Steps for designing a Digital Butterworth low pass filter

step ① Determine the analog frequency from from given specification.

(a) impulse invariant transformation.

$$\Omega_p = \frac{\omega_p}{T} \qquad \Omega_s = \frac{\omega_s}{T}.$$

(b) bilinear transformation

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2}.$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2}.$$

(prewarping).

step②: From the given specification determine the order of the filter N and round off it to next highest integer

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)}$$

or

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log(\Omega_s/\Omega_p)}$$

step③ : Calculate the cut-off frequency Ω_c

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{\Omega_p}{\left(\sqrt{10^{0.1\alpha_p} - 1}\right)^{1/N}}.$$

step④ : Find the normalized analog transfer (for $\Omega_c=1$) function $H_N(s)$ for the value of N
[use table for denominator of $H(s)$]

step⑤ : Find the transfer function of analog filter $H_a(s)$ for the obtained value of Ω_c by substituting $s \longrightarrow s/\Omega_c$ in $H_N(s)$

i.e $\quad H_a(s) = H_N(s)\Big|_{s \to s/\Omega_c}.$

step⑥ : Convert the analog filter with transfer function $H_a(s)$ to digital filter by using.

(a) Impulse invariant method.

(i) express analog transfer function of the form:

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}.$$

(ii) select sampling rate T in sec.

(iii) Compute the transfer function of digital filter by using the formula

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

for high sampling rate (T very small)

$$H(z) = \sum_{k=1}^{N} \frac{T c_k}{1 - e^{p_k T} z^{-1}}$$

(b) Bilinear Transformation.

(i) substitute $s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]$ into the transfer function $H_a(s)$ to get $H(z)$

ie $H(z) = H_a(s) \Big|_{s = \frac{2}{T}\left[\frac{1 - z^{-1}}{1 + z^{-1}}\right]}$

Q) Design a digital Butterworth filter satisfying the constraints pg- 5.19

$$0.707 \le |H(e^{j\omega})| \le 1 \quad \text{for } 0 \le \omega \le \pi/2.$$

$$|H(e^{j\omega})| \le 0.2 \quad \text{for } \frac{3\pi}{4} \le \omega \le \pi.$$

with T=1 sec using.

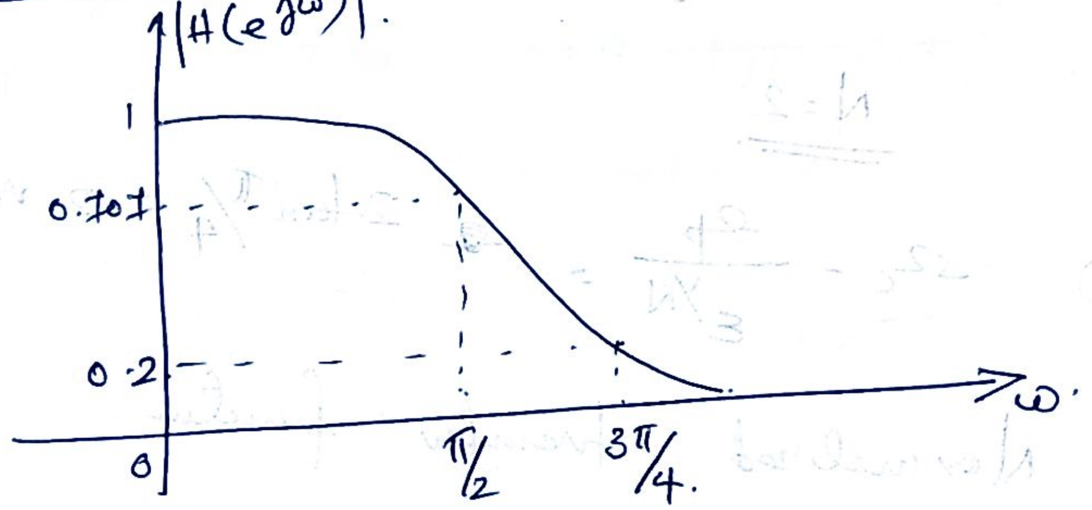
(a) Bilinear transformation.

(b) Impulse invariant transformation.

(Ramesh Babu)

Ans: Realise the filter given:

~~Bilinear transformation~~



given: $\omega_p = \pi/2$ $\quad \omega_s = 3\pi/4.$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.707 \quad , \quad \frac{1}{\sqrt{1+\lambda^2}} = 0.2.$$

$$\Rightarrow \varepsilon = 1 \quad , \quad \lambda = 4.898.$$

(a) Bilinear transformation (prewarping).

① $\Omega = \frac{2}{T} \tan \omega/2.$

$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2} = 2 \tan \pi/4.$

$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2} = 2 \tan \frac{3\pi}{8}.$

$\frac{\Omega_s}{\Omega_p} = \frac{2 \tan 3\pi/8}{2 \tan \pi/4} = 2.414.$

② $N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)} \geq \underline{\underline{1.803}}$

$\underline{\underline{N = 2}}$

③ $\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = 2 \tan \pi/4 = 2$ rad/sec.

④ Normalized transfer function

$H_N(s) = \frac{1}{s^2 + \sqrt{2}\, s + 1}.$

⑤ $H_a(s)$ for $\Omega_c = 2$ rad/sec.

$s \rightarrow \frac{s}{\Omega_c} = \frac{s}{2}.$

$$\therefore H_a(s) = \frac{1}{(s/2)^2 + \sqrt{2}\, s/2 + 1}$$

$$= \frac{1}{s^2/4 + \frac{s}{\sqrt{2}} + 1}.$$

$$= \frac{4}{s^2 + 2.828\, s + 4}.$$

using bilinear transformation

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}.$$

$$\therefore H(z) = \frac{4}{s^2 + 2.828 s + 4}\Bigg|_{s = 2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}.$$

$$= \frac{4}{\left[2\left[\frac{1-z^{-1}}{1+z^{-1}}\right]\right]^2 + 2.828 \times 2\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 4}.$$

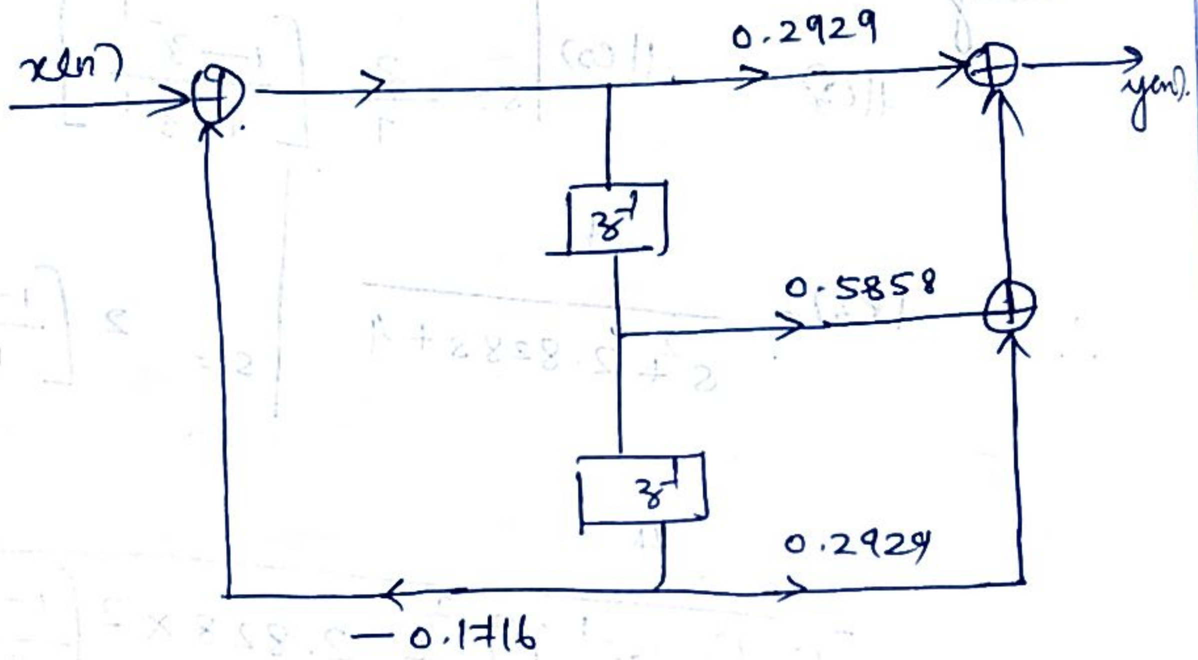
$$= \frac{4(1+z^{-1})^2}{4(1-z^{-1})^2 + 5.656(1-z^{-2}) + 4(1+z^{-1})^2}$$

$$= \frac{0.2929(1+z^{-1})^2}{1 + 0.1716\, z^{-2}}.$$

$$H(z) = \frac{0.2929(1+2z^{-1}+z^{-2})}{1+0.1716z^{-2}}$$

$$= \frac{0.2929 + 0.5858z^{-1} + 0.2929z^{-2}}{1+0.1716z^{-2}}$$

The above system function can be realized in direct form II.



(b) Impulse invariant

$\Omega = \omega/T$. $T = 1$

$\Omega_p = \omega_p = \pi/2$ $\Omega_s = \omega_s = 3\pi/4$.

$$N > \frac{\log \lambda/\varepsilon}{\log \Omega_s/\Omega_p}$$ $N \geq 3.924$

$N = 4$.

$$H_N(s) = \frac{1}{(s^2 + 0.7653\, s + 1)(s^2 + 1.84775\, s + 1)}$$

$$\Omega_c = 0.5\pi$$

$$H_a(s) = H_N(s)\Big|_{s = \frac{s}{\Omega_c}}$$

$$H_a(s) = \frac{(1.57)^4}{(s^2 + 1.202s + 2.465)(s^2 + 2.902s + 2.465)}$$

(from text)

(b) Impulse Invariant Method

Solution

The relationship between analog & digital frequencies in Impulse invariant method is $\omega = \Omega T$.

From the given data $T = 1$ sec i.e., $\omega = \Omega$

$$\Rightarrow \quad \Omega_p = \omega_p; \quad \Omega_s = \omega_s$$

We know $\lambda = 4.898$; $\varepsilon = 1$.

The order of the filter

$$N \geq \frac{\log \frac{\lambda}{\varepsilon}}{\log \frac{\Omega_s}{\Omega_p}} = \frac{\log 4.989}{\log \frac{3\pi/4}{\pi/2}}$$

$$N \geq 3.924$$

i.e., $N = 4$

The transfer function of a fourth order normalized Butterworth filter is

$$H(s) = \frac{1}{(s^2 + 0.76537 + 1)(s^2 + 1.8477s + 1)}$$

As $\varepsilon = 1$: $\quad \Omega_p = \Omega_c = 0.5\pi = 1.57$

$$H_a(s) = H(s)\Big|_{s \to \frac{s}{1.57}}$$

$$= \frac{(1.57)^4}{(s^2 + 1.202s + 2.465)(s^2 + 2.902s + 2.465)}$$

$H_a(s)$ in the partial fraction form is given by

$$H_a(s) = \frac{A}{(s+1.45+j0.6)} + \frac{A^*}{(s+1.45-j0.6)} + \frac{B}{(s+0.6+j1.45)} + \frac{B^*}{(s+0.6-j1.45)}$$

$$A = (s+1.45+j0.6)\left.\frac{(1.57)^4}{(s+1.45+j0.6)(s+1.45-j0.6)(s^2+1.202s+2.465)}\right|_{s=-1.45-j0.6}$$

$$= \frac{(1.57)^4}{(-j0.6-0.6)\,[(-1.45-j0.6)^2+1.202(-1.45-j0.6)+2.465]}$$

$$= \frac{(1.57)^4}{-j(1.2)\,[1.7425+1.74j-1.7429-j0.7212+2.465]}$$

$$= \frac{(1.57)^4}{-j(1.2)(2.465+j1.0188)}$$

$$= \frac{5.063}{1.0188-j2.465} = \frac{5.063(1.0188+j2.465)}{7.114}$$

$$= 0.7116(1.0188+j2.465) = 0.7253+j1.754$$

$$B = (s+0.6+j1.45)\left.\frac{(1.57)^4}{(s+0.6+j1.45)(s+0.6-j1.45)(s^2+2.902s+2.465)}\right|_{s=-0.6-j1.45}$$

$$= \frac{(1.57)^4}{-j(2.9)\,[(-0.6-j1.45)^2+2.902(-0.6-j1.45)+2.465]}$$

$$= \frac{(1.57)^4}{-j(2.9)\,[-1.7425+j1.74-1.7412-j4.208+2.465]}$$

$$= \frac{2.095}{-j\,[-1.0187-j2.468]}$$

$$= \frac{2.095}{-2.468+j1.0187} = \frac{2.095\,[-2.468-j1.0187]}{7.1287}$$

$$= 0.29388\,[-2.468-j1.0187] = -0.7253-0.3j$$

$$H_a(s) = \frac{0.7253+j1.754}{s-(-1.45-j0.6)} + \frac{0.7253-j1.754}{s-(-1.45+j0.6)} + \frac{-0.7253-0.3j}{s-(-0.6-j1.45)} + \frac{-0.7253+0.3j}{s-(-0.6+j1.45)}$$

We know for $T = 1$ sec

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k}z^{-1}}$$

Therefore

$$H_a(s) = \frac{0.7253 + j1.754}{1 - e^{-1.45}e^{-j0.6}z^{-1}} + \frac{0.7253 - j1.754}{1 - e^{-1.45}e^{j0.6}z^{-1}}$$
$$+ \frac{-(0.7253 + 0.3j)}{1 - e^{-0.6}e^{-j1.45}z^{-1}} + \frac{-0.7253 + 0.3j}{1 - e^{-0.6}e^{j1.45}z^{-1}}$$
$$= \frac{1.454 + 0.1839z^{-1}}{1 - 0.387z^{-1} + 0.055z^{-2}} + \frac{-1.454 + 0.2307z^{-1}}{1 - 0.1322z^{-1} + 0.301z^{-2}}$$

This can be realized using parallel form as shown in Fig. 5.63.

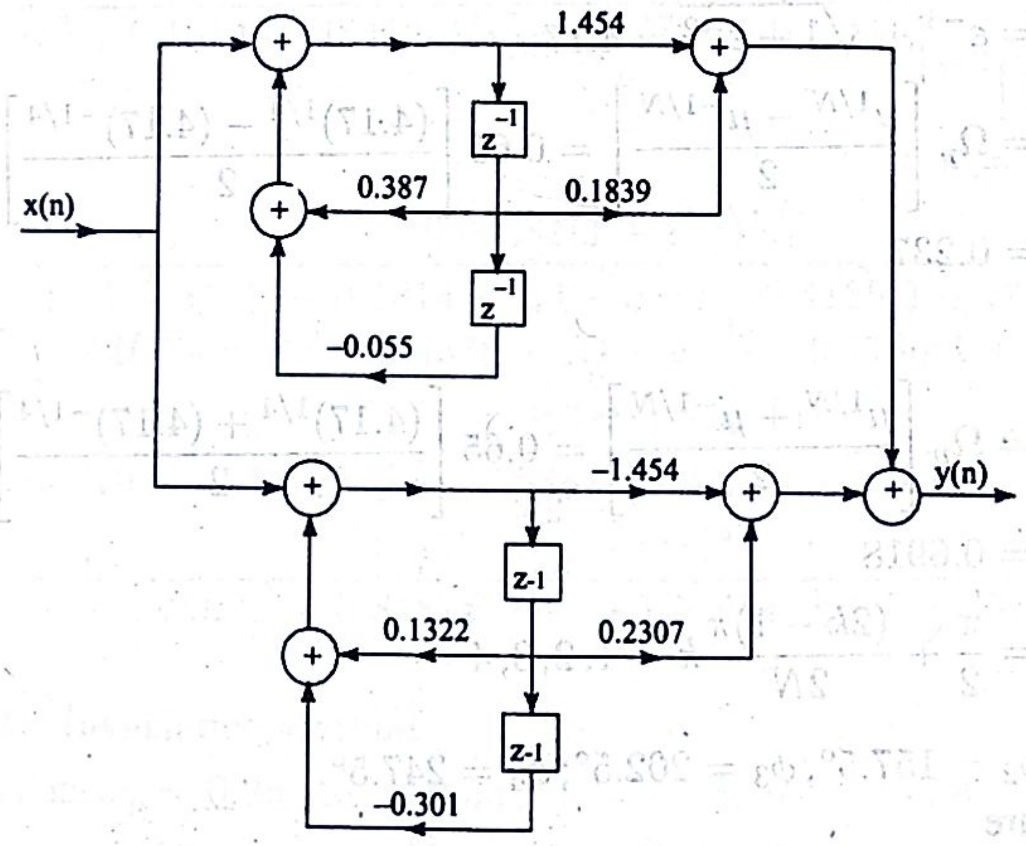


Fig. 5.63