
KTU
NOTES

MODULE - II

A field which consists of both electric and magnetic components is called an electromagnetic field (EM field). In static EM fields, electric and magnetic fields are independent of each other.

In a dynamic EM field, the two fields are interdependent ie a time-varying electric field will have a time-varying magnetic field associated with it. Time-varying electric & magnetic fields are represented as $E(x,y,z,t)$ and $H(x,y,z,t)$.

* Electrostatic fields are produced by static electric charges.

* Magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles).

* time-varying fields or waves are due to accelerated charges or time varying currents such as shown in figure. Any pulsating current will produce radiation (time-varying fields).

Stationary charges - electrostatic fields

Steady currents - Magnetostatic fields

time-varying currents - electromagnetic fields or waves.

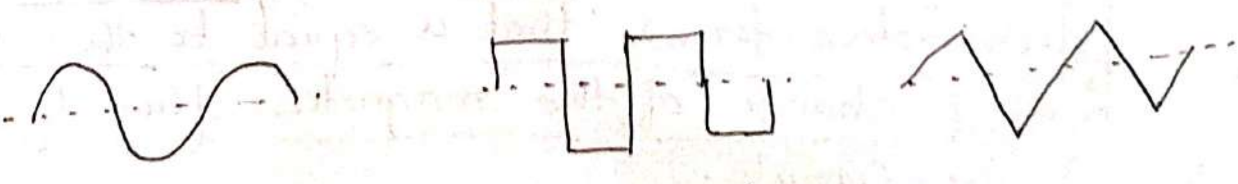


Fig: Time-Varying current.

Maxwell's equations

Maxwell's equations are based on three fundamental laws.

1) Ampere's law
2) Faraday's law
3) Gauss's law.

1. From Gauss's law.

Electric flux $\psi = \oint_s \vec{D} \cdot ds = Q_{enc} = \int_v \rho_v dv$ —— (1)

And on applying divergence

$$\boxed{\oint_s \vec{D} \cdot ds = \int_v \nabla \cdot \vec{D}\, dv} \quad \text{—— (2)}$$

Comparing (1) and (2)

$\therefore \int_v \rho_v dv = \int_v \nabla \cdot \vec{D}\, dv$ → Integral form of Maxwell's 1st equation

$\therefore \boxed{\rho_v = \nabla \cdot \vec{D}}$ → Point form / Differential form.

Similarly,

$\boxed{\nabla \cdot B = 0}$ → point form / Differential form

$\boxed{\oint_s \vec{B} \cdot ds = 0}$ → Integral form

2. From Faraday's law.

In 1820, Oersted discovered that a steady current produces a magnetic field. Conversely, Michael Faraday discovered that a time-varying magnetic field would produce an electric current.

Faraday's law states that a time-varying magnetic field induces a voltage (called electromotive force) that is equal to the time rate of change of the magnetic flux linkage by the circuit.

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\phi_m}{dt}$$

- For a circuit with a single turn,

$$V_{emf} = -\frac{d\phi_m}{dt}$$

$$= -\frac{d}{dt}\int_S \vec{B}\cdot ds$$

$$= -\int_S \frac{\partial \vec{B}}{\partial t}\cdot ds.$$

The emf in a circuit can be represented as the line integral of the electric field around a closed path

$$\boxed{V_{emf} = \oint_L \vec{E}\cdot dl = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot ds}$$

From above equation, it is clear that in a time varying situation, both electric and magnetic fields are present and are interrelated.

By applying Stoke's theorem

$$\oint_L \vec{E}\cdot dl = \int_S (\nabla\times\vec{E})\, ds$$

∴ Equating expressions for V_{emf}

$$V_{emf} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot ds = \int_S (\nabla\times\vec{E})\cdot ds$$

ie $\boxed{\nabla\times\vec{E} = \frac{-\partial \vec{B}}{\partial t}}$ Differential form of Maxwell's equation.

$\boxed{\oint_L \vec{E}\cdot dl = -\int_S \frac{\partial \vec{B}}{\partial t}}$ Integral form of Maxwell's equation.

From Ampere's Circuit law.

$$\nabla \times \vec{H} = \vec{J}$$

But by the Vector identity, divergence of the Curl of any Vector is zero.

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J} \quad \text{——— (1)}$$

The continuity equation states that,

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \quad \text{——— (2)}$$

Equations (1) and (2) are incompatible for time-Varying conditions.

Hence, Maxwell introduced a modification to Ampere's Circuit law.

i.e $$\nabla \times \vec{H} = \vec{J} + \vec{J_d}$$

Now, taking divergence on both sides.

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \nabla \cdot \vec{J_d} = 0$$

$$\therefore \nabla \cdot \vec{J_d} = -\nabla \cdot \vec{J} = \frac{\partial \rho_v}{\partial t}$$

$$= \frac{\partial \nabla \cdot \vec{D}}{\partial t}$$

$$= \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

From this we get,

$$\boxed{\vec{J_d} = \frac{\partial \vec{D}}{\partial t}}$$

This is known as the displacement current density and is different from $\vec{J}$ which is known as conduction current density.

Displacement current $I_d = \int_S \vec{J_d} \cdot ds$

$$= \int_S \frac{\partial \vec{D}}{\partial t} \cdot ds$$

Hence for time-Varying conditions,

$$\boxed{\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$ Differential form of Maxwell's eqn.

$$\boxed{\oint_L \vec{H} \cdot dl = I_{enc} = \int_S \vec{J} \cdot ds + \int_S \frac{\partial \vec{D}}{\partial t} \cdot ds}$$

Integral form of Maxwell's equation.

Conduction Current density $\boxed{J_c = \sigma E}$ — σ Conductivity.

Maxwell's equation can be summarized as,

~~Differe~~ Law's	Differential form	Integral form
Gauss's law	$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot ds = \int_V \rho_v dv$
	$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot ds = 0$ Nonexistence of Magnetic monopole
Faraday's Law	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_L \vec{E} \cdot dl = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot ds$
Ampere's law	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_L \vec{H} \cdot dl = \int_S \vec{J} ds + \int_S \frac{\partial \vec{D}}{\partial t} \cdot ds$

Solutions of Wave Equation

Waves are means of transporting energy or information. Wave is a physical phenomenon that occurs at one place at a given time and is reproduced at other place at a later time.

Examples of EM waves include radio. waves, TV signals, light rays etc.

EM wave equation in the following media is solved using maxwell's equation.

1. free space ($\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$)
2. Lossless dielectric ($\sigma = 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$ or $\sigma << \omega\varepsilon$)
3. Lossy dielectric ($\sigma \neq 0$, $\varepsilon = \varepsilon_0 \varepsilon_r$, $\mu = \mu_0 \mu_r$)
4. Good conductors ($\sigma = \infty$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0 \mu_r$ or $\sigma >> \omega\varepsilon$).

where ω is the angular frequency of the wave.

Wave Equations (For free-space or Lossless or Non-conducting medium)

Consider EM wave in free-space, in a perfect dielectric medium containing no charges and no conduction currents. Then $\rho_v = 0, \sigma = 0$.

We know,

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\vec{J} = \sigma \vec{E}$$

where ε, μ and σ are the permittivity, Permeability and conductivity of the medium. Then, In any electromagnetic phenomenon, the following basic maxwell's equation must be satisfied. (7)

$$\nabla \cdot \vec{B} = 0 \quad \text{——} (1)$$

$$\nabla \cdot \vec{D} = \rho_v \, , \quad \nabla \cdot \vec{D} = 0 \text{ —} (2) \because \rho_v = 0$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \, , \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \text{ —} (3) \because \vec{J} = 0.$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{——} (4)$$

Taking equation (3)

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} = \frac{\varepsilon_0 \partial \vec{E}}{\partial t}$$

Applying Curl Operation on both sides,

$$\nabla \times \nabla \times \vec{H} = \varepsilon_0 \nabla \times \frac{\partial \vec{E}}{\partial t} \quad \text{——} (5)$$

From maxwell's equation, $\nabla \times \vec{E} = \frac{-\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

differentiating both sides,

$$\therefore \nabla \times \frac{\partial \vec{E}}{\partial t} = -\mu_0 \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{——} (6)$$

Substituting (6) in (5)

$$\therefore \nabla \times \nabla \times H = -\varepsilon_0 \mu_0 \frac{\partial^2 H}{\partial t^2}$$

By Vector identity,

$$\nabla \times \nabla \times \vec{H} = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}$$

$$\therefore \nabla(\nabla\cdot\vec{H}) - \nabla^2\vec{H} = -\mu_0\varepsilon_0 \frac{\partial^2\vec{H}}{\partial t^2} \quad \text{———} (7)$$

$$\nabla\cdot\vec{H} = \frac{1}{\mu_0}\nabla\cdot\vec{B} = 0, \quad \because \boxed{\nabla\cdot B = 0}$$

$$\because \vec{B} = \mu_0\vec{H}$$

$\therefore$ Above equation becomes,

$$-\nabla^2\vec{H} = -\mu_0\varepsilon_0 \frac{\partial^2\vec{H}}{\partial t^2}$$

$$\boxed{\nabla^2\vec{H} = \mu_0\varepsilon_0 \frac{\partial^2\vec{H}}{\partial t^2}} \quad \text{———} (8)$$

Taking a similar approach with $\vec{E}$, From equation (4)

$$\nabla\times\vec{E} = \frac{-\partial\vec{B}}{\partial t} = -\mu_0\frac{\partial\vec{H}}{\partial t} \quad \text{———} (9)$$

Applying curl operation on both sides,

$$\nabla\times\nabla\times\vec{E} = -\mu_0\nabla\times\frac{\partial\vec{H}}{\partial t} \quad \text{———} (10)$$

We, know $\nabla\times\vec{H} = \varepsilon_0\frac{\partial\vec{E}}{\partial t}$

Differentiating, above equation

$\therefore$

$$\nabla\times\frac{\partial\vec{H}}{\partial t} = \varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{———} (11)$$

Substituting (11) in (10)

$$\nabla\times\nabla\times\vec{E} = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\mu_0\varepsilon_0 \frac{\partial^2 E}{\partial t^2} \quad (9)$$

$$\nabla\cdot\vec{E} = \frac{1}{\varepsilon_0}\nabla\cdot\vec{D} = 0$$

$$\therefore -\nabla^2\vec{E} = -\mu_0\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\boxed{\therefore \nabla^2\vec{E} = \mu_0\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}} \quad (12)$$

Equation (8) and (12) are called the wave equations. Thus a wave is a function of both space and time.

Electromagnetic Wave Equation (for a Conducting Medium)

Consider an homogeneous, Isotropic, linear, media in which charge density $\rho_v = 0$. Since there are no net charge within the conductor, ~~although~~

Maxwell's Basic Equations,

$$\nabla \cdot \vec{B} = 0 \quad \text{--- (1)}$$

$$\nabla \cdot \vec{D} = 0 \quad \text{--- (2)} \quad \because \rho_v = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{--- (3)}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{--- (4)} \qquad \nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Since the medium has a conductivity σ (mho/meter) the conduction current density

$$\vec{J} = \sigma \vec{E}$$

From eq (4),

$$\nabla \times \vec{H} = \sigma \vec{E} + \frac{\varepsilon \partial \vec{E}}{\partial t}$$

Taking curl on both sides,

$$\nabla \times \nabla \times \vec{H} = \nabla \times \sigma \vec{E} + \nabla \times \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$= \sigma \nabla \times \vec{E} + \varepsilon \left(\nabla \times \frac{\partial \vec{E}}{\partial t} \right)$$

$$= \sigma \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) + \varepsilon \cdot \left(-\mu \frac{\partial^2 \vec{H}}{\partial t^2} \right)$$

$$\nabla \times \nabla \times \vec{H} = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{--- (5)}$$

By Vector identity

$$\nabla \times \nabla \times \vec{H} = \nabla(\nabla \cdot \vec{H}) - \nabla^2 H$$

∴ Equation (5) becomes,

$$\nabla(\nabla \cdot \vec{H}) - \nabla^2 H = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} \quad \text{--- (6)}$$

But here $\nabla \cdot \vec{H} = \frac{1}{\mu} \nabla \cdot \vec{B} = 0$

Equation (6) becomes,

$$-\nabla^2 \vec{H} = -\mu\sigma \frac{\partial \vec{H}}{\partial t} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\boxed{\nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}} \quad \text{--- (7)}$$

Similarly,

From equation (3)

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}$$

Taking curl on both sides,

$$\nabla\times\nabla\times\vec{E} = -\mu\,\nabla\times\frac{\partial\vec{H}}{\partial t} \quad \text{——— } (8)$$

We know,

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}$$

Differentiating both sides,

$$\nabla\times\frac{\partial\vec{H}}{\partial t} = \sigma\frac{\partial\vec{E}}{\partial t} + \varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{——— } (9)$$

Substituting (9) in (8)

$$\nabla\times\nabla\times\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{——— } (10)$$

By Vector identity

$$\nabla\times\nabla\times\vec{E} = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$$

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla\cdot\vec{E} = \frac{1}{\varepsilon}(\nabla\cdot\vec{D}) = 0 \quad \text{——— } (11)$$

$$\therefore\ -\nabla^2\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

or

$$\boxed{\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}} \quad \text{——— } (12)$$

The equation (7) and (12) is known as wave equation in a conducting medium.

Boundary Conditions.

If the electric field exists in a region consisting of two different media, the conditions that the electric field must satisfy at the interface separating the media are called boundary conditions.

These conditions are helpful in determining the field on one side of the boundary. If the field on the other side is known.

We shall consider the boundary conditions at an interface separating,

(1) dielectric - dielectric boundary
(2) Conductor - dielectric boundary
(3) Conductor - free space boundary.

(1) Dielectric strength.

It is the maximum electric field that dielectric can tolerate or withstand.

2) Dielectric Constant.

It is also called relative permittivity. It is the ratio of permittivity of dielectric to that of free space.

$$\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} \qquad \varepsilon_0 = 8.854 \times 10^{-12}\ F/m.$$

To determine Boundary Conditions, we use maxwe equations,

$$\oint_L \vec{E}\, dl = 0$$

$$\oint_S \vec{D} \cdot ds = Q_{enc}.$$

Also, we need to decompose the electric field intensity $\vec{E}$ into two orthogonal components

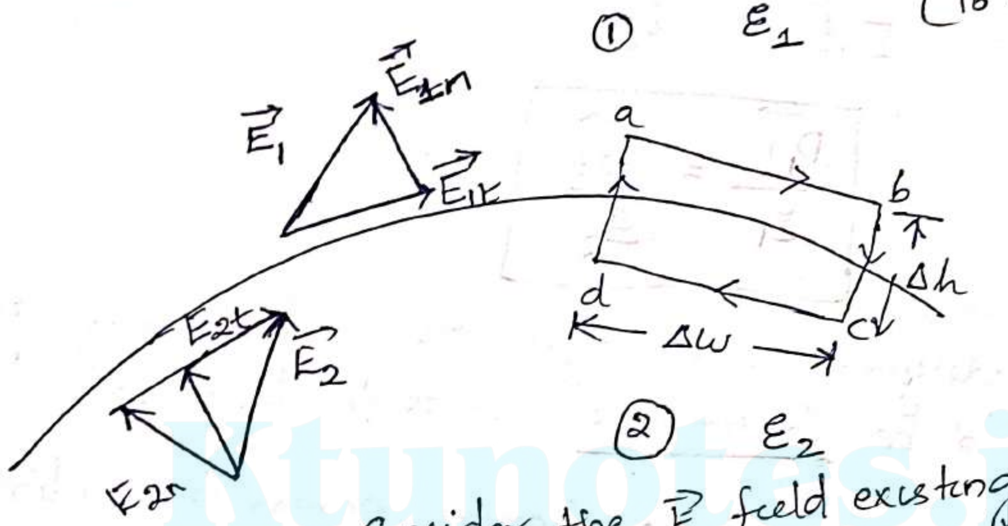
$$\vec{E} = \vec{E_t} + \vec{E_n}$$

$\vec{E_t}$ is the tangential component of $\vec{E}$

$\vec{E_n}$ is the normal component of $\vec{E}$.

Dielectric - Dielectric Boundary Conditions.

(To determine $\vec{E_{1t}} = \vec{E_{2t}}$)



Consider the $\vec{E}$ field existing in a region consisting of two different dielectrics characterised by $\varepsilon_1 = \varepsilon_0 \varepsilon_{r1}$ and $\varepsilon_2 = \varepsilon_0 \varepsilon_{r2}$. $\vec{E_1}$ and $\vec{E_2}$ in media 1 and 2, can be decomposed as

$$\vec{E_1} = \vec{E_{1t}} + \vec{E_{1n}} \quad \text{—— (1)}$$

$$\vec{E_2} = \vec{E_{2t}} + \vec{E_{2n}} \quad \text{—— (2)}$$

$$\oint_L \vec{E} \cdot dl = 0 \quad \text{—— (a)}$$

Applying eqn (a) to the closed path abcda, (assuming path is very small)

$$\therefore \vec{E_{1t}} \Delta w - \vec{E_{1n}} \frac{\Delta h}{2} - \vec{E_{2n}} \frac{\Delta h}{2} - \vec{E_{2t}} \Delta w + \vec{E_{2n}} \frac{\Delta h}{2} + \vec{E_{1n}} \frac{\Delta h}{2} = 0$$

$$\vec{E_{1t}}\,\Delta w - \vec{E_{2t}}\,\Delta w = 0$$

As $\Delta h \to 0$, $\boxed{\vec{E_{1t}} = \vec{E_{2t}}}$

Tangential components of $\vec{E}$ are the same on the two sides of the boundary.

Since $\vec{D} = \varepsilon\vec{E}$, $\vec{D} = \vec{D_t} + \vec{D_n}$

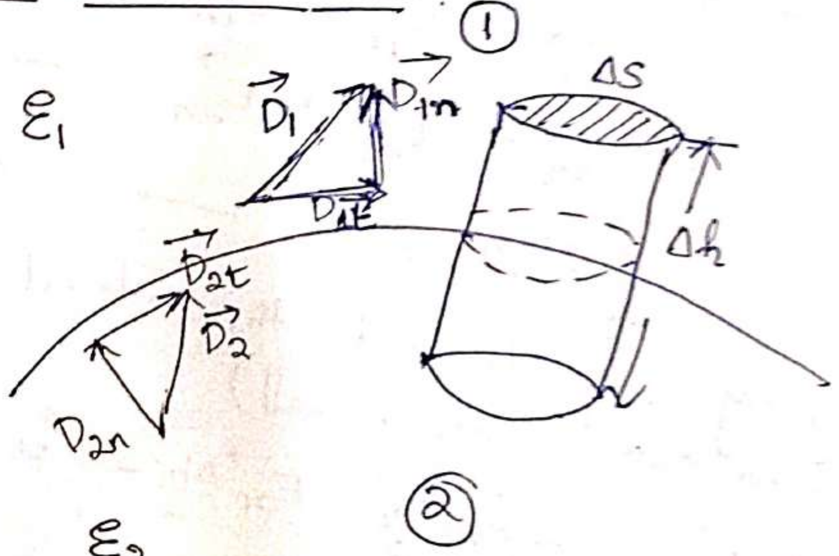
Since $\vec{E_{1t}} = \vec{E_{2t}}$

$$\boxed{\frac{\vec{D_{1t}}}{\varepsilon_1} = \frac{\vec{D_{2t}}}{\varepsilon_2}}$$

$\therefore \vec{E_t}$ undergoes no changes on the boundary and it is said to be Continuous across the boundary

$\vec{D_t}$ undergoes some changes across the interface. Hence D_t is said to be discontinuous across the boundary.

To determine $\vec{D_{1n}} = \vec{D_{2n}}$



Applying the relation

$\oint_S \vec{D} \cdot ds = Q_{enc}$ to the pillbox (Cylindrical Gaussian Surface)

Allowing $\Delta h \to 0$, Contribution from the Cylinder side Vanishes,

$$\therefore \vec{D_{1n}} \Delta S - \vec{D_{2n}} \Delta S = \Delta Q$$

$$\vec{D_{1n}} \Delta S - \vec{D_{2n}} \Delta S = \rho_S \Delta S$$

$$\left(\vec{D_{1n}} - \vec{D_{2n}} \right) \cancel{\Delta S} = \rho_S \cancel{\Delta S}$$

$$\therefore \vec{D_{1n}} - \vec{D_{2n}} = \rho_s$$

Where ρ_s is free charge density at the boundary.

If $\rho_S = 0$, the boundary is free from all charges, then,

$$\vec{D_{1n}} - \vec{D_{2n}} = 0$$

$$\boxed{\vec{D_{1n}} = \vec{D_{2n}}}$$

Since $\vec{D} = \varepsilon \vec{E}$

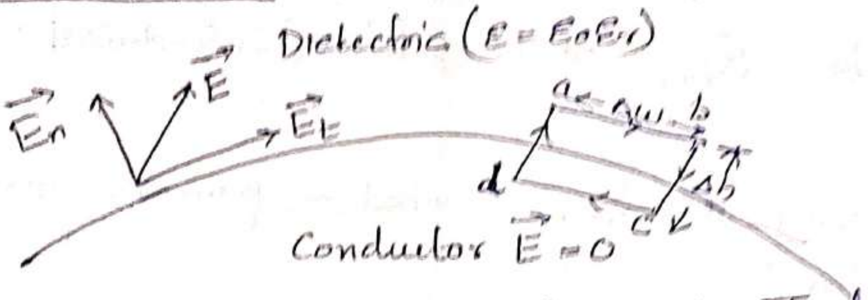
$$\varepsilon_1 \vec{E_{1n}} = \varepsilon_2 \vec{E_{2n}}$$

$$\boxed{\frac{\vec{E_{1n}}}{\vec{E_{2n}}} = \frac{\varepsilon_2}{\varepsilon_1}}$$

Hence Normal component of $\vec{D}$ is continuous at the boundary ($\rho_s = 0$)

Normal Component of $\vec{E}$ is discontinuous.

Conductor – Dielectric Boundary conditions



Assuming perfect Conductor ($\sigma = \infty$). To determine boundary conditions, Consider a closed path abcda, also considering $\vec{E} = 0$ inside the Conductor.

Applying the relation (To determine E_t)

$$\oint_L \vec{E} \cdot dl = 0$$

$$E_t \Delta w - \vec{E_n} \frac{\Delta h}{2} - 0 + \vec{E_n} \frac{\Delta h}{2} = 0$$

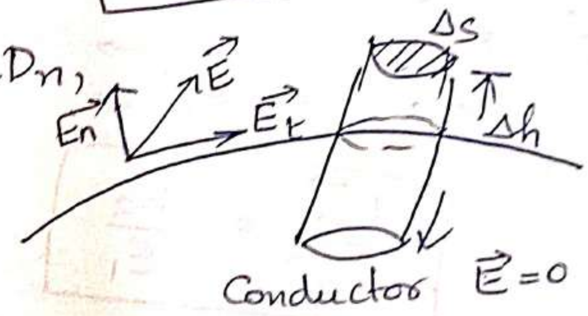
$$E_t \Delta w = 0$$

As $\Delta h \to 0$, $\boxed{E_t = 0}$

Since $D = \varepsilon \vec{E}$

$$\boxed{D_t = \varepsilon \vec{E_t} = 0}$$

To determine D_n,



Applying the relation

$$\oint_S \vec{D} \cdot ds = Q_{enc} = \rho_s \Delta S$$ to the cylindrical Surface

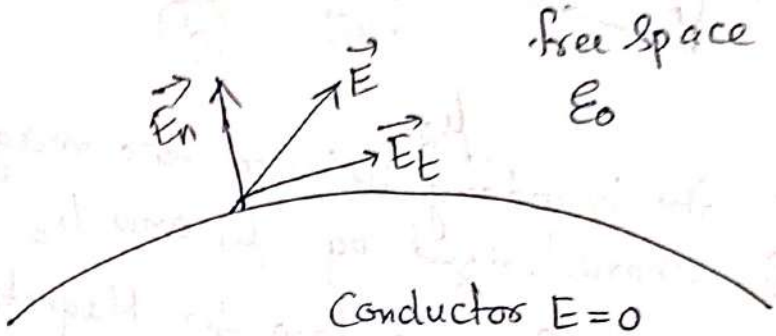
$$\vec{D_n} \Delta S - 0 \cdot \Delta S = \Delta Q$$

$$\vec{D_n} \cancel{\Delta S} = \rho_s \cancel{\Delta S}$$

$$\boxed{\therefore \vec{D_n} = \rho_s}$$

Summary
No electric field exist within a conductor, under static conditions.

Conductor - Free Space boundary Conditions



For free space $\varepsilon_r = 1$,

$\therefore$ We have

$\boxed{\vec{E_t} = 0}$ (For Conductor - dielectric) is applicable

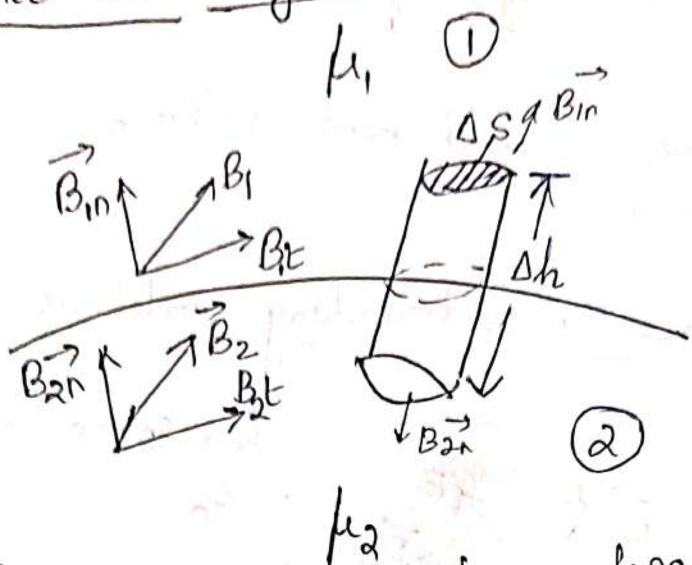
$$\boxed{\vec{D_t} = \varepsilon_0 \vec{E_t} = 0}$$

$$\vec{D_n} = \rho_s$$

$$\boxed{\varepsilon_0 \vec{E_n} = \rho_s.}$$

Free space can be treated as a special dielectric with $\varepsilon_r = 1$.

Magnetic Boundary Conditions [Two different media)



Consider the boundary between two magnetic media ① and ②, characterized by μ_1 and μ_2.

Applying Gauss's law for Magnetic field, $\oint_S \vec{B} \cdot ds = 0$ to the Cylindrical surface

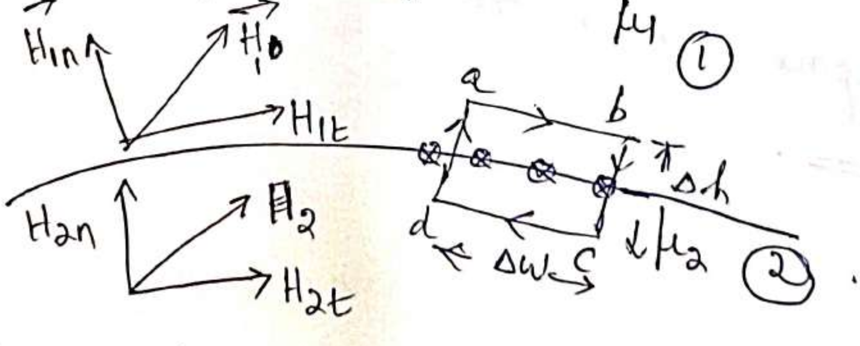
$$\vec{B_{1n}} \Delta S - \vec{B_{2n}} \Delta S = 0$$

$$\therefore \boxed{\vec{B_{1n}} = \vec{B_{2n}}}$$

$$B = \mu H \quad \therefore \boxed{\mu_1 \vec{H_{1n}} = \mu_2 \vec{H_{2n}}}$$

Normal Component of $\vec{B}$ is said to be continuous

Normal Component of $\vec{H}$ is said to be discontinuous



Applying $\oint_L \vec{H} \cdot dl = I_{enc}$ to the closed path abcda,

$$\vec{H_{1t}}\Delta w - \cancel{H_{1n}\frac{\Delta h}{2}} - \cancel{H_{2n}\frac{\Delta h}{2}} - \vec{H_{2t}}\Delta w + \cancel{H_{2n}\frac{\Delta h}{2}} + \cancel{H_{1n}\frac{\Delta h}{2}}$$

$$= I_{enc}$$

$$\vec{H_{1t}}\Delta w - \vec{H_{2t}}\Delta w = I_{enc}$$

$\because$ Total Surface Current
$I = k \cdot \Delta w$

$$\therefore \vec{H_{1t}}\Delta w - \vec{H_{2t}}\Delta w = k \cdot \Delta w$$

$$\therefore \vec{H_{1t}} - \vec{H_{2t}} = k$$

$$\frac{\vec{B_{1t}}}{\mu_1} - \frac{\vec{B_{2t}}}{\mu_2} = k.$$

If the Boundary is free of Surface current, then $k = 0$

$$\therefore \boxed{\vec{H_{1t}} = \vec{H_{2t}}}$$ — continuous across the Boundary

$$\boxed{\frac{\vec{B_{1t}}}{\mu_1} = \frac{\vec{B_{2t}}}{\mu_2}}$$ — Discontinuous.

Maxwell's Equations Using Phasor Notation [Time Varying Fields].

In practice, most generator produce Voltages and currents that Vary sinusoidally with time. Hence the electric and magnetic fields also Vary sinusoidally with time. This sinusoidal time factor can be expressed by using the phasor notation.

The wave equation in phasor form.

ie $\vec{E}(z,t) = Re\{\vec{E}e^{j\omega t}\}$ at some point z in Space,

Maxwell's first equation,

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} = \sigma\vec{E} + \frac{\varepsilon \partial \vec{E}}{\partial t}$$

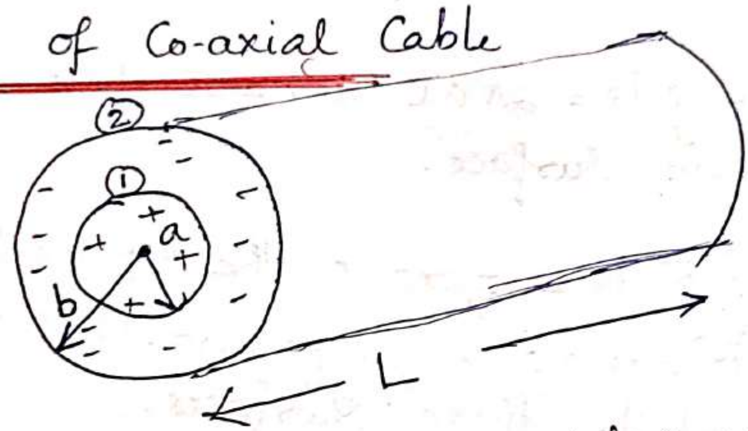
Capacitance:

The capacitance between two conductors, is defined by the relation.

$$C = \frac{Q}{V}$$

Where V is the potential difference between the conductors due to equal and opposite charges on them of magnitude Q.

Capacitance of Co-axial Cable



consider length L of co-axial conductor, inner radius, a and outer radius b, (b>a).

Let space between the conductors be filled with homogeneous dielectric with permittivity, ε.

Assume conductor 1 and 2 carrying +Q and -Q charge respectively.

For an arbitrary cylindrical surface of radius ρ, and length L, the electric ~~field intensity~~ flux density

$$\boxed{\vec{D} = \frac{\rho_L}{2\pi\rho}}$$

ρ_L is in Colombs per meter

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho}}$$

Considering the co-axial conductor as a Gaussian cylindrical surface with radius ρ $(a<\rho<b)$.

The potential $V = -\int \vec{E}\cdot d\vec{l}$

potential corresponding to the figure.

$$V = -\int_a^b \vec{E} \cdot d\vec{l}$$

Here $\vec{E} = \dfrac{\rho_L}{2\pi\rho\varepsilon}$ $\quad d\vec{l} = d\vec{\rho}$

$$\therefore V = -\int_a^b \frac{\rho_L}{2\pi\varepsilon\rho} d\rho$$

$$= \frac{-\rho_L}{2\pi\varepsilon} \int_a^b \frac{1}{\rho} d\rho$$

$$= \frac{-\rho_L}{2\pi\varepsilon} \Big[\ln(\rho) \Big]_a^b$$

$$= \frac{-\rho_L}{2\pi\varepsilon} \Big[\ln(a) - \ln(b) \Big]$$

$$V = \frac{\rho_L}{2\pi\varepsilon} \Big[\ln(b) - \ln(a) \Big]$$

$$= \frac{\rho_L \ln(b/a)}{2\pi\varepsilon}$$

$$C = \frac{Q}{V} = \frac{\rho_L \cdot L}{V} \quad \text{for } L = 1 \text{ Unit meter } Q = \frac{\rho_L}{V}$$

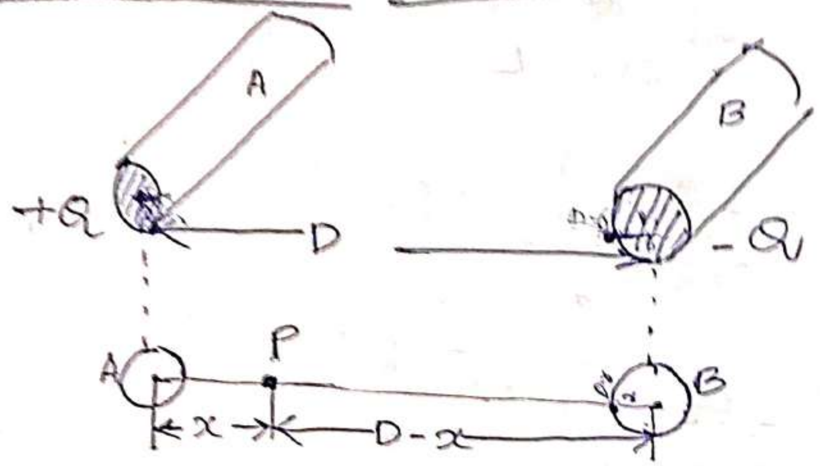
$$= \frac{\cancel{\rho_L}}{\frac{\cancel{\rho_L}}{2\pi\varepsilon} \ln(b/a)}$$

$$= \frac{2\pi\varepsilon}{\ln(b/a)}$$

∴ Capacitance per unit length $(L=1)$

$$\boxed{C = \frac{2\pi\varepsilon}{\ln(b/a)}}.$$

Capacitance of two wire transmission line.



Assume that $+Q$ and $-Q$ are the charges in the wires A and B, spaced D meters apart. Radius of each wire is rm, $D >>> r$.

To determine the Capacitance between A and B, consider the potential difference between them, which can be evaluated by considering the straight line path of integration from B to A

$$V_{BA} = \int_B^A \vec{E_x}\, dx.$$

where $\vec{E_x}$ is the electric field intensity at any point P at distance x from A and $D-x$ from B. The field at P is the sum of the fields due to A & B.

The field due to a charge $+Q$ per meter length at a distance x,

$$\vec{E_x} = \frac{\rho_L}{2\pi\varepsilon_0 x}.$$

∴ The resultant field at P,

$$\vec{E_x} = \frac{\rho_L}{2\pi\varepsilon_0 x} + \frac{\rho_L}{2\pi\varepsilon_0 (D-x)}$$

$$\therefore V_{BA} = -\int_{D-r}^{r} \vec{E_x}\, dx \quad \text{——} (1).$$

The field at P can be Calculated when x lies between r and D-r.

(1) becomes,

$$V_{BA} = -\int_{D-r}^{r} \frac{\rho_L}{2\pi\varepsilon_0 x} + \frac{\rho_L}{2\pi\varepsilon_0 (D-x)}\, dx$$

$$V = \frac{-\rho_L}{2\pi\varepsilon_0} \int_{D-r}^{r} \frac{1}{x} + \frac{1}{D-x}\, dx.$$

$\int \frac{1}{x} = \ln x$

$\int \frac{1}{D-x} = \frac{\ln(D-x)}{-1}$

$$V = -\frac{\rho_L}{2\pi\varepsilon_0}\left[\ln(x) + -\ln(D-x)\right]_{D-r}^{r}$$

$$= \frac{\rho_L}{2\pi\varepsilon_0}\left[\ln(r) - \ln(D-r) - [\ln(D-r) - \ln(\cancel{D}-(\cancel{D}-r))]\right]$$

$$= \frac{-\rho_L}{2\pi\varepsilon_0}\left[\ln(r) - \ln(D-r) - \ln(D-r) + \ln(r)\right]$$

$$= \frac{-\cancel{2}\rho_L}{\cancel{2}\pi\varepsilon_0}\left[\ln(r) - \ln(D-r)\right]$$

$$= \frac{\rho_L}{\pi\varepsilon_0}\left[\ln(D-r) - \ln(r)\right]$$

$$\therefore V = \frac{\rho_L}{\pi\varepsilon_0}\left[\ln\left(\frac{D-r}{r}\right)\right]$$

As $D >>> r$.

$$\frac{D-r}{r} \simeq \frac{D}{r}$$

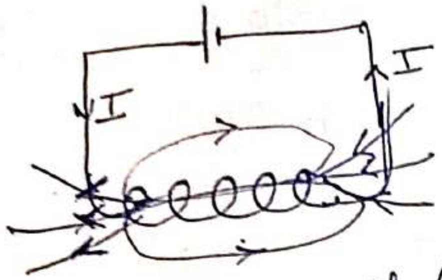
$$\therefore \quad V = \frac{\rho_L}{\pi \varepsilon_0} \ln\left(\frac{D}{r}\right)$$

$$C = \frac{Q}{V} = \frac{\rho_L \cdot l}{V} \qquad l = 1$$

$$C = \frac{\cancel{\rho_L}}{\frac{\cancel{\rho_L}}{\pi \varepsilon_0} \ln\left(\frac{D}{r}\right)}$$

$$\boxed{C = \frac{\pi \varepsilon_0}{\ln\left(D/r\right)} \text{ F/m.}}$$

Inductance



Magnetic field produced by a Circuit.

A circuit (or closed conducting path) carrying current I produces a magnetic field B that causes a flux $\psi_m = \int_s \vec{B} \cdot ds$ to pass through each turn of the circuit. If the circuit has N identical turns, ~~[illegible]~~, then the flux linkage λ,

$$\lambda = N \psi_M$$

Also flux linkage λ is proportional to the current I producing it,

$$\lambda \propto I$$

$$\lambda = LI.$$

Where L is a constant of proportionality called the inductance of the circuit.

Inductance L of an inductor is defined as the ratio of the magnetic flux linkage λ to the current I through the inductor.

$$L = \frac{\lambda}{I} = \frac{N\psi_m}{I}$$ Unit is Henry (H) or weber per ampere.

$$L = \frac{d\lambda}{dI}$$ or

Inductance of a Co-axial Cable.

Consider a co-axial line of radius 'a' for the inner conductor and 'b' for the outer conductor. The current on the inner conductor is I and on the outer conductor is of same magnitude with opposite direction. The flux density between the two conductors,

$$\vec{B} = \frac{\mu I d}{2\pi\rho}\hat{a_\phi} \qquad a \le \rho \le b$$

using cylindrical Co-ordinate System, the total flux linkage of the Co-axial line of length l will be,

$$\psi_m = \int_S \vec{B}\cdot ds = \frac{\mu I l}{2\pi}\int_a^b \frac{1}{\rho}\, d\rho.$$

$$= \frac{\mu I l}{2\pi}\left[\ln \rho\right]_a^b$$

$$= \frac{\mu I l}{2\pi}\ln(b/a)$$

Inductance per Unit length of the Coaxial line is

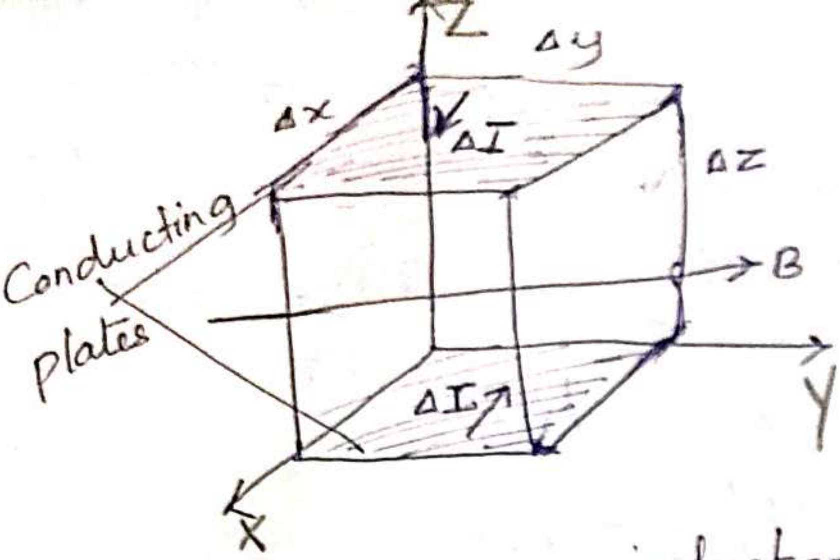
$$L = \frac{\psi_m}{I} = \frac{\mu}{2\pi}\ln(b/a)$$

Energy Stored in Magnetostatic Field.

The magnetic energy in the field of an inductor

$$W_M = \frac{1}{2}LI^2.$$

To express this equation in terms of $\vec{B}$ or $\vec{H}$, Consider a differential volume in a magnetic field as shown in figure. Let the Volume be Covered with Conducting Sheets at the top and bottom Surfaces with Current ΔI.



Each Volume has a inductance

$$\Delta L = \frac{\Delta \psi_m}{\Delta I}$$

$$\psi_m = \int_s \vec{B} \cdot ds = \therefore \Delta \psi_m = \vec{B}\, \Delta x\, \Delta z = \mu \vec{H}\, \Delta x \Delta z$$

$$\oint_L \vec{H} \cdot dl = I \quad \therefore \quad \Delta I = \vec{H} \cdot \Delta y$$

($\vec{B}$ is along y $\therefore \vec{H}$ is also along y direction)

$$W_M = \frac{1}{2} L I^2$$

$$\Delta W_M = \frac{1}{2} \Delta L\, \Delta I^2$$

$$= \frac{1}{2} \frac{\Delta \psi_m}{\Delta I} \cdot \Delta I^2$$

$$= \frac{1}{2} \Delta \psi_m\, \Delta I$$

$$= \frac{1}{2} \vec{B}\, \Delta x \Delta z\, (\vec{H} \cdot \Delta y)$$

$$= \frac{1}{2} \vec{B}\, \Delta x\, \Delta y \cdot \Delta z\, \vec{H}$$

$$= \frac{1}{2} \mu \vec{H}\, \Delta x \Delta y \Delta z\, \vec{H}$$

$$= \frac{1}{2} \mu \vec{H}^2 \Delta V$$

$$\boxed{W_m = \frac{1}{2} \int_v \mu \vec{H}^2 dv = \frac{1}{2} \int \vec{B} \cdot \vec{H}\, dv}$$

Inductance of two wire transmission line.

Consider the same dimensions taken for calculating capacitance.

Then at point P, $\vec{B}$ will be the sum of the fields due to wires A and B.

$$\vec{B} = \frac{\mu I l}{2\pi}\left(\frac{1}{x} + \frac{1}{D-x}\right)$$

$$\phi_m = \int_S \vec{B}\cdot ds$$

$$= -\int_{D-r}^{r} \frac{\mu I l}{2\pi}\left(\frac{1}{x} + \frac{1}{D-x}\right) dx.$$

over a length l.

$$= -\frac{\mu I l}{2\pi}\int_{D-r}^{r}\left(\frac{1}{x} + \frac{1}{D-x}\right) dx.$$

$$= \frac{-\mu I l}{2\pi}\left[\ln(x) + \frac{\ln(D-x)}{-1}\right]_{D-r}^{r}$$

$$= -\frac{\mu I l}{2\pi} \cdot 2\ln\frac{r}{D-r}$$

$D-r >>> r$

$$= \frac{\mu I l}{\pi}\ln\frac{(D-r)}{r}$$

$$= \frac{\mu I l}{\pi}\ln\left(\frac{D}{r}\right)$$

Magnetic Scalar and Vector potentials.

Just as the electric potential V is related to electric field intensity $\vec{E}$ as $\vec{E} = -\nabla V$, we can define a potential associated with magnetostatic field.

The magnetic potential can be scalar or Vector.

Magnetic Scalar potential

V_m is related to $\vec{H}$ as

$$\boxed{\vec{H} = -\nabla V_m}$$

Two important Vector identities are.

$$\begin{cases} \nabla \times (\nabla V) = 0 & V\text{-scalar} \\ \nabla \cdot (\nabla \times \vec{A}) = 0 & A\text{-Vector} \end{cases}$$

I^{st} identity
II^{nd} ,,

We know, $\nabla \times \vec{H} = \vec{J}$

In terms of magnetic scalar potential,

$$\nabla \times (-\nabla V_m) = \vec{J} = 0 \quad (\text{by } I^{st} \text{ identity})$$

$$\therefore \boxed{\nabla^2 V_m = 0}$$ is known as Laplace equation in magnetostatic field.

Magnetic Vector potential.

$$\boxed{\vec{B} = \nabla \times \vec{A}}$$

$\vec{A}$ is called the magnetic Vector potential.

given as,

$$\vec{A} = \int_L \frac{\mu_0 I dl}{4\pi R} \text{ for line Current}$$

$$\vec{A} = \int_S \frac{\mu_0 J ds}{4\pi R} \text{ for Surface Current}$$

$$\vec{A} = \int_V \frac{\mu_0 k dv}{4\pi R}, \text{ for Volume Current.}$$

Q. Given the magnetic Vector potential $\vec{A} = -\rho^2/4 \; a_z$ wb/m, Calculate the total magnetic flux crossing the surface $\phi - \pi/2$, $1 \leq \rho \leq 2m$, $0 \leq z \leq 5m$.

$$B = \nabla \times \vec{A} = \frac{-\partial A_z}{\partial \rho} a_\phi = \frac{\rho}{2} a_\phi$$

$$\therefore ds = d\rho dz \, a_\phi$$

Hence $\psi_m = \int \vec{B} \cdot ds = \frac{1}{2} \int_{z=0}^{5} \int_{\rho=1}^{2} \rho \, d\rho dz$

$$= \left[\frac{1}{4}\rho^2\right]_1^2 (5) = \underline{\underline{15/4}}$$

$$\therefore \psi_m = \underline{\underline{3.75 wb}}.$$

Q. A current distribution gives rise to the Vector magnetic potential $\vec{A} = x^2 y \, a_x + y^2 x \, a_y - 4xyz \, a_z$ wb/m. Calculate

(a) $\vec{B}$ at $(-1, 2, 5)$

(b) The flux through the surface defined by $z = 1$, $0 \leq x \leq 1$, $-1 \leq y \leq 4$.