Discrete Fourier Transform

Who is Fourier?

- Jean Baptiste Joseph Fourier
  - (France, 1768-1830).
- French Mathematician and physicist

Why Fourier Analysis?

- Fourier Analysis
  - Tool to connect the time domain and frequency domain.
- In the frequency domain
  - Simplify the calculation

Relationship Between Time Properties of a Signal and the Appropriate Fourier Representation

Time Property	*Periodic*	*Nonperiodic*
Continuous	Fourier Series (FS)	Fourier Transform (FT)
Discrete	Discrete-Time Fourier Series (DTFS)	Discrete-Time Fourier Transform (DTFT)

Time Domain	*Periodic*	*Nonperiodic*	
Continuous	Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$ $X[k] = \frac{1}{T}\int_{\langle T\rangle} x(t)e^{-jk\omega_o t}\,dt$ $x(t)$ has period T $\omega_o = \frac{2\pi}{T}$	Fourier Transform $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$ $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	*Nonperiodic*
Discrete	Discrete-Time Fourier Series $x[n] = \sum_{k=\langle N\rangle} X[k]e^{jk\Omega_o n}$ $X[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\Omega_o n}$ $x[n]$ and $X[k]$ have period N $\Omega_o = \frac{2\pi}{N}$	Discrete-Time Fourier Transform $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega})\, e^{j\Omega n}\,d\Omega$ $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$ $X(e^{j\Omega})$ has period 2π	*Periodic*
	Discrete	*Continuous*	*Frequency Domain*

Discrete-time Fourier Transform

- Defined for discrete time nonperiodic signal
- DTFT is continuous and periodic with a period 2

DTFT Pair

$$x[n] \longleftrightarrow X(e^{j\omega})$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

– Analysis Equation
– FT

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

– Synthesis Equation
– Inverse FT

Discrete Fourier Transform(DFT)

- DFT of a finite duration sequence x[n] is obtained by sampling DTFT at N equally spaced points over the interval $0 \leq \omega \leq 2\pi$ with spacing $2\pi / N$
- The N points should be located at

$$\omega_k = \frac{2\pi}{N} k, \qquad k = 0,1,2.....N-1$$

Discrete Fourier Transform(DFT)

- These N equally spaced frequency samples of the DTFT are known as DFT denoted by X(k) is

$$X(k) = X(e^{jw})\Big|_{\omega=\frac{2\pi}{N}k}$$

$$where$$

$$k = 0,1,2.....N-1$$

Frequency-domain Sampling of Fourier Transform

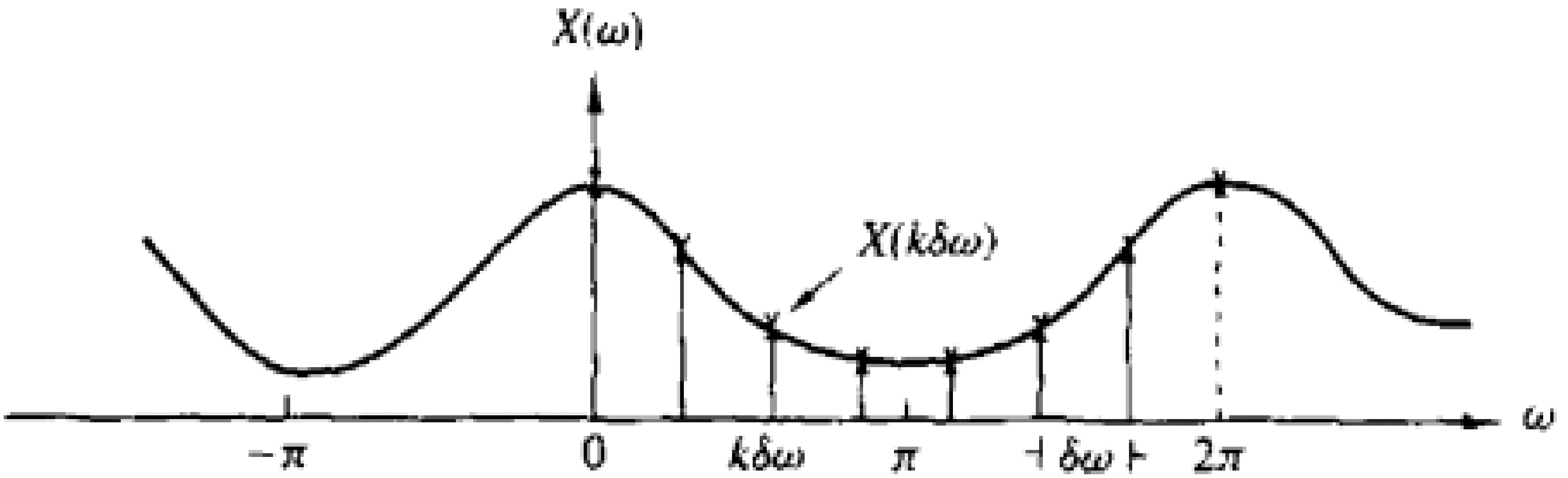


Figure 5.1 Frequency-domain sampling of the Fourier transform

Discrete Fourier Transform(DFT)

- N point DFT of a finite duration sequence x[n] of length L(L< or =N) can be calculated as

$$X(k)=\sum_{n=0}^{N-1} x(n)e^{\frac{-j2\pi}{N}kn}$$

$$k=0,1,2.....N-1$$

Question Q1

- Find DFT of the sequence $x[n]=\{1,1,0,0\}$

Solution Q1

Let us assume $N = L = 4$.

We have $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} \quad k = 0, 1, \ldots, N-1$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi n/2} = x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

$$= 1 - j$$

Solution Q1….contd

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi$$

$$= 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j3n\pi/2} = x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 + j$$

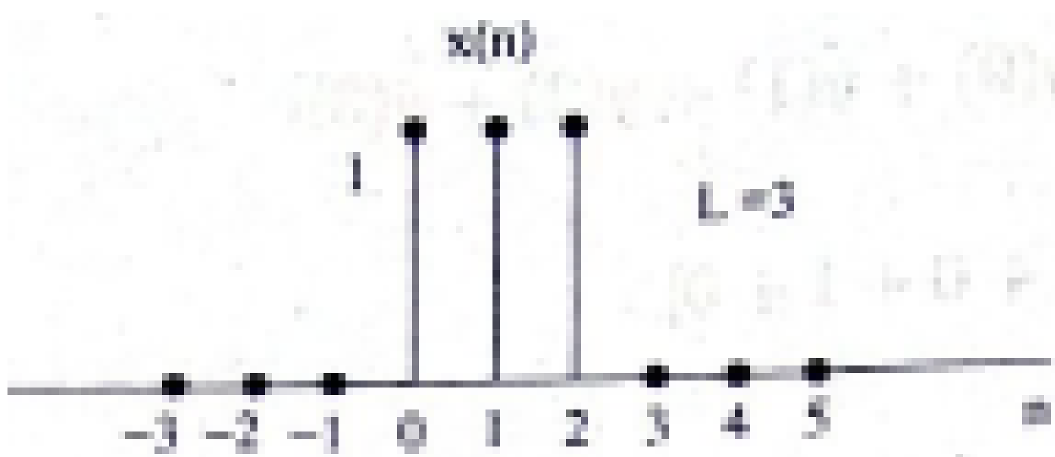
$$X(k) = \{2, 1 - j, 0, 1 + j\}$$

Question Q2

Example 3.2 Find the DFT of a sequence

$$x(n) = 1 \quad \text{for } 0 \leq n \leq 2$$
$$= 0 \quad \text{otherwise}$$

for (i) $N = 4$ (ii) $N = 8$. Plot $|X(k)|$ and $\angle X(k)$. Comment on the result.



Solution Q2

Solution

Given $L = 3$. For $N = 4$, the periodic extension of $x(n)$ shown in Fig. 3.5 can be obtained by adding one zero (i.e., $N - L$ zeros).

We have

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad k = 0, 1, \ldots, N-1$$

Solution Q2… contd
Peridic extension of x(n) for N=4

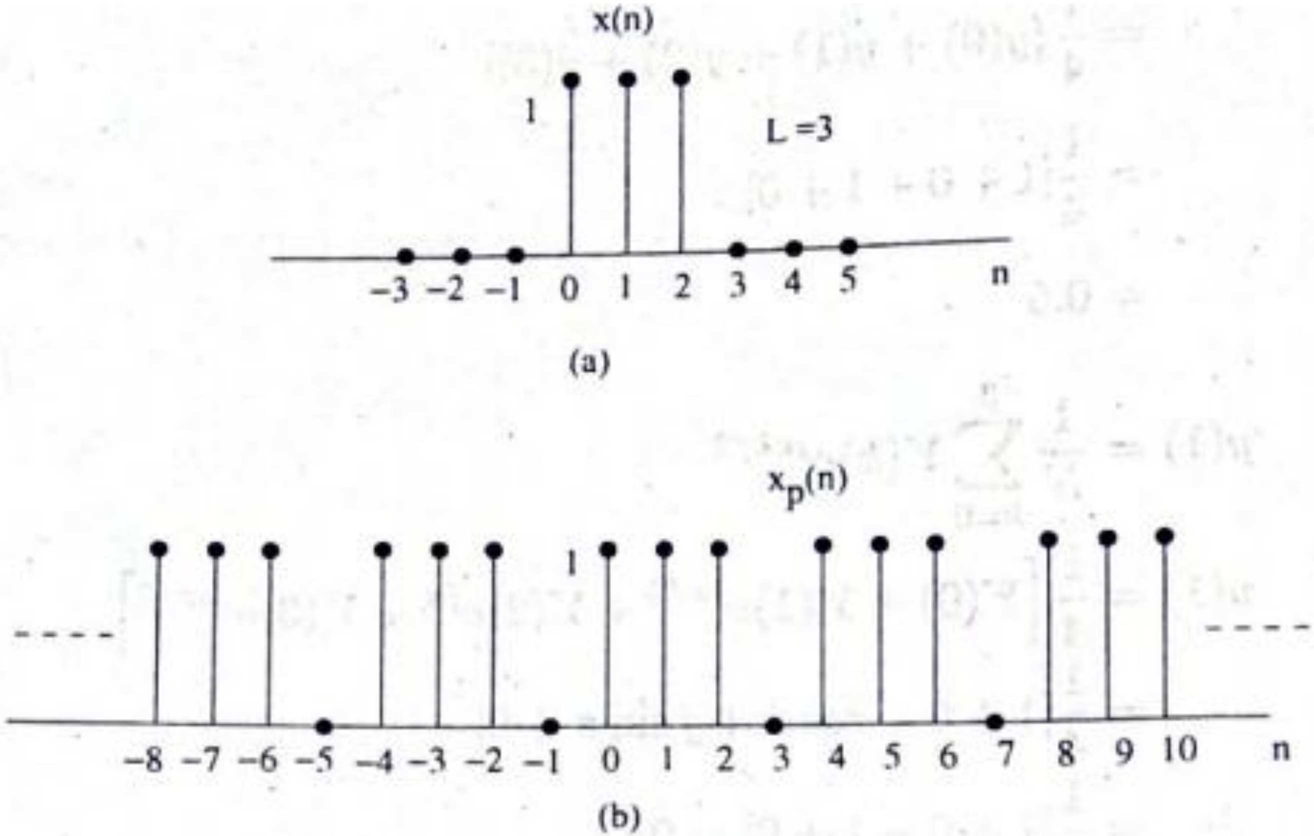


Fig. 3.5 (a) The sequence given in example 3.2 (b) Periodic extension of the sequence for $N = 4$.

From Fig. 3.5b we find

$$x(0) = 1; x(1) = 1; \qquad x(2) = 1; x(3) = 0$$

For $N = 4$

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\pi nk/2} \quad k = 0, 1, 2, 3$$

For $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$
$$= 3$$

Therefore, $|X(0)| = 3, \angle X(0) = 0$
For $k = 1$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n/2}$$
$$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$
$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi + 0$$
$$= 1 - j - 1 = -j$$

Therefore,

$$|X(1)| = 1, \quad \angle X(1) = \frac{-\pi}{2}$$

For $k = 2$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n}$$

$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi + 0$$

$$= 1 - 1 + 1 = 1$$

Therefore,

$$|X(2)| = 1, \quad \angle X(2) = 0$$

For $k = 3$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j3\pi n/2}$$

$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi + 0$$

$$= 1 + j - 1 = j$$

Therefore $|X(3)| = 1, \quad \angle X(3) = \frac{\pi}{2}$

$$|X(k)| = \{3, 1, 1, 1\}$$

$$\angle X(k) = \left\{0, -\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$$

The plot of $|X(k)$ and $\angle X(k)$ for $N = 4$ is shown in Fig. 3.6.

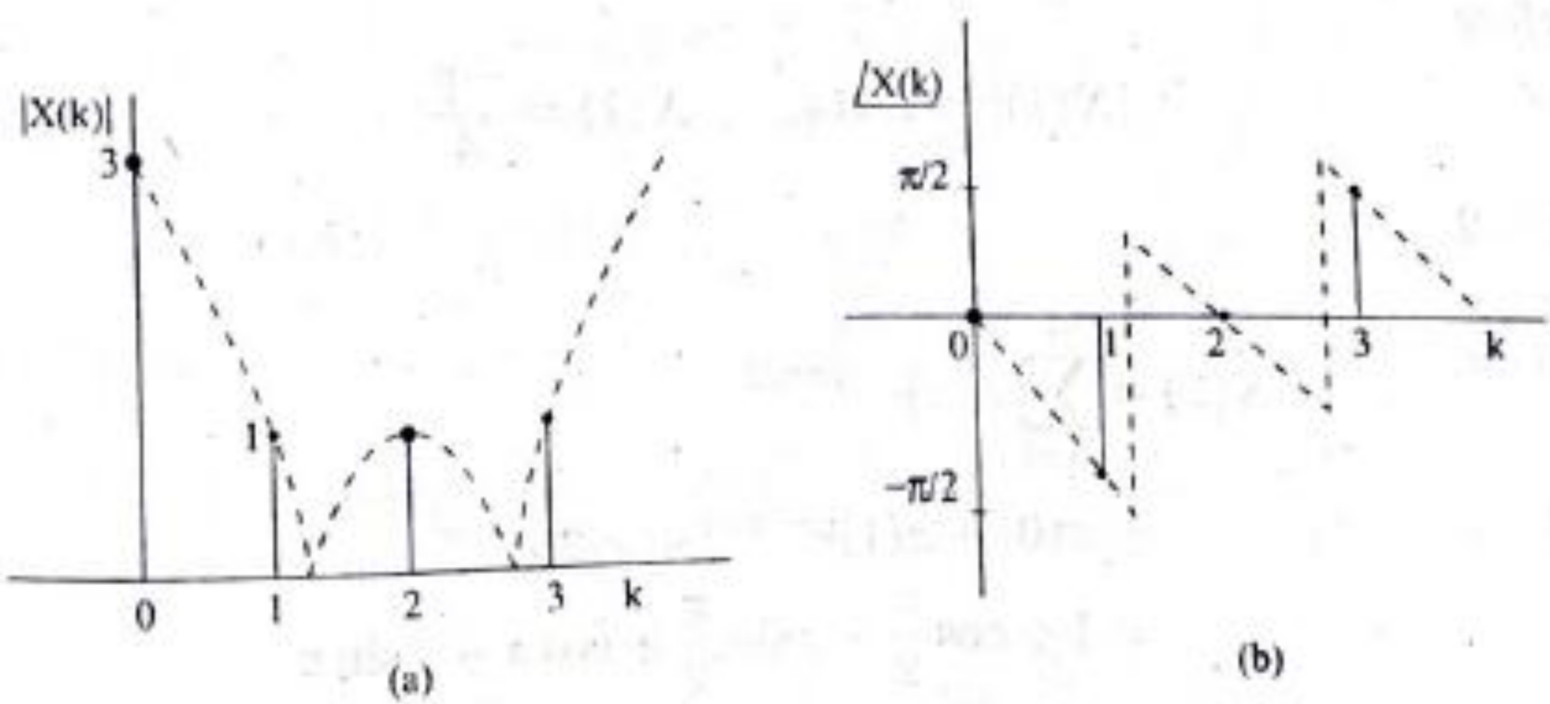


Fig. 3.6 Frequency response of $x(n)$ for N = 4

Solution Q2… contd
Peridic extension of x(n) for N=8

For $N = 8$ the periodic extension of $x(n)$ shown in Fig. 3.7 can be obtained by adding five zeros ($\because$ $N - L$ zeros).

$$x(0) = x(1) = x(2) = 1 \text{ and } x(n) = 0 \text{ for } 3 \le n \le 7$$

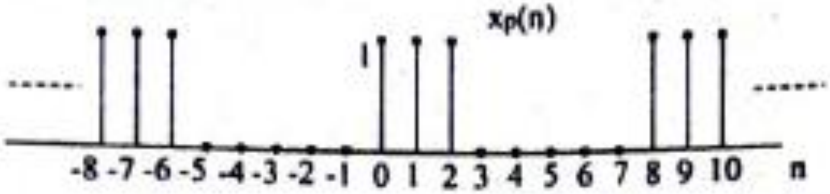


Fig. 3.7 Periodic extension of the sequence $x(n)$ for N = 8

We have $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N}$

For $N = 8$

$$X(k) = \sum_{n=0}^{7} x(n)e^{-j\pi nk/4} \quad k = 0, 1 \ldots, 7$$

For $k = 0$

$$X(0) = \sum_{n=0}^{7} x(n)$$

$$X(0) = 1 + 1 + 1 + 0 + 0 + 0 + 0 + 0 = 3$$

Therefore, $|X(0)| = 3 \quad \angle X(0) = 0$

For $k = 1$

$$\begin{aligned} X(1) &= \sum_{n=0}^{7} x(n)e^{-j\pi n/4} \\ &= x(0) + x(1)e^{-j\pi/4} + x(2)e^{-j\pi/2} \\ &= 1 + 0.707 - j0.707 + 0 - j \\ &= 1.707 - j1.707 \end{aligned}$$

Therefore,

$$|X(1)| = 2.414, \quad \angle X(1) = \frac{-\pi}{4}$$

For $k = 2$

$$X(2) = \sum_{n=0}^{7} x(n)e^{-j\pi n/2}$$

$$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi$$

$$= 1 - j - 1 = -j$$

Therefore

$$|X(2)| = 1, \quad \angle X(2) = \frac{-\pi}{2}$$

For $k = 3$

$$X(3) = \sum_{n=0}^{7} x(n)e^{-j3\pi n/4}$$

$$= x(0) + x(1)e^{-j3\pi/4} + x(2)e^{-j3\pi/2}$$

$$= 1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 - 0.707 - j0.707 + j$$

$$= 0.293 + j0.293$$

Therefore, $|X(3)| = 0.414, \quad \angle X(3) = \frac{\pi}{4}.$

For $k = 4$

$$X(4) = \sum_{n=0}^{7} x(n)e^{-j\pi n}$$

$$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi}$$

$$= 1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi$$

$$= 1 - 1 + 1 = 1$$

Therefore, $|X(4)| = 1, \quad \angle X(4) = 0$

For $k = 5$

$$X(5) = \sum_{n=0}^{7} x(n)e^{-j5\pi n/4}$$
$$= x(0) + x(1)e^{-j5\pi/4} + x(2)e^{-j5\pi/2}$$
$$= 1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2}$$
$$= 1 - 0.707 + j0.707 - j$$
$$= 0.293 - j0.293$$
$$|X(5)| = 0.414, \quad \angle X(5) = -\frac{\pi}{4}$$

For $k = 6$

$$X(6) = \sum_{n=0}^{7} x(n)e^{-j3\pi n/2}$$
$$= x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi}$$
$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi$$
$$= 1 + j - 1 = j$$
$$|X(6)| = 1, \quad \angle X(6) = \frac{\pi}{2}$$

For $k = 7$

$$X(7) = \sum_{n=0}^{7} x(n)e^{-j7\pi n/4}$$

$$= 1 + e^{-j7\pi/4} + e^{-j7\pi/2}$$

$$= 1 + \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2}$$

$$= 1 + 0.707 + j0.707 + j$$

$$= 1.707 + j1.707$$

$$|X(7)| = 2.414, \quad \angle X(7) = \frac{\pi}{4}$$

$$|X(k)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$$

$$\angle X(k) = \left\{0, -\frac{\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, 0\frac{-\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\right\}$$

The plot of $|X(k)|$ and $\angle X(k)$ vs. k for $N = 8$ is shown in Fig. 3.8.

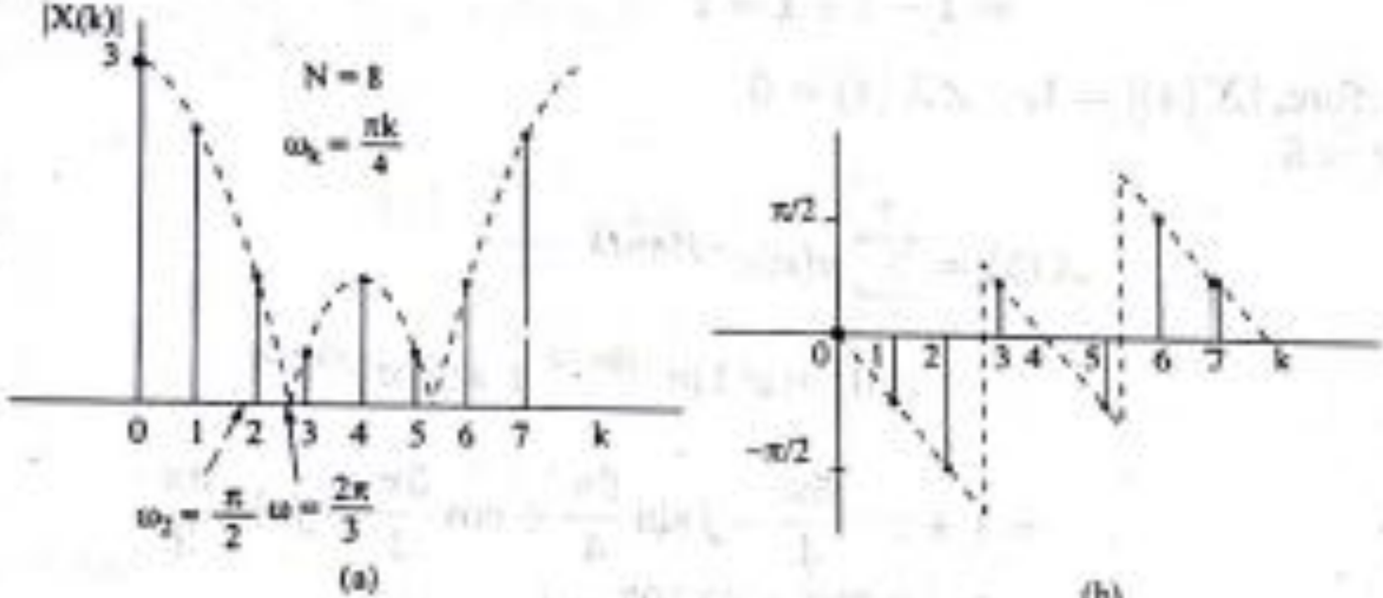


Fig. 3.8 Frequency response of $x(n)$ for N = 8

Comments: Based on the Fig. 3.6 and Fig. 3.8 we can observe the following.

From Fig. 3.6 we can observe that, with $N = 4$, it is difficult to extrapolate the entire frequency spectrum. For low values of N, the spacing between successive samples is high, which results in poor resolution. On the other hand when $N = 8$, from Fig. 3.8 we can observe that it is possible to extrapolate the frequency of spectrum. That is zero padding gives a high density spectrum and provides a better displayed version for plotting.

Inverse Discrete Fourier Transform(IDFT)

- IDFT of X(k) length can be calculated as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi}{N} kn}$$

$$n = 0,1,2\ldots.N-1$$

Q. Find IDFT of the sequence

$$X(k) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$$

We have $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad n = 0, 1, \ldots, N-1$

For $N = 8$

$$x(n) = \frac{1}{8} \sum_{k=0}^{N-1} X(k) e^{j\pi kn/4} \quad n = 0, 1, \ldots, 7$$

For $n = 0$

$$x(0) = \frac{1}{8} \left[\sum_{k=0}^{7} X(k) \right] = \frac{1}{8}[5 + 0 + 1 - j + 0 + 1 + 0 + 1 + j + 0] = 1$$

$$x(1) = \frac{1}{8} \left[\sum_{k=0}^{7} X(k) e^{j\pi k/4} \right] = \frac{1}{8}[5 + (1 - j)(j) + 1(-1) + (1 + j)(-j)]$$

$$= \frac{1}{8}[6] = 0.75$$

$$x(2) = \frac{1}{8} \left[\sum_{k=0}^{7} X(k) e^{j\pi k/2} \right] = \frac{1}{8}[5 + (1 - j)(-1) + 1(1) + (1 + j)(-1)]$$

$$= \frac{1}{8}[4] = 0.5$$

$$x(3) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)e^{j3\pi k/4}\right] = \frac{1}{8}[5 + (1-j)(-j) + 1(-1) + (1+j)(j)]$$

$$= \frac{1}{8}[2] = 0.25$$

$$x(4) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)e^{j\pi k}\right] = \frac{1}{8}[5 + (1-j)(1) + 1(1) + (1+j)(1)]$$

$$= 1$$

$$x(5) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)e^{j5\pi k/4}\right] = \frac{1}{8}[5 + (1-j)(j) + (1)(-1) + (1+j)(-j)]$$

$$= \frac{1}{8}[6] = 0.75$$

$$x(6) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)e^{j3\pi k/2}\right] = \frac{1}{8}[5 + (1-j)(-1) + 1(1) + (1+j)(-1)]$$

$$= \frac{1}{8}[4] = 0.5$$

$$x(7) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)e^{j7\pi k/4}\right] = \frac{1}{8}[5 + (1-j)(-j) + 1(-1) + (1+j)(j)]$$

$$= \frac{1}{8}[2] = 0.25$$

$$x(n) = \{1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25\}$$