GRAM SCHMIDT ORTHOGONALIZATION PROCEDURE

The Gram schmidt orthogonalissation procedure is used to construct a set of orthonormal basis functions or orthonormal waveforms from an available set of finite energy signal waveforms. So that by finding out the co-ordinate or coefficients of the signal we can find the vector model of the s/l.

Suppose that we have a set of finite energy s/l denoted by $s_1(t), s_2(t) \ldots s_m(t)$.

Starting with $s_1(t)$ the first basis function is

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

This $\phi_1(t)$ should be orthogonal to other basis function it will As it is $(\phi_1(t))$ is the first basis function. Then $\phi_2(t)$ must be the reference basis function.

be orthogonal to $\phi_1(t)$.

Like that when we form the basis function $\phi_3(t)$ it must be orthogonal to both $\phi_2(t)$ and $\phi_1(t)$

Like that we have to proceed in the case of $\phi_4, \phi_5(t) \ldots$ and so on.

In all cases the norm value of the basis function should be one.

$$\phi_1(t) = \frac{s_1(t)}{\| s_1(t) \|} \rightarrow \text{norm of } s_1(t)$$

$\| s_1(t) \| =$ square root of the energy of the $s_1(t)$

$$\| s_1(t) \| = \sqrt{\mathcal{E}_1}$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\mathcal{E}_1}} \quad \text{— (1)}$$

$\rightarrow$ when $s_1(t)$ devided by it's energy; the resulting $\phi_1(t)$ will be having unit energy (normalised).

$$s_1(t) = \sqrt{\mathcal{E}_1} \cdot \phi_1(t) \quad \text{— (2)}$$

from the synthesis equation we know that

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\, \phi_j(t) \quad \text{— (3)}$$

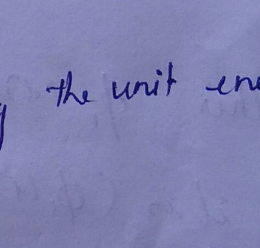
$\therefore$ comparing eqn (2) and (3)

$$s_1(t) = s_{11}\, \phi_1(t)$$

$$s_{11} = \sqrt{\mathcal{E}_1}$$

and $\phi_1(t)$ having the unit energy.

The second basis function $\phi_2(t)$ is formed from $s_2(t)$ for that we have to calculate the projection of $s_2(t)$ on $\phi_1(t)$ which is denoted by s_{21}



$$s_{21} = \langle s_2(t), \phi_1(t) \rangle$$

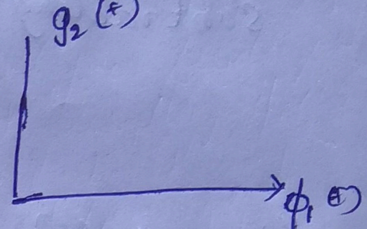
$$= \int_0^T s_2(t)\, \phi_1(t)\, dt \rightarrow \text{inner product}$$

$\phi_1(t)$, $\hat{\phi}_1(t)$ is same because $\phi_1(t)$ is a real valued function

Consider an intermediate function $g_2(t)$ which is obtained by subtracting $s_{21}\,\phi_1(t)$ from $s_2(t)$

$$g_2(t) = s_2(t) - s_{21}\,\phi_1(t) \quad \text{---} \quad (4)$$

This $g_2(t)$ will be orthogonal to $\phi_1(t)$



inorder to find out $\phi_2(t)$

we should make the norm value of $g_2(t)$ equal to one.

ie; we have to normalise $g_2(t)$ ($g_2(t)$ devided by norm value of $g_2(t)$)

for that calculate

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}}$$

$$\sqrt{\int_0^T g_2^2(t)\,dt}$$

$$= \sqrt{\int_0^T [s_2(t) - s_{21}\,\phi_1(t)]^2\,dt}$$

$\hookrightarrow (A-B)^2$

$$= \sqrt{\int_0^T s_2^2(t) - 2\,s_2(t) \cdot s_{21}\phi_1(t) + s_{21}^2\,\phi_1^2(t)\,dt}$$

$$= \sqrt{E_2 - 2\int_0^T s_2(t)\,s_{21}\,\phi_1(t)\,dt + \int_0^T s_{21}^2\,\phi_1^2(t)\,dt}$$

$$= \sqrt{E_2 - 2 s_{21}\underbrace{\int_0^T s_2(t)\,\phi_1(t)\,dt}_{\to s_{21}} + s_{21}^2 \underbrace{\int_0^T \phi_1^2(t)\,dt}_{\to 1}}$$

$$= \sqrt{E_2 - 2 s_{21}^2 + s_{21}^2}$$

$$= \sqrt{E_2 - s_{21}^2}$$

$$\boxed{\phi_2(t) = \frac{s_2(t) - s_{21}\,\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}}$$

w.k. that if $\phi_2(t)$ is a orthonormal basis function it's energy must be normalised to one.

ie; $\int_0^T \phi_2^2(t)\,dt = 1$ and

inner product of $\phi_1(t)$ and $\phi_2(t) = 0$

$$\int_0^T \phi_1(t) \cdot \phi_2(t)\,dt = 0.$$

Using the Same procedure we have to calculate $\phi_3(t)$. For that we have to define another intermediate function $g_3(t)$ which is obtained by subtracting S_{31} and S_{32} from $S_3(t)$.

$S_{31} \rightarrow$ Projection of $S_3(t)$ on $\phi_1(t)$

$S_{32} \rightarrow$ Projection of $S_3(t)$ on $\phi_2(t)$

This $g_3(t)$ must be orthogonal to $\phi_1(t)$ and $\phi_2(t)$

So inorder to find out $\phi_3(t)$ we can devide this $g_3(t)$ by square root of energy of $g_3(t)$.

In this manner we can calculate the remaining orthonormal basis functions.

So generally we can write; if we want to calulate $\phi_i(t)$ first find

$$g_i(t) = S_i(t) - \sum_{j=1}^{i-1} s_{ij}\,\phi_j(t) \quad \text{—— (5)}$$

where $s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt ; j = 1, 2 \ldots i-1$

$\rightarrow$ projection of $s_i(t)$ on $\phi_j(t)$

eqn 4 is a case of eqn 5 where $i = 2$.

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,dt}} \qquad i = 1, 2, \ldots N$$

$$N \leq m$$

There fore using this equation we can calculate the set of orthonormal basis function from a set of s/ls $s_1(t)$, $s_2(t)$ .. $s_m(t)$.

Further, the dimension N is less than or equal to the number of given signals, M depending on one of two possibilities:

(*i*) The signals $s_1(t)$, $s_2(t)$, ..., $s_M(t)$ form a linearly independent set, in which case $N = M$.

(*ii*) The signals $s_1(t)$, $s_2(t)$, ..., $s_M(t)$ are not linearly independent, in which case, $N < M$, and the intermediate function $g_i(t)$ is zero for $i > N$.

Important Note: *It may be noted that the conventional Fourier series expansion or a periodic signals is an example of a particular expansion of this type. Also, the representation of a bandlimited signal in terms of its samples taken at the Nyquist rate may be viewed as another example of a particular expansion of this type. There are, however, two important following distinctions that should be made:*

(*i*) The form of the basis function $\phi_1(t)$, $\phi_2(t)$ ... $\phi_N(t)$ has not been specified. This means that unlike the Fourier series expansion of a periodic signal or the sampled representation of a band-limited signal, we have not restricted the Gram-Schmidt orthogonalization procedure to be in terms of sinusoidal functions or sinc functions of time.

(*ii*) The expansion of the signal $s_i(t)$ in terms of a finite number of terms is not an approximation wherein only the the first N terms are significant but rather an exact expression where N and only N terms are significant.