5.12.3 *Design of IIR Filter Using Bilinear Transformation*

The bilinear transformation is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Furthermore, all points in the LHP of 's' are mapped inside the unit circle in the z-plane and all points in the RHP of 's' are mapped into corresponding points outside the unit circle in the z-plane.

Let us consider an analog linear filter with system function

$$H(s) = \frac{b}{s+a} \tag{5.82}$$

which can be written as

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$\text{so} \quad sY(s) + aY(s) = bX(s) \tag{5.83}$$

This can be characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \tag{5.84}$$

$y(t)$ can be approximated by the trapezoidal formula.

Thus

$$y(t) = \int_{t_o}^{t} y'(\tau)d\tau + y(t_o) \tag{5.85}$$

where $y'(t)$ denotes the derivative of $y(t)$.

The approximation of the integral in Eq.(5.85) by the trapezoidal formula at $t = nT$ and $t_0 = nT - T$ yields

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T) \tag{5.86}$$

From the differential Eq.(5.84) we obtain

$$y'(nT) = -ay(nT) + bx(nT) \tag{5.87}$$

Substituting Eq.(5.87) in Eq.(5.86) we get

$$y(nT) = \frac{T}{2}[-ay(nT) + bx(nT) - ay(nT-T) + bx(nT-T)] + y(nT-T)$$

which implies

$$y(nT) + \frac{aT}{2}y(nT) - \left(1 - \frac{aT}{2}\right) y(nT-T) = \frac{bT}{2}[x(nT) + x(nT-T)] \quad (5.88)$$

With $y(n) = y(nT)$ and $x(n) = x(nT)$ we obtain the result

$$\left(1 + \frac{aT}{2}\right) y(n) - \left(1 - \frac{aT}{2}\right) y(n-1) = \frac{bT}{2}[x(n) + x(n-1)]$$

The z-transform of this difference equation is

$$\left(1 + \frac{aT}{2}\right) Y(z) - \left(1 - \frac{aT}{2}\right) z^{-1}Y(z) = \frac{bT}{2}[1 + z^{-1}]X(z)$$

The system function of the digital filter is

$$\begin{aligned} H(z) = \frac{Y(z)}{X(z)} &= \frac{\frac{bT}{2}(1+z^{-1})}{1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right) z^{-1}} \\ &= \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})} \end{aligned}$$

Dividing numerator and denominator by $\frac{T}{2}(1 + z^{-1})$ we get

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a} \quad (5.89)$$

Comparing Eq.(5.82) and Eq.(5.89), the mapping from s-plane to the z-plane can be obtained as

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad (5.90)$$

This relationship between s and z is known as bilinear transformation.

Let $z = re^{j\omega}$ and

$$s = \sigma + j\Omega \quad (5.90a)$$

then Eq.(5.90) can be expressed as

$$\begin{aligned} s &= \frac{2(z-1)}{T(z+1)} \\ &= \frac{2}{T}\left[\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right] = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right] \\ &= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right] \\ &= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{(r\cos\omega + 1)^2 + r^2\sin^2\omega}\right] \\ &= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{1 + r^2\cos^2\omega + 2r\cos\omega + r^2\sin^2\omega}\right] \end{aligned}$$

Separating imaginary and real parts, we have

$$s = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right] \tag{5.90b}$$

Comparing Eq. (5.90a) and Eq. (5.90b), we have

$$\sigma = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right]; \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right] \tag{5.91}$$

From Eq. (5.91), we find that if $r \leq 1$, then $\sigma < 0$ and if $r > 1$, then $\sigma > 0$. Consequently the LHP in 's' maps into the inside of the unit circle in the z-plane and the RHP in the 's' maps into the outside of the unit circle. When $r = 1$, then $\sigma = 0$ and

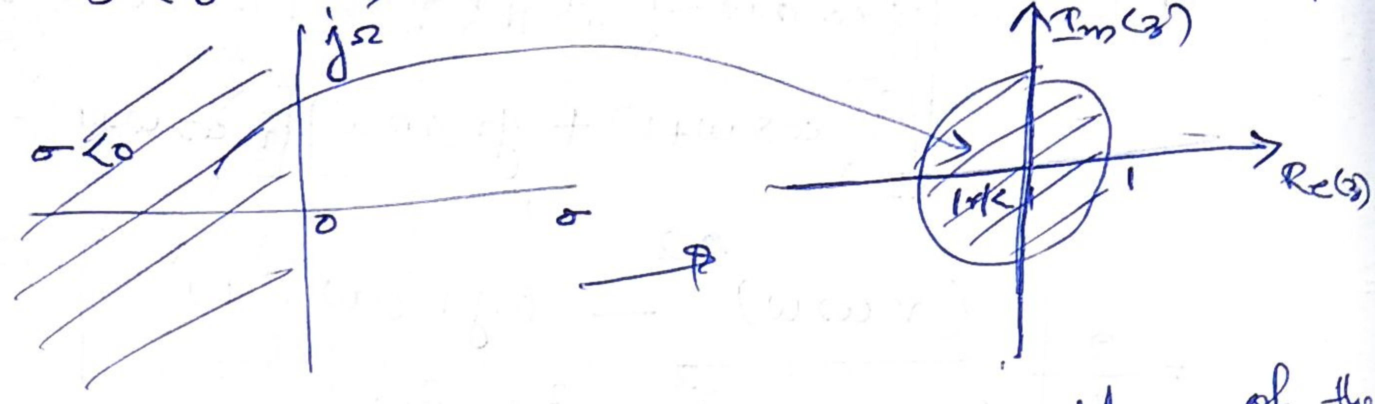
$$\Omega = \frac{2}{T}\frac{\sin\omega}{1 + \cos\omega} = \frac{2}{T}\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}}$$

$$= \frac{2}{T}\tan\frac{\omega}{2} \tag{5.92}$$

or

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \tag{5.93}$$

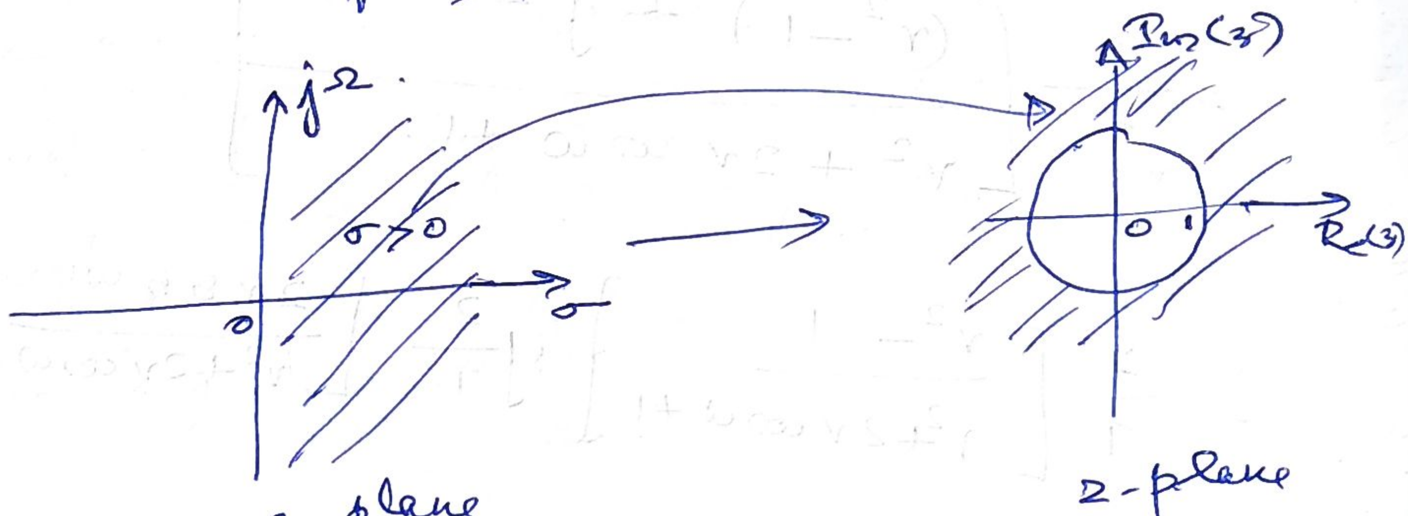
① $\sigma < 0 \Longrightarrow r^2 - 1 < 0 \Longrightarrow r^2 < 0 \Longrightarrow |r| < 1$



ie LHP of s maps to inside of the unit circle.

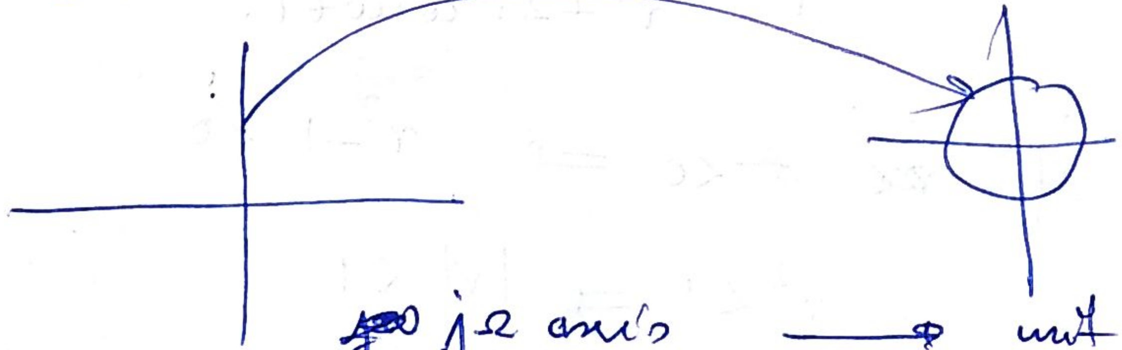
② if $\sigma > 0 \Longrightarrow r^2 - 1 > 0$.

$r^2 > 1 \Longrightarrow |r| > 1$



RHP ⟶ outside unit circle.

③ $\sigma = 0 \Longrightarrow r^2 - 1 = 0 \Longrightarrow |r| = 1$.



~~for~~ jΩ axis ⟶ unit circle.

when $r = 1$ and $\sigma = 0$

$$\Omega = \frac{2}{T} \frac{2 \sin \omega}{1 + 1 + 2 \cos \omega}$$

$$= \frac{2}{T} \frac{2 \sin \omega}{2 \left[1 + \cos \omega \right]}$$

$$= \frac{2}{T} \left[\frac{2 \sin \omega/2 \, \cos \omega/2}{2 \cos^2 \omega/2} \right]$$

$$= \frac{2}{T} \tan(\omega/2)$$

$$\therefore \boxed{\omega = 2 \tan^{-1} \frac{\Omega T}{2}}$$

The above equation represents the relation between relation b/w the frequency variables in two domain for bilinear transformation.

Ω — analog frequency.

$\omega \longrightarrow$ digital frequency

The warping effect

Let Ω and ω represent the frequency variables in the analog filter and the derived digital filter respectively. From Eq. (5.92) we have

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

For small value of ω

$$\boxed{\because \quad \text{for small value of } \theta \quad \tan\theta = \theta}$$

$$\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T}$$

$$\omega = \Omega T \tag{5.94}$$

For low frequencies the relationship between Ω and ω are linear, as a result, the digital filter have the same amplitude response as the analog filter. For high frequencies, however, the relationship between ω and Ω becomes non-linear (see Fig. 5.24) and distortion is introduced in the frequency scale of the digital filter to that of the analog filter. This is known as the warping effect.

The influence of the warping effect on the amplitude response is shown in Fig. 5.25 by considering an analog filter with a number of passbands centered at regular intervals. The derived digital filter will have same number of passbands. But the center frequencies and bandwidth of higher frequency passband will tend to reduce disproportionately.

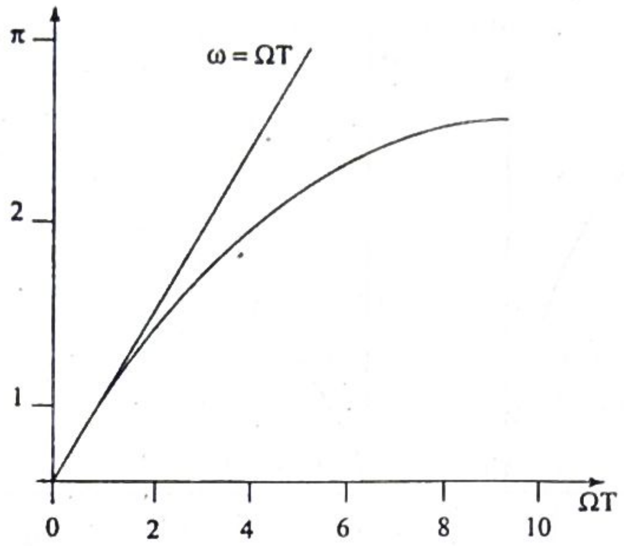


Fig. 5.24 Relationship between Ω and ω.

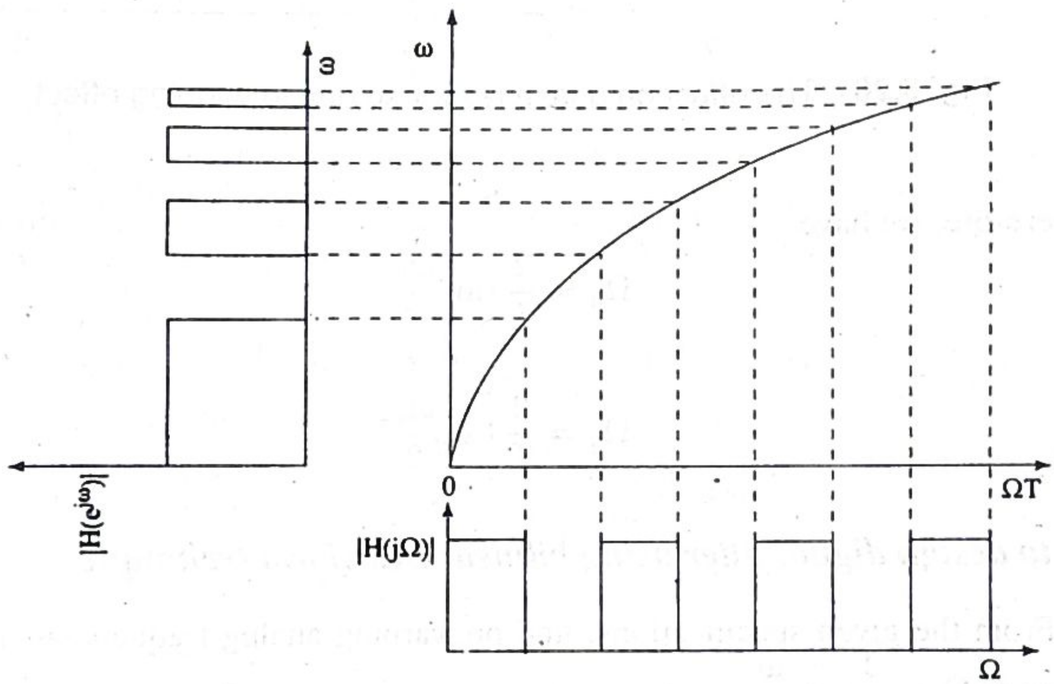


Fig. 5.25 The effect on magnitude response due to warping effect.

The influence of the warping effect on the phase response is shown in Fig. 5.26. Considering an analog filter with linear phase response, the phase response of the derived digital filter will be non-linear.

Prewarping

The warping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies using the formula

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2} \tag{5.95}$$

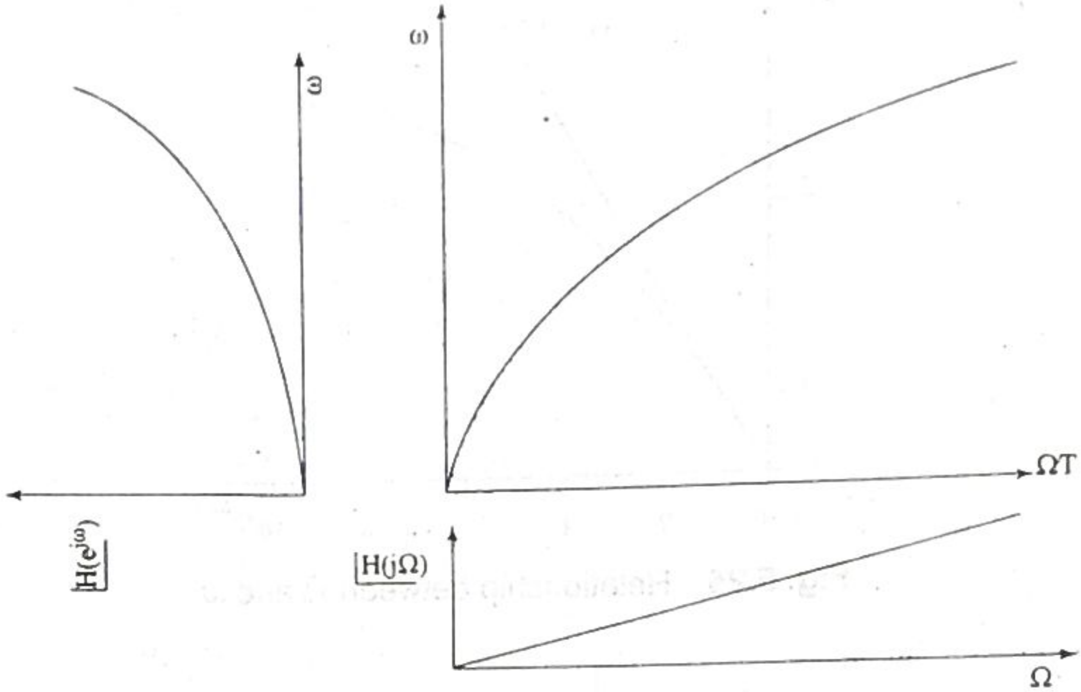


Fig. 5.26 The effect on phase response due to warping effect

Therefore, we have

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} \tag{5.96a}$$

and

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} \tag{5.96b}$$

Steps to design digital filter using bilinear transform technique.

1. From the given specifications, find prewarping analog frequencies using formula $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$.
2. Using the analog frequencies find $H(s)$ of the analog filter.
3. Select the sampling rate of the digital filter, call it T seconds per sample.
4. Substitute $s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$ into the transfer function found in step 2.

Example 5.16 Apply bilinear transformation to $H(s) = \dfrac{2}{(s+1)(s+2)}$ with $T = 1$ sec and find $H(z)$.

Solution

$$\text{Given } H(s) = \frac{2}{(s+1)(s+2)}$$

$$\text{Substitute } s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \text{ in } H(s) \text{ to get } H(z)$$

$$\begin{aligned} H(z) &= H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \\ &= \frac{2}{(s+1)(s+2)}\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \end{aligned}$$

Given $T = 1$ sec

$$\begin{aligned} H(z) &= \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}} \\ &= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)} \\ &= \frac{(1+z^{-1})^2}{6-2z^{-1}} \\ &= \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})} \end{aligned}$$