INFORMATION THEORY & CODING

LECTURE 7

CONTENTS

- Quick recap
- Capacity of Binary symmetric channel
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Symmetric channel

eg:

a) $P(A/B) = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

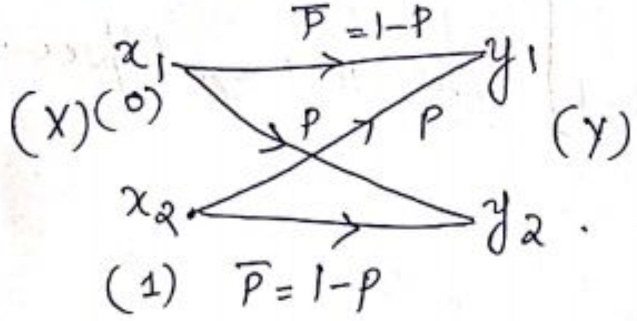
it is symmetric as rows and column are identical except for permutations.

Binary symmetric channel(BSC)

Binary symmetric channel (BSC) (special case of symmetric

→ it is one of the most commonly & widely used channel.

→ channel diagram of BSC



$$P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \end{array}$$

→ Let $P(x_1) = \omega$.

$P(x_2) = \bar{\omega} = 1-\omega$

→ Let 'p' be the probability of error.
i.e. the probability of reception of '1' when '0' is transmitted
the probability of reception of '0' when '1' is transmitted

→ x_1 is encoded as '0' and x_2 as '1'

→ $\omega + \bar{\omega} = 1$ & $p + \bar{p} = 1$. (where 'ω' → i/p probability)
& $p, \bar{p}$ is the conditional prob of channel)

Channel matrix
$$P(Y/X) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} P(y_1/x_1) & P(y_2/x_1) \\ P(y_1/x_2) & P(y_2/x_2) \end{bmatrix} \end{array}$$

$$P(Y/X) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} \bar{p} & p \\ p & \bar{p} \end{bmatrix} \end{array}$$

$$P(Y/X) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \end{array} & \begin{bmatrix} \bar{p} & p \\ p & \bar{p} \end{bmatrix} \end{array}$$

equivocation $H(Y/X) = \sum_{j=1}^{2} p_j \log \frac{1}{p_j}$

$$h := H(Y/X) = \bar{p} \log \frac{1}{\bar{p}} + p \log \frac{1}{p}.$$

find $H(Y)$

$$H(Y) = \sum_{j=1}^{2} P(y_j) \log \frac{1}{P(y_j)}$$

$$H(Y) \cdot P(y_1) \log \frac{1}{P(y_1)} + P(y_2) \log \frac{1}{P(y_2)}.$$

Equivocation

$$H(Y/X) = \sum_{i=1}^{2} \sum_{j=1}^{2} P(x_i, y_j) \log \frac{1}{P(y_j/x_i)}.$$

$$= \sum_{i} \sum_{j} P(x_i) \cdot P(y_j/x_i) \log \frac{1}{P(y_j/x_i)}$$

$$= \underbrace{\sum_{i=1}^{2} P(x_i)}_{} \sum_{j=1}^{2} P(y_j/x_i) \log \frac{1}{P(y_j/x_i)}$$

$$= 1 \sum_{j=1}^{2} P_j \log \frac{1}{P_j}$$

$$\boxed{H(Y/X) = \sum_{j=1}^{2} P_j \log \frac{1}{P_j}}$$

$P(y_1)$ & $P(y_2)$ are calculated using theorem of total probability as

$$P(y_1) = P(x_1)\cdot P(y_1/x_1) + P(x_2)\cdot P(y_1/x_2).$$

$$\boxed{P(y_1) = \omega\cdot\bar{p} + \bar{\omega}\, p.}$$

$$P(y_2) = P(x_1)\cdot P(y_2/x_2) + P(x_2)\cdot P(y_2/x_2)$$

$$= \boxed{\omega\cdot p + \bar{\omega}\,\bar{p}}$$

$$H(Y) = (\omega\bar{p} + \bar{\omega}p)\log\frac{1}{\omega\bar{p}+\bar{\omega}p} + (\omega p + \bar{\omega}\bar{p})\log\frac{1}{(\omega p+\bar{\omega}\bar{p})} \rightarrow (1)$$

$$I(x,y) = I(y,x) = H(Y) - H(Y/X)$$

$$I(x,y)\left[(\omega\bar{p} + \bar{\omega}p)\log\frac{1}{(\omega\bar{p}+\bar{\omega}p)} + (\omega p + \bar{\omega}\bar{p})\log\frac{1}{(\omega p+\bar{\omega}\bar{p})}\right] -$$

$$\left(\bar{p}\log\frac{1}{\bar{p}} + p\log\frac{1}{p}\right) \rightarrow (2)$$

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$$H(B/A) = h$$

$$I(A, B) = H(B) - H(B/A)$$

$$= H(B) - h.$$

The channel capacity for $r_s = 1$ message symbol/sec

$$C = \text{Max}\ I[(A, B)]$$

$$= \max\left[H(B) - h\right]$$

$$= \max\left[H(B)\right] - h$$

$$\boxed{C = \log s - h}$$

The entropy H(B) becomes max when symbols become equiprobable, and there are s number of output symbols

Since Binary symmetric Channel with $r_s = 1$ message symbol/sec

$C = \log s - h \quad (s = 2)$

$= \log_2 2 - h$

$$\boxed{C = 1 - h} \rightarrow (3)$$

or $$\boxed{C = 1 - \left[\bar{p}\log\frac{1}{\bar{p}} + p\log\frac{1}{p}\right] \text{ bits/sec}} \rightarrow (4)$$

when $w = \bar{w} = \frac{1}{2}$, sub in (1).

$$H(Y) = \frac{1}{2}(p+\bar{p})\log\frac{1}{\frac{1}{2}(p+\bar{p})} + \frac{1}{2}(p+\bar{p})\log\frac{1}{\frac{1}{2}(p+\bar{p})}$$

$$= \frac{1}{2}\log_2 2 + \frac{1}{2}\log_2 2 \qquad \because (p+\bar{p}) = 1$$

$$\boxed{H(Y) = 1}$$

$$I(x,y) = H(Y) - H(Y/X)$$

$$= 1 - h$$

$$\boxed{I(x,y) = 1 - \left(\bar{p}\log\frac{1}{\bar{p}} + p\log\frac{1}{p}\right)} \text{ bits/msg symbol.}$$

ie. When inputs are equiprobable, mutual information maximizes & becomes equal to channel capacity C.

EXAMPLE

Eg: A BSC has following noise matrix with source prob of $p(x_1) = \frac{2}{3}$ $p(x_2) = 1/3$.

$$P(Y/X) = \begin{array}{c|c} & \begin{matrix} y_1 & y_2 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \end{matrix} & \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \end{array}$$

(a) Determine $H(X)$, $H(Y)$, $H(X, Y)$, $H(Y/X)$, $H(X/Y)$, $I(X, Y)$

b) find channel capacity C

c) find channel efficiency & redundancy

$p(y_1/x_1) = 3/4 = \bar{p}$

$p(y_2/x_1) = 1/4 = p$.

$$\begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & \bar{p} & p \\ x_2 & p & \bar{p} \end{array}$$

$p(x_1) = \omega = \frac{2}{3}$, $p(x_2) = \bar{\omega} = \frac{1}{3}$

ii) $H(x) = p(x_1)\log\frac{1}{p(x_1)} + p(x_2)\log\frac{1}{p(x_2)}$

$= \frac{2}{3}\log\frac{3}{2} + \frac{1}{3}\log 3 = \underline{\underline{0.9183}}$ bits/msg symbol

ii) $H(Y) = \sum_{i=1} p(y_i) \log \frac{1}{p(y_i)}$

$P(y_1) = \omega \bar{P} + \bar{\omega} P$

$P(y_2) = \omega p + \bar{\omega} \bar{P}$

$\left(\frac{2}{3} \times \frac{3}{4}\right) + \left(\frac{1}{3} \times \frac{1}{4}\right)$

$P(y_1) = \frac{1}{2} + \frac{1}{12} \quad \frac{6+1}{12} = \frac{7}{12}$

$P(y_2) = \left(\frac{2}{3} \times \frac{1}{4}\right) + \left(\frac{1}{3} \times \frac{3}{4}\right)$

$= \left(\frac{1}{6} + \frac{1}{4}\right) = \frac{2+3}{12} = \frac{5}{12}.$

$H(Y) = \frac{7}{12} \log \frac{12}{7} + \frac{5}{12} \log \frac{12}{5}$

$= 0.9799$ bits/ msg symbol

3) $H(x,y) = H(x) + H(y/x)$.

where $H(y/x) = h = \sum_{l=1}^{2} p_j \log \frac{1}{p_h}$

$$= \bar{p} \log \frac{1}{\bar{p}} + p \log \frac{1}{p}$$

$$= \frac{3}{4} \log \frac{4}{3} + \frac{1}{4} \log 4$$

$h = 0.8113$ bits/msg symbol.

CONNECTED

$$\therefore \; H(x,y) = H(x) + H(y/x)$$
$$= 0.9183 + 0.8113 = \underline{\underline{1.7296}} \text{ bits/msg symbol}$$

$$H(X/Y) = ?$$

$$H(x,y) = H(Y) + H(X/Y)$$

$$\text{or } H(X/Y) = H(X,Y) - H(Y)$$
$$= (1.7296 - 0.9799) \text{ bits/msg}$$
$$= \underline{\underline{0.7497}} \text{ bits/msg symbol}$$

$$I(x,y) = H(X) + H(Y) - H(x,y)$$
$$= (0.9183 + 0.9799) - 1.7296$$
$$= \underline{0.1686 \text{ bits}} \text{/msg}$$

Channel capacity, C

$C = 1 - h$

$= 1 - 0.8113$

$= \underline{0.1887}$ bit/msg symbol

Channel effeciency, $\eta_{ch} = \frac{I(x,y)}{C}$

$= \frac{0.1686}{0.1887} = \underline{89.35\%}$

Channel redundancy, $R_{\eta ch} = 1 - \eta_{ch}$

$= 1 - 0.8935$

$= \underline{10.65\%}$

CONCLUSION

- Define Binary symmetric channel
- Capacity C= 1-h
- Practice questions

THANK YOU