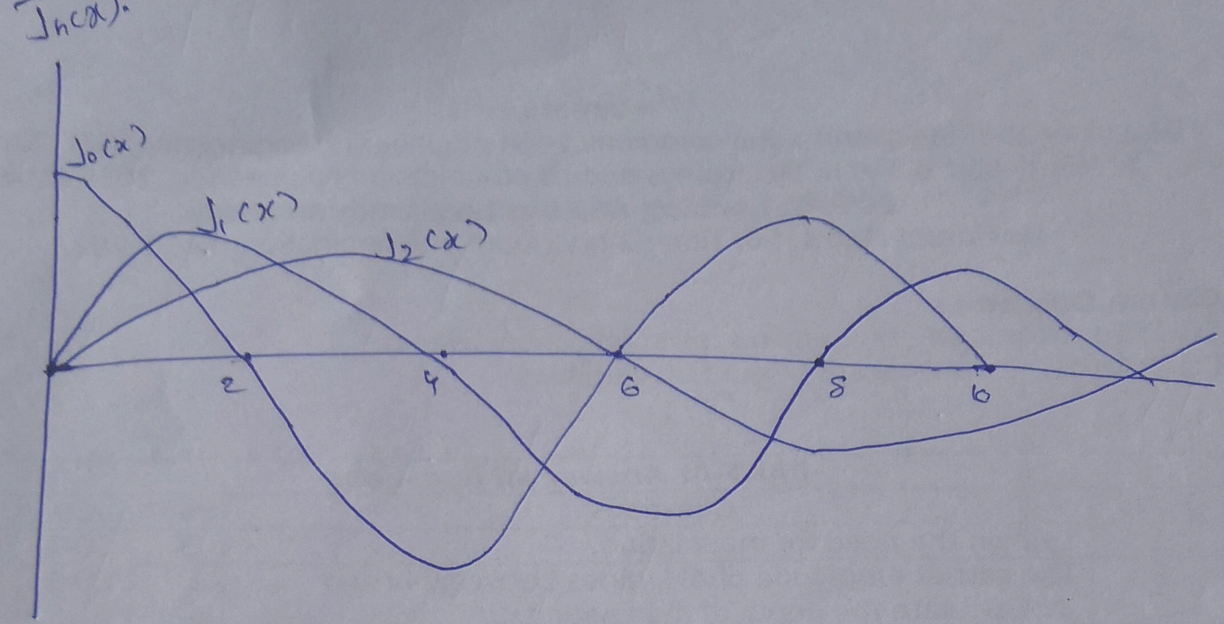
(2) $J_{-n}(x) = (-1)^n J_n(x)$

$$J_{-n}(x) = J_n(x) \quad \text{when} \quad n = \text{even}$$

$$J_{-n}(x) = -J_n(x) \quad \text{when} \quad n = \text{odd.}$$

(3). $\sum_{n=-\infty}^{\infty} J_n^2(x) = 1$

(4) $J_n(x)$ always results in real quantity.



We know that the equation for a single tone FM wave for any value of β is

$$S_{FM}(t) = A_c \cos\left[2\pi f_c t + \beta \sin 2\pi f_m t\right]$$

we can write $\cos\theta = Re\left[e^{j\theta}\right]$ $\because e^{j\theta} = \cos\theta + j\sin\theta$.

$\therefore$

$$S_{FM}(t) = A_c\, Re\left[e^{j 2\pi f_c t + \beta \sin 2\pi f_m t}\right]$$

$$= A_c\, Re\left[e^{j 2\pi f_c t} \cdot e^{j\beta \sin 2\pi f_m t}\right]$$

when we write; the real part of some signal multiplied by the $e^{j2\pi f_c t}$; this becomes a complex base band equivalent signal.

i.e., from the above equation

$$Re\left[A_c\, e^{j\beta \sin 2\pi f_m t} \times e^{j2\pi f_c t}\right]$$

The passband signal $x_p(t)$ is equal to the real part of complex baseband signal multiplied by $e^{j2\pi f_c t}$

$$x_p(t) = Re\left\{ x_b(t)\, e^{j2\pi f_c t} \right\}$$

In the equation of $S_{Fm}(t)$ above

the term $Re\left[A_c\, e^{j\beta \sin 2\pi f_m t}\right]$ is the complex base band equivalent signal and it can be denoted by $\tilde{S}(t)$ ~~and~~ It can retain complete information of the fm wave.

$$S_{Fm}(t) = Re\left[\tilde{S}(t)\, e^{j2\pi f_c t}\right]$$

This complex envelop is a periodic function of time with fundamental frequency f_m

and $f_m = \frac{1}{T}$

As it is a periodic s/l it's spectrum can be obtained by taking the discrete fourier series

we have to find out the base band spectrum because passband spectrum is simply a translation of it.

∴ spectrum of $\tilde{s}(t)$ is given by $\left[\tilde{s}(t) = A_c e^{j\beta \sin 2\pi f_m t} \right]$

$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} C_n e^{j 2\pi n f_m t}$. It contains infinite no: of harmonics and its fundamental frequency is f_m (As ~~n~~ can take values from $-\infty$ to ∞)

Now we have to find the coefficients

the n^{th} coefficient of discrete source is given by

$$C_n = \frac{1}{T} \int_0^T \tilde{s}(t) e^{-j 2\pi n f_m t}$$

$$= \frac{1}{T} \int_0^T A_c e^{j\beta \sin 2\pi f_m t} e^{-j 2\pi f_m t} dt$$

$$= \frac{1}{T} \int_0^T e^{j(\beta \sin 2\pi f_m t - 2\pi f_m t)} dt$$

Assume $2\pi f_m t = x$

$$dt = \frac{dx}{2\pi f_m}$$ integration limit changes to 0 to 2π

$$C_n = \frac{1}{T} \int_0^{2\pi} A_c e^{j\beta \sin x - nx} \frac{dx}{2\pi f_m}$$

W.k. that $T = \frac{1}{f_m}$; $T f_m = 1$

$$C_n = A_c \cdot \underbrace{\frac{1}{2\pi}\int_0^{2\pi} e^{i(\beta \sin x - nx)}\, dx}_{\text{Bessel function}}$$

$C_n = A_c J_n(\beta) \rightarrow$ n^{th} coefficient of the discrete fourier series of the complex baseband signal

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_m t}$$

$$\boxed{\tilde{s}(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi n f_m t}}$$

From this we can derive the passband signal

Pass band s/l

$$S_{FM}(t) = Re\left\{ \tilde{s}(t) e^{j2\pi f_c t} \right\}$$

$$= Re\left\{ \sum_{n=-\infty} A_c J_n(\beta) e^{j2\pi n f_m t} \times e^{j2\pi f_c t} \right\}$$

$$= Re\left\{ \sum_{n=-\infty}^{\infty} A_c J_n(\beta) e^{j2\pi (f_c + n f_m) t} \right\}$$

$$\boxed{s(t) = \sum_{n=\infty}^{\infty} A_c J_n(\beta) \cos(2\pi f_c + n f_m) t}$$

This is the pass band s/l.

We know that fourier transform of $\cos 2\pi f_0 t = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$

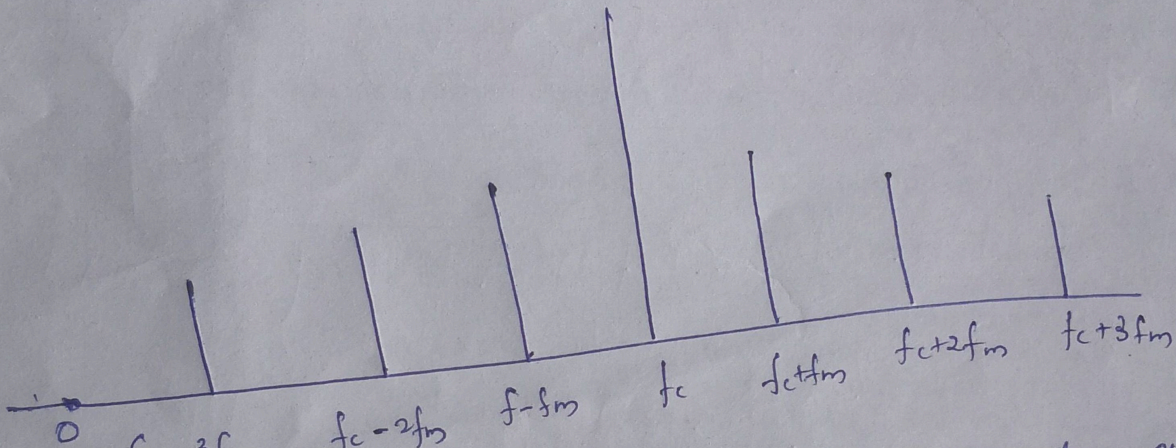
Therefore in the equation of $s(t)$ each $\cos(2\pi f_c + n f_m) t$ is an impulse; if we take the Spectrum of that

∴ We will get impulses of amplitude $\frac{1}{2}$ at $f_c + n f_m$ and impulses of amplitude $\frac{1}{2}$ at $f_c - n f_m$

$$\therefore \quad s(t) = \sum_{n=-\infty}^{\infty} A_c J_n(\beta) \cos(2\pi f_c + n f_m) t$$

Spectrum is given by

$$\boxed{S(f) = \frac{A_c}{2} \sum_{n=-\infty}^{\infty} J_n(\beta) \left[\delta(f - f_c - n f_m) + \delta(f + f_c + n f_m) \right]}$$



It contains carrier frequency component and infinite no: of USB and LSB

Actual Bandwidth of WBFM is infinite

For WBFM strength of higher order sidebands goes decreasing and finally becomes zero.

So for WBFM lower order sidebands are said to be significant and higher order sidebands are insignificant