DFT as Linear Transformation (Linear Operation)

DFT : $X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$; $k = 0, 1 \cdots N-1$.

IDFT : $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}$; $n = 0, 1 \cdots N-1$

where $W_N = e^{-j\frac{2\pi}{N}}$

W_N is ~~also~~ known as Twiddle factor.

$e^{-j2\pi} = \cos(2\pi) - j\sin(2\pi) = 1 - j.0 = 1$

$\therefore e^{-j\frac{2\pi}{N}} = (e^{-j2\pi})^{1/N} = (1)^{1/N}$

$\therefore W_N = e^{-j\frac{2\pi}{N}}$ is N^{th} root of unity

Let $x_N \longrightarrow$ N point column vector of sequence $x(n)$.

ie $x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1}$ $\longrightarrow$ Has N rows and 1 column.

$X_N \longrightarrow$ N point column vector of DFT of $X(k)$.

ie $X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}_{N\times 1}$ $\longrightarrow$ Has N rows and 1 column.

and

$W_N \longrightarrow$ Matrix of Linear Transformation and is $N \times N$ dimension matrix.

$$W_N = \left[W_N^{nk} \right]_{N \times N}$$

$$W_N = \begin{array}{c|cccc} & n=0 & n=1 & \cdots & n=N-1 \\ \hline k=0 & & & & \\ k=1 & & & & \\ \vdots & & & & \\ k=N-1 & & & & \end{array}$$

$$W_N = \begin{array}{c} k=0 \rightarrow \\ k=1 \rightarrow \\ \vdots \\ k=N-1 \end{array} \begin{bmatrix} W_N^{0.0} & W_N^{1.0} & \cdots & W_N^{(N-1).0} \\ W_N^{0.1} & W_N^{1.1} & \cdots & W_N^{(N-1).1} \\ \vdots & \vdots & & \\ W_N^{0(N-1)} & W_N^{1.(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

$$\therefore W_N = \begin{bmatrix} W_N^{0} & W_N^{0} & \cdots & W_N^{0} \\ W_N^{0} & W_N^{1} & \cdots & W_N^{N-1} \\ \vdots & \vdots & & \\ W_N^{0} & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Thus DFT can be calculated as linear transformation as.

$$X_N = W_N x_N$$

ie
$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & W_N^{N-1} \\ \vdots & \vdots & & \\ W_N^0 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

The above transformation can be written in equ form as.

$$X(0) = x(0) W_N^0 + x(1) W_N^0 \cdots x(N-1) W_N^0$$

$$X(1) = x(0) W_N^0 + x(1) W_N^1 + \cdots x(N-1) W_N^{N-1}$$

$$\vdots$$

$$X(N-1) = x(0) W_N^0 + x(1) W_N^{N-1} + \cdots x(N-1) W_N^{(N-1)(N-1)}$$

Similar to our previous equations.

Two basic properties of Twiddle factor

① Symmetry property: $W_N^{k+N/2} = -W_N^k$.

eg: $W_8^6 = W_8^{2+4} = -W_8^2$.

② Periodicity property: $W_N^{k+N} = W_N^{k}$.

eg: $W_8^{10} = W_8^{2+8} = W_8^{2}$

Q) Compute the DFT of the 4 point ~~DFT~~ ~~of the~~ sequence $x(n) = \{0, 1, 2, 3\}$.

Ans: Here $N = 4$

we have to find.

$$X_N = X_4 = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

Given $$x_4 = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$ (given in question).

Here

$$W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^{9} \end{bmatrix} \quad \text{——} ①$$

By applying symmetry property and periodicity property.

$W_4^4 = W_4^{0+4} = W_4^0$ (periodicity property)

$W_4^6 = W_4^{2+4} = W_4^2$ (periodicity property)

$W_4^9 = W_4^{5+4} = W_4^5 = W_4^{1+4} = W_4^1$ (periodicity property).

sub in ①

$$\therefore W_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^0 & W_4^2 \\ W_4^0 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}.$$

$W_4^0 = e^{-j\frac{2\pi}{4}\cdot 0} = e^0 = 1$

$W_4^1 = e^{-j\frac{2\pi}{4}\cdot 1} = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = 0 - j$

$= -j$

$W_4^2 = e^{-j\frac{2\pi}{4}\cdot 2} = e^{-j\pi} = \cos\pi - j\sin\pi = -1 - 0.j$

$= -1$

$W_4^3 = e^{-j\frac{2\pi}{4}\cdot 3} = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = j$

$$\therefore W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

equ to find DFT

$$X_4 = W_4 \, x_4$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 + 1 + 2 + 3 \\ 0 - j - 2 + 3j \\ 0 - 1 + 2 - 3 \\ 0 + j - 2 - 3j \end{bmatrix} = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$\therefore$ DFT $X(k) = \underline{\{6, -2+2j, -2, -2-2j\}}$.

<u>IDFT as Linear Transformation</u>

IDFT: $x(n) = \dfrac{1}{N} \sum_{k=0}^{N-1} X(k) \, W_N^{-nk}$.

$$\therefore \boxed{x_N = \frac{1}{N} W_N^{*} X_N} \quad \text{——— ②}$$

$W_N^{*} \longrightarrow$ Complex conjugate of matrix W_N

ie to get W_4^* → take complex conjugate of each entry.

$$\therefore W_4^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

we have DFT eqn.

$$X_N = W_N x_N$$

pre multiplying both sides by W_N^{-1}

$$W_N^{-1} X_N = W_N^{-1} W_N x_N.$$

we have $W_N^{-1} W_N = I$ → Identity matrix

$$\Rightarrow W_N^{-1} X_N = I x_N = x_N$$

$$\Rightarrow x_N = W_N^{-1} X_N \text{ ——— (3)}$$

from (2) and (3) $W_N^{-1} = \frac{1}{4} W_N^*$.

Q) Perform IDFT on the above answer

ie $X(k) = \{6, -2+2j, -2, -2-2j\}$.

using eqn (2). and $N=4$

$$x_4 = \frac{1}{4} W_4^* X_4.$$

$$x_4 = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 6 + -2+2j + -2 + -2-2j \\ 6+(-2j-2)+2 \; + +2j - 2 \\ 6+2-2j-2 \; +2+2j \\ 6+2j+2 \; +2 \; -2j+2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 0 \\ 4 \\ 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\therefore \; x(n) = \{0, 1, 2, 3\}.$$