Applications of FFT Algorithms:

Here ~~we~~ illustrate how to enhance the efficiency of FFT algorithm by forming complex valued sequences prior to the computation of the DFT.

in two ways.

① Efficient computation of the DFT of two real sequences:

In view of the fact that the algorithm can handle complex valued input sequences, we can exploit this capability in the computation of the DFT of two real valued sequences.

Suppose that $x_1(n)$ and $x_2(n)$ are two real valued sequences of length N and whose N point DFTs $X_1(k)$ and $X_2(k)$ has to be computed.

Now let $x(n)$ be a complex valued sequence defined as.

$$x(n) = x_1(n) + j\, x_2(n) \quad \text{———} \quad (1)$$

$$0 \le n \le N-1.$$

The DFT operation is linear and hence DFT of $x(n)$ can be expressed as.

$$X(k) = X_1(k) + j\, X_2(k) \quad \text{———} \quad (2).$$

from eqn (1)

$$x^*(n) = x_1(n) - j\, x_2(n) \quad \text{———} \quad (3)$$

$$(1) + (3) \Longrightarrow \quad x_1(n) = \frac{x(n) + x^*(n)}{2} \quad \text{———} \quad (4)$$

$$(1) - (3) \Longrightarrow \quad x_2(n) = \frac{x(n) - x^*(n)}{2j} \quad \text{———} \quad (5)$$

Hence from eqn (4) DFT of $x_1(n)$

$$\cancel{X_1(k)} = \cancel{X(k)} + \cancel{X^*}$$

$$X_1(k) = \frac{1}{2}\Big[\text{DFT}\left[x(n)\right] + \text{DFT}\left[x^*(n)\right]\Big] \quad \text{———} \quad (6)$$

from eqn (5) DFT of $x_2(n)$.

$$X_2(k) = \frac{1}{2j}\Big[\text{DFT}\left[x(n)\right] - \text{DFT}\left[x^*(n)\right]\Big] \quad \text{———} \quad (7)$$

From complex conjugate property of DFT we have

$$\text{DFT}\left[x^*(n)\right] = X^*(N-k).$$

$\therefore$ eqn (6) $\Rightarrow$

$$\boxed{\begin{aligned} X_1(k) &= \frac{1}{2}\left[X(k) + X^*(N-k)\right] \quad \text{--- (8)} \\ X_2(k) &= \frac{1}{2j}\left[X(k) - X^*(N-k)\right] \quad \text{--- (9)} \end{aligned}}$$

eqn (7) $\Rightarrow$

Thus by performing a single DFT on the complexvalued sequence $x(n)$ we have obtained DFT of two real valued sequence $x_1(n)$ and $x_2(n)$

Q). Compute DFT of two sequence

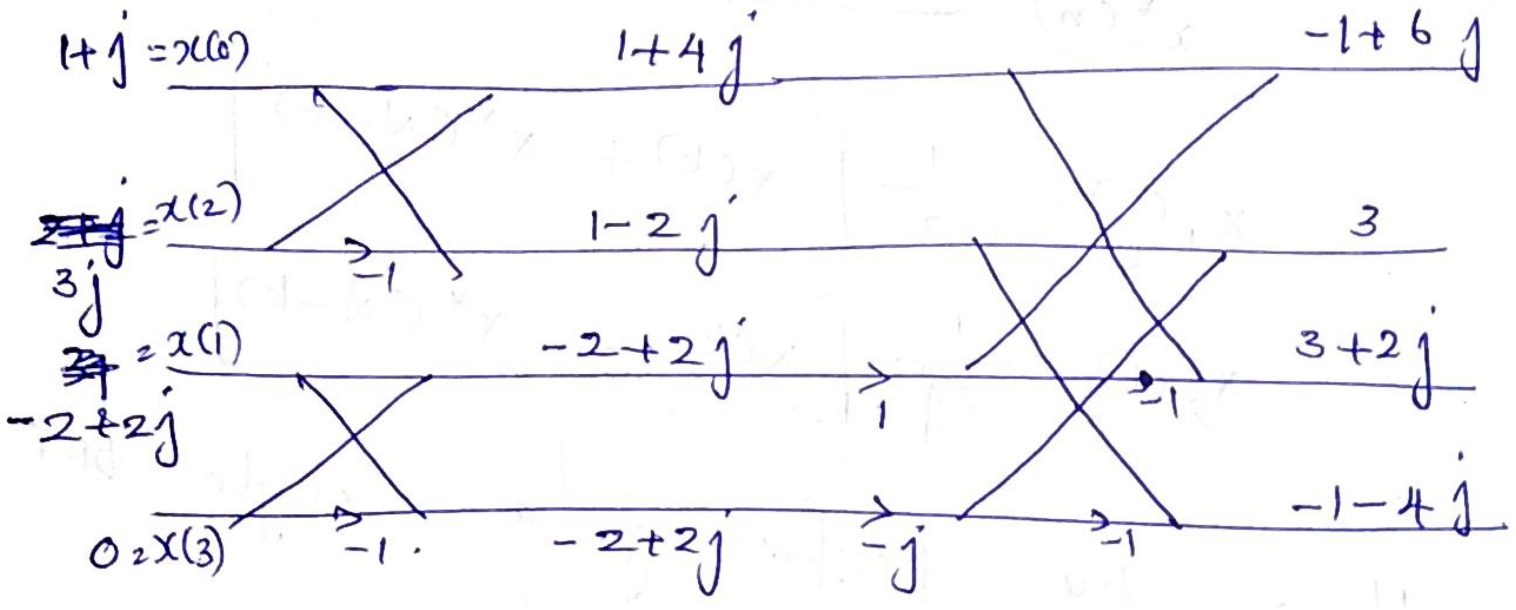
$x_1(n) = \{1, -2, 0, 0\}$ and $x_2(n) = \{1, 2, 3, 0\}$

by using 4 point radix-2 FFT signal flow graph only once.

A: Assume $x(n) = x_1(n) + j\, x_2(n)$.

$x(n) = \{1+j, -2+2j, 3j, 0\}$.

Use 4 point DIT-FFT Butterfly algorithm.



$$x(k) = \{-1+6j, 3, 3+2j, -1-4j\}.$$

we have
$$x_1(k) = \frac{1}{2}\left[x(k) + x^*(N-k)\right]$$

$$x_1(0) = \frac{1}{2}\left[x(0) + x^*(4-0)\right]$$

$$= \frac{1}{2}\left[(-1+6j) + (-1-6j)\right]$$

$$= \frac{1}{2}\left[-2\right] = \underline{\underline{-1}}.$$

$$x_1(1) = \frac{1}{2}\left[x(1) + x^*(4-1)\right].$$

$$= \frac{1}{2}\left[3 + -1 + 4j\right] = 1+2j$$

$$x_1(2) = \frac{1}{2}\left[x(2) + x^*(4-2)\right]$$

$$= \frac{1}{2}\left[3+2j + 3-2j\right] = 3$$

$$x_1(3) = \frac{1}{2}\left[x(3) + x^*(4-3)\right]$$

$$= \frac{1}{2}\left[-1-4j + 3\right] = 1-2j$$

$$\therefore \quad X_1(k) = \{-1,\ 1+2j,\ 3,\ 1-2j\}.$$

$$X_2(k) = \frac{1}{2j}\left[X(k) - X^*(N-k)\right].$$

$$X_2(0) = \frac{1}{2j}\left[X(0) - X^*(4)\right].$$

$$= \frac{1}{2j}\left[-1+6j - (-1-6j)\right].$$

$$= \frac{1}{2j}\left[12j\right] = 6$$

$$X_2(1) = \frac{1}{2j}\left[X(1) - X^*(3)\right]$$

$$= \frac{1}{2j}\left[3 - (-1+4j)\right]$$

$$= \frac{1}{2j}\left[4-4j\right] = -2-2j$$

$$X_2(2) = \frac{1}{2j}\left[X(2) - X^*(2)\right] = \frac{1}{2j}\left[\right.$$

$$= \frac{1}{2j}\left[3+2j) - (3-2j)\right] = +2.$$

$$X_2(3) = \frac{1}{2j}\left[-1-4j-3\right] = -2+2j.$$

$X_2(k) = \{6, -2-2j, 2, -2+2j\}$

HW: Show how 8 point radix-2 signal flow graph can be used to compute the DFT of

$x_1(n) = \{1, 4, 3, 2, 2, 3, 2, 2\}$ and

$x_2(n) = \{1, 4, 1, 2, 2, 3, 4, 2\}$

simultaneously using signal flow graph only once and hence compute $X_1(k)$ and $X_2(k)$.

(2) Efficient computation of the DFT of a 2N point Real sequence:

Let $g(n)$ be a real valued sequence of 2N points.

To obtain the 2N-point DFT of $g(n)$ from computation of one N-point DFT

- Divide the 2N point real sequence $g(n)$ into two N-point real sequences as even and odd indexed.

even indexed: $x_1(n) = g(2n)$.

odd indexed: $x_2(n) = g(2n+1)$.

From previous section if we combine $x_1(n)$ and $x_2(n)$ to form $x(n)$

as $x(n) = x_1(n) + j\, x_2(n)$.

their DFTs

$$X_1(k) = \frac{1}{2}\left[X(k) + X^*(N-k)\right] \quad (1)$$

$$X_2(k) = \frac{1}{2j}\left[X(k) - X^*(N-k)\right] \quad (2)$$

Now combine $X_1(k)$ and $X_2(k)$ to get $G(k)$ as.

we have

$$\cancel{G(k)} = \quad g(n) \xrightarrow[2N]{DFT} G(k)$$

$$\therefore G(k) = \sum_{n=0}^{2N-1} g(n)\, W_{2N}^{nk}.$$

$$= \sum_{n=0}^{N-1} g(2n)\, W_{2N}^{2nk} + \sum_{n=0}^{N-1} g(2n+1)\, W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} x_1(n)\, W_N^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} x_2(n)\, W_N^{nk}$$

$$\therefore \boxed{G(k) = X_1(k) + W_{2N}^{k} X_2(k)} \quad \text{——— (3)}.$$

$k = 0, 1 \ldots 2N-1$

where $G(k) \longrightarrow 2N$ point DFT

$X_1(k), X_2(k) \longrightarrow N$ point DFTs.

$\therefore$ eqn (3) can be rewritten as.

$$\left.\begin{aligned} G(k) &= X_1(k) + W_{2N}^{k} X_2(k). \quad ; \; k = 0, 1 \ldots N-1 \\ G(k+N) &= X_1(k) - W_{2N}^{k} X_2(k). \; ; \; k = 0, 1 \ldots N-1 \end{aligned}\right\}$$

——— (4).

Thus we have computed the DFT of a 2N point real sequence from N point DFT and some additional computations.

Q) Compute 8 point DFT of the sequence $g(n) = \{1, 2, 2, 1, 1, 2, 2, 1\}$.

by using 4 point Butterfly only once

Ans: $\{12, 0, -2-2j, 0, 0, 0, -2+2j, 0\}$.

Answer:

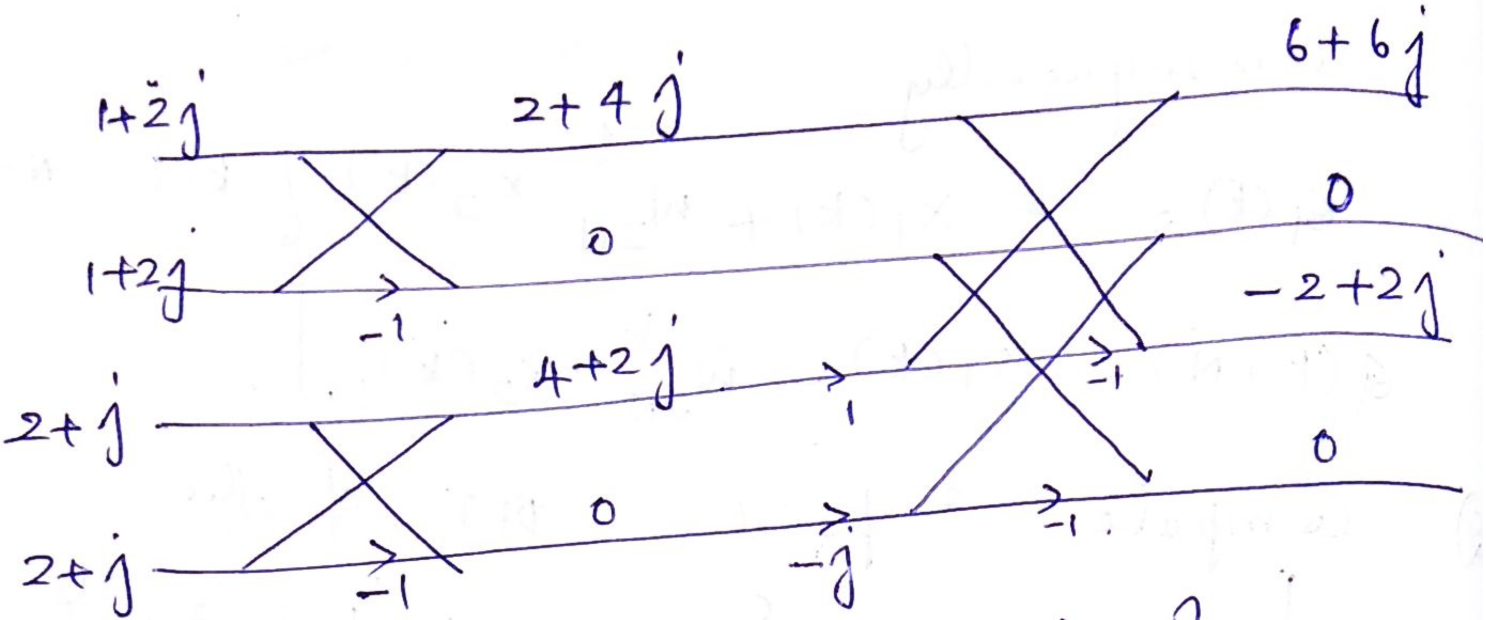
even indexed $x_1(n) = \{1, 2, 1, 2\}$

odd indexed $x_2(n) = \{2, 1, 2, 1\}$.

$x(n) = x_1(n) + j\, x_2(n)$.

$x(n) = \{1+2j, 2+j, 1+2j, 2+j\}$.

Find $X(k)$ using 4 point DIT FFT Butterfly diagram.



$$X(k) = \{6+6j, 0, -2+2j, 0\}$$

$$X_1(k) = \frac{1}{2}\left[X(k) + X^*(N-k)\right]$$

$$X_1(0) = \frac{1}{2}\left[X(0) + X^*(4)\right]$$

$$= \frac{6+6j + 6-6j}{2} = 6$$

$$X_1(1) = \frac{1}{2}\left[X(1) + X^*(3)\right]$$

$$= \frac{1}{2}\left[0 + 0\right] = 0$$

$$X_1(2) = \frac{1}{2}\left[X(2) + X^*(2)\right] = \underline{\underline{-2}}$$

$$X_1(3) = \frac{1}{2}\left[X(3) + X^*(1)\right] = 0$$

$$X_2(0) = \frac{1}{2j}\left[X(0) - X^*(4)\right]$$

$$= \frac{1}{2j}\left[6+6j - (6-6j)\right] = 6$$

$$X_2(1) = \frac{1}{2j}\left[X(1) - X^*(3)\right] = 0$$

$$X_2(2) = \frac{1}{2j}\left[X(2) - X^*(2)\right] =$$

$$= \frac{1}{2j}\left[-2+2j - (-2-2j)\right] = 2$$

$$X_2(3) = \frac{1}{2j}\left[0-0\right] = 0$$

$$X_1(k) = \{6, 0, -2, 0\}$$

$$X_2(k) = \{6, 0, 2, 0\}$$

$$G(k) = X_1(k) + W_{2N}^{k} X_2(k)$$

$$= X_1(k) + W_8^{k} X_2(k)$$

$$G(0) = 6 + 1 \times 6 = 12$$

$$G(1) = 0 + W_8^{1} \times 0 = 0$$

$$G(2) = X_1(2) + W_8^{2} X_2(2)$$

$$= -2 + -j \times 2 = -2-2j$$

$$G(3) = X_1(3) + W_8^{3} X_2(3)$$

$$= 0 + W_8^{3} \cdot 0 = 0$$

$G(4) = x_1(4) + W_8^4 \, x_2(4)$

$= x_1(0) - W_8^0 \, x_2(0)$

$= 6 - 6 = \underline{\underline{0}}.$

$G(5) = x_1(5) + W_8^5 \, x_2(5)$

$= x_1(1) - W_8^1 \, x_2(1) = 0$

$G(6) = x_1(2) - W_8^2 \, x_2(2)$

$= -2 - (-j)2 = -2 + 2j$

$G(7) = x_1(3) - W_8^3 \, x_2(3)$

$= 0 - 0 = 0.$

$G(k) = \{12, 0, -2-2j, 0, 0, 0, -2+2j, 0\}$