
KTU
NOTES

MODULE - III

Reflection and Refraction of Electromagnetic Wave.

Poynting Vector.

Poynting vector, denoted by $\vec{S}$ is given by the vector product of $\vec{E}$ & $\vec{H}$.

$$\vec{S} = \vec{E} \times \vec{H}$$

$$|\vec{S}| = |\vec{E}|\,|\vec{H}| \sin(\vec{E}, \vec{H})$$

Poynting vector measures the rate of flow of energy of the wave as it propagates. The direction of $\vec{S}$ represents the direction of power flow and it is perpendicular to the plane containing $\vec{E}$ and $\vec{H}$.

Poynting's Theorem:

When electromagnetic wave propagates through space, there will be a transfer of energy. There exist a simple and direct relation between the rate of this energy transfer and the amplitudes of E & H.

Statement:

The vector product of electric field intensity and magnetic field intensity at any point is a measure of the rate of energy flow per unit area at the point.

Derivation:

Consider the field intensities $\vec{E}$ and $\vec{H}$.

From Maxwell's equations.

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}}$$

$$\nabla \times \vec{H} = \vec{J} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t}$$

~~We know the~~

$$= \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

We know the poynting Vector

$$\vec{S} = \vec{E} \times \vec{H}$$

Taking Dot product on both sides

$$\nabla \cdot \vec{S} = \nabla \cdot (\vec{E} \times \vec{H})$$

By Vector identity

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\therefore \nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H})$$

Substituting the Values,

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot \left(-\mu \frac{\partial \vec{H}}{\partial t}\right) - \vec{E} \cdot \left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$= -\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} - \sigma \vec{E}^2 - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \quad \text{—— (1)}$$

$$\text{Let, } \frac{\partial (\vec{H} \cdot \vec{H})}{\partial t} = \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \quad \text{(product rule)}$$

$$= 2 \vec{H} \cdot \frac{\partial \vec{H}}{\partial t}$$

$$\therefore \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{1}{2} \frac{\partial (\vec{H} \cdot \vec{H})}{\partial t} = \frac{1}{2} \frac{\partial \vec{H}^2}{\partial t} \quad \text{—— (A)}$$

Substituting (A) in equation (1).

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\frac{\mu}{2}\frac{\partial H^2}{\partial t} - \sigma E^2 - \frac{\varepsilon\,\partial E^2}{2\partial t} \qquad \left(\because \vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}\right)$$

Taking Volume Integral on both sides,

$$\int_v \nabla\cdot(\vec{E}\times\vec{H})\,dv = -\frac{d}{dt}\int_v\left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right]dv - \int_v \sigma E^2\,dv$$

Applying the divergence theorem to L.H.S

$$\oint_s (\vec{E}\times\vec{H})\cdot ds = -\frac{d}{dt}\int_v\left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right]dv - \int_v \sigma E^2\,dv$$

Total power leaving the Volume = Rate of decrease in energy stored in electric and magnetic fields − ohmic power dissipated.

This equation is referred to as poynting theorem.

The quantity $\vec{E}\times\vec{H}$ is known as poynting Vector.

$$\boxed{P = \vec{E}\times\vec{H}}$$

Poynting's theorem states that the net power flowing out of a given Volume V is equal to the time rate of decrease in the energy stored within V minus the Conduction losses.

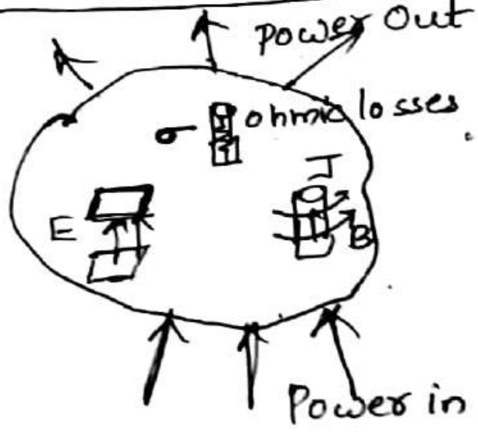
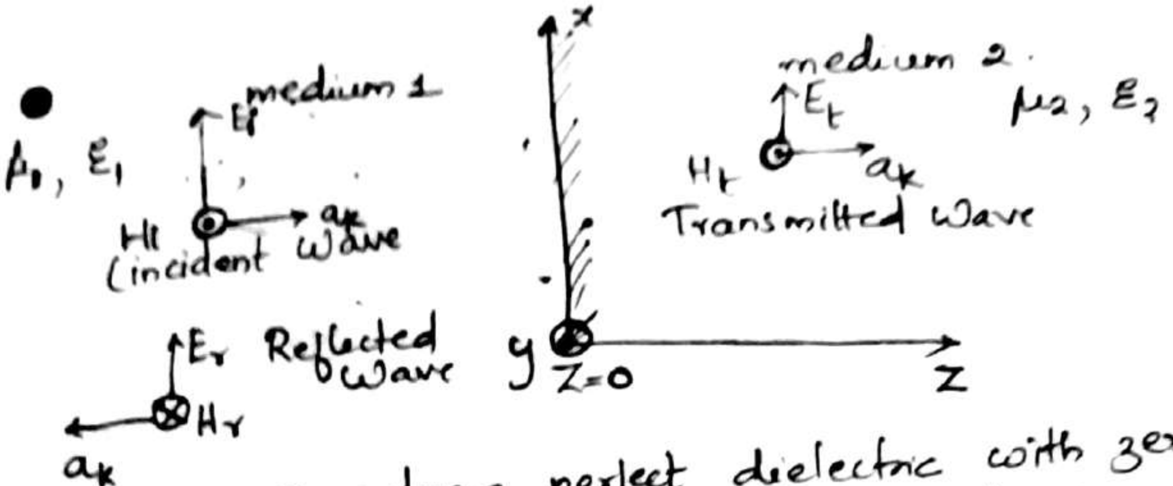


figure: Illustration of power balance for EM fields.

Reflection of Plane Wave At Normal Incidence

(1) Reflection By dielectric - Normal incidence.



Consider a perfect dielectric with zero conductivity. Let E_i be the electric field strength incident wave, E_r is the reflected wave, E_t is the electric field strength transmitted wave.

If μ_1, ε_1 are constants of the medium 1

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

and for the second medium

$$\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$$

We know, $\dfrac{E_i}{H_i} = \eta_1 \quad \therefore E_i = \eta_1 H_i$

Similarly

$$\frac{E_r}{H_r} = -\eta_1$$

$$\therefore E_r = -\eta_1 H_r$$

and $$\frac{E_t}{H_t} = \eta_2$$

$$\therefore E_t = \eta_2 H_t$$

From Boundary condition, tangential components of Electric field and magnetic field are continuous across the boundary. ie $E_{1t} = E_{2t}$ $H_{1t} = H_{2t}$

$$\therefore E_t = E_i + E_r$$

$$H_t = H_i + H_r.$$

We need to find the expressions, $\frac{E_t}{E_i}, \frac{E_r}{E_i}, \frac{H_t}{H_i}, \frac{H_r}{H_i}$

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Reflection Co-efficient.

$$\Gamma = \frac{E_r}{E_i}$$

$$H_i + H_r = H_t = \frac{E_t}{\eta_2} \quad \text{—— (1)}$$

Again, $$H_i + H_r = \frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} \quad \text{—— (2)}$$

Combining (1) = (2)

$$\frac{E_i}{\eta_1} - \frac{E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

$$= \frac{E_i + E_r}{\eta_2}$$

$$(E_i - E_r)\eta_2 = (E_i + E_r)\eta_1$$

$$\eta_2 E_i - \eta_2 E_r = \eta_1 E_i + \eta_1 E_r$$

Dividing by E_i

$$\eta_2 - \eta_2 \frac{E_r}{E_i} = \eta_1 + \eta_1 \frac{E_r}{E_i}$$

$$\eta_2 - \eta_1 = \eta_1 \frac{E_r}{E_i} + \eta_2 \frac{E_r}{E_i}$$

$$\boxed{\frac{\eta_2 - \eta_1}{\eta_1 + \eta_2} = \frac{E_r}{E_i}}$$ → reflection Co-efficient

Transmission Co-efficient.

$$E_t = E_i + E_r$$

$$\frac{E_t}{E_i} = \frac{E_i + E_r}{E_i}$$

$$= 1 + \frac{E_r}{E_i}$$

$$= 1 + \frac{\eta_2 - \eta_1}{\eta_1 + \eta_2}$$

$$= \frac{\cancel{\eta_1} + \eta_2 + \eta_2 - \cancel{\eta_1}}{\eta_1 + \eta_2}$$

$$\boxed{\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2}}$$ Transmission Coefficient.

$$\frac{H_r}{H_i} = -\left[\frac{E_r}{E_i}\right]$$

$$\boxed{\frac{H_r}{H_i} = \frac{\eta_1 - \eta_2}{\eta_1 + \eta_2}}$$

Reflection - Coefficent of magnetic field

$$\frac{H_t}{H_i} = \frac{E_t/\eta_2}{E_i/\eta_1} = \frac{E_t}{E_i} \cdot \frac{\eta_1}{\eta_2}$$

$$= \frac{2\cancel{\eta_2}}{\eta_1+\eta_2} \cdot \frac{\eta_1}{\cancel{\eta_2}}$$

$$\therefore \boxed{\frac{H_t}{H_i} = \frac{2\eta_1}{\eta_1+\eta_2}}$$

Transmission Co-efficient of magnetic field

The permeabilities of all known insulators donot differ much from the free space.

$$\mu_1 = \mu_2 = \mu_0.$$

$$\therefore \eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_1}} \quad \& \quad \eta_2 = \sqrt{\frac{\mu_0}{\varepsilon_2}}$$

Re-writing all expressions using this,

$$\frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\frac{\mu_0}{\varepsilon_2}} - \sqrt{\frac{\mu_0}{\varepsilon_1}}}{\sqrt{\frac{\mu_0}{\varepsilon_2}} + \sqrt{\frac{\mu_0}{\varepsilon_1}}}$$

$$\therefore \boxed{\frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}}$$

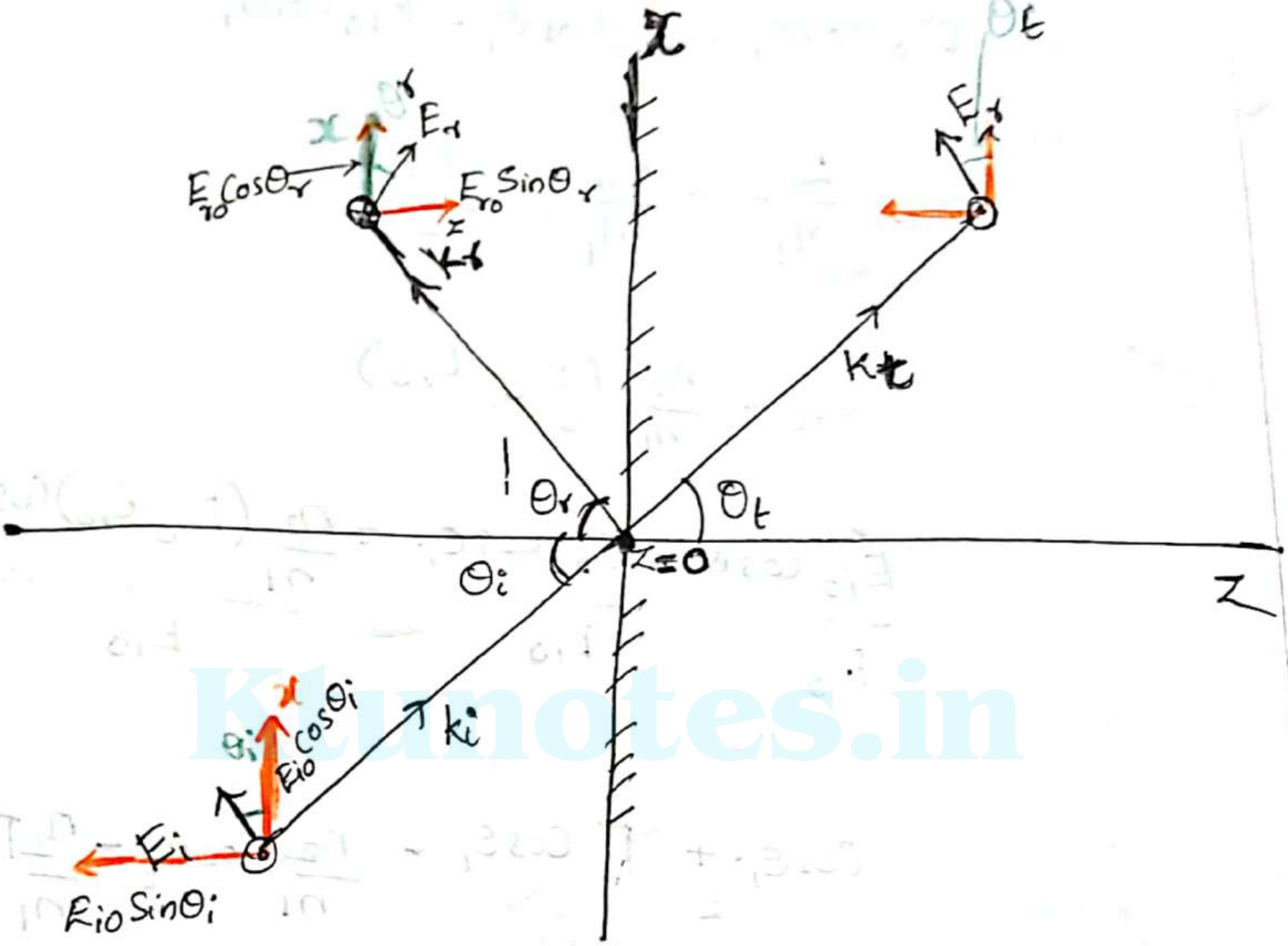
Reflection Co-efficient

Similarly

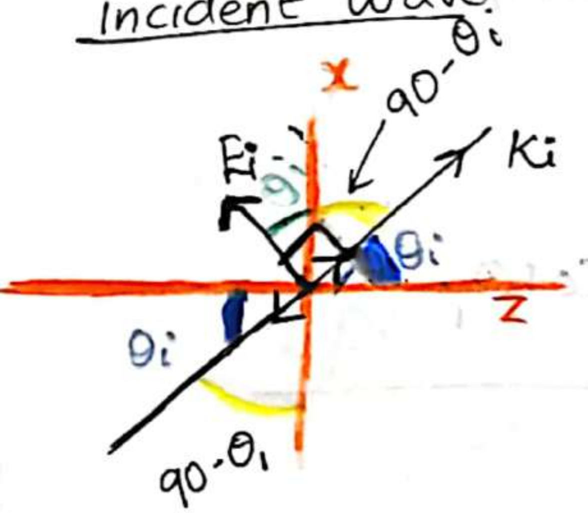
$$\boxed{\frac{E_t}{E_i} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}}$$

$$\boxed{\frac{H_r}{H_i} = \frac{\sqrt{\varepsilon_2}-\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}}$$

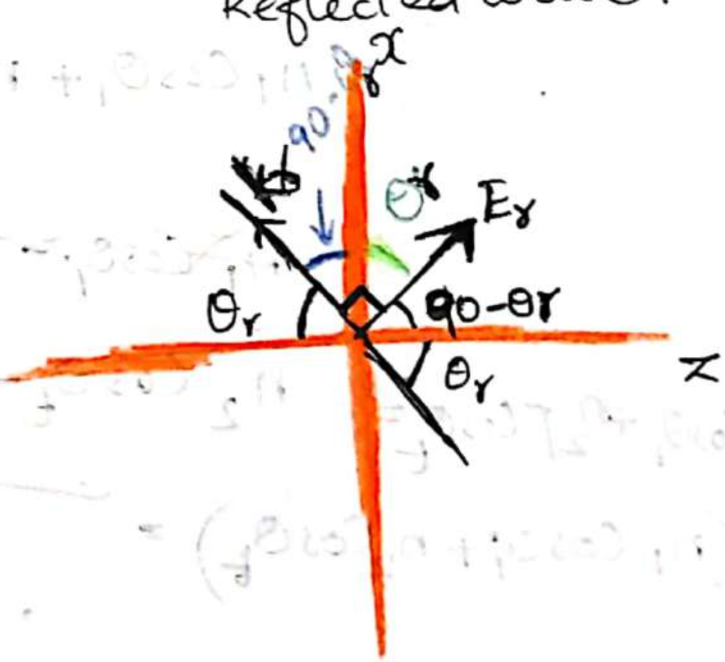
$$\boxed{\frac{H_t}{H_i} = \frac{2\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}+\sqrt{\varepsilon_2}}}$$



Incident wave



Reflected wave.



Parallel Polarization.

→ Electric field $\vec{E}$ lies in the x-z plane, the plane of incidence.

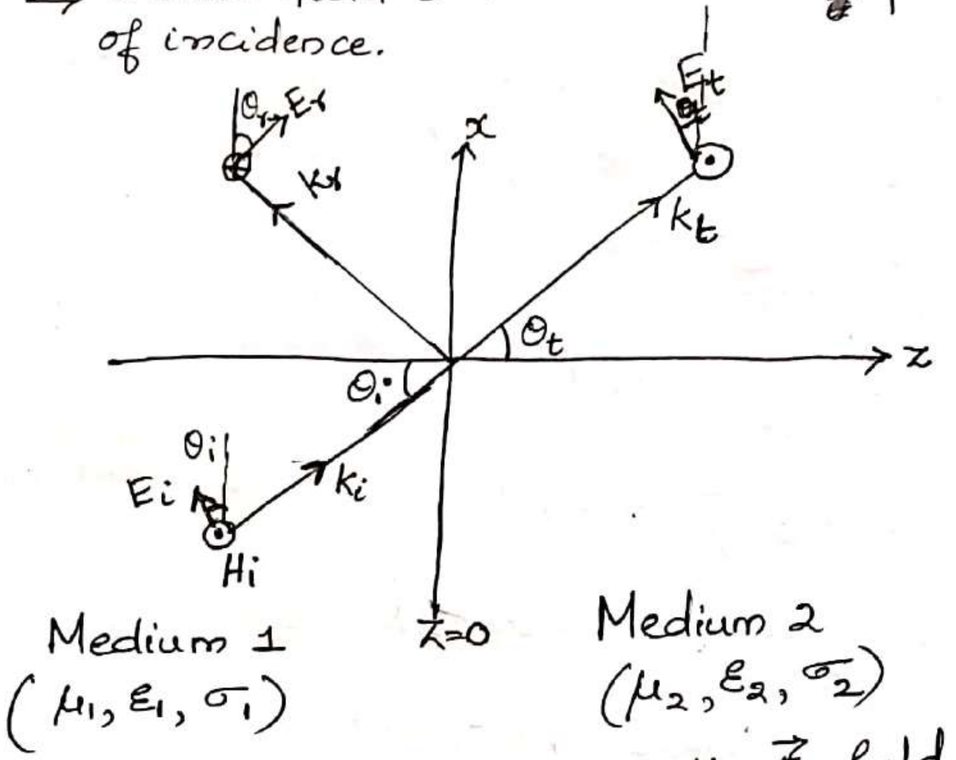


Figure: Oblique incidence with $\vec{E}$ field **parallel** to the plane of incidence.

In Medium 1, We have both incident and reflected fields given by,

$$E_{is} = \left[E_{i0} \cos\theta_i \hat{a_x} - E_{i0} \sin\theta_i \hat{a_z}\right] \cdot e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$= E_{i0}\left(\cos\theta_i \hat{a_x} - \sin\theta_i \hat{a_z}\right) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \quad \text{(1)}$$

$$H_{is} = \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \hat{a_y} \quad \text{(2)}$$

$$E_{rs} = E_{r0}\left[\cos\theta_r \hat{a_x} + \sin\theta_r \hat{a_z}\right] e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \quad \text{(3)}$$

$$H_{r0} = -\frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \hat{a_y} \quad \text{(4)}$$

where $\beta_1 = \omega\sqrt{\mu_1 \varepsilon_1}$

Equation (1) is obtained by substituting the x and z component of $\vec{k}$ and $\vec{E}$.

i.e. $\vec{E_{is}} = E_{i0} e^{-j(k \cdot r)}$

$= E_{i0} e^{-j(\vec{k_i} \cdot \vec{r})}$

$= E_{i0} e^{-j(k_{ix} x + k_{iz} z)}$, i.e. $\vec{k_i} \cdot \vec{r} = k_{ix} x + k_{iz} z$

$= E_{i0} e^{-j(x \beta_1 \sin\theta_i + z \beta_1 \cos\theta_i)}$

where $k_{ix} = \beta_1 \sin\theta_i$

$k_{iz} = \beta_1 \cos\theta_i$.

$\vec{E_{is}} = E_{i0} (\cos\theta_i \hat{a_x} - \sin\theta_i \hat{a_z}) e^{-j(x \beta_1 \sin\theta_i + z\beta_1 \cos\theta_i)}$

Medium 2, has only transmitted fields given by

$\vec{E_{ts}} = E_{t0} (\cos\theta_t \hat{a_x} - \sin\theta_t \hat{a_z}) e^{-j\beta_2(x \sin\theta_t + z \cos\theta_t)}$ —— (5)

$\vec{H_{ts}} = \frac{E_{to}}{\eta_2} e^{-j(x\beta_2 \sin\theta_t + z\beta_2 \cos\theta_t)}$

$= \frac{E_{to}}{\eta_2} e^{-j\beta_2(x \sin\theta_t + z \cos\theta_t)}$ —— (6)

where $\beta_2 = \sqrt{\mu_2 \varepsilon_2}$

By the law of reflection $\theta_i = \theta_r$

and the tangential components of $\vec{E}$ and $\vec{H}$ are Continuous at the boundary $z=0$, $E_{1tan} = E_{2tan}$

$$\vec{E}_{is}(z) + \vec{E}_{rs}(z) = \vec{E}_{t}(z)$$

$$\vec{E}_{is}(0) + \vec{E}_{rs}(0) = \vec{E}_{ts}(0)$$

Substituting in ①, ③ and ⑤ gives,

$$E_{io} \cos\theta_i \, \hat{a_x} \cdot e^{-j\beta_1 x \sin\theta_i} + E_{ro} \cos\theta_r \, e^{-j\beta_1 x \sin\theta_r} \hat{a_x}$$

$$= E_{to} \cos\theta_t \, e^{-j\beta_2 x \sin\theta_t} \hat{a_x}.$$

As $\theta_i = \theta_r$

Above equation becomes,

$$E_{io} \cos\theta_i \, e^{-j\beta_1 x \sin\theta_i} + E_{ro} \cos\theta_i \, e^{-j\beta_1 x \sin\theta_i}$$

$$= E_{to} \cos\theta_t \, e^{-j\beta_2 x \sin\theta_t}$$

$$\cos\theta_i \, e^{-j\beta_1 x \sin\theta_i} \left[E_{io} + E_{ro}\right] = E_{to} \cos\theta_t \, e^{-j\beta_2 x \sin\theta_t}$$

from Snell's Law, $\beta_1 x \sin\theta_i = \beta_2 x \sin\theta_t$

$$\therefore \cos\theta_i \, \cancel{e^{-j\beta_1 x \sin\theta_i}} \left[E_{io} + E_{ro}\right] = E_{to} \cos\theta_t \, \cancel{e^{-j\beta_2 x \sin\theta_t}}$$

$$\cos\theta_i \left[E_{io} + E_{ro}\right] = E_{to} \cos\theta_t \quad (7)$$

$$\frac{1}{\eta_1}\left[E_{io} - E_{ro}\right) = \frac{1}{\eta_2} E_{to} \quad \text{--- (8)}$$

By solving equation (7) and (8) we will get reflection Co-efficient and Transmission Co-efficent,

<u>Reflection Co-efficent</u>

$$\Gamma = \frac{E_{ro}}{E_{io}}$$

Equation (8) $\Rightarrow$ $\frac{1}{\eta_1}\left[E_{io} - E_{ro}\right) = \frac{1}{\eta_2} E_{to}$

$$\therefore \quad E_{to} = \frac{\eta_2}{\eta_1}\left(E_{io} - E_{ro}\right)$$

Substituting E_{to} in equation (7)

$$\cos\theta_i\left(E_{io} + E_{ro}\right) = \frac{\eta_2}{\eta_1}\left(E_{io} - E_{ro}\right)\cos\theta_t$$

$$E_{io}\cos\theta_i + E_{ro}\cos\theta_i = \frac{\eta_2}{\eta_1} E_{io}\cos\theta_t - \frac{\eta_2}{\eta_1} E_{ro}\cos\theta_t$$

Dividing each term with E_{io} above eqn becomes,

$$\cos\theta_i + \Gamma\cos\theta_i = \frac{\eta_2}{\eta_1}\cos\theta_t - \frac{\eta_2}{\eta_1}\Gamma\cos\theta_t$$

$$\eta_1\cos\theta_i + \eta_1\Gamma\cos\theta_i = \frac{\eta_2}{\cancel{\eta_1}}\cos\theta_t - \eta_2\Gamma\cos\theta_t$$

$$\therefore \eta_1 \Gamma \cos\theta_i + \eta_2 \Gamma \cos\theta_t = \eta_2 \cos\theta_t - \eta_1 \cos\theta_i$$

$$\Gamma(\eta_1 \cos\theta_i + \eta_2 \cos\theta_t) = \eta_2 \cos\theta_t - \eta_1 \cos\theta_i$$

$$\boxed{\Gamma = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}}$$

Transmission Co-efficient 'τ'

τ ~~is~~ defined as the ratio of transmitted wave to incident wave.

$$\tau = \frac{E_{to}}{E_{io}}$$

from equation (8), $\frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{E_{to}}{\eta_2}$

Dividing each term by E_{io}

$$\frac{1}{\eta_1}(1 - \Gamma) = \frac{1}{\eta_2} \frac{E_{to}}{E_{io}}$$

$$\therefore \frac{E_{to}}{E_{io}} = \frac{\eta_2}{\eta_1}(1 - \Gamma)$$

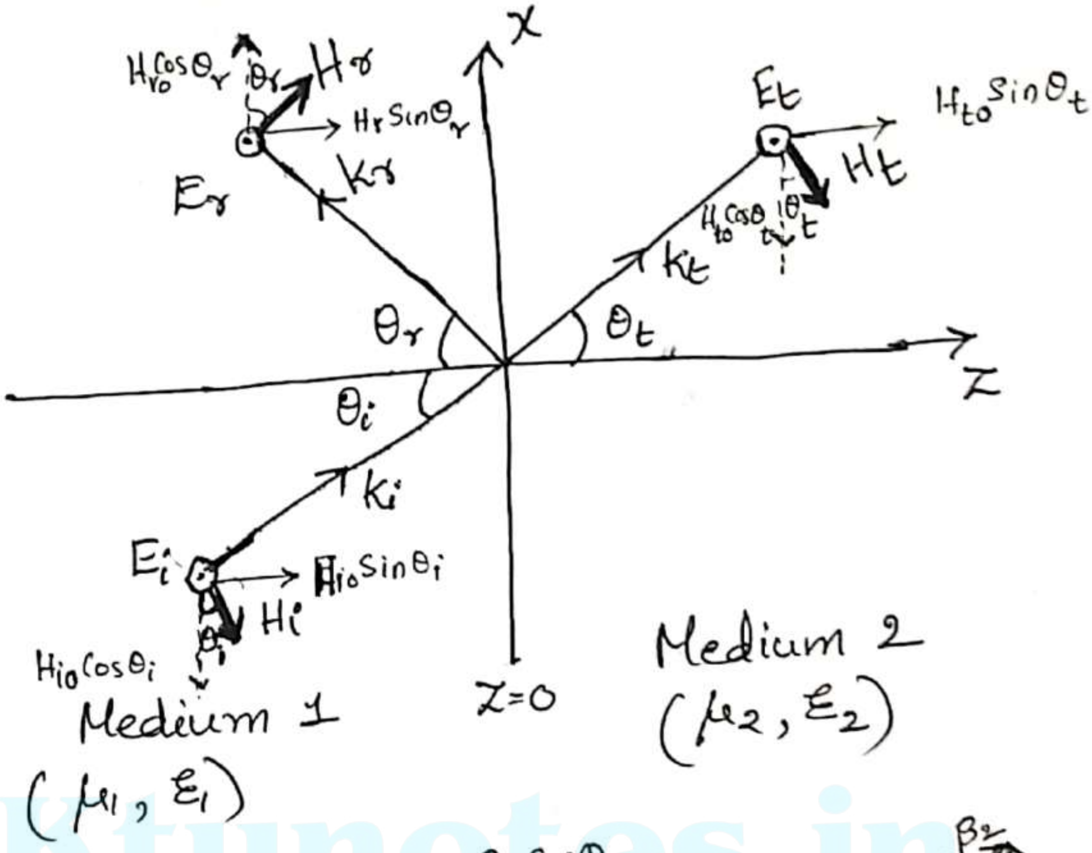
$$= \frac{\eta_2}{\eta_1}\left(1 - \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}\right)$$

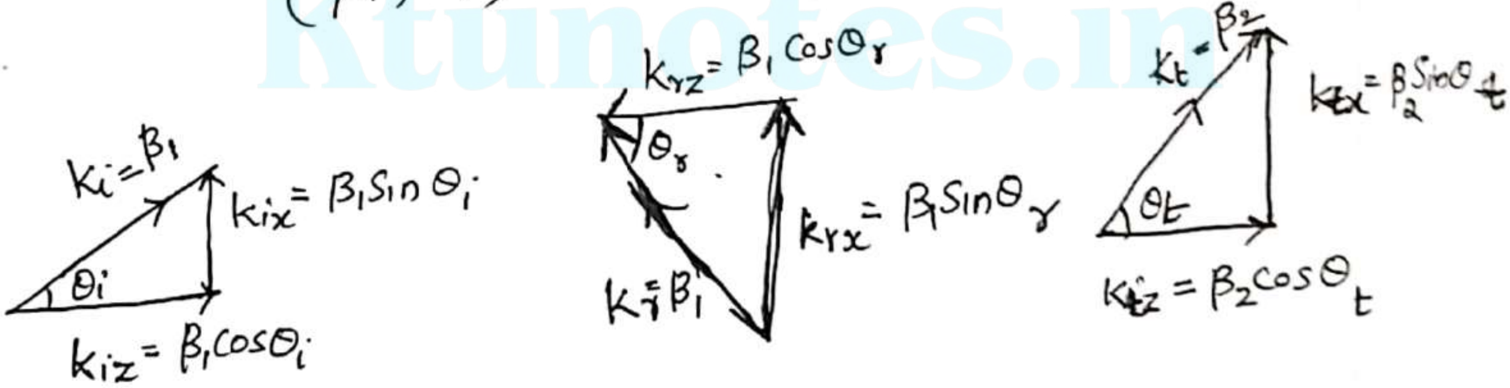
$$\frac{E_{to}}{E_{io}} = \frac{\eta_2}{\eta_1}\left(\frac{\cancel{\eta_2 \cos\theta_t} + \eta_1 \cos\theta_i - \cancel{\eta_2 \cos\theta_t} + \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}\right)$$

$$= \frac{2\eta_2 \cancel{\eta_1} \cos\theta_i}{\cancel{\eta_1}\left(\eta_2 \cos\theta_t + \eta_1 \cos\theta_i\right)}$$

$$\boxed{\tau = \frac{E_{to}}{E_{io}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}}$$

II Perpendicular Polarization.





In perpendicular polarization, electric field $\vec{E}$ is perpendicular to the plane of incidence (the x-z plane) as shown above, where Magnetic field $\vec{H}$ is parallel to the plane of incidence.

In medium 1, we have both incident and reflected wave given by,

$$\vec{E_{is}} = E_{io}\, e^{-j(\vec{k_i}\cdot\vec{r})}\, \hat{a_y}$$

$$= E_{io}\, e^{-j(k_{ix}\cdot x + k_{iz}\cdot z)}\, \hat{a_y}$$

$$= E_{io}\, e^{-j(\beta_1 \sin\theta_i \cdot x + \beta_1 \cos\theta_i \cdot z)}\, \hat{a_y}$$

$$= E_{io}\, e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\, \hat{a_y} \quad \text{---} \quad (1)$$

$$\vec{H_{is}} = \vec{H_{io}}\, e^{-j(\vec{k_i}\cdot\vec{r})}$$

$$= \left(-H_{io}\cos\theta_i\, \hat{a_x} + H_{io}\sin\theta_i\, \hat{a_z}\right) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$= H_{io}\left(-\cos\theta_i\, \hat{a_x} + \sin\theta_i\, \hat{a_z}\right) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$= \frac{E_{io}}{\eta_1}\left(-\cos\theta_i\, \hat{a_x} + \sin\theta_i\, \hat{a_z}\right) e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)} \quad \text{---} \quad (2)$$

$$\vec{E_{rs}} = E_{ro}\, e^{-j(\vec{k_r}\cdot\vec{r})}\, \hat{a_y}$$

$$= E_{ro}\, e^{-j(\beta_1 \sin\theta_r \cdot x - \beta_1 \cos\theta_r \cdot z)}\, \hat{a_y}$$

$$= E_{ro}\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}\, \hat{a_y} \quad \text{---} \quad (3)$$

$$\vec{H_{rs}} = \vec{H_{ro}}\, e^{-j(\vec{k_r}\cdot\vec{r})}$$

$$= \left(H_{ro}\cos\theta_r\, \hat{a_x} + H_{ro}\sin\theta_r\, \hat{a_z}\right) e^{-j(\beta_1 \sin\theta_r x - z\cos\theta_r \beta_1)}$$

$$= H_{ro}\left(\cos\theta_r\, \hat{a_x} + \sin\theta_r\, \hat{a_z}\right) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$= \frac{E_{ro}}{\eta}\left(\cos\theta_r\, \hat{a_x} + \sin\theta_r\, \hat{a_z}\right) e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)} \quad \text{---} \quad (4)$$

while the transmitted wave is in medium 2 given by,

$$\vec{E_{ts}} = E_{to}\, e^{-j(\vec{k_t}\cdot\vec{r})}\, \hat{a_y}$$

$$= E_{to}\, e^{-j(k_{tx}x + k_{tz}z)}\, \hat{a_y}$$

$$= E_{to}\, e^{-j(\beta_2 \sin\theta_t \cdot x + \beta_2 \cos\theta_t \cdot z)}\, \hat{a_y}$$

$$= E_{to}\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\, \hat{a_y} \quad \text{—— (5)}$$

$$\vec{H_{ts}} = \vec{H_{to}}\, e^{-j(\vec{k_t}\cdot\vec{r})}$$

$$= (-H_{to}\cos\theta_t\, \hat{a_x} + H_{to}\sin\theta_t\, \hat{a_z})\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$= H_{to}\, (-\cos\theta_t\, \hat{a_x} + \sin\theta_t\, \hat{a_z})\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$= \frac{E_{to}}{\eta_2}\, (-\cos\theta_t\, \hat{a_x} + \sin\theta_t\, \hat{a_z})\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} \quad \text{—— (6)}$$

By the law of reflection, $\theta_i = \theta_r$ and the tangential components of $\vec{E}$ and $\vec{H}$ are continuous at the boundary $z=0$, $\vec{E_{1tan}} = \vec{E_{2tan}}$ and $\vec{H_{1tan}} = \vec{H_{2tan}}$.

Substituting in equation (1), (3) and (5) gives

$$\vec{E_{is}}(z=0) + \vec{E_{rs}}(z=0) = \vec{E_{ts}}(z=0)$$

$$E_{io}\, e^{-j\beta_1 x\sin\theta_i} + E_{ro}\, e^{-j\beta_1 x\sin\theta_r} = E_{to}\, e^{-j\beta_2 x\sin\theta_t} \quad \text{—— (7)}$$

As $\theta_i = \theta_r$ Equation (7) becomes,

$$E_{io}\, e^{-j\beta_1 x \sin\theta_i} + E_{ro}\, e^{-j\beta_1 x \sin\theta_i} = E_{to} e^{-j\beta_2 x \sin\theta_t}$$

$$e^{-j\beta_1 x \sin\theta_i}\left(E_{io} + E_{ro}\right) = E_{to}\, e^{-j\beta_2 x \sin\theta_t}$$

By Snell's law,

$$\beta_1 x \sin\theta_i = \beta_2 x \sin\theta_t$$

$$\therefore \cancel{e^{-j\beta_1 x \sin\theta_i}}\left(E_{io} + E_{ro}\right) = E_{to}\, \cancel{e^{-j\beta_1 x \sin\theta_i}}$$

$$\therefore E_{io} + E_{ro} = E_{to} \quad \text{——— (8)}$$

Similarly by substituting in equation (2), (4) and (6) gives

$$\vec{H_{is}}(z=0) + \vec{H_{rs}}(z=0) = \vec{H_{ts}}(z=0)$$

$$\frac{E_{io}}{\eta_1}\left(-\cos\theta_i \hat{a_x}\right) e^{-j\beta_1 x \sin\theta_i} + \frac{E_{ro}}{\eta_1} \cos\theta_r \hat{a_x}\, e^{-j\beta_1 x \sin\theta_r}$$

$$= \frac{E_{to}}{\eta_2}\left(-\cos\theta_t \hat{a_x}\right) e^{-j\beta_2 x \sin\theta_t} \quad \text{——— (9)}$$

As $\theta_i = \theta_r$, equation (9) becomes,

$$\frac{E_{io}}{\eta_1} - \cos\theta_i \hat{a_x}\, e^{-j\beta_1 x \sin\theta_i} + \frac{E_{ro}}{\eta_1} \cos\theta_i \hat{a_x}\, e^{-j\beta_1 x \sin\theta_i}$$

$$= \frac{E_{to}}{\eta_2} - \cos\theta_t \hat{a_x}\, e^{-j\beta_2 x \sin\theta_t}$$

$$\left[-\frac{E_{i0}}{\eta_1}+\frac{E_{r0}}{\eta_1}\right]\cos\theta_i e^{-j\beta_1 x\sin\theta_i} = -\frac{E_{t0}}{\eta_2}\cos\theta_t e^{-j\beta_2 x\sin\theta_t}$$

By Snell's law $\beta_1 x\sin\theta_i = \beta_2 x\sin\theta_t$

$$\left(-\frac{E_{i0}}{\eta_1}+\frac{E_{r0}}{\eta_1}\right)\cos\theta_i = \frac{-E_{t0}}{\eta_2}\cos\theta_t$$

$$\frac{1}{\eta_1}\left(E_{i0}-E_{r0}\right)\cos\theta_i = \frac{1}{\eta_2}E_{t0}\cos\theta_t \quad \text{—— (10)}$$

By solving equation (8) and (10) we will get Reflection Co-efficient and transmission Co-efficient.

<u>Reflection Co-efficient</u> '$\Gamma_\perp$'.

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}}$$

From equation (8), $E_{i0}+E_{r0}=E_{t0}$,

Substituting equation (8) in (10).

$$\frac{1}{\eta_1}\left(E_{i0}-E_{r0}\right)\cos\theta_i = \frac{1}{\eta_2}\left(E_{i0}+E_{r0}\right)\cos\theta_t$$

Dividing each term by E_{i0},

$$\frac{1}{\eta_1}\left(1-\Gamma_\perp\right)\cos\theta_i = \frac{1}{\eta_2}\left(1+\Gamma_\perp\right)\cos\theta_t$$

$$\frac{1}{\eta_1}\cos\theta_i - \frac{1}{\eta_1}\Gamma_\perp \cos\theta_i = \frac{1}{\eta_2}\cos\theta_t + \frac{1}{\eta_2}\Gamma_\perp \cos\theta_t$$

$$\frac{1}{\eta_1}\cos\theta_i - \frac{1}{\eta_2}\cos\theta_t = \frac{1}{\eta_2}\Gamma_\perp \cos\theta_t + \frac{1}{\eta_1}\Gamma_\perp \cos\theta_i$$

$$\frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\cancel{\eta_1\eta_2}} = \Gamma_\perp\left(\frac{\eta_1\cos\theta_t + \eta_2\cos\theta_i}{\cancel{\eta_1\eta_2}}\right)$$

$$\therefore \boxed{\Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_1\cos\theta_t + \eta_2\cos\theta_i}}$$

Transmission Co-efficient, $\tau_\perp$

$$\tau_\perp = \frac{E_{to}}{E_{io}}$$

Equation (3), $E_{io} + E_{ro} = E_{to}$

Dividing each term by E_{io}

$$\frac{E_{io}}{E_{io}} + \frac{E_{ro}}{E_{io}} = \frac{E_{to}}{E_{io}}$$

$$1 + \Gamma_\perp = \tau_\perp$$

$$1 + \left(\frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_1\cos\theta_t + \eta_2\cos\theta_i}\right) = \tau_\perp$$

$$\frac{\cancel{\eta_1\cos\theta_t} + \eta_2\cos\theta_i + \eta_2\cos\theta_i - \cancel{\eta_1\cos\theta_t}}{\eta_1\cos\theta_t + \eta_2\cos\theta_i}$$

$$\boxed{\tau_{\perp} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}}$$

Polarization of Plane Waves:

The polarization of Uniform plane wave is defined as time Varying behaviour of the electric field intensity $\vec{E}$ at some fixed point in Space, along the direction of propagation

Consider a Uniform plane wave travelling in positive 'z' direction. Then the fields $\vec{E}$ and $\vec{H}$ lie in the x-y plane, which is perpendicular to the direction of propagation.

Being an electromagnetic wave (EM), as it travels in a Space, both the fields undergo some Variations with respect to time.

There are three different types of polarization
1) Linear polarization
2) Elliptical polarization
3) Circular polarization.

Linear Polarization:

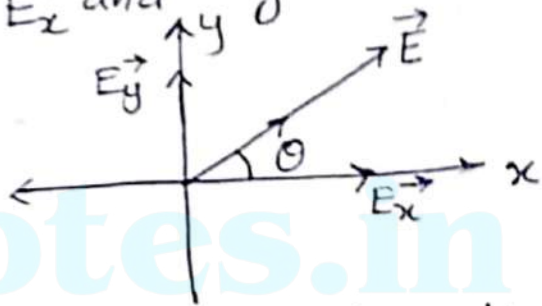
Consider that the electric field $\vec{E}$ has only x component and y component of $\vec{E}$ is zero. Then looking from the direction of propagation, the wave is said to be linearly polarized in x-direction.

Similarly if only the y component of $\vec{E}$ is present and x - Component of $\vec{E}$ is zero, then the wave is said to be linearly polarized in y-direction.

Let us assume that both the components of $\vec{E}$ are present denoted by $\vec{E_x}$ and $\vec{E_y}$. The electric field is the resultant of $\vec{E_x}$ and $\vec{E_y}$ and the direction of it depends on the relative magnitude of $\vec{E_x}$ and $\vec{E_y}$

$$\vec{E} = \sqrt{E_x^2 + E_y^2}$$

Angle made by $\vec{E}$ $\theta = \tan^{-1}\frac{E_y}{E_x}$ where E_x and E_y are the magnitudes of $\vec{E_x}$ and $\vec{E_y}$ respectively.

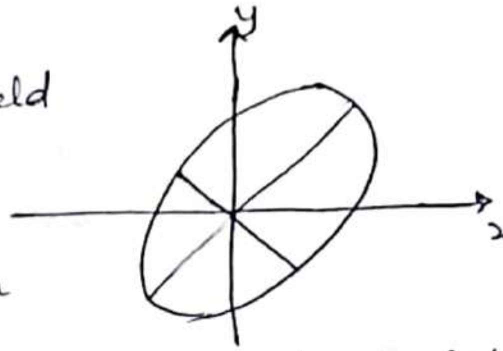


Linear polarization.

This angle 'θ' is constant with respect to time. In other words, the resultant vector $\vec{E}$ is oriented in a direction which is constant with time, thus the wave is said to be <u>linearly polarized</u>.

When both the components have same amplitudes ie $E_x = E_y$, then the polarization is called Linear polarization with constant angle of 45°.

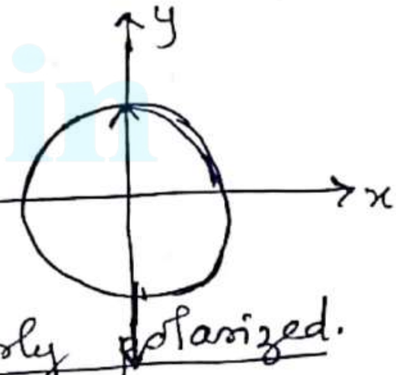
Elliptical Polarization



Elliptical polarization

Consider that the $\vec{E}$ electric field has both the components $\vec{E_x}$ and $\vec{E_y}$ with different amplitudes and are not in phase, the direction of the resultant field $\vec{E}$ varies with time. If the locus of the end points of $\vec{E}$ is traced, it is observed that $\vec{E}$ moves elliptically. Then such a wave is said to be elliptically polarized.

Circular Polarization



Let us consider that $\vec{E}$ has two components, $\vec{E_x}$ and $\vec{E_y}$ of equal amplitudes but the phase difference between them is exactly 90°. Such a wave is said to be circularly polarized.

Condition for the polarization of a sinusoidal wave

Two components of $\vec{E}$ electric field can be expressed in phasor form as,

$$\vec{E_x} = E_1 e^{j(\omega t - \beta z)} \quad \text{---}(1)$$

$$\vec{E_y} = E_2 e^{j(\omega t - \beta z - \delta)} \quad \text{---}(2)$$

where δ is the phase difference between the two components.

As the electric field vector $\vec{E}$ is the resultant of $\vec{E_x}$ and $\vec{E_y}$

$$\vec{E} = \vec{E_x} + \vec{E_y}$$

$$\vec{E} = E_1 e^{j(\omega t - \beta z)} \hat{a_x} + E_2 e^{j(\omega t - \beta z - \delta)} \hat{a_y} \quad \text{——} (3)$$

At $z=0$, above equation becomes,

$$\vec{E} = E_1 e^{j\omega t} \hat{a_x} + E_2 e^{j(\omega t - \delta)} \quad \text{——} (4)$$

$$\therefore \vec{E} = E_1 [\cos \omega t + j \sin \omega t] \hat{a_x} + E_2 [\cos(\omega t - \delta) + j \sin(\omega t - \delta)] \quad \text{——} (5)$$

In general, the electric field vector can be represented with its two components as,

$$\vec{E} = E_x \hat{a_x} + E_y \hat{a_y} \quad \text{——} (6).$$

Equating right hand side of equation (6) with the real part of the equation (5).

$$E_x \hat{a_x} + E_y \hat{a_y} = E_1 \cos \omega t \, \hat{a_x} + E_2 \cos(\omega t - \delta) \hat{a_y} \quad \text{—} (7)$$

Comparing x and y components,

$$E_x = E_1 \cos \omega t \quad \text{——} (8)$$

$$E_y = E_2 \cos(\omega t - \delta) \quad \text{——} (9)$$

From eq (8),

$$\left(\frac{E_x}{E_1}\right) = \cos \omega t \quad \text{——} (9\text{-}a).$$

$$\therefore \left(\frac{E_x}{E_1}\right)^2 = \cos^2 \omega t = 1 - \sin^2 \omega t \quad \text{——} (10)$$

$$\sin^2 \omega t = 1 - \left(\frac{E_x}{E_1}\right)^2$$

$\therefore \sin\omega t = \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2}$ ——— (10a)

From (9),
$$\frac{E_y}{E_2} = \cos(\omega t - \delta)$$

$$\therefore \frac{E_y}{E_2} = \cos\omega t \cos\delta + \sin\omega t \sin\delta \quad \text{——— (11)}$$

Substituting the values of $\cos\omega t$ and $\sin\omega t$ from equation (9a) and (10a), in equation (11).

$$\frac{E_y}{E_2} = \frac{E_x}{E_1}\cos\delta + \sqrt{1-\left(\frac{E_x}{E_1}\right)^2}\sin\delta$$

$$\therefore \frac{E_y}{E_2} - \frac{E_x}{E_1}\cos\delta = \sqrt{1-\left(\frac{E_x}{E_1}\right)^2}\sin\delta$$

Squaring both sides,
$$\left(\frac{E_y}{E_2} - \frac{E_x}{E_1}\cos\delta\right)^2 = \left[1-\left(\frac{E_x}{E_1}\right)^2\right]\sin^2\delta$$

$(a-b)^2 = a^2-2ab+b^2$

$$\left(\frac{E_y}{E_2}\right)^2 - 2\frac{E_y}{E_2}\frac{E_x}{E_1}\cos\delta + \left(\frac{E_x}{E_1}\right)^2\cos^2\delta = \sin^2\delta - \left(\frac{E_x}{E_1}\right)^2\sin^2\delta$$

Simplifying above equation

$$\left(\frac{E_x}{E_1}\right)^2\left[\cos^2\delta + \sin^2\delta\right] + \left(\frac{E_y}{E_2}\right)^2 - 2\frac{E_y}{E_2}\frac{E_x}{E_1}\cos\delta = \sin^2\delta$$

$$\boxed{\therefore \left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right)\cos\delta = \sin^2\delta} \quad \text{——— (12)}$$

Above equation is the equation for polarization of Sinusoidal wave. By applying different conditions to equation for the polarization of Sinusoidal wave, We get different types of the polarization.

Condition 1 : $\vec{E_x}$ and $\vec{E_y}$ are in phase and $\delta = 0$.

Substituting this condition in equation (12),

$$\left(\frac{E_x}{E_1}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right) + \left(\frac{E_y}{E_2}\right)^2 = 0$$

$$\therefore \left(\frac{E_x}{E_1} - \frac{E_y}{E_2}\right)^2 = 0$$

$$\therefore \frac{E_x}{E_1} - \frac{E_y}{E_2} = 0$$

$$\boxed{\text{or} \quad E_x = \left(\frac{E_1}{E_2}\right) E_y} \quad \text{——— (A)}$$

If the amplitudes of $\vec{E_x}$ and $\vec{E_y}$ are constant Then the ratio $\frac{E_1}{E_2}$ is also constant. The above equation is similar to the equation of a straight line passing through origin $y = mx$. The wave is said to be linearly polarized wave.

Condition 2

$\vec{E_x}$ and $\vec{E_y}$ components of unequal amplitudes with a phase difference $\delta \neq 0$, Let us assume $\delta = \pi/2$

Applying condition to equation (12),

$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right)\cos\pi/2 - \sin^2\pi/2$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta), \quad \therefore \frac{1}{2}(1-\cos 2\times\pi/2)$$

$$= \frac{1}{2}(1--1) = 1$$

Then equation becomes,

$$\boxed{\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1}$$

This equation represents equation for an ellipse. Hence the wave satisfies above equation is called elliptically polarized.

Condition 3:

Let the amplitude of $\vec{E_x}$ and $\vec{E_y}$ be equal and phase difference between them $\delta = 90^\circ$, Applying conditions to equation (12) gives,

$$\boxed{E_x^2 + E_y^2 = 1}$$

equation represents circle. The wave satisfies above equation is called circullarly polarized.

$\mu_1 \qquad \mu_2$

Maxwell's Equations Using Phasor Notation (Time Varying Fields).

In practice, most generator produce voltages and currents that vary sinusoidally with time. Hence the electric and magnetic fields also vary sinusoidally with time. This sinusoidal time factor can be expressed by using the phasor notation.

The wave equation in phasor form.

ie $\vec{E}(z,t) = Re\left\{\vec{E}e^{j\omega t}\right\}$ at some point z in space,

Maxwell's first equation,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \frac{\varepsilon \partial \vec{E}}{\partial t}$$

Let $\vec{E} = E_0 e^{j\omega t}$

$\therefore \frac{\partial \vec{E}}{\partial t} = j\omega E_0 e^{j\omega t}$

$= \underline{\underline{j\omega \vec{E}}}$

Similarly, $\frac{\partial^2 \vec{E}}{\partial t^2} = j\omega \times j\omega \vec{E} = -\omega^2 \vec{E}$

$\therefore \frac{\partial}{\partial t}$ can be replace by $j\omega$ and $\frac{\partial^2}{\partial t^2}$ by $-\omega^2$

Hence Maxwell's equation in phasor form,

$$\nabla \times \vec{H} = J + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\boxed{\nabla \times \vec{H} = \sigma \vec{E} + j\omega\varepsilon \vec{E}} \quad \boxed{\nabla \times \vec{H} = J + j\omega \vec{D}}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$= -\mu j\omega \vec{H}$$

$$\boxed{\begin{aligned} \therefore \nabla \times \vec{E} &= -j\omega\mu \vec{H} \\ \nabla \times \vec{E} &= -j\omega \vec{B} \end{aligned}}$$

$$\boxed{\begin{aligned} \nabla \cdot \vec{D} &= \rho_v \\ \nabla \cdot \vec{B} &= 0 \end{aligned}}$$

Q Write expressions for wave equations of $\vec{E}$ and $\vec{H}$ field in free space.

The wave equation of $\vec{E}$ field in free space is

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$= \frac{1}{u^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

where $u = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$ u is known as wave velocity in free space = Velocity of light = 3×10^8 m/s.

Also
Wave equation of $\vec{H}$ field in free space

$$\nabla^2 \vec{H} = \frac{1}{u^2} \frac{\partial^2 \vec{H}}{\partial t^2}$$

MODULE 3

Propagation of plane EM wave in perfect dielectric, lossy medium, good conductor, media-attenuation, phase velocity, group velocity, skin depth.

Reflection and refraction of plane electromagnetic waves at boundaries for normal & oblique incidence (parallel and perpendicular polarization), Snell's law of refraction, Brewster angle.

Electromagnetic wave propagation. (plane wave) (2)

① Different media are

1) Lossy dielectric : $\sigma \neq 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$

2) Lossless dielectric : $\sigma = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$, $\omega\varepsilon \gg \sigma$

3) free space : $\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

4) Good Conductors : $\sigma = \infty$, $\varepsilon = \varepsilon_0 \varepsilon_r$ $\mu = \mu_0 \mu_r$, $\omega\varepsilon \ll \sigma$

I wave propagation of plane wave in Lossy dielectric / partially conducting medium

A lossy dielectric is a medium in which an EM wave loses power as it propagates due to poor conduction.

Here we consider a linear, isotropic, homogeneous medium and is charge free $\rho_v = 0$.

Linear - D varies with E
Isotropic - D & E in same direction
homogeneous - E doesnot vary

In phasor notation,

$$\nabla \cdot \vec{E} = 0 \quad \text{— ①}$$

$$\nabla \cdot \vec{H} = 0 \quad \text{— ②}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad \text{— ③} \quad \because \sigma \neq 0,$$

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E} \quad \text{— ④}$$

$$(\sigma + j\omega\varepsilon)\vec{E}$$

Taking curl on both sides of eq ③

$$\nabla \times \nabla \times \vec{E} = -j\omega\mu \, \nabla \times \vec{H}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}$$

$$\nabla^2 \vec{E} = j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}$$

$$\nabla^2 \vec{E} - j\omega\mu(\sigma + j\omega\varepsilon)\vec{E} = 0$$

$$\boxed{\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0} \quad \text{——— (5)}$$

where $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$ and γ is called the propagation constant of the medium.

By a similar procedure for $\vec{H}$

$$\boxed{\nabla^2 \vec{H} - \gamma^2 \vec{H} = 0} \quad \text{——— (6)}$$

(5) and (6) are known as Vector Helmholtz' equation

Since γ is a complex quantity, It can be expressed as

$$\boxed{\gamma = \alpha + j\beta}$$

where α is called attenuation constant.

β is called phase shift constant.

Attenuation Constant (α) unit Np/m (Nepers/m)

It denotes how the amplitude of the wave diminishes during the propagation through the medium

phase shift constant (β) radian/m

It specifies the change in phase of the wave at a fixed time of observation in the direction of propagation.

Expression for α and β

$$\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

Squaring both sides.

$$(\alpha + j\beta)^2 = [j\omega\mu(\sigma + j\omega\varepsilon)]$$

$$\alpha^2 + 2j\alpha\beta - \beta^2 = j\omega\mu\sigma - \omega^2\mu\varepsilon$$

$$\alpha^2 - \beta^2 + 2j\alpha\beta = j\omega\mu\sigma - \omega^2\mu\varepsilon$$

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

$$2\alpha\beta = \omega\mu\sigma$$

Using the identity,

$$\alpha^2 + \beta^2 = \sqrt{(\alpha^2-\beta^2)^2 + 4\alpha^2\beta^2}$$

$$\alpha^2 + \beta^2 = \sqrt{(-\omega^2\mu\varepsilon)^2 + \omega^2\mu^2\sigma^2}$$

$$= \sqrt{\omega^4\mu^2\varepsilon^2 + \omega^2\mu^2\sigma^2}$$

$$= \sqrt{\omega^2\mu^2(\omega^2\varepsilon^2 + \sigma^2)}$$

$$= \omega\mu\sqrt{\omega^2\varepsilon^2 + \sigma^2}$$

Solving these equations,

$$\boxed{\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} \quad \text{(7)}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}} \quad \text{(8)}$$

Phase Velocity, V_p

The phase velocity of a wave is the rate at which the phase of the wave propagates in space

$$V_p = \frac{\omega}{\beta} = \lambda f = \frac{1}{\sqrt{\mu\varepsilon}}.$$

The Solution of the Helmholtz equation [In General]

$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$ will be of the form.

$$\vec{E}(z) = E_0 e^{-\gamma z} + E_0' e^{\gamma z}$$

wave

Considering the wave propagation in +z direction so we can neglect the second term,

Assuming $\vec{E}$ has only the x-component

$$\therefore \vec{E}(z) = E_0 e^{-\gamma z} \hat{a_x}.$$

Incorporating the time factor

$$\vec{E}(z,t) = Re\left\{ \vec{E}(z) e^{j\omega t} \hat{a_x} \right\} \qquad \because \gamma = \alpha + j\beta$$

$$= Re\left\{ E_0 e^{-\gamma z} e^{j\omega t} \hat{a_x} \right\}$$

$$= Re\left\{ E_0 e^{-\alpha z} e^{j(\omega t - \beta z)} \hat{a_x} \right\}$$

$$= E_0 e^{-\alpha z} Re\left\{ e^{j(\omega t - \beta z)} \right\} \hat{a_x}$$

$$= \underline{E_0 e^{-\alpha z} \cos(\omega t - \beta z) \hat{a_x}}$$

If we solve for the magnetic field using maxwell's equation, we get

$$\boxed{\vec{H}(z,t) = \frac{E_0}{\eta} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \hat{a_y}}$$

where η is the intrinsic impedance of the medium & θ_η is phase difference between $\vec{E}$ and $\vec{H}$

$$\boxed{\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}} \quad \& \quad \boxed{\tan 2\theta_n = \frac{\sigma}{\omega\varepsilon}}$$

η is a complex quantity $|\eta| \angle \theta_n$ or $|\eta| e^{j\theta_n}$

... good Conductors.

II Wave Propagation In free Space

In this Case

$$\sigma = 0, \quad \varepsilon = \varepsilon_0, \quad \mu = \mu_0$$

$\therefore$ α-attenuation Constant $= 0$ (Sub: $\sigma = 0$ in eq (7))

$\beta \rightarrow$ phase constant $= \omega\sqrt{\mu_0 \varepsilon_0} = \dfrac{\omega}{u}$

$\mu_0 = 1.257\times10^{-6}$ H/m
$\varepsilon_0 = 8.854\times10^{-12}$ F/m

$$u = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$$

ie EM waves travel in free Space at the speed of light

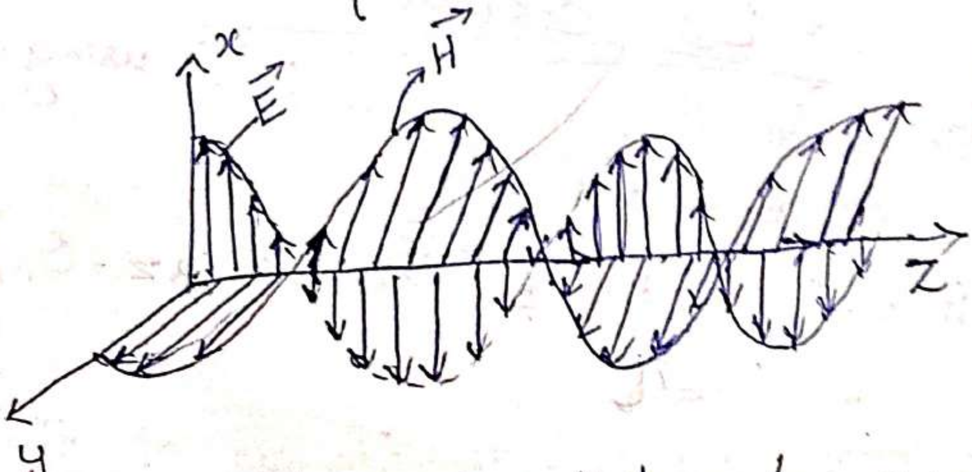
$$\eta = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi = 377\Omega$$

Where η is known as characteristic / Intrinsic Impedance.

$$\tan 2\theta_n = 0 \quad \therefore \theta_n = 0$$

$$\therefore \vec{E} = E_0 \cos(\omega t - \beta z)\, \hat{a_x}$$

$$\vec{H} = \frac{E_0}{\eta} \cos(\omega t - \beta z)\, \hat{a_y}$$



Both $\vec{E}$ & $\vec{H}$ fields are every where normal to the direction of propagation

Such a Wave is called a transvere electromagnetic Wave (TEM)

II Wave Propagation in good Conductors.

A perfect or good Conductor is one in which σ is very high.

ie $\boxed{\sigma >> \omega\varepsilon \text{ or } \frac{\sigma}{\omega\varepsilon} >>> 1 \text{ or } \sigma \simeq \infty}$

$$\therefore \quad \alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu \sigma} \quad \because \omega = 2\pi f$$

$$\boxed{\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ}$$

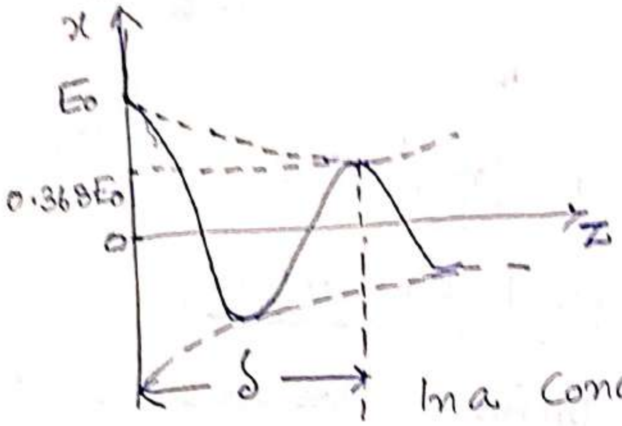
$$\boxed{\tan 2\theta_n = \infty \therefore \theta_n = 45^\circ}$$

thus $\vec{E}$ leads $\vec{H}$ by 45°.

$$\boxed{\begin{aligned} \vec{E} &= E_0 e^{-\alpha z} \cos(\omega t - \beta z)\hat{a_x} \\ \vec{H} &= \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ)\hat{a_y} \end{aligned}}$$

Therefore as ($\vec{E}$ and $\vec{H}$) wave travels in a Conducting medium its amplitude is attenuated by a factor $e^{-\alpha z}$. The distance δ, through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called Skin depth or penetration depth of the medium

Skin depth [Depth of Penetration].



$$\vec{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$$

In a conductive medium, wave gets progressively attenuated as it penetrates the medium. Thus the depth of penetration is defined as the depth at which the wave has been attenuated or the wave amplitude decreases by a factor e^{-1} (approximate 37%) of its original Value.

The skin depth is a measure of the depth to which an EM wave can penetrate the medium.

The distance δ through which the amplitude decreases by a factor e^{-1} (about 37%) is called depth of penetration or skin depth

$$E = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$$

distance

$$E_0 e^{-\alpha z} = E_0 e^{-\alpha \delta} = \frac{1}{e} E_0$$

$$e^{-\alpha\delta} = e^{-1}$$

$$\therefore \alpha\delta = 1$$

$$\delta = \frac{1}{\alpha}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$$ neglecting 1

$\frac{\sigma}{\omega\varepsilon} >> 1$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\sqrt{\frac{\sigma^2}{\omega^2\varepsilon^2}}}$$

$$= \omega\sqrt{\frac{\mu\cancel{\varepsilon}}{2}\frac{\sigma}{\omega\cancel{\varepsilon}}}$$

$$\boxed{\alpha = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{\cancel{2}\pi f\mu\sigma}{\cancel{2}}} = \sqrt{\pi f\mu\sigma}}$$

$$\boxed{\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}}}$$

For Good Conductors. $\frac{\sigma}{\omega\varepsilon} >> 1$

① Propagation Constant

$$\therefore \gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$= \sqrt{j\omega\mu\sigma\left(1 + j\frac{\omega\varepsilon}{\sigma}\right)} \quad \frac{\omega\varepsilon}{\sigma} << 1$$

$$\gamma = \sqrt{j\omega\mu\sigma}$$

$\boxed{\text{Propagation Constant, } \gamma = \sqrt{j\omega\mu\sigma}}$

② $\boxed{\text{Attenuation Constant, } \alpha = \sqrt{\frac{\omega\mu\sigma}{2}}}$

③ $\boxed{\text{Phase Constant, } \beta = \sqrt{\frac{\omega\mu\sigma}{2}}}$

④ Phase Velocity or Wave Velocity U_p or V or u

$$\boxed{U_p = \frac{\omega}{\beta}}$$

(5) Intrinsic Impedence $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$

$$\therefore = \sqrt{\frac{j\omega\mu}{\sigma\left(1 + \frac{j\omega\varepsilon}{\sigma}\right)}} \qquad \because \frac{\omega\varepsilon}{\sigma} <<< 1$$

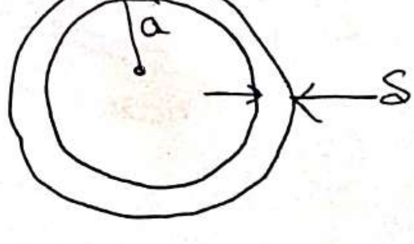
$$\boxed{\eta = \sqrt{\frac{j\omega\mu}{\sigma}}} \text{ or } \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ .$$

Skin depth. (Note)

The phenomenon whereby field intensity in a conductor rapidly decreases is known as skin effect. It is a tendency of charges to migrate from the bulk of the conducting material to the surface, resulting in higher resistance. The fields and associated currents are confined to a very thin layer (the skin) of the conductor surface.

Thus at high frequencies it is a good approximation to assume that all the current flows in a circular ring of thickness δ for a wire of radius 'a'.

It is for this reason, that hollow tubular conductors are used instead of solid conductors in outdoor television antennas



(31) (16) (1)

I EM wave or Plane Waves Propagation in Lossless Dielectric or Perfect dielectric

In a lossless dielectric,

$\sigma = 0, \quad \varepsilon = \varepsilon_0 \varepsilon_r, \quad \mu = \mu_0 \mu_r$

① Attenuation Constant $\boxed{\alpha = 0}$

② Phase Constant $\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$

③ Phase Velocity $\boxed{v_p \text{ or } u = \dfrac{\omega}{\beta} = \dfrac{1}{\sqrt{\mu\varepsilon}}}$ $\lambda = \dfrac{2\pi}{\beta}$

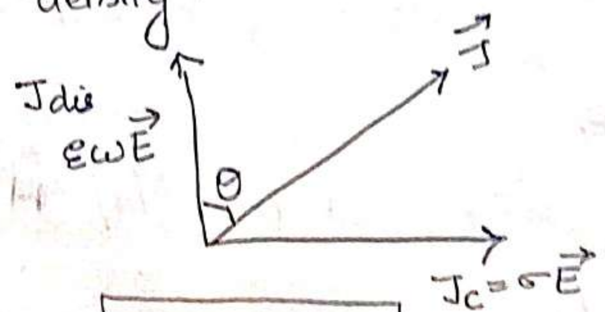
④ Intrinsic impedance, $\boxed{\eta = \sqrt{\dfrac{\mu}{\varepsilon}} \angle 0^\circ}$

and thus $\vec{E}$ and $\vec{H}$ fields are in time phase with each other

⑤
$$\boxed{\begin{aligned} \vec{E} &= E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a_x} \\ \vec{H} &= \frac{E_0}{\eta} e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a_y} \end{aligned}}$$
$\because \theta_n = 0$

★ Loss tangent.

The loss tangent is a measure of impurity of the given dielectric material or medium. It is the ratio of conduction current density to displacement current density.



$\tan\theta = \dfrac{J_c}{J_d} = \dfrac{\sigma\vec{E}}{\varepsilon\omega\vec{E}}$

$\boxed{\tan\theta = \dfrac{\sigma}{\varepsilon\omega}}$. This expression is termed as loss tangent.

Q. A Uniform plane wave propagating in a medium has

$$\vec{E} = 2\, e^{-\alpha z} \sin\left(10^8 t - \beta z\right) a_y \ \text{V/m}$$

If the medium is characterized by $\varepsilon_r = 1$, $\mu_r = 20$ and $\sigma = 3$ mhos/m. find α, β and H.

Sol: we need to determine $\frac{\sigma}{\omega\varepsilon}$ to know whether the medium is dielectric or a good Conductor.

$$\frac{\sigma}{\omega\varepsilon} = \frac{3}{10^8 \times 1 \times 8.85 \times 10^{-12}} = .3389.83 \times 10^4 = \underline{3390}$$

$$\frac{\sigma}{\omega\varepsilon} >>> 1$$

Thus the medium is a good conductor,

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{10^8 \times 4\pi \times 10^{-7} \times 20}{3}}$$

$$= 61.4$$

$$\therefore \alpha = 61.4 \text{ Neper/m}, \quad \beta = 61.4 \text{ rad/m}$$

$$|\eta| = \sqrt{\frac{\mu\omega}{\sigma}} = \left[\frac{4\pi \times 10^{-7} \times 20 \times 10^8}{3}\right]^{1/2}$$

$$= \sqrt{\frac{800\pi}{3}} = \underline{28.94\,\Omega}$$

$$\tan 2\theta_n = \frac{\sigma}{\omega\varepsilon} = 3390 \longrightarrow \theta_n = 45^\circ = \pi/4$$

Hence $H = H_0 e^{-\alpha z} \sin(\omega t - \beta z - \pi/4)\, a_H$

and $H_0 = \frac{E_0}{|\eta|} = \frac{2}{\sqrt{\frac{800\pi}{3}}} = \underline{69.1 \times 10^{-3}}$

A medium that is regarded as a good Conductor at low frequencies may be a Good dielectric at high frequencies $\theta = 2\theta_n$

$$H = -69.1e^{-61.4z}\sin(10^8 t - 61.42z - \pi/4)a_x \text{ mA/m}$$

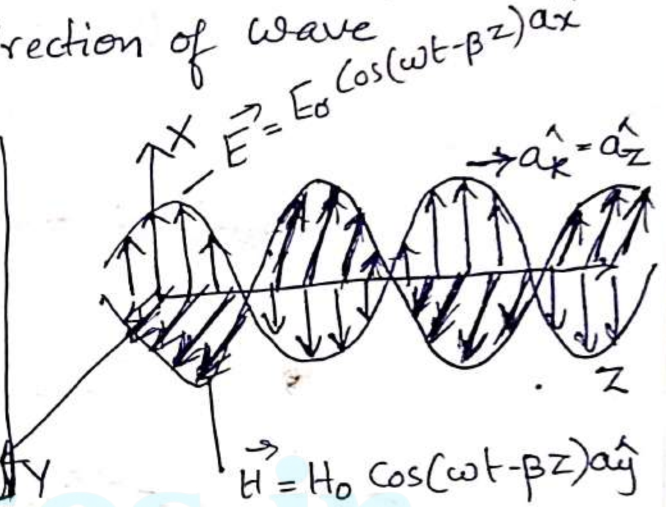
where $\hat{a}_H = \hat{a}_K \times \hat{a}_E$

$= \hat{a}_z \times \hat{a}_y = -\hat{a}_x$

Note

If $\hat{a}_E$, $\hat{a}_H$ and $\hat{a}_K$ are unit Vectors along the $\vec{E}$ field, $\vec{H}$ field and the direction of wave propagation then

$$\boxed{\begin{aligned} \hat{a}_K \times \hat{a}_E &= \hat{a}_H \\ \hat{a}_K \times \hat{a}_H &= -\hat{a}_E \\ \text{or} \quad \hat{a}_E \times \hat{a}_H &= \hat{a}_K \end{aligned}}$$



Both $\vec{E}$ and $\vec{H}$ fields (or EM waves) are everywhere normal to the direction of wave propagation, $\hat{a}_K$. That means, the fields lie in a plane that is transverse or orthogonal to the direction of propagation. They form an EM wave that has ~~no direction of propagation~~ no electric & magnetic field components along the direction of propagation. Such a wave is called a transverse electromagnetic wave. [TEM)

If $\vec{E}$ field and $\vec{H}$ field has constant amplitude or has same magnitude throughout any transverse plane, defined $z =$ constant. Then EM wave is called Uniform Plane Wave

Q A lossy dielectric has an intrinsic impedance of $200\angle 30^\circ\,\Omega$ at a particular frequency. If, at that frequency, the plane wave propagating through the dielectric has the magnetic field component

$$\vec{H} = 10e^{-\alpha x}\cos(\omega t - \tfrac{1}{2}x)\,\hat{a_y}\ \text{A/m}$$

find $\vec{E}$, α and skin depth.

$$H_0 = 10, \quad \frac{E_0}{H_0} = \eta \quad \therefore \eta = 200\angle 30^\circ$$

$$\therefore E_0 = 2000\angle 30^\circ = 2000\,e^{j\pi/6}$$

wave travels in x direction, $\hat{a_k} = \hat{a_x}$, $a_H = \hat{a_y}$

$$\hat{a_k} \times \hat{a}_H = -\hat{a}_E$$

$$\therefore \hat{a_x} \times \hat{a_y} = -\hat{a_z}$$

$$\therefore \vec{E} = -2000\,e^{-\alpha x}\cos(\omega t - \tfrac{1}{2}x + \pi/6)\,a_z\ \text{V/m}.$$

$$\beta = \frac{1}{2}$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+(\sigma/\omega\varepsilon)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+(\sigma/\omega\varepsilon)^2} + 1\right]}$$

$$\therefore \frac{\alpha}{\beta} = \left[\frac{\left(\sqrt{1+(\sigma/\omega\varepsilon)^2} - 1\right)}{\left(\sqrt{1+(\sigma/\omega\varepsilon)^2} + 1\right)}\right]^{1/2}$$

$$\frac{\sigma}{\omega\varepsilon} = \tan 2\theta_n = \tan 60 = \sqrt{3}$$

$$\therefore \frac{\alpha}{\beta} = \left[\frac{2-1}{2+1}\right]^{1/2} = 1/\sqrt{3}$$

$$\alpha = \frac{\beta}{\sqrt{3}} = \frac{1}{2\sqrt{3}} = 0.2887\ \text{Np/m}$$

$$\delta = \frac{1}{\alpha} = 2\sqrt{3} = \underline{3.464\,\text{m}}$$

Q. A plane wave propagating through a medium with $\varepsilon_r = 8$, $\mu_r = 2$ has $\vec{E} = 0.5e^{-z/3}\sin(10^8 t - \beta z)\hat{a}_x$ V/m. Determine a) β, b) The loss tangent c) wave impedance d) wave Velocity e) H field.

Ans Let $x_0 = \sqrt{1+(\sigma/\omega\varepsilon)^2}$, $\alpha = 1/3$

$$\alpha = \omega\sqrt{\frac{\varepsilon_0\varepsilon_r\mu_0\mu_r}{2}(x_0-1)} = \frac{\omega}{c}\sqrt{\frac{16}{2}}\cancel{\sqrt{x_0-x}}$$

Since $c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8$ m/s

$$\alpha = \frac{\omega}{c}\sqrt{\frac{16}{2}(x_0-1)}$$

$$\text{or } \sqrt{x_0-1} = \frac{\alpha c}{\omega\sqrt{8}} = \frac{1/3\times3\times10^8}{10^8\sqrt{8}} = \frac{1}{\sqrt{8}}$$

$$\therefore x_0 = 9/8$$

$$\therefore x_0^2 = \left[1+(\sigma/\omega\varepsilon)^2\right] = (9/8)^2$$

$$1+(\sigma/\omega\varepsilon)^2 = 81/64$$

$$\therefore \sigma/\omega\varepsilon = 0.5154$$

$$\tan 2\theta_n = 0.5154 \quad \therefore \theta_n = 13.63^\circ$$

$$\frac{\beta}{\alpha} = \sqrt{\frac{x_0+1}{x_0-1}} = \sqrt{17}$$

a) $\beta = \alpha\sqrt{17} = 1.374$ rad/m.

b) $\sigma/\omega\varepsilon = 0.5154$

c) $\eta = \sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\varepsilon}}$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\left[1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{1/4}}$$

$$= \frac{\sqrt{\mu/\varepsilon}}{\sqrt{x_0}} = \underline{177.72}$$

d) Velocity $= \omega/\beta = \underline{7.278 \times 10^7 \, m/s}$

e)

$a_K \times a_E = a_H$

$a_z \times \hat{a_x} = \underline{\hat{a_y}}$

$$H = \frac{0.5}{177.5} e^{-z/3} \underline{\sin(10^8 t - \beta z - 13.63^\circ)} \hat{a_y}$$

Q: Find the skin depth, δ at a frequency of 1.6 MHz in aluminium, where $\sigma = 38.2$ MS/m and $\mu_r = 1$. Also find the propagation constant γ and the wave velocity v.

Brewster Angle θ_{Bi}

$$\Gamma_{11} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$ — reflection co-efficient of parallel polarization

From above equation, it is evident that it is possible that $\Gamma_{11} = 0$ because the numerator is the difference of two terms.

Under this condition there is no reflection ($E_{ro} = 0$) and the incident angle at which this takes place is called the Brewster angle θ_{Bi}. The Brewster angle is also known as the polarizing angle.

The Brewster angle is obtained by setting $\theta_i = \theta_{Bi}$ where $\Gamma_{11} = 0$ in equation (10).

i.e $\eta_2 \cos\theta_t = \eta_1 \cos\theta_{Bi}$ ——— (1)

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t}$$

$$\therefore \quad \eta_2\sqrt{1-\sin^2\theta_t} = \eta_1\sqrt{1-\sin^2\theta_{Bi}} \quad \text{——— (2)}$$

Squaring both sides,

$$\eta_2^2\left(1-\sin^2\theta_t\right) = \eta_1^2\left(1-\sin^2\theta_{Bi}\right) \quad \text{——— (3)}$$

By Snell's law,

$$n_1\sin\theta_i = n_2\sin\theta_t$$

$$\therefore \quad \sin\theta_t = \frac{n_1}{n_2}\sin\theta_i$$

$$\sin^2\theta_t = \frac{n_1^2}{n_2^2}\sin^2\theta_i \quad \text{——— (4)}$$

Substituting (4) in (3)

$$\eta_2^2\left(1-\frac{n_1^2}{n_2^2}\sin^2\theta_i\right) = \eta_1^2\left(1-\sin^2\theta_{Bi}\right) \quad \text{——— (5)}$$

For lossless dielectric, $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$, $\eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}}$

$n_1 = c\sqrt{\mu_1\varepsilon_1}$, $n_2 = c\sqrt{\mu_2\varepsilon_2}$

$\therefore$ Equation (5) becomes.

$$\therefore \quad \frac{\mu_2}{\varepsilon_2}\left(1-\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i\right) = \frac{\mu_1}{\varepsilon_1}\left(1-\sin^2\theta_{Bi}\right)$$

As $\theta_i = \theta_{Bi}$

$$\frac{\mu_2}{\varepsilon_2}\left(1-\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_{Bi}\right) = \frac{\mu_1}{\varepsilon_1}\left(1-\sin^2\theta_{Bi}\right)$$

$$\frac{\mu_2}{\varepsilon_2} - \frac{\mu_2 \mu_1 \varepsilon_1}{\mu_2 \varepsilon_2 \varepsilon_2} \sin^2\theta_{Bi} = \frac{\mu_1}{\varepsilon_1} - \frac{\mu_1}{\varepsilon_1} \sin^2\theta_{Bi}$$

$$\left(\frac{\mu_1}{\varepsilon_1} - \frac{\cancel{\mu_2} \mu_1 \varepsilon_1}{\cancel{\mu_2} \varepsilon_2 \varepsilon_2}\right) \sin^2\theta_{Bi} = \frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}$$

$$\sin^2\theta_{Bi} = \frac{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_2}{\varepsilon_2}}{\frac{\mu_1}{\varepsilon_1} - \frac{\mu_1 \varepsilon_1}{\varepsilon_2^2}}$$

$$= \frac{\cancel{\frac{\mu_1}{\varepsilon_1}}\left(1 - \frac{\mu_2 \varepsilon_1}{\mu_1 \varepsilon_2}\right)}{\cancel{\frac{\mu_1}{\varepsilon_1}}\left(1 - \frac{\varepsilon_1}{\cancel{\mu_1}} \frac{\cancel{\mu_1} \varepsilon_1}{\varepsilon_2^2}\right)}$$

$$= \frac{\left(1 - \frac{\mu_2 \varepsilon_1}{\mu_1 \varepsilon_2}\right)}{1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}} \qquad \text{As } \mu_1 = \mu_2 = \mu_0$$

$$= \frac{1 - \frac{\varepsilon_1}{\varepsilon_2}}{1 - \frac{\varepsilon_1^2}{\varepsilon_2^2}} = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2^2 - \varepsilon_1^2} \cdot \varepsilon_2$$

$$= \frac{\cancel{\varepsilon_2 - \varepsilon_1}}{(\varepsilon_2 + \varepsilon_1)\cancel{(\varepsilon_2 - \varepsilon_1)}} \cdot \varepsilon_2 = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}$$

$$\sin^2\theta_{B1} = \frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \quad \text{or} \quad \boxed{\sin\theta_{Bi} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1 + \varepsilon_2}}}$$

$$\boxed{\text{or} \quad \tan\theta_{Bi} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1}}$$

Showing that there is a Brewster angle for any Combination of ε_1 and ε_2