Module-4

Discrete-time Fourier series (DTFS)

For discrete time periodic signal $x(n)$, the discrete time Fourier Series representation

$$x(n) = \sum_{k=\langle N \rangle} X_k e^{jk\omega_0 n}$$

Where the Fourier Series Coefficient

$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-jk\omega_0 n}$$

or

$$X_k = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-jk\omega_0 n}$$

Q. Determine the Fourier Series Coefficients of the signal $x(n)$ and plot its magnitude and phase spectrum.

$$x(n) = 1 + \sin\left(\frac{2\pi}{N}n\right) + 3\cos\left(\frac{2\pi}{N}n\right) + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

The given signal is periodic with period N and frequency $\omega_0 = \frac{2\pi}{N}$

$$\therefore x(n) = 1 + \sin\omega_0 n + 3\cos\omega_0 n + \cos\left(2\omega_0 n + \frac{\pi}{2}\right)$$

$$= 1 + \left(\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right) + 3\left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right) + \frac{e^{j(2\omega_0 n + \pi/2)} + e^{-j(2\omega_0 n + \pi/2)}}{2}$$

$$= 1 + \left(\frac{3}{2} + \frac{1}{2j}\right)e^{j\omega_0 n} + \left(\frac{3}{2} - \frac{1}{2j}\right)e^{-j\omega_0 n} + \left(\frac{1}{2}e^{j\pi/2}\right)e^{j2\omega_0 n} + \left(\frac{1}{2}e^{-j\pi/2}\right)e^{-j2\omega_0 n}$$

Comparing with general DTFS equation

$$x(n) = X_0 + X_1 e^{j\omega_0 n} + X_{-1}e^{-j\omega_0 n} + X_2 e^{j2\omega_0 n} + X_{-2}e^{-j2\omega_0 n} + \cdots$$

$\therefore \quad X_0 = 1$

$X_1 = \frac{3}{2} + \frac{1}{2j} \qquad = \frac{3}{2} - \frac{1}{2}j$

$X_{-1} = \frac{3}{2} - \frac{1}{2j} \qquad = \frac{3}{2} + \frac{1}{2}j$

$X_2 = \frac{1}{2}j$

$X_{-2} = -\frac{1}{2}j$

magnitude

$|X_0| = 1 \qquad |X_1| = |X_{-1}| = \frac{\sqrt{10}}{2}$

$|X_2| = |X_{-2}| = \frac{1}{2}$

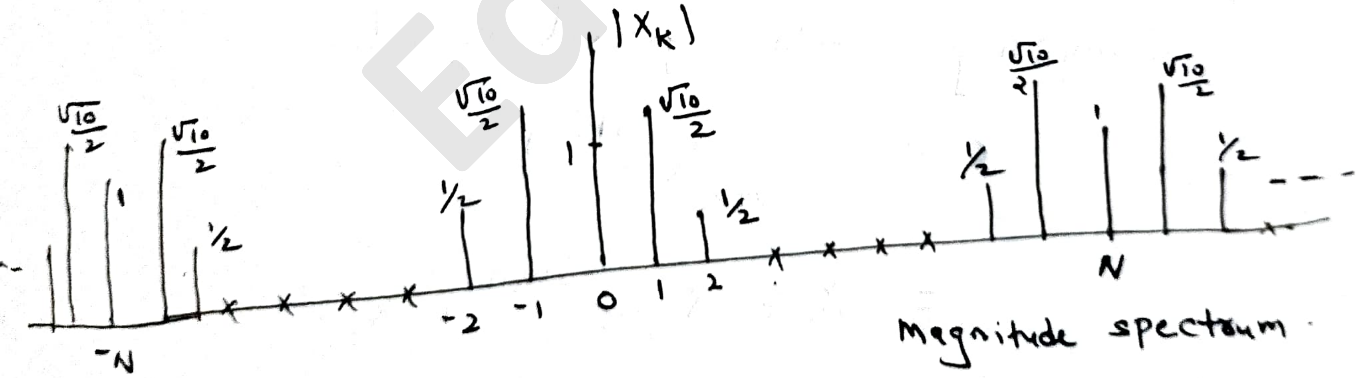
phase

$\angle X_0 = 0 \qquad \angle X_1 = -\tan^{-1}(1/3)$

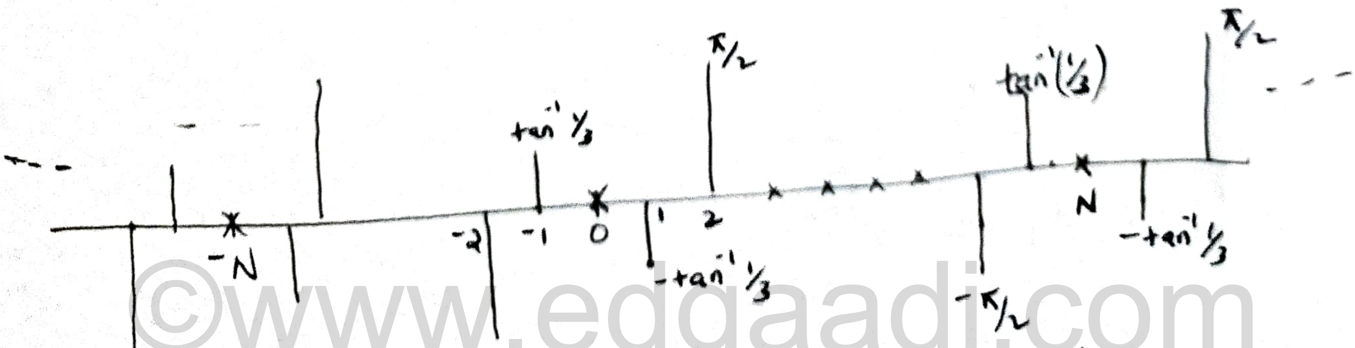
$\angle X_{-1} = \tan^{-1}(1/3)$

$\angle X_2 = \pi/2$

$\angle X_{-2} = -\pi/2$



Magnitude spectrum



Phase spectrum

Q.) Determine the Fourier series coefficient of the signal

$$x(n) = 2 + \cos\left(\frac{\pi}{3}n + \frac{\pi}{4}\right)$$

$$x(n) = 2 + \frac{e^{j(\frac{\pi}{3}n + \frac{\pi}{4})} + e^{-j(\frac{\pi}{3}n + \frac{\pi}{4})}}{2}$$

$$= 2 + \frac{e^{j\pi/4}}{2} e^{j\frac{\pi}{3}n} + \frac{e^{-j\pi/4}}{2} e^{-j\frac{\pi}{3}n}$$

$$N = \frac{2\pi}{\pi/3} m$$

$$= 6m$$

for $m = 1$ $N = 6$

$$\therefore \omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$$

Comparing with general equation

$$x(n) = \sum_{k=\langle N\rangle} X_k e^{jk\omega_0 n}$$

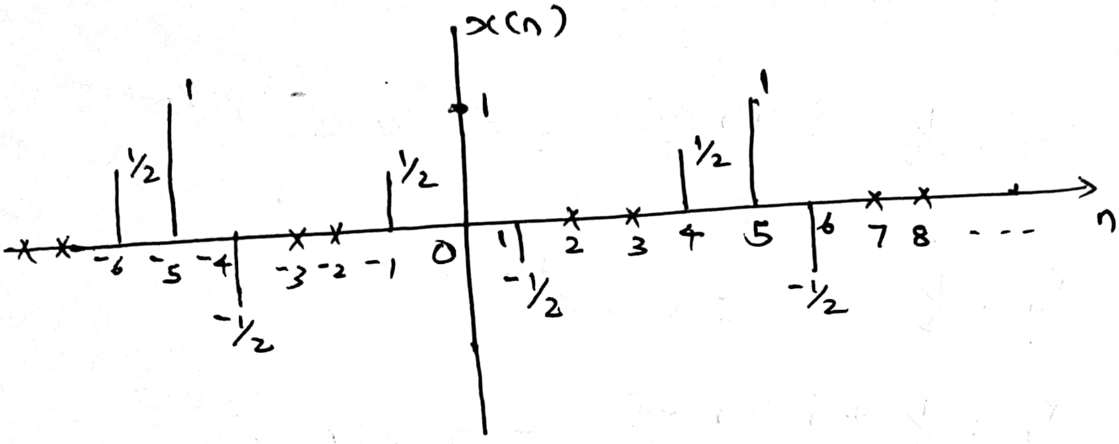
$$= X_0 + X_{-1} e^{-j\omega_0 n} + X_1 e^{j\omega_0 n} + \cdots$$

$X_0 = 2$

$X_{-1} = \frac{e^{-j\pi/4}}{2}$

$X_1 = \frac{e^{j\pi/4}}{2}$

9. Find the frequency domain representation of the signal



Here $N = 5$

$\therefore \omega_0 = \frac{2\pi}{5}$

signal has odd symmetry

$\therefore$ 'n' varies from -2 to 2

$$X_K = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\omega_0 n}$$

$$= \frac{1}{5} \sum_{n=-2}^{2} x(n)\, e^{-jk\frac{2\pi}{5}n}$$

$$= \frac{1}{5} \left[x(-2)\, e^{jk4\pi/5} + x(-1)\, e^{jk2\pi/5} + x(0) + x(1)\, e^{-jk2\pi/5} + x(2)\, e^{-jk4\pi/5} \right]$$

$$= \frac{1}{5} \left[1 + \frac{1}{2} e^{jk2\pi/5} - \frac{1}{2} e^{-jk2\pi/5} \right]$$

$$= \frac{1}{5} \left[1 + j\sin\left(k\frac{2\pi}{5}\right) \right]$$

Here k also varies from -2 to 2.

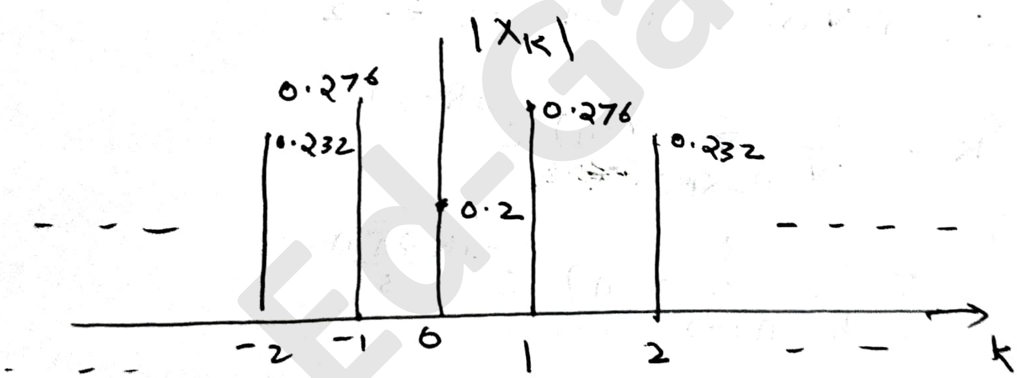
$$X_{-2} = \frac{1}{5} - j\frac{\sin 4\pi/5}{5} = 0.232\, e^{-j0.531}$$

$$X_{-1} = \frac{1}{5} - j\frac{\sin \frac{2\pi}{5}}{5} = 0.276\, e^{-j0.76}$$

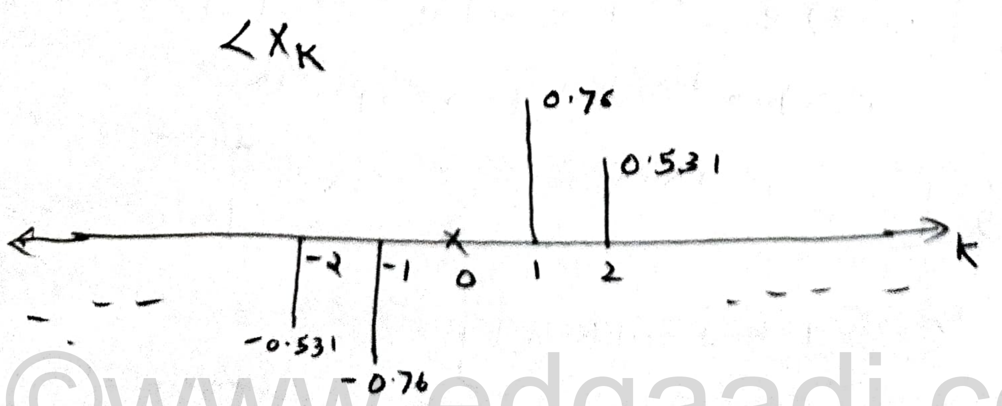
$$X_0 = \frac{1}{5} = 0.2\, e^{j0}$$

$$X_1 = \frac{1}{5} + j\frac{\sin 2\pi/5}{5} = 0.276\, e^{j0.76}$$

$$X_2 = \frac{1}{5} + j\frac{\sin 4\pi/5}{5} = 0.232\, e^{j0.531}$$



magnitude spectrum



phase spectrum.

Properties of DTFS.

1. Linearity

If $x(n)$ and $y(n)$ denote two periodic signals with period N, and

$$x(n) \longleftrightarrow X_k$$

$$y(n) \longleftrightarrow Y_k$$

then $z(n) = ax(n) + by(n) \longleftrightarrow Z_k = aX_k + bY_k$

proof

$$Z_k = \frac{1}{N} \sum_{n=\langle N \rangle} z(n) e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} \left[ax(n) + by(n)\right] e^{-jk\omega_0 n}$$

$$= a \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-jk\omega_0 n} + b \frac{1}{N} \sum_{n=\langle N \rangle} y(n) e^{-jk\omega_0 n}$$

$$= aX_k + bY_k.$$

2. Time scaling

If $x(n) \longleftrightarrow X_k$

the $y(n) = x(n-n_0) \longleftrightarrow Y_k = X_k e^{-jk\omega_0 n_0}$

proof

$$Y_k = \frac{1}{N} \sum_{n=\langle N \rangle} y(n) e^{-jk\omega_0 n_0}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} y(n) e^{-jk\omega_0 n_0}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n-n_0)\, e^{-jk\omega_0 n}$$

put $m = (n - n_0)$

as $n \to 0 \quad m \to -n_0$

$n \to N-1 \quad m \to N-1-n_0$

$$\therefore Y_k = \frac{1}{N} \sum_{m=-n_0}^{N-1-n_0} x(m)\, e^{-jk\omega_0 m}\, e^{-jk\omega_0 n_0}$$

$$= X_k\, e^{-jk\omega_0 n_0}$$

3. Frequency shifting

If $x(n) \longleftrightarrow X_k$

then $y(n) = e^{jM\omega_0 n} x(n) \longleftrightarrow Y_k = X_{k-m}$

proof

$$Y_k = \frac{1}{N} \sum_{n=\langle N \rangle} y(n)\, e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} y(n)\, e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} e^{jM\omega_0 n}\, e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j(k-m)\omega_0 n}$$

$$= X_{k-m}$$

4) Time reversal

If $x(n) \longleftrightarrow X_k$

then $y(n) = x(-n) \longleftrightarrow Y_k = X_{-k}$

Proof

$$Y_k = \frac{1}{N} \sum_{n=\langle N \rangle} y(n) e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} y(n) e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} x(-n) e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{m=-(N-1)}^{0} x(m) e^{-j(-k)\omega_0 m}$$

$$= X_{-k}$$

If $x(n)$ is even

ie, $x(-n) = x(n)$

then

$$X_{-k} = X_k$$

If $x(n)$ is odd

ie, $x(-n) = -x(n)$

then $X_{-k} = -X_k$.

5) Parseval's Relation

If $x(n) \longleftrightarrow X_k$

then $\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \sum_{k=\langle N \rangle} |X_k|^2$

Proof

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2 = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, x^*(n)$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n) \left(\sum_{k=\langle N \rangle} X_k e^{jkw_0 n} \right)^*$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n) \sum_{k=\langle N \rangle} X_k^* e^{-jkw_0 n}$$

$$= \sum_{k=\langle N \rangle} X_k^* \left(\frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-jkw_0 n} \right)$$

$$= \sum_{k=\langle N \rangle} X_k^* \, X_k$$

$$= \sum_{k=\langle N \rangle} |X_k|^2$$

Discrete-time Fourier Transform (DTFT)

The Fourier transform of any discrete-time signal $x(n)$ is given as

$$F[x(n)] = X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

The inverse DTFT of $X(w)$ is

$$F^{-1}[X(e^{jw})] = x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn}\, dw$$

Existence of DTFT

The Fourier transform exists for a discrete-time sequence $x(n)$ if and only if the sequence is absolutely summable.

ie, $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$

1) Find the DTFT of

a) $x(n) = \delta(n)$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-jwn}$$

$$= 1$$

b) $x(n) = u(n)$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} u(n)\, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (e^{-jw})^n$$

$$= \frac{1}{1-e^{-jw}}$$

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad a<1}$$

c) $x(n) = a^n u(n)$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n)\, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} a^n\, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (a e^{-jw})^n = \frac{1}{1-ae^{-jw}}$$

d) $x(n) = a^{|n|}$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} a^{|n|}\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{-1} a^{-n}\, e^{-jwn} + \sum_{n=0}^{\infty} a^{n}\, e^{-jwn}$$

$$= \sum_{n=1}^{\infty} a^{n}\, e^{jwn} + \sum_{n=0}^{\infty} a^{n}\, e^{-jwn}$$

$$= \sum_{n=1}^{\infty} (a e^{jw})^{n} + \sum_{n=0}^{\infty} (a e^{-jw})^{n}$$

$$= a e^{jw}\left[\frac{1}{1-a e^{jw}}\right] + \frac{1}{1-a e^{-jw}}$$

$$= \frac{1-a^2}{1-2a\cos w + a^2}$$

e) $x(n) = 3^{n} u(n)$

The given sequence is not absolutely summable. Therefore, DTFT does not exist.

f) $x(n) = \{1, -2, 2, 3\}$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=0}^{3} x(n)\, e^{-jwn}$$

$$= x(0)\, e^{0} + x(1)\, e^{-jw} + x(2)\, e^{-j2w} + x(3)\, e^{-j3w}$$

$$= 1 + -2e^{-jw} + 2e^{-j2w} + 3e^{-j3w}$$

g) $x(n) = (0.5)^n u(n) + 2^n u(-n-1)$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} (0.5)^n u(n)\, e^{-jwn} + \sum_{n=-\infty}^{\infty} 2^n u(-n-1)\, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (0.5)^n e^{-jwn} + \sum_{n=-\infty}^{-1} 2^n e^{-jwn}$$

$$= \sum_{n=0}^{\infty} (0.5)^n e^{-jwn} + \sum_{n=1}^{\infty} 2^{-n} e^{jwn}$$

$$= \sum_{n=0}^{\infty} (0.5e^{-jw})^n + \sum_{n=1}^{\infty} (2^{-1} e^{jw})^n$$

$$= \frac{1}{1-0.5e^{-jw}} + \frac{1}{1-2^{-1}e^{jw}}$$

h) $x(n) = \cos w_0 n \, u(n)$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} \cos w_0 n \, u(n) e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \cos w_0 n \, e^{-jwn}$$

$$= \sum_{n=0}^{\infty} \left[\frac{e^{jw_0 n} + e^{-jw_0 n}}{2}\right] e^{-jwn}$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{jw_0 n} e^{-jwn} + \sum_{n=0}^{\infty} e^{-jw_0 n} e^{-jwn}\right]$$

$$= \frac{1}{2}\left[\sum_{n=0}^{\infty} e^{j(w_0-w)} + \sum_{n=0}^{\infty} e^{-j(w_0+w)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-e^{j(w_0-w)}} + \frac{1}{1-e^{-j(w_0+w)}}\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-j(w_0+w)} + 1-e^{j(w_0-w)}}{1+e^{-j2w} - e^{-jw}(e^{jw_0}+e^{-jw_0})}\right]$$

$$= \frac{1-e^{-jw}\cos w_0}{1-2e^{-jw}\cos w_0 + e^{-j2w}}$$

Find the DTFT of

$$x(n) = \begin{cases} A, & |n| \le N \\ 0, & |n| > N \end{cases}$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-jwn}$$

$$= \sum_{n=-N}^{N} A e^{-jwn}$$

$$= \sum_{n=-N}^{-1} A e^{-jwn} + \sum_{n=0}^{N} A e^{-jwn}$$

$$= \sum_{n=1}^{N} A e^{jwn} + \sum_{n=0}^{N} A e^{-jwn}$$

$$= A e^{jw} \sum_{n=0}^{N-1} e^{jwn} + A \sum_{n=0}^{N} e^{-jwn}$$

$$= A e^{jw} \left[\frac{1 - e^{jwN}}{1 - e^{jw}} \right] + A \left[\frac{1 - e^{-jw(N+1)}}{1 - e^{-jw}} \right]$$

$$= A \left[\frac{e^{jw} - e^{jw(N+1)}}{1 - e^{jw}} \right] + A \left[\frac{1 - e^{-jw(N+1)}}{1 - e^{-jw}} \right]$$

$$= A \left[\frac{e^{jw} - 1 - e^{jw(N+1)} + e^{jwN} + 1 - e^{+jw} - e^{-jw(N+1)} + e^{-jwN}}{1 + 1 - e^{jw} - e^{-jw}} \right]$$

$$= A\left[\frac{\left(e^{jwN}+e^{-jwN}\right)-\left(e^{jw(N+1)}+e^{-jw(N+1)}\right)}{2-\left(e^{jw}+e^{-jw}\right)}\right]$$

$$= A\left[\frac{2\cos wN - 2\cos w(N+1)}{2-2\cos w}\right]$$

$$= A\left[\frac{2\sin w(N+1/2)\sin w/2}{2\sin^2 w/2}\right]$$

$$= \frac{A\sin w(N+1/2)}{\sin w/2}$$

Properties of DTFT

1. Linearity

If $x_1(n) \longleftrightarrow X_1(e^{jw})$

$x_2(n) \longleftrightarrow X_2(e^{jw})$

then $a x_1(n) + b x_2(n) \longleftrightarrow a X_1(e^{jw}) + b X_2(e^{jw})$

proof

$$FT\left[a x_1(n) + b x_2(n)\right] = \sum_{n=-\infty}^{\infty} \left(a x_1(n) + b x_2(n)\right) e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} a x_1(n)\, e^{-jwn} + \sum_{n=-\infty}^{\infty} b x_2(n)\, e^{-jwn}$$

$$= a X_1(e^{jw}) + b X_2(e^{jw})$$

2. Time shifting

If $x(n) \longleftrightarrow X(e^{jw})$

then $x(n-n_0) \longleftrightarrow X(e^{jw})\, e^{-jwn_0}$.

Proof

$$FT\left[x(n-n_0)\right] = \sum_{n=-\infty}^{\infty} x(n-n_0)\, e^{-jwn}$$

put $n - n_0 = m$

as $n \to -\infty \quad m \to -\infty$

$n \to +\infty \quad m \to +\infty$

$$\therefore FT[x(n-n_0)] = \sum_{m=-\infty}^{\infty} x(m)\, e^{-jw(m+n_0)}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\, e^{-jwm}\, e^{-jwn_0}$$

$$= e^{-jwn_0}\, X(e^{jw})$$

3) Frequency shifting

$$x(n) \longleftrightarrow X(e^{jw})$$

then $y(n) = x(n)\, e^{jw_0 n} \longleftrightarrow Y(e^{jw}) = X(e^{j(w-w_0)})$

proof

$$FT[x(n) e^{jw_0 n}] = \sum_{n=-\infty}^{\infty} x(n)\, e^{jw_0 n}\, e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(w-w_0)n}$$

$$= X(e^{j(w-w_0)})$$

4) Time reversal

If $x(n) \longleftrightarrow X(e^{jw})$

then $x(-n) \longleftrightarrow X(e^{-jw})$

proof

$$FT[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-jwn}$$

Put $-n = m \Rightarrow Y(e^{jw}) = \sum_{m=-\infty}^{\infty} x(m) e^{jwm}$

$$\therefore Y(e^{jw}) = \sum_{m=-\infty}^{\infty} x(m) e^{-j(-w)m}$$

$$= X(e^{-jw})$$

5) Differentiation in Frequency domain

If $x(n) \leftrightarrow X(e^{jw})$

then $-jn\,x(n) \longleftrightarrow \dfrac{dX(e^{jw})}{dw}$

or

$$n\,x(n) \longleftrightarrow j\frac{dX(e^{jw})}{dw}$$

Proof

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn}$$

By differentiating w.r.t w

$$\frac{dX(e^{jw})}{dw} = \sum_{n=-\infty}^{\infty} x(n) e^{-jwn} (-jn)$$

$$= \sum_{n=-\infty}^{\infty} \left(-jn\,x(n)\right) e^{-jwn}$$

$$= FT\left[-jn\,x(n)\right]$$

$$FT\left[n\,x(n)\right] = j\frac{dX(e^{jw})}{dw}$$

6) Convolution property

If $x_1(n) \xleftrightarrow{FT} X_1(e^{jw})$

and $x_2(n) \longleftrightarrow X_2(e^{jw})$

then $x_1(n) * x_2(n) \longleftrightarrow X_1(e^{jw})\, X_2(e^{jw})$

proof

$$FT\left[x_1(n) * x_2(n)\right] = \sum_{n=-\infty}^{\infty}\left(x_1(n) * x_2(n)\right) e^{-jwn}$$

$$= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right) e^{-jwn}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\left(\sum_{n=-\infty}^{\infty} x_2(n-k)\, e^{-jwn}\right)$$

using time shifting property

$$= \sum_{k=-\infty}^{\infty} x_1(k)\; X_2(e^{jw})\, e^{-jwk}$$

$$= X_2(e^{jw}) \sum_{k=-\infty}^{\infty} x_1(k)\, e^{-jwk}$$

$$= X_2(e^{jw})\; X_1(e^{jw})$$

7) parseval's relation

If $x(n) \longleftrightarrow X(e^{jw})$

then $\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{jw})|^2 dw$

proof

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{2\pi} \int_{2\pi} X(e^{jw})\, e^{jwn} dw \right)^*$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{1}{2\pi} \int_{2\pi} X^*(e^{jw})\, e^{-jwn} dw \right)$$

$$= \frac{1}{2\pi} \int_{2\pi} X^*(e^{jw}) \left(\sum_{n=-\infty} x(n)\, e^{-jwn} \right) dw$$

$$= \frac{1}{2\pi} \int_{2\pi} X^*(e^{jw})\, X(e^{jw})\, dw$$

$$= \frac{1}{2\pi} \int_{2\pi} |X(e^{jw})|^2 dw$$

Q). Find the DTFT

a) $x(n) = n\left(\frac{1}{2}\right)^n u(n)$

$$\left(\frac{1}{2}\right)^n u(n) \longleftrightarrow \frac{1}{1-\frac{1}{2}e^{-jw}}$$

using differentiation in frequency domain property

$$n\left(\frac{1}{2}\right)^n u(n) \longleftrightarrow j\frac{d}{dw}\left(\frac{1}{1-\frac{1}{2}e^{-jw}}\right)$$

$$= j\left[\frac{0-1\left(-\frac{1}{2}e^{-jw}\right)(-j)}{\left(1-\frac{1}{2}e^{-jw}\right)^2}\right]$$

$$= \frac{\frac{1}{2}e^{-jw}}{\left(1-\frac{1}{2}e^{-jw}\right)^2}$$

b) $\delta(n-2) - \delta(n+2)$

$$FT\left[\delta(n-2) - \delta(n+2)\right]$$

$$= FT\left[\delta(n-2)\right] - FT\left[\delta(n+2)\right]$$

using time shifting property

$$= e^{-jw(2)}\,1 - e^{-jw(-2)}(1)$$

$$= e^{-j2w} - e^{j2w}$$

$$= -2j\sin 2w$$

c) $x(n) = e^{j3n} u(n)$

$$u(n) \longleftrightarrow \frac{1}{1-e^{-jw}}$$

using frequency shifting property

$$e^{j3n} u(n) \longleftrightarrow \frac{1}{1-e^{-j(w-3)}}$$

d) $u(-n)$

$$u(n) \longleftrightarrow \frac{1}{1-e^{-jw}}$$

Time reversal property

$$u(-n) \longleftrightarrow \frac{1}{1-e^{jw}}$$

Analysis of LTI system using DTFT

If $x(n)$ and $y(n)$ are input and output of an LTI system with impulse response $h(n)$

then $y(n) = x(n) * h(n)$

Apply DTFT

$$Y(e^{jw}) = X(e^{jw}) H(e^{jw})$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

$H(e^{jw})$ is the frequency response of the system.

$h(n) = F^{-1}[H(e^{jw})]$ is known as the impulse response.

1). Consider a causal LTI system that is characterized by the difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 2x(n)$$

a) Find the frequency response $H(e^{jw})$ and the impulse response $h(n)$ of the system.

b) Find the response $y(n)$ if the input to this system is

$$x(n) = \left(\frac{1}{4}\right)^n u(n)$$

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a) $$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 2x(n)$$

Taking DTFT

$$Y(e^{jw}) - \frac{3}{4}Y(e^{jw})e^{-jw} + \frac{1}{8}Y(e^{jw})e^{-j2w} = 2X(e^{jw})$$

$$Y(e^{jw})\left[1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-j2w}\right] = 2X(e^{jw})$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-j2w}}$$

Frequency response.

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}$$

$$H(e^{j\omega}) = \frac{2e^{j2\omega}}{(e^{j\omega}-\frac{1}{2})(e^{j\omega}-\frac{1}{4})}$$

$$\frac{H(e^{j\omega})}{e^{j\omega}} = \frac{2e^{j\omega}}{(e^{j\omega}-\frac{1}{2})(e^{j\omega}-\frac{1}{4})}$$

$$= \frac{A}{e^{j\omega}-\frac{1}{2}} + \frac{B}{e^{j\omega}-\frac{1}{4}}$$

$$A = \cancel{(e^{j\omega}-\tfrac{1}{2})} \frac{2e^{j\omega}}{\cancel{(e^{j\omega}-\frac{1}{2})}(e^{j\omega}-\frac{1}{4})}\Bigg|_{e^{j\omega}=\frac{1}{2}}$$

$$= \underline{4}$$

$$B = \cancel{(e^{j\omega}-\tfrac{1}{4})} \frac{2e^{j\omega}}{(e^{j\omega}-\frac{1}{2})\cancel{(e^{j\omega}-\frac{1}{4})}}\Bigg|_{e^{j\omega}=\frac{1}{4}}$$

$$= \underline{\underline{2}}$$

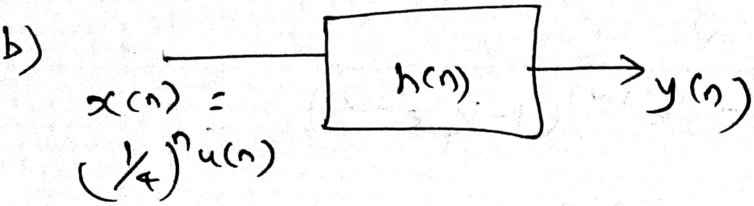
$$\therefore \frac{H(e^{j\omega})}{e^{j\omega}} = \frac{4}{e^{j\omega}-\frac{1}{2}} + \frac{2}{e^{j\omega}-\frac{1}{4}}$$

$$H(e^{j\omega}) = \frac{4e^{j\omega}}{e^{j\omega}-\frac{1}{2}} + \frac{2e^{j\omega}}{e^{j\omega}-\frac{1}{4}}$$

Impulse response

$$\therefore h(n) = \underline{\underline{4\left(\tfrac{1}{2}\right)^n u(n) + 2\left(\tfrac{1}{4}\right)^n u(n)}}$$

b)



$$y(n) = x(n) * h(n)$$

$$Y(e^{jw}) = X(e^{jw})\, H(e^{jw})$$

$$= \frac{1}{(1 - \frac{1}{4}e^{-jw})} \left(\frac{2}{1 - \frac{3}{4}e^{-jw} + \frac{1}{8}e^{-j2w}} \right)$$

$$= \frac{2}{(1 - \frac{1}{4}e^{-jw})^2 (1 - \frac{1}{2}e^{-jw})}$$

$$= \frac{A}{1 - \frac{1}{2}e^{-jw}} + \frac{B}{1 - \frac{1}{4}e^{-jw}} + \frac{C}{(1 - \frac{1}{4}e^{-jw})^2}$$

on solving

$A = 8$

$B = -4$

$C = -2$

$$\therefore Y(e^{jw}) = \frac{8}{1 - \frac{1}{2}e^{-jw}} + \frac{-4}{1 - \frac{1}{4}e^{-jw}} \frac{-2}{(1 - \frac{1}{4}e^{-jw})^2}$$

$$\therefore y(n) = 8(1/2)^n u(n) - 4(1/4)^n u(n) - 2(n+1)(1/4)^n u(n)$$

2) Suppose that we want to design a discrete-time LTI system with the property that if the input is

$$x(n) = (1/2)^n u(n) - 1/4\,(1/2)^{n-1} u(n-1)$$

then output is

$$y(n) = (1/3)^n u(n).$$

a) Find the impulse response and frequency response.

b) Find a difference equation relating input and output.

$$Y(e^{jw}) = \frac{1}{1 - 1/3\, e^{-jw}}$$

$$X(e^{jw}) = \frac{1}{1 - 1/2\, e^{-jw}} - 1/4 \left(\frac{1}{1 - 1/2\, e^{-jw}} \right) e^{-jw}$$

$$= \frac{1 - 1/4\, e^{-jw}}{1 - 1/2\, e^{-jw}}$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})}$$

$$= \frac{1 - 1/2\, e^{-jw}}{(1 - 1/3\, e^{-jw})(1 - 1/4\, e^{-jw})}$$

$$H(e^{jw}) = \frac{e^{jw}\left(e^{jw}-\frac{1}{2}\right)}{\left(e^{jw}-\frac{1}{3}\right)\left(e^{jw}-\frac{1}{4}\right)}$$

$$\frac{H(e^{jw})}{e^{jw}} = \frac{A}{e^{jw}-\frac{1}{3}} + \frac{B}{e^{jw}-\frac{1}{4}}$$

$$A = \cancel{\left(e^{jw}-\frac{1}{3}\right)} \frac{\left(e^{jw}-\frac{1}{2}\right)}{\cancel{\left(e^{jw}-\frac{1}{3}\right)}\left(e^{jw}-\frac{1}{4}\right)}\Bigg|_{e^{jw}=\frac{1}{3}}$$

$$= \frac{\frac{1}{3}-\frac{1}{2}}{\frac{1}{3}-\frac{1}{4}} = \underline{\underline{-2}}$$

$$B = \cancel{\left(e^{jw}-\frac{1}{4}\right)} \frac{\left(e^{jw}-\frac{1}{2}\right)}{\left(e^{jw}-\frac{1}{3}\right)\cancel{\left(e^{jw}-\frac{1}{4}\right)}}\Bigg|_{e^{jw}=\frac{1}{4}}$$

$$= \underline{\underline{3}}$$

$$\therefore H(e^{jw}) = -2\frac{e^{jw}}{e^{jw}-\frac{1}{3}} + 3\frac{e^{jw}}{e^{jw}-\frac{1}{4}}$$

$$h(n) = \underline{\underline{-2\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{4}\right)^n u(n)}}$$

b) $$H(e^{jw}) = \frac{1-\frac{1}{2}e^{-jw}}{\left(1-\frac{1}{3}e^{-jw}\right)\left(1-\frac{1}{4}e^{-jw}\right)}$$

$$\frac{Y(e^{jw})}{X(e^{jw})} = \frac{1-\frac{1}{2}e^{-jw}}{1-\frac{7}{12}e^{-jw}+\frac{1}{12}e^{-j2w}}$$

$$Y(e^{jw}) - \frac{7}{12}e^{-jw}Y(e^{jw}) + \frac{1}{12}e^{-j2w}Y(e^{jw}) = X(e^{jw}) - \frac{1}{2}e^{-jw}X(e^{jw})$$

Taking Inverse

$$y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) - \frac{1}{2}x(n-1)$$

magnitude and phase response

Fourier transform $X(e^{jw})$ is a complex function

$$X(e^{jw}) = X_R(e^{jw}) + j\,X_I(e^{jw})$$

$X_R(e^{jw})$ is real part

$X_I(e^{jw})$ is imaginary part

$|X(e^{jw})|$ is called magnitude spectrum

$\angle X(e^{jw})$ is known as phase spectrum.

$$|X(e^{jw})| = \sqrt{X_R^2(e^{jw}) + X_I^2(e^{jw})}$$

$$\angle X(e^{jw}) = \tan^{-1}\left[\frac{X_I(e^{jw})}{X_R(e^{jw})}\right]$$

1) plot the magnitude and phase response of

$$x(n) = a^n u(n) \quad ; \quad |a| < 1$$

$$FT[a^n u(n)] = \frac{1}{1 - ae^{-jw}}$$

$$= \frac{1 - ae^{jw}}{(1 - ae^{-jw})(1 - ae^{jw})}$$

$$= \frac{1 - a\cos w - ja\sin w}{1 - 2a\cos w + a^2}$$

$$= \frac{1 - a\cos w}{1 - 2a\cos w + a^2} - j\frac{a\sin w}{1 - 2a\cos w + a^2}$$

$$= X_R(e^{jw}) + jX_I(e^{jw})$$

magnitude spectrum

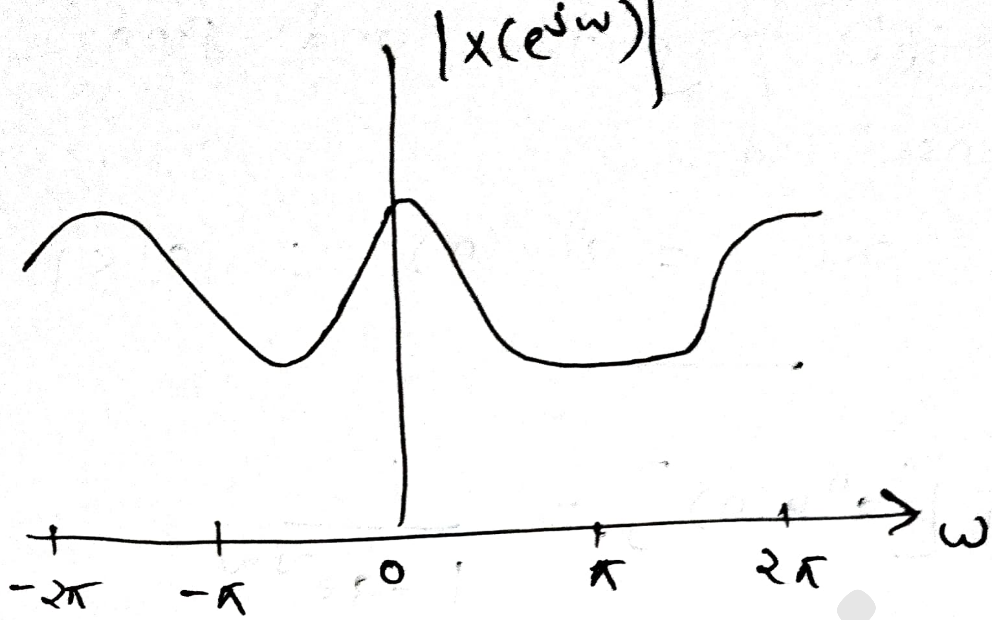
$$|X(e^{jw})| = \sqrt{X_R^2(e^{jw}) + X_I^2(e^{jw})}$$

$$= \frac{1}{\sqrt{1 - 2a\cos w + a^2}}$$

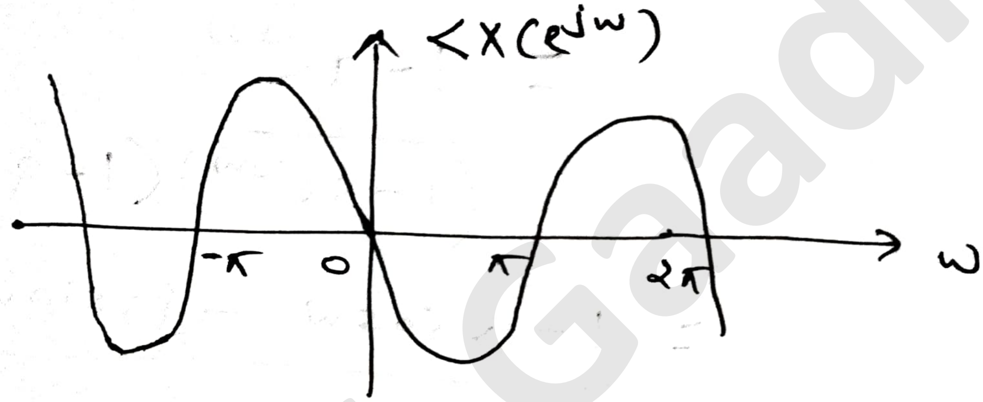
phase spectrum

$$\angle X(e^{jw}) = \tan^{-1}\left(\frac{X_I(e^{jw})}{X_R(e^{jw})}\right)$$

$$= \tan^{-1}\left[\frac{a\sin w}{1 - a\cos w}\right]$$



magnitude spectrum



phase spectrum.