# Design of FIR filters — Module III.

## Symmetric and Antisymmetric FIR filters

— An FIR filter with length M with input $x(n)$ and output $y(n)$ is described by the difference equation:

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

$$= b_0 x(n) + b_1 x(n-1) + \dots b_{M-1} x(n-M+1)$$

where $\{b_k\}$ is the set of filter coefficients.

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)$$

$$= h(n) * x(n).$$

$$b_k = h(k).$$

The filter can also be characterized by its system function:

$$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}.$$

— An FIR filter has linear phase its unit sample response satisfy the condn:

$$h(n) = \pm h(M-1-n). \qquad n = 0, 1 \ldots M-1$$

$+ \longrightarrow$ symmetry.

$- \longrightarrow$ ~~ay~~ antisymmetry.

For $M$ - odd. ~~the~~ for anti symmetric.

$$h(n) = -h(M-1-n).$$

the center point of the antisymmetric $h(n)$ is $n = \frac{M-1}{2}$.

$$\therefore h\left(\frac{M-1}{2}\right) = 0.$$

- The choice of a symmetric or ~~anti~~ antisymmetric unit sample response is suitable for some applications, and ~~anter~~ when symmetry and antisymmetry condition is incorporated into $H(z)$

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \ldots h(M-1)z^{-(M-1)}$$

## 6.6 Design of FIR filters using windows

The desired frequency response $H_d(e^{j\omega})$ of a filter is periodic in frequency and can be expanded in a Fourier series. The resultant series is given by

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)e^{-j\omega n} \tag{6.70}$$

where

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{6.71}$$

and known as Fourier coefficents having infinite length. One possible way of obtaining FIR filter is to truncate the infinite Fourier series at $n = \pm\left(\frac{N-1}{2}\right)$, where $N$ is the length of the desired sequence. But abrupt truncation of the Fourier series results in oscillation in the passband and stopband. These oscillations are due to slow convergence of the Fourier series and this effect is known as the Gibbs phenomenon. To reduce these oscillations, the Fourier coefficients of the filter are modified by multiplying the infinite impulse response with a finite weighing sequence $w(n)$ called a window where

$$\begin{aligned} w(n) = w(-n) &\neq 0 \quad \text{for} \quad |n| \leq \left(\frac{N-1}{2}\right) \\ &= 0 \quad \text{for} \quad |n| > \left(\frac{N-1}{2}\right) \end{aligned} \tag{6.72}$$

After multiplying window sequence $w(n)$ with $h_d(n)$, we get a finite duration sequence $h(n)$ that satisfies the desired magnitude response

$$h(n) = h_d(n)w(n) \quad \text{for all} \quad |n| \leq \left(\frac{N-1}{2}\right)$$
$$= 0 \quad \text{for} \quad |n| > \left(\frac{N-1}{2}\right) \tag{6.73}$$

The frequency response $H(e^{j\omega})$ of the filter can be obtained by convolution of $H_d(e^{j\omega})$ and $W(e^{j\omega})$ given by

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \tag{6.74}$$
$$= H_d(e^{j\omega}) * W(e^{j\omega}) \tag{6.75}$$

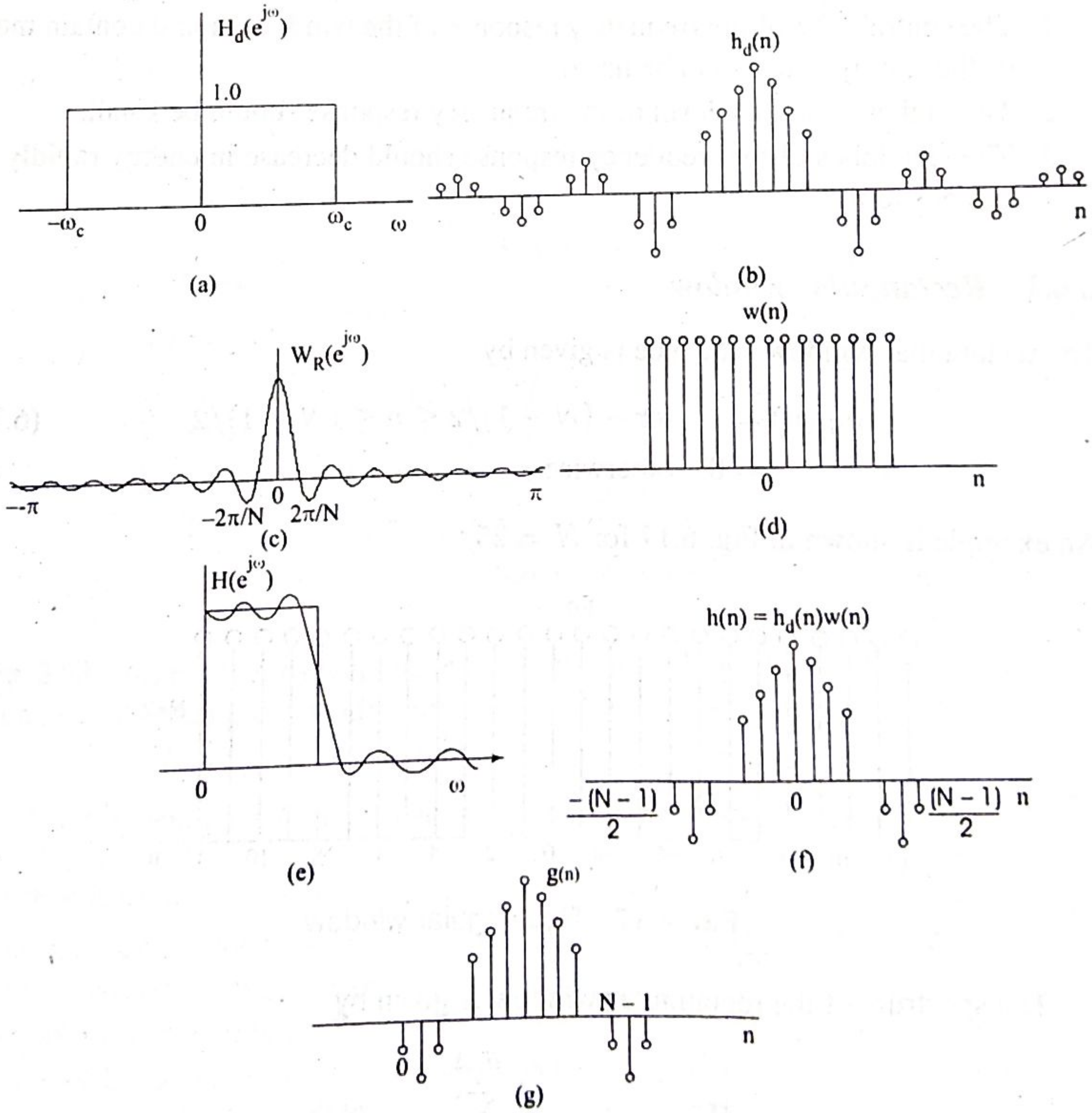



**Fig. 6.16** Windowing technique

Because both $H_d(e^{j\omega})$ and $W(e^{j\omega})$ are periodic functions the operation is often called as periodic convolution. The windowing technique is shown in Fig. 6.16. The desired frequency response and its Fourier coefficients are shown in Fig. 6.16a and 6.16b respectively. The Fig. 6.16c and 6.16d show a finite window sequence $w(n)$ and its Fourier transform $W(e^{j\omega})$. The Fourier transform of a window consists of a central lobe and side lobes. The central lobe contains most of the energy of the window. To get an FIR filter, the sequence $h_d(n)$ and $w(n)$ are multiplied and a finite length of non-causal sequence $h(n)$ is obtained. The Fig. 6.16f and 6.16e show $h(n)$ and its Fourier transform $H(e^{j\omega})$. The frequency response $H(e^{j\omega})$ is obtained using Eq.(6.74). The realizable sequence $g(n)$ in Fig. 6.16g can be obtained by shifting $h(n)$ by $\alpha$ number of samples, where $\alpha = \frac{N-1}{2}$.

From Eq.(6.74) we find that the frequency response of the filter $H(e^{j\omega})$ depends on the frequency response of window $W(e^{j\omega})$. Therefore, the window, chosen for truncating the infinite impulse response should have some desirable characteristics. They are

1. The central lobe of the frequency response of the window should contain most of the energy and should be narrow.
2. The highest side lobe level of the frequency response should be small.
3. The side lobes of the frequency response should decrease in energy rapidly as $\omega$ tends to $\pi$.

### 6.6.1 *Rectangular window*

The rectangular window sequence is given by

$$\begin{aligned} w_R(n) &= 1 \quad \text{for} -(N-1)/2 \le n \le (N-1)/2 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{6.76}$$

An example is shown in Fig. 6.17 for $N = 25$.

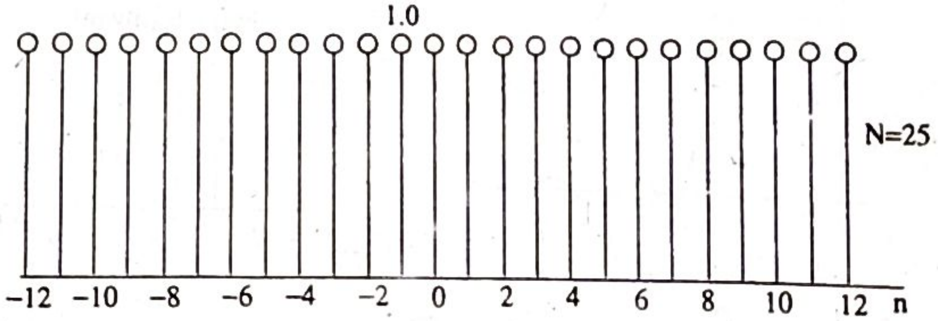



**Fig. 6.17** Rectangular window

The spectrum of the rectangular window is given by

$$W_R(e^{j\omega}) = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} e^{-j\omega n}$$

$$
\begin{aligned}
&= e^{j\omega(N-1)/2} + \ldots + e^{j\omega} + 1 + e^{-j\omega} + \ldots + e^{-j\omega(N-1)/2} \\
&= e^{j\omega(N-1)/2}\left[1 + e^{-j\omega} + \ldots + e^{-j\omega(N-1)}\right] \\
&= e^{j\omega(N-1)/2}\left[\frac{1-e^{j\omega N}}{1-e^{-j\omega}}\right] \qquad \boxed{1 + a + a^2 \ldots a^{N-1} = \frac{1-a^N}{1-a}} \\
&= \frac{e^{j\omega N/2}(1-e^{-j\omega N})}{e^{j\omega/2}(1-e^{-j\omega})} \\
&= \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} \\
&= \frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}} \qquad (6.77)
\end{aligned}
$$

The frequency spectrum for $N = 25$ is shown in Fig. 6.18.

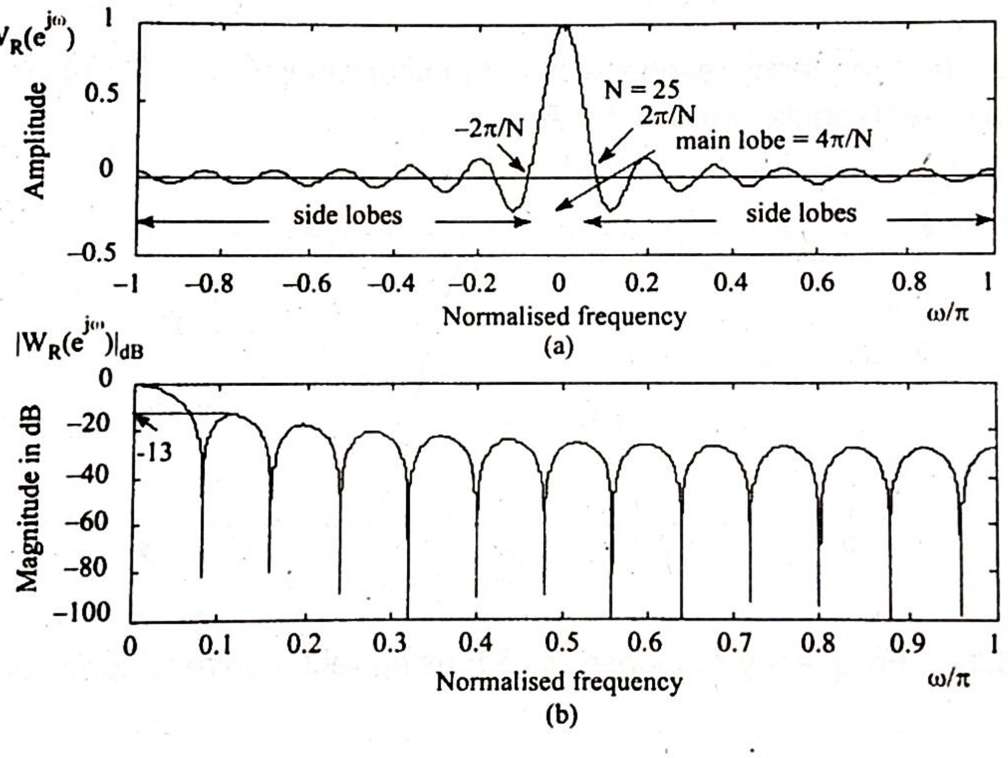



**Fig. 6.18** (a) Frequency response of rectangular window $N = 25$ (b) Log magnitude response of rectangular window for $N = 25$.

## Design of FIR filter using window method

**step 1**: From the given desired frequency response $H_d(e^{j\omega})$ find desired impulse response by finding Inverse fourier transform

ie $$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega .$$

**step 2**: Truncate the $h_d(n)$ to $h(n)$ using appropriate window function of length N

$$h(n) = h_d(n) \cdot w(n)$$

**step 3**: Obtain the transfer function of the corresponding filter by calculating z transform of $h(n)$.

$$h(n) \xrightarrow{\;z\;} H(z)$$

$$H(z) = \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} h(n)\, z^{-n}.$$

(which is not realizable or non causal.)

**step 4**: To get the corresponding realizable transfer function

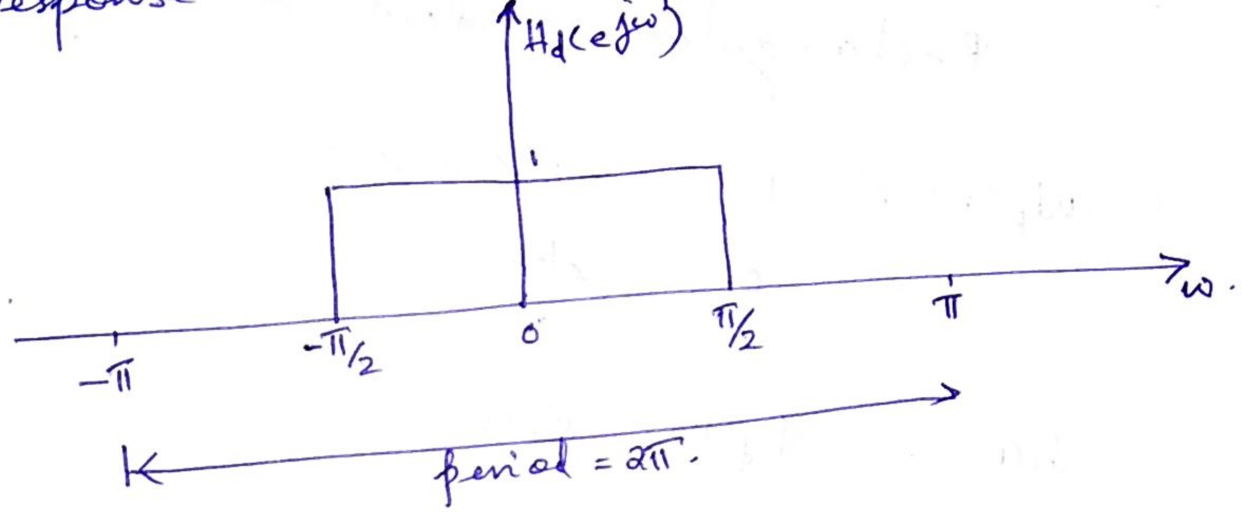
$$H'(z) = z^{-\left(\frac{N-1}{2}\right)} H(z).$$

Q) Design an ideal lowpass filter with frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & \text{for } -\pi/2 \leq \omega \leq \pi/2. \\ 0 & \text{for } \pi/2 \leq |\omega| \leq \pi \end{cases}$$

using rectangular window of length $N = 11$.

Ans: The given lowpass filter frequency response can be plotted as.




step 1: Find desired impulse response.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot e^{j\omega n}\, d\omega.$$

$$= \frac{1}{2\pi} \left[ \frac{e^{j\omega n}}{jn} \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi j n}\left[e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right]$$

$$= \frac{1}{\pi n}\left[\frac{(\cos\frac{\pi}{2}n + j\sin\frac{\pi}{2}n) - (\cos\frac{\pi}{2}n - j\sin\frac{\pi}{2}n)}{2j}\right]$$

$$h_d(n) = \frac{1}{\pi n}\sin(\pi/2)n \quad \text{——— ①}$$

<u>step 2</u>:

Rectangular window of length 11

$$w_R(n) = \begin{cases} 1 & -5 \le n \le 5 \\ 0 & \text{else} \end{cases}$$

$$h(n) = h_d(n) \cdot w_R(n)$$

$$\therefore \ h(n) = \begin{cases} h_d(n) & -5 \le n \le 5 \\ 0 & \text{else} \end{cases}$$

$$\therefore \ h(0) = h_d(0)$$

from eqn ①

$$h(0) = \frac{1}{\pi \cdot 0}\sin(\pi/2) \cdot 0$$

(divide by zero)

so apply L' Hospitals rule.

$$(\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1)$$

$$h(0) = \lim_{n \to 0} \frac{\sin \frac{\pi}{2} n}{\pi n}$$

$$= \frac{1}{2} \lim_{n \to 0} \frac{\sin(\frac{\pi}{2} n)}{(\frac{\pi}{2} n)}$$

$$= \frac{1}{2} \times 1 = \underline{\underline{0.5}}$$

$$h(1) = h(-1) = h_d(1) = \frac{1}{\pi} \sin(\pi/2) = 0.3183.$$

$$h(2) = h(-2) = h_d(2) = \frac{1}{2\pi} \sin(\pi) = 0.$$

$$h(3) = h(-3) = h_d(3) = \frac{1}{3\pi} \sin(3\pi/2) = -0.106$$

$$h(4) = h(-4) = h_d(4) = \frac{1}{4\pi} \sin(4\pi/2) = 0.$$

$$h(5) = h(-5) = h_d(5) = \frac{1}{5\pi} \sin(5\pi/2) = 0.0636$$

<u>step3</u>: corresponding transfer function.

$$H(z) = \sum_{n=-5}^{5} h(n)\, z^{-n}.$$

$$H(z) = h(-5)\, z^{5} + h(-4)\, z^{4} + h(-3)\, z^{3} + h(-2)\, z^{2}$$

$$h(-1)\, z^{1} + h(0)\, z^{0} + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3}$$

$$+ h(4)\, z^{-4} + h(5)\, z^{-5}$$

$$H(z) = 0.0636\, z^{5} + 0 + -0.106\, z^{3} + 0 + 0.3183\, z^{1}$$

$$+ 0.5 + 0.3183\, z^{-1} + 0 - 0.106\, z^{-3} + 0 + 0.0636\, z^{-5}$$

Since some powers of $z$ are positive the above filter is not realizable.

step 4: to get corresponding realizable transfer function.

$$H'(z) = z^{-5} H(z)$$

$$H'(z) = 0.0636 - 0.106 z^{-2} + 0.3183 z^{-4} + 0.5 z^{-5} + 0.3183 z^{-6} - 0.106 z^{-8} + 0.0636 z^{-10} \quad (2)$$

$$H'(z) = 0.0636 [1 + z^{-10}] - 0.106 [z^{-2} + z^{-8}] + 0.3183 [z^{-4} + z^{-6}] + 0.5 z^{-5}$$

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By taking Inverse Z transform

$$H'(z) = \sum_{n=0}^{10} h'(n) z^{-n} \quad (3)$$

Comparing equ (2) and (3)

The filter coefficients of realizable filter $h'(n)$ is

$h'(0) = h'(10) = 0.0636$

$h'(2) = h'(8) = -0.106$

$h'(4) = h'(6) = 0.3183$

$h'(5) = 0.5$

remaining all $h'(n) = 0$.