
KTUNOTES

Integrating bothsides w.r.t. s,

from $s \to \infty$,

$$\int_s^\infty X(s)\,ds = \int_s^\infty \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt\cdot ds$$

$$= \int_{-\infty}^{\infty} x(t) \left[\int_s^\infty e^{-st}\,ds\right] dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot \left[e^{-st}/-t\right]_s^\infty dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[e^{-st}/t\right] dt$$

$$\int_s^\infty X(s)\,ds = \int_s^\infty \left[\frac{x(t)}{t}\right] e^{st}\,dt = LT\left[\frac{x(t)}{t}\right]$$

$$\therefore \frac{x(t)}{s} \xleftrightarrow{LT} \int_s^\infty X(s)\,ds$$

# MODULE - IV

## ANALYSIS OF LTI SYSTEM

(Continuous time)

### * DIFFERENTIAL EQUATION:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \quad \dots (1)$$

CT → differential eqn $dy(t)/dt$

DT → difference eqn $y[n]-y[n-1]$

When $N = M = 1$

$$\sum_{k=0}^{1} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{1} b_k \frac{d^k x(t)}{dt^k}$$

$x(t) \to \boxed{h(t)} \to y(t)$

$x[n] \to \boxed{h[n]} \to y[n]$

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} = b_0 x(t) + b_1 \frac{dx(t)}{dt}$$

### • FREQUENCY RESPONSE:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{FT\{o/p\}}{FT\{i/p\}}$$

transfer fn: $H(s) = \frac{Y(s)}{X(s)} = \frac{L\{o/p\}}{L\{i/p\}}$ with all initial conditions 0

The frequency response of (1) is given by,

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$

(by fourier transforming on bothsides).

Now, $Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right]$

Now,

$$H(j\omega) = Y(j\omega)/X(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$ → frequency response

$H(j\omega) = FT\{h(t)\}$.

? Consider a stable LTI system characterised by the differential equation

$$\frac{dy(t)}{dt} + a\,y(t) = x(t).$$

Calculate the impulse response? And frequency response.

Ans. We've frequency response,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Also, $\frac{dy(t)}{dt} + a\,y(t) = x(t)$ .... (a)

FT of (a) is,

$$j\omega\, Y(j\omega) + a\,Y(j\omega) = X(j\omega)$$

$$Y(j\omega)\,[j\omega + a] = X(j\omega)$$

$$\therefore H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a+j\omega}$$



Now, $h(t) \xleftrightarrow{FT} H(j\omega)$

We know that, $h(t) = IFT\{H(j\omega)\}$

$$\therefore h(t) = e^{-at}\,u(t)$$

? Consider a stable LTI system characterised by the differential eqn,

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

Calculate freq. response and the impulse response.

Ans. We've $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$ (1)

Now, FT of (1),

$$(j\omega)^2 Y(j\omega) + 4Y(j\omega)j\omega + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega)$$

$$Y(j\omega)\left[(j\omega)^2 + 4j\omega + 3\right] = X(j\omega)\left[2 + j\omega\right]$$

Now, $H(j\omega) = Y(j\omega)/X(j\omega) = \frac{2+j\omega}{4j\omega + 3 + (j\omega)^2}$

$$= \frac{2+j\omega}{(j\omega)^2 + 4j\omega + 3} = \frac{j\omega + 2}{(j\omega+1)(j\omega+3)}$$

Now, $h(t) = \cancel{\frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega) e^{-j\omega t} d\omega}$

$$= \frac{A}{(1+j\omega)} + \frac{B}{j\omega+3}$$

$A|_{j\omega=-1} = \frac{1}{2}$ and $B|_{j\omega=-3} = \frac{1}{2}$

$$\therefore H(j\omega) = \frac{1/2}{(1+j\omega)} + \frac{1/2}{(j\omega+3)}$$

$$\therefore h(t) = \frac{1}{2} e^{-t}u(t) + \frac{1}{2} e^{-3t}u(t)$$

? For the above Example, find the output of the system for the input $x(t) = e^{-4t}u(t)$

Ans. $\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{d\,x(t)}{dt} + 2x(t)$

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

$$x(t) = e^{-4t}u(t).$$

Now, we've to find $y(t)$

$y(t) \xleftrightarrow{FT} Y(j\omega)$

$\therefore y(t) = IFT\{Y(j\omega)\}$

$Y(j\omega) = X(j\omega) \cdot H(j\omega)$ (property)

Now,

$$Y(j\omega) = \frac{1}{4+j\omega} \cdot \frac{j\omega+2}{(j\omega+1)(j\omega+3)}$$

$$= \frac{A}{4+j\omega} + \frac{B}{1+j\omega} + \frac{C}{3+j\omega}$$

$$\therefore A|_{j\omega=-4} = \frac{-4+2}{(-4+1)(-4+3)} = \frac{-2}{-3 \cdot -1} = \frac{-2}{3} = -\frac{2}{3}$$

$$B|_{j\omega=-1} = \frac{-1+2}{(4-1)(-1+3)} = \frac{1}{3 \cdot 2} = \frac{1}{6}$$

$$C|_{j\omega=-3} = \frac{-3+2}{(4-3)(-3+1)} = \frac{-1}{1 \cdot -2} = \frac{1}{2}$$

$$\therefore Y(j\omega) = \frac{-2/3}{4+j\omega} + \frac{1/6}{1+j\omega} + \frac{1/2}{3+j\omega}$$

Now, $y(t) = -\frac{2}{3} e^{-4t}u(t) + \frac{1}{6} e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$

★ General Eqns :

$$\left.\begin{array}{l} \frac{d}{dt}x(t) \xleftrightarrow{FT} j\omega X(j\omega) \\ \frac{d^2}{dt^2}x(t) \xleftrightarrow{FT} (j\omega)^2 X(j\omega) \\ \frac{d^k}{dt^k}x(t) \xleftrightarrow{FT} (j\omega)^k X(j\omega) \end{array}\right\} FT$$

$\frac{d}{dt}x(t) \xleftrightarrow{BLT} s\,X(s)$

$\frac{d^2}{dt^2}x(t) \xleftrightarrow{BLT} s^2 X(s)$ } BLT

$\frac{d^K}{dt^K}x(t) \xleftrightarrow{BLT} s^K X(s)$

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$\frac{d}{dt}x(t) \xleftrightarrow{ULT} sX(s) - x(0^-)$

$\frac{d^2}{dt^2}x(t) \xleftrightarrow{ULT} s^2 X(s) - s\,x(0^-) - x'(0^-)$

$\frac{d^3}{dt^3}x(t) \xleftrightarrow{ULT} s^3 X(s) - s^2 x(0^-) - s\,x'(0^-) - x''(0^-)$

$\frac{d^K}{dt^K}x(t) \xleftrightarrow{ULT} s^K X(s) - s^{K-1}x(0^-) - s^{K-2}x'(0^-) \ldots\ldots\ldots x^{K-1}(0^-)$.

? Consider a second order system whose diff eqn is given by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 2\frac{dx(t)}{dt} - 3x(t)$$

(i) Determine transfer fn?.

(ii) Impulse response

(iii) Output $y(t)$ for the i/p $x(t) = e^{-3t}u(t)$.

(iv) O/p $y(t)$ for $x(t) = \delta(t)$

(v) O/p $y(t)$ for i/p $x(t) = u(t)$.

Ans We've $\frac{d^2y(t)}{dt^2} + \frac{3\,dy(t)}{dt} + 2y(t) = 2\frac{dx(t)}{dt}$

(i) LT on both sides, we get,

$s^2 \cancel{x} Y(s) + 3sY(s) + 2Y(s) = 2sX(s) - 3X(s)$

$Y(s)[s^2 + 3s + 2] = X(s)[2s - 3]$.

$H(s) = \frac{Y(s)}{X(s)} = \frac{2s-3}{s^2+3s+2}$ = Transfer function

(ii) Now, $h(t) = ILT\{H(s)\}$

$\frac{2s-3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$

$A|_{s=-1} = \frac{-2-3}{-1+2} = -\frac{5}{1} = -5$

$B|_{s=-2} = \frac{-4-3}{-1} = \frac{-7}{-1} = 7$

$\therefore H(s) = \frac{-5}{s+1} + \frac{7}{s+2}$

$\therefore h(t) = -5e^{-t}u(t) + 7e^{-2t}u(t)$

(iii) Now, $x(t) = e^{-3t}u(t)$

$y(t) = x(t) * h(t)$

or $Y(s) = X(s) \cdot H(s)$

$$Y(s) = \frac{1}{s+3} \cdot \frac{2s-3}{(s+1)(s+2)}$$

$$= \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A\Big|_{s=-3} = \frac{-6-3}{(-3+1)(-3+2)} = \frac{-9}{-2 \cdot -1}$$

$$= -9/2 = -9/2$$

$$B\Big|_{s=-1} = \frac{-2-3}{(-1+3)\cdot(-1+2)} = \frac{-5}{2\cdot 1} = -5/2$$

$$C\Big|_{s=-2} = \frac{-4-3}{(-2+1)\cdot(-2+3)} = \frac{-7}{-1\cdot 1} = +7$$

$$\therefore Y(s) = \frac{-9/2}{s+3} + \frac{-5/2}{s+1} + \frac{7}{s+2}$$

$$\therefore y(t) = -9/2\, e^{-3t} u(t) - 5/2\, e^{-t} u(t) + 7 e^{-2t} u(t)$$

(iv) $y(t) = ?$

$x(t) = \delta(t)$

Now, $X(s) = \mathcal{L}\{x(t)\}$

$= \mathcal{L}\{\delta(t)\} = 1.$

$\mathcal{L}\{\delta(t)\} = 1$

$\int_{-\infty}^{\infty} \delta(t) e^{-st} dt$

$= 1$

$(\because \int x(t)\delta(t) = x(0)$

$= e^{0} = 1)$

Now, $H(s) = Y(s)/X(s) = \frac{2s-3}{s^2+3s+2}$

$$Y(s) = \frac{2s-3}{s^2+3s+2}$$

$$\therefore Y(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$Y(s) = \frac{-5}{s+1} + \frac{7}{s+2}$$

$$\therefore y(t) = -5e^{-t}u(t) + 7e^{-2t}u(t) = h(t).$$

(v) $x(t) = u(t)$

$\therefore X(s) = 1/s$

$$Y(s) = \left[\frac{2s-3}{(s^2+3s+2)}\right] \times \frac{1}{s}$$

$$= \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A\Big|_{s=0} = -3/2$$

$$B\Big|_{s=-1} = \frac{-5}{-1} = 5$$

$$C\Big|_{s=-2} = \frac{-7}{-2} = 7/2$$

$$\therefore Y(s) = \frac{-3/2}{s} + \frac{5}{s+1} + \frac{7/2}{s+2}$$

$$\therefore y(t) = -3/2\, u(t) + 5e^{-t}u(t) + 7/2\, e^{-2t}u(t)$$

$$= u(t)\left[-3/2 + 5e^{-t} + 7/2\, e^{-2t}\right]$$

? Solve the 2nd order linear differential Equation $\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 8y(t) = 2x(t)$ for initial condition $y(0^-) = 1$ & $y'(0^-) = -2$. ~~Determine~~ the transfer fn and impulse response, o/p $y(t)$ for the i/p $x(t) = u(t)$. o/p $y(t)$ for i/p $x(t) = e^{-3t}u(t)$.

Ans. $H(s) = Y(s)/X(s)$

(a) Taking LT on both sides we get,

$$s^2 Y(s) - s\,y(0^-) - y'(0^-) + 6\,s\,Y(s) - 6y(0^-) + 8\,Y(s) = 2\,X(s)$$

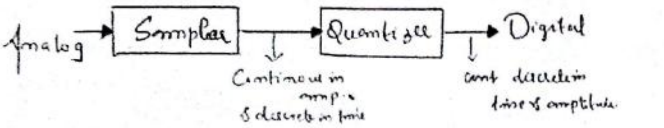
$$Y(s)[s^2 + 8 + 6s] - s\times 1 - -2 - 6 = 2\,X(s)$$

$$Y(s)[s^2+6s+8] - s - 4 = 2\,X(s)$$

# ★ SAMPLING:



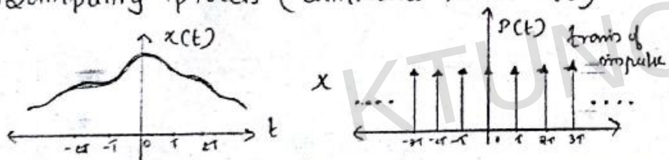
Analog Signal (Continous in time & amp) $\xrightarrow{\text{Sampling}}$ Digital Signal

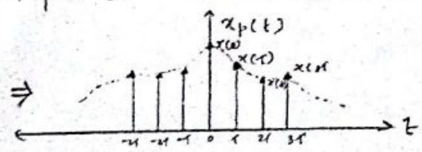



→ There are two blocks Sampler and Quantiser which helps in the conversion of analog to digital signal.

## ⊙ Sampling process (Continous to discrete):

TIME DOMAIN




$x(t) \cdot P(t) \Rightarrow x_p(t)$

(simple Sampling process).

train of impulse can be expressed as,

$$P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



$$= \cdots + \delta(t+3T) + \delta(t+2T) + \delta(t+T) + \delta(t) + \delta(t-T) + \delta(t-2T) + \delta(t-3T) + \cdots$$

Now, $x_p(t) = x(t) \cdot P(t)$

$$= x(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{k=-\infty}^{\infty} x(t) \cdot \delta(t - nT)$$

$$x_p(t) = \sum_{k=-\infty}^{\infty} x(nT) \cdot \delta(t - nT)$$

Since, $x(t)\, \delta(t - t_0) = x(t_0)\, \delta(t - t_0)$

## ⊙ FREQUENCY DOMAIN:

$x(t) \longleftrightarrow X(j\omega)$

$\cancel{x_p(t)}\ \cdot P(t) \longleftrightarrow 2\pi/T \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) = P(j\omega)$

$(t - nT)$

$x_p(t) \longleftrightarrow X_p(j\omega)$

$x_p(t) = x(t) \cdot P(t)$.

$$X_p(j\omega) = \frac{1}{2\pi} \left[ X(j\omega) * P(j\omega) \right]$$

$$= \frac{1}{2\pi} \left[ X(j\omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \right]$$

We've $x(t) * \delta(t - t_0) = x(t - t_0)$

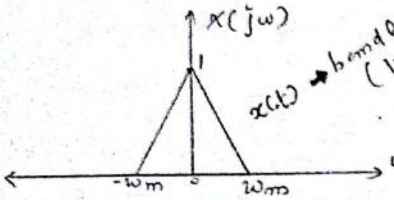
$x(t) \cdot \delta(t - t_0) = x(t_0)\, \delta(t - t_0)$

$$\int_{-\infty}^{\infty} x(t)\cdot \delta(t-t_0)\,dt = x(t_0)$$



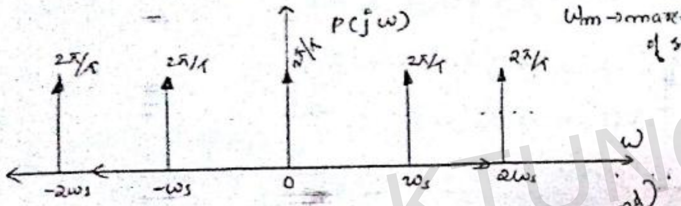
By first property,

$$\boxed{X_p(j\omega) = \frac{1}{T}\left[\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))\right]}$$

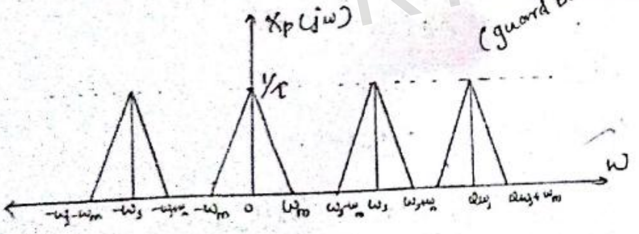



$\omega_s \rightarrow$ Sampling frequency

$\omega_m \rightarrow$ max. freq of signal $x(t)$




(guard band)




(figure 1)

Sample signal Spectrum



## * Sampling Theorem :

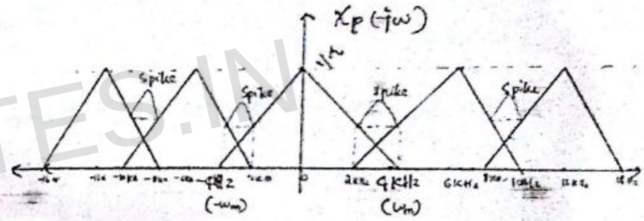
→ For the conversion of discrete signal to continuous signal, we have to obey certain conditions

(i) $x(t)$ should be bandlimited

(ii) $\omega_s \geq 2\omega_m$ (Sampling frequency should be greater than or equal to the max. frequency of the signal.)

→ Now, if $\omega_s < 2\omega_m$

• Let's assume $\omega_m = 4KHz$,
$\omega_s = 6KHz$.

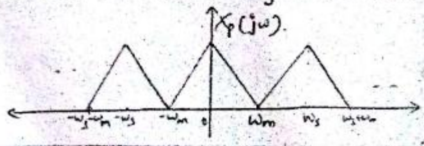



Here, overlapping occurs. This condition is called Aliasing.

→ If $\omega_s > 2\omega_m$, then we get spectrum like figure 1. This condition is called Guard band.

→ If $\omega_s = 2\omega_m$

There is no guard band.



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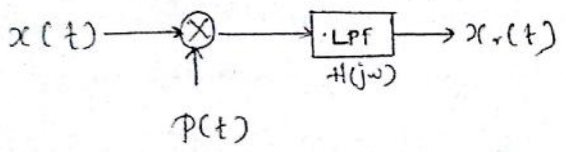
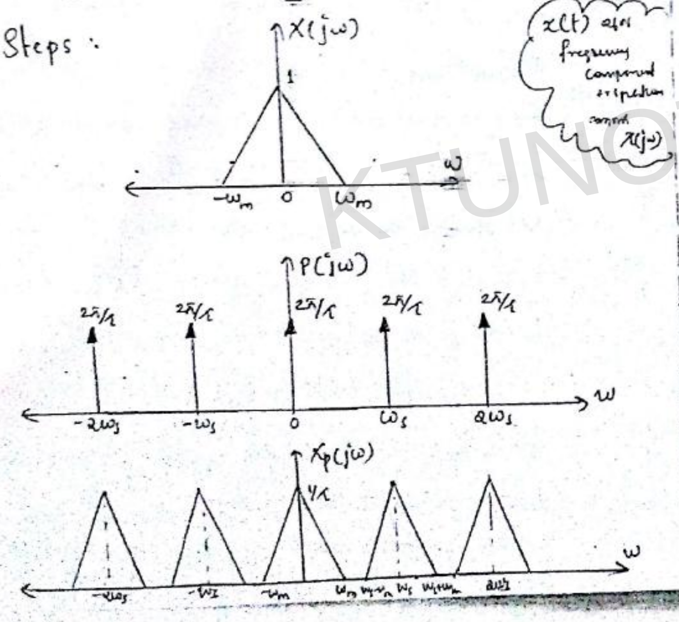
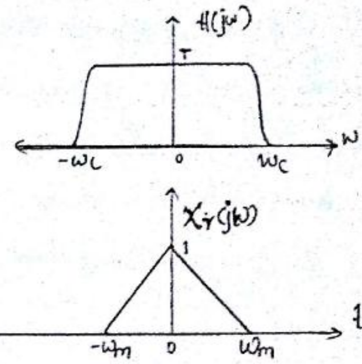
* Inorder to recover the original signal $x(t)$ from the sampled signal $x_p(t)$, is




We need to pass the sampled signal $x_p(t)$ through a low pass filter, whose frequency response is given by $H(j\omega)$ and cut off frequency $\omega_m < \omega_c < \omega_s - \omega_m$.

Steps :




## * MAGNITUDE AND PHASE RESPONSE :

For a continuous LTI system, we have,

$$y(t) = x(t) * h(t)$$

$$Y(j\omega) = X(j\omega) \cdot H(j\omega)$$

o/p response

$$\therefore H(j\omega) = Y(j\omega) / X(j\omega)$$

ie, $|Y(j\omega)|\, e^{j\theta_Y(j\omega)} = |X(j\omega)|\, e^{j\theta_X(j\omega)} \cdot |H(j\omega)|\, e^{j\theta_H(j\omega)}$

By comparing,

$$|Y(j\omega)| = |X(j\omega)| \cdot |H(j\omega)|$$

$$\theta_Y(j\omega) = \theta_H(j\omega) + \theta_X(j\omega)$$

→ Magnitude response of o/p is the ~~sum~~ prod[t] of the magnitude of response of frequency response and input.

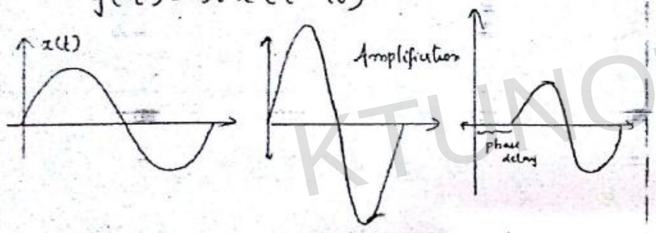
→ phase of output is the ~~phase~~ sum of the phase response of frequency response & input response.

$$\text{ie, } \theta_Y(j\omega) = \theta_H(j\omega) + \theta_X(j\omega)$$

DISTORTIONLESS TRANSMISSION:

Condition for distortionless transmission is

$$y(t) = K\,x(t - t_0).$$




In these cases (amplification & phase delay), there is no distortion. Distortion happens only when the shape changes.

$$\text{ie, } y(t) = K\,x(t - t_0)$$

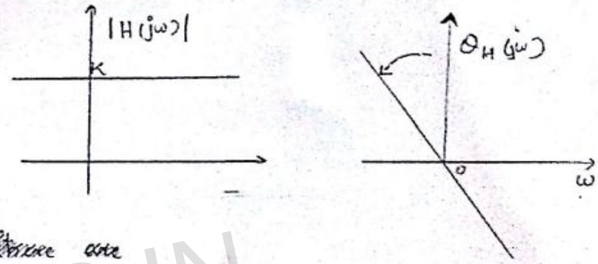
By fourier transforming,

$$Y(j\omega) = K\,e^{-j\omega t_0}\,X(j\omega)$$

$$H(j\omega) = K\,e^{-j\omega t_0} \qquad (A\,e^{j\theta})$$

⇒ • Amplitude $|H(j\omega)|$ = K (constant)
• phase = $-\omega t_0$ (linear)

Condition for distortionless transmission




Note

→ For distortionless transmission through an LTI system, we require the exact input signal shape to be reproduced at the output although its amplitude may be different and it may be delayed in time.

$$y(t) = K\,x(t - t_0).$$

→ There are two types of distortions.

(i) Amplitude distortion:

when amplitude spectrum $|H(j\omega)|$ of the system is not constant within the frequency band of interest, the frequency components of the i/p signal are transmitted with a different amount of gain or attenuation. This is known as amplitude distortion.

(ii) phase distortion:

→ When the phase spectrum $\theta_H(j\omega)$ is not linear within the frequency band of interest the output signal has a different waveform than the input signal bcz of different delays in passing through the system for different frequency components of input signal. This is known as phase distortion.

~~Prod~~

? Given an RC network with i/p, $x(t) = u(t)$, o/p $y(t) = (1 - e^{-t/RC})\, u(t)$. and impulse response $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$.

Calculate the magnitude & phase response of the system.

we've to find $|H(j\omega)|$ & $\theta_H(j\omega)$

Ans. We've $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$



$e^{-at} u(t) \longleftrightarrow \frac{1}{a + j\omega}$

Now, $H(j\omega) = \frac{1}{RC} \cdot \frac{1}{\frac{1}{RC} + j\omega}$

$$= \frac{1/RC}{1/RC + j\omega}$$

$$|H(j\omega)| = \frac{1/RC}{\sqrt{(1/RC)^2 + \omega^2}}$$

$$\theta_H(j\omega) = -\tan^{-1}(\omega RC)$$

when $\omega = 0$, $|H(j\omega)| = \frac{1/RC}{1/RC} = 1$

when $\omega = 1/RC$, $|H(j\omega)| = 1/\sqrt{2}$

when $\omega = -1/RC$, $|H(j\omega)| = 1/\sqrt{2}$.