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MODULE - IV
TRANSMISSION LINES

Introduction:

Wave propagation can be in Unbounded medium or by guided structures.

Eg: of such Guided structure is a transmission line.

Transmission lines are commonly used in power distribution (low frequency) and in communication (high frequency).

Transmission lines such as twisted pair or co-axial cable are used in computer networks.

Basic Transmission Line.

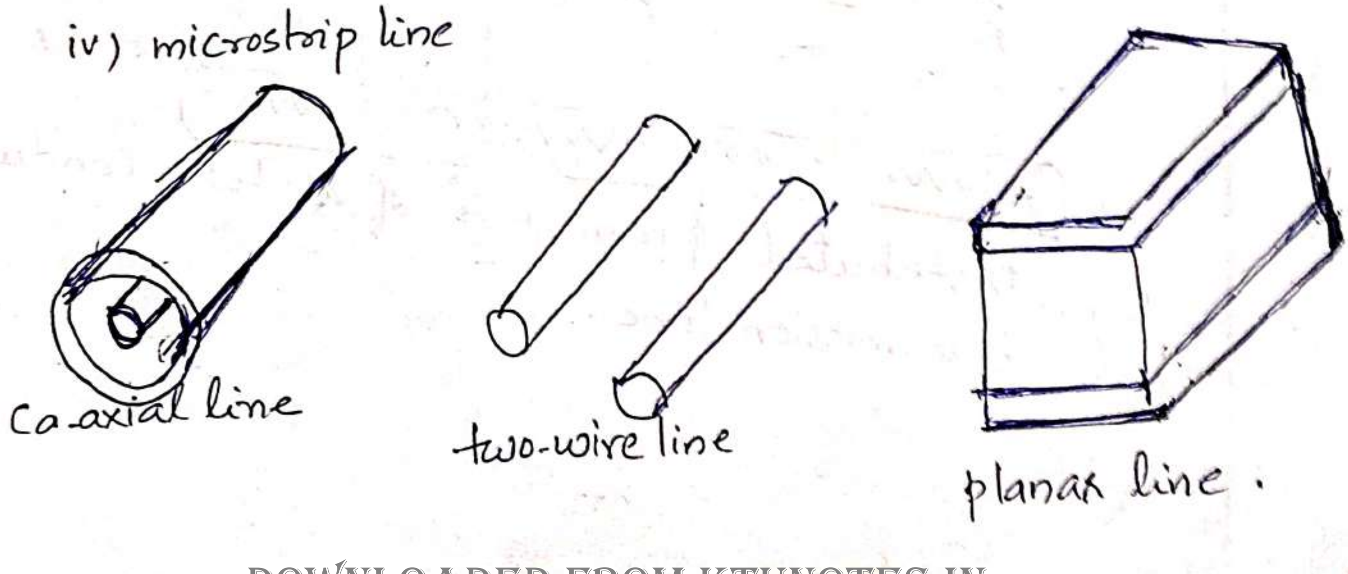
Basically, a transmission line (TL) consists of 2 or more parallel conductors used to connect a source to load.

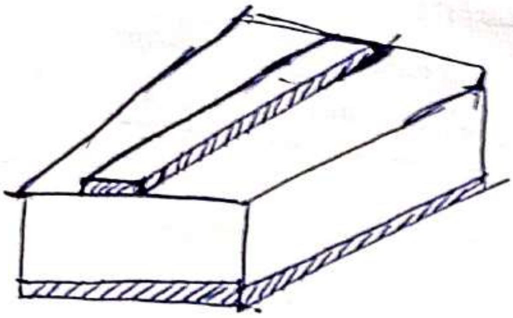
The source can be a hydroelectric generator or a transmitter or an Oscillator. The load can be a factory or an antenna or an Oscilloscope.

Typical transmission Line.

i) Co-axial cable ii) two wire line iii) planar line

iv) microstrip line





Micro-strip line.

Transmission lines can be analysed using EM field theory or electric circuit theory. Here we use circuit theory because it is easier to deal with mathematically.

Transmission Line parameters:

It is convenient to describe a transmission line in terms of its line parameters. They are

1) Resistance per unit length (R)
2) Inductance per unit length (L)
3) Conductance per unit length (G)
4) Capacitance per unit length (C).

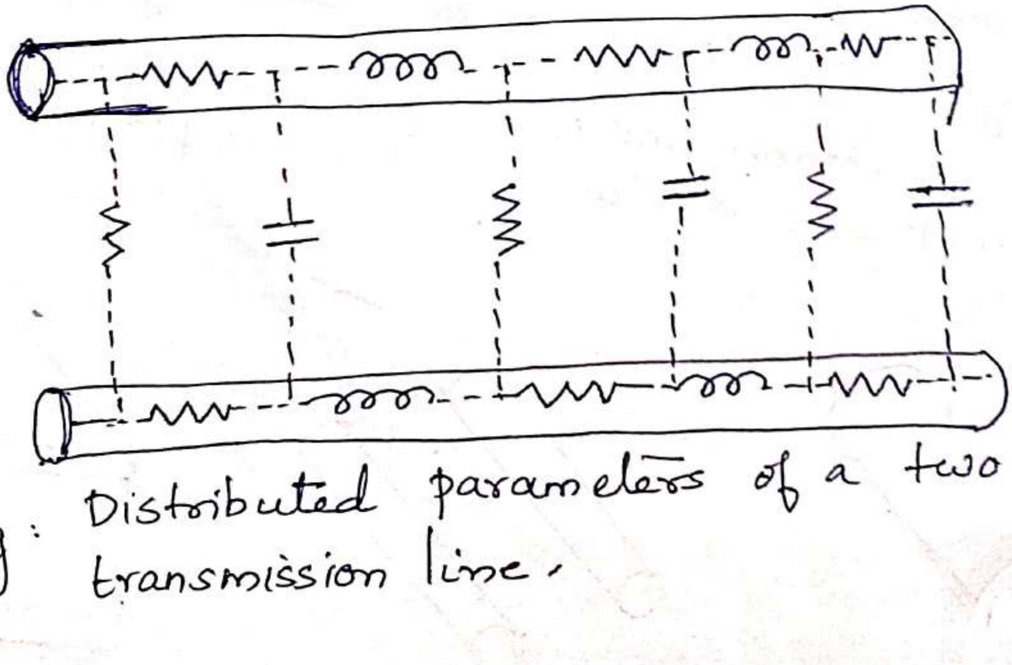


Fig: Distributed parameters of a two conductor transmission line.

characteristics of line parameters:

1. The line parameters R, L, G and C are distributed along the entire length of the line.
2. For each line, Conductors are characterized by σ_c, μ_c, $\varepsilon_c = \varepsilon_0$ and the homogeneous dielectric separating the Conductors is characterized by σ, μ, ε.
3. R is the ac resistance per unit length and G is the conductance per Unit length.
4. For each line, $LC = \mu\varepsilon$ and $\frac{G}{C} = \frac{\sigma}{\varepsilon}$.
5. L is the external inductance per unit length.

Transmission line equations.

A two-conductor transmission line Supports a TEM wave, that is, the $\vec{E}$ & $\vec{H}$ fields are perpendicular to the direction of wave propagation.

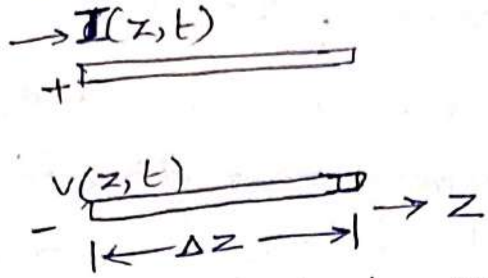
For a TEM wave, $\vec{E}$ & $\vec{H}$ are related to Voltage V and current I by

$$V = -\int \vec{E} \cdot dl \quad \text{and} \quad I = \oint_L \vec{H} \cdot dl$$

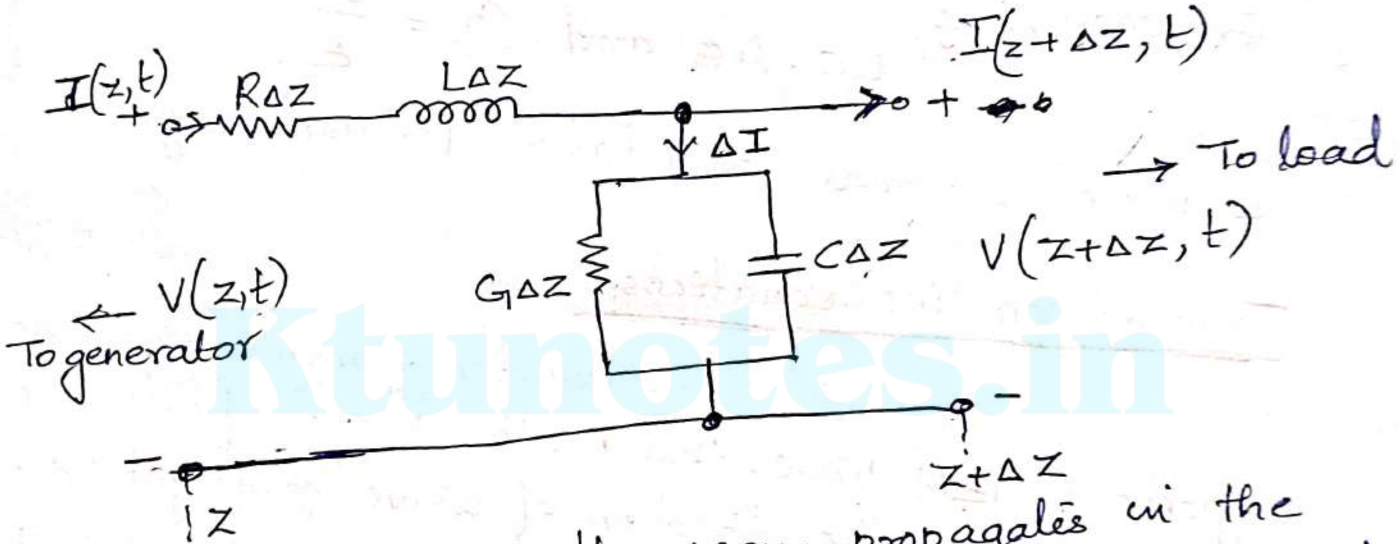
Instead of solving field quantities $\vec{E}$ and $\vec{H}$ (solving Maxwell's eq and boundary conditions), we use Circuit quantities V and I for solving transmission line equations.

Circuit model is simpler and more convenient.

Consider an incremental portion of length Δz of a two-conductor transmission line.



The equivalent circuit of this portion of the line can be modelled using the <u>L-type equivalent circuit</u> as shown:



We assume the wave propagates in the positive z-direction, from the generator to the load.

Applying Kirchoff's Voltage law to the outer loop of the circuit.

$$V(z,t) = R\Delta z\, I(z,t) + L\Delta z \frac{\partial I(z,t)}{\partial t} + V(z+\Delta z,t)$$

$$\therefore V(z,t) - V(z+\Delta z,t) = R\Delta z\, I(z,t) + L\Delta z \frac{\partial I(z,t)}{\partial t}$$

$$\frac{V(z,t) - V(z+\Delta z,t)}{\Delta z} = R\, I(z,t) + L\frac{\partial I(z,t)}{\partial t}$$

As $\Delta z \to 0$ leads to,

$$-\frac{\partial V(z,t)}{\partial z} = R\,I(z,t) + L\frac{\partial I(z,t)}{\partial t} \quad \text{--- (1)}$$

Similarly, applying kirchoff's current law to the main node of the Circuit.

$$I(z,t) = \Delta I + I(z+\Delta z, t)$$

$$= G\Delta z \cdot V(z+\Delta z, t) + C\Delta z \frac{\partial}{\partial t} V(z+\Delta z, t) + I(z+\Delta z, t)$$

$$\frac{I(z,t) - I(z+\Delta z, t)}{\Delta z} = G\,V(z+\Delta z, t) + C\frac{\partial V(z+\Delta z, t)}{\partial t}$$

As $\Delta z \to 0$

$$-\frac{\partial I(z,t)}{\partial z} = G\,V(z,t) + C\frac{\partial}{\partial t}V(z,t) \quad \text{--- (2)}$$

If we assume that Voltage and current have a time harmonic dependence,

$$V(z,t) = Re\left[V_s e^{j\omega t}\right]$$

$$I(z,t) = Re\left[I_s e^{j\omega t}\right]$$

$\because \frac{\partial}{\partial t} = j\omega$.

$\therefore$ (1) becomes,

$$-\frac{\partial}{\partial z} Re\left[V_s e^{j\omega t}\right] = R \cdot Re\left[I_s e^{j\omega t}\right] + j\omega L\, Re\left[I_s e^{j\omega t}\right]$$

$$-\frac{\partial}{\partial z} Re\left[V_s e^{j\omega t}\right] = (R + j\omega L)\, Re\left[I_s e^{j\omega t}\right]$$

$$-\frac{\partial}{\partial z} V_s \cos\omega t = (R + j\omega L)(I_s \cos\omega t)$$

$$\cancel{\cos\omega t} - \frac{dV_s}{dt} = (R+j\omega L) I_s \cancel{\cos\omega t}$$

$$\therefore -\frac{dV_s}{dz} = (R+j\omega L) I_s \quad \text{——— (3)}$$

Similarly, the equation (2) becomes,

$$-\frac{dI_s}{dz} = (G+j\omega C) V_s \quad \text{——— (4)}$$

The differential equations (3) & (4) are coupled. To separate them, we take double derivative.

$\therefore$ Double derivative of (3) gives,

$$-\frac{d^2V_s}{dz^2} = (R+j\omega L)\frac{dI_s}{dz}$$

Substituting (4) in above equation,

$$-\frac{d^2V_s}{dz^2} = -(R+j\omega L)(G+j\omega C) V_s$$

or $$\frac{d^2V_s}{dz^2} - \gamma^2 V_s = 0 \quad \text{——— (5)}$$

Where $\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}$

Similarly, taking double derivative of equation (4)

$$\frac{d^2I_s}{dz^2} - \gamma^2 I_s = 0 \quad \text{——— (6)}$$

$\therefore$ (5) and (6) are the wave equations for voltage and current respectively and γ is the propagation constant.

α is the attenuation constant in Np/m

β is the phase constant in rad/m.

Thus we obtain the Voltage and current equations as

$$\boxed{\begin{aligned} \frac{d^2V_s}{dz^2} - \gamma^2 V_s = 0 \\ \frac{d^2I_s}{dz^2} - \gamma^2 I_s = 0 \end{aligned}}$$ (7)

These are transmission line equations and are similar to wave equations for plane waves.

Wavelength, Wave Velocity & Characteristic Impedance from transmission line equations.

(1) Wavelength of the signal through the transmission line $\boxed{\lambda = \frac{2\pi}{\beta}}$

(2) Wave Velocity $\boxed{u = \frac{\omega}{\beta} = f\lambda}$

Note - The solution of wave equation or transmission line equation of the form,

$\frac{d^2V_s}{dz^2} - \gamma^2 V_s = 0$ and $\frac{d^2I_s}{dz^2} - \gamma^2 I_s = 0$ are,

$$\boxed{V_s(z) = \underbrace{V_0^+ e^{-\gamma z}}_{\rightarrow +z} + \underbrace{V_0^- e^{\gamma z}}_{-z \leftarrow}} \quad \text{------} \quad (7)$$

Also $$\boxed{I_s(z) = \underbrace{I_0^+ e^{-\gamma z}}_{\rightarrow +z} + \underbrace{I_0^- e^{\gamma z}}_{-z \leftarrow}} \quad \text{------} \quad (8)$$

where V_0^+, V_0^-, I_0^+ and I_0^- are wave amplitudes, the + and - signs denote wave traveling along +z and -z directions respectively.

Characteristic Impedance (Z_0)

Definition: The characteristic impedance of a line is the ratio of the positively traveling voltage to the current wave at any point on the line.

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-}$$

This is analogous to the intrinsic impedance of the medium of wave propagation.

Substituting equation (7) ie, $V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$ in equation (3) ie $-\frac{dV_s(z)}{dz} = (R+j\omega L) I_s$.

$$\therefore -\frac{d}{dz}\left[V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}\right] = (R+j\omega L) I_s$$

$$\therefore \gamma\left[V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right) = (R+j\omega L) I_s$$

$$\text{or } I_s = \frac{\gamma\left(V_0^+ e^{-\gamma z} - V_0^- e^{\gamma z}\right)}{(R+j\omega L)}$$

Comparing the co-efficients of the exponential term with equation, $I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$

$$\therefore I_0^+ = \frac{\gamma V_0^+}{R+j\omega L}$$

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{R+j\omega L}{\gamma} = \frac{R+j\omega L}{\sqrt{(R+j\omega L)(G+j\omega C)}}$$

$$\therefore \boxed{Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}}$$

propagation constant γ and characteristic impedance Z_0 are important properties of a transmission line because they both depend on line parameters R, L, G and C and the frequency of operation.

Above equations are derived for a general lossy line, ie the conductors of the line have some resistance R and the dielectric between the conductors have some conductivity G.

Now considering two special cases:

I Lossless line $(R=0,\ G=0)$

A transmission line is said to be lossless if the conductors of the line are perfect $(\sigma_c = \infty)$ and the dielectric medium separating them is lossless $(\sigma = 0)$

$$\boxed{R = 0 = G}$$

– This is a necessary condition for a line to be lossless.

Propagation constant γ

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \quad \because R=0, G=0$$

$$= \sqrt{(j\omega L)(j\omega C)}$$

$$= \sqrt{j^2 \omega^2 LC}$$

$$= j\omega\sqrt{LC}$$

we know $\gamma = \alpha + j\beta$

on comparing with expression obtained for γ

$$\therefore \boxed{\alpha = 0.}$$

and $j\beta = j\omega\sqrt{LC}$

$$\text{ie } \boxed{\beta = \omega\sqrt{LC}}$$

$$Z_0 = \sqrt{\frac{j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}}$$

$\because$ characteristic impedance $Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$

$$\therefore \boxed{Z_0 = \sqrt{L/C}}$$

Wave Velocity $u = \frac{\omega}{\beta} = \frac{\cancel{\omega}}{\cancel{\omega}\sqrt{LC}}$

$$\boxed{u = \frac{1}{\sqrt{LC}}}$$

III Distortionless line $\left(\frac{R}{L} = \frac{G}{C}\right)$.

A signal normally consists of a band of frequencies. Amplitudes of different frequency components will be attenuated differently in a lossy line, as α is frequency dependent. This results in distortion.

A distortion less line is one in which the attenuation constant, α is frequency independent while the phase constant β is linearly dependent on frequency.

The general criteria for a distortion less line is $\boxed{\frac{R}{L} = \frac{G}{C}}$.

Thus for a distortionless line,

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

$$= \sqrt{RG\left(1+\frac{j\omega L}{R}\right)\left(1+\frac{j\omega C}{G}\right)} \qquad \because \frac{R}{L} = \frac{G}{C}$$

$$= \sqrt{RG\left(1+\frac{j\omega C}{G}\right)\left(1+\frac{j\omega C}{G}\right)} \qquad \therefore \frac{L}{R} = \frac{C}{G}$$

$$= \sqrt{RG\left(1+\frac{j\omega C}{G}\right)^2}$$

$$= \sqrt{RG}\left(1+\frac{j\omega C}{G}\right)$$

$$= \sqrt{RG} + \sqrt{RG}\,\frac{j\omega C}{G}$$

$$= \alpha + j\beta$$

$$\therefore \boxed{\alpha = \sqrt{RG}}$$

$$\beta = \sqrt{RG}\,\frac{\omega C}{G}$$

$$= \sqrt{\frac{R}{G}}\cdot \omega C$$

$$= \sqrt{\frac{L}{C}}\cdot \omega C \qquad \left(\because \frac{R}{L} = \frac{G}{C}\right)$$

$$\boxed{\beta = \omega\sqrt{LC}}$$

which shows that α is independent on frequency and β is linear function of frequency.

characteristic Impedance

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$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

$$= \sqrt{\frac{R(1+j\omega L/R)}{G(1+j\omega C/G)}}$$

$$= \sqrt{\frac{R}{G}\left(\frac{1+j\omega L/R}{1+j\omega L/R}\right)}$$

$$\boxed{Z_0 = \sqrt{\frac{R}{G}} = \sqrt{L/C}}$$

Wave Velocity, $u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{LC}} = \frac{1}{\sqrt{LC}} = f\lambda.$

Transmission line characteristics.

Case	Propagation Constant $\gamma = \alpha + j\beta$	Characteristic Impedance $Z_0 = R_0 + jX_0$
General	$\sqrt{(R+j\omega L)(G+j\omega C)}$	$\sqrt{\frac{R+j\omega L}{G+j\omega C}}$
Lossless	$0 + j\omega\sqrt{LC}$	$\sqrt{\frac{L}{C}} + j0$
Distortionless	$\sqrt{RG} + j\omega\sqrt{LC}$	$\sqrt{\frac{L}{C}} + j0$

Q1. An air line has a characteristic impedance of 70Ω and phase constant of 3 rad/m at 100 MHz. Calculate the inductance/m & Capacitance per meter of the line.

Sol: $Z_0 = 70\Omega$, $\beta = 3 rad/m$, $f = 100MHz$

(13)

$L = ?$

$C = ?$

air line can be regarded as a lossless line.
with $\sigma_c = \infty$ & $\sigma = 0$

For a lossless line $R = G = 0$

$\alpha = 0$

$$Z_0 = \sqrt{\frac{L}{C}} = R_0 = 70\Omega$$

$$\sqrt{\frac{L}{C}} = 70$$

$$\therefore \frac{L}{C} = 4900$$

$$L = 4900C$$

$$\beta = \omega\sqrt{LC} = 3 rad/m$$

$f = 100MHz$

$$\omega = 2\pi f = 2\pi \times 100 \times 10^6$$
$$= 628318530.7$$

$$\therefore 628318530.7\sqrt{4900C^2} = 3$$

$$C = 6.8209 \times 10^{-11} F/m$$
$$= \underline{\underline{68.20 \times 10^{-12} f/m}}$$

$$L = 4900C$$
$$= 4900 \times 68.2 \times 10^{-12}$$
$$= 334.2 \times 10^{-9} H/m$$
$$= \underline{\underline{334.2 n H/m}}$$

2) A distortionless line has $Z_0 = 60\Omega$, $\alpha = 20 mNp/m$, $u = 0.6C$, where C is the speed of light in a Vaccum. Find R, L, G, C and λ at 100 MHz. (14)

Ans For a distortionless line,

$$\frac{R}{L} = \frac{G}{C} \quad \text{or} \quad G = \frac{RC}{L} \quad \text{or} \quad L = \frac{R}{G} \cdot C$$

and hence $Z_0 = \sqrt{L/C}$

$$\alpha = \sqrt{RG} = 20 \times 10^{-3} \; Np/m$$

$$RG = (20 \times 10^{-3})^2 \quad \text{——— (1)}$$

$$Z_0 = \sqrt{L/C} = 60\Omega$$

$$L/C = 60^2$$

$$Z_0 = \sqrt{R/G} = 60$$

$$R/G = 60^2 \quad \text{——— (2)} \quad \therefore R = 3600\,G$$

Sub (2) in (1)

$$3600\,G \times G = (20 \times 10^{-3})^2$$

$$3600\,G^2 = (20 \times 10^{-3})^2$$

$$G = 3.33 \times 10^{-4} \; S/m$$

$$R = 3600 \times G = 1.2\,\Omega/m$$

$$v = \frac{1}{\sqrt{LC}} = 0.6\,C$$

$$\frac{1}{\sqrt{LC}} = 0.6 \times 3 \times 10^8$$

$$\frac{1}{LC} = 3.24 \times 10^{16}$$

$$L = \frac{R}{G} \cdot G \text{ or } 3600 \cdot C$$

$$v = \frac{1}{3600 \cdot C^2} = 3.24 \times 10^{16}$$

$$\underline{\underline{C = 92.5\,pF/m}}$$

$$L = 3600 \cdot C = \underline{3.33 \times 10^{-7}\,H/m}$$

$$v = f\lambda$$

$$0.6 \times 3 \times 10^{8} = 100 \times 10^{6}\,\lambda$$

$$\underline{\underline{\lambda = 1.8\,m}}.$$

3) A lossless transmission line is 80cm long and operates at a frequency of 600MHz. The line parameters are $L = 0.25\,\mu H/m$ & $C = 100\,pf/m$. Find characterstic impedance, phase constant, phase velocity.

$f = 600\,MHz$

$L = 0.25\,\mu H/m$

$C = 100\,pf$

Z_0 for a lossless line $= \sqrt{L/C}$

$$Z_0 = \sqrt{\frac{0.25 \times 10^{-6}}{100 \times 10^{-12}}} = \sqrt{2500}$$

$$Z_0 = \underline{\underline{50\Omega}}$$

phase constant $\beta = \omega\sqrt{LC} = 2\pi f \sqrt{LC} = 2\pi \times 600 \times 10^{6} \sqrt{0.25 \times 10^{-6} \times 100 \times 10^{-12}}$

$$\beta = 18.849\,rad/m.$$

phase velocity $v = \lambda f = \frac{1}{\sqrt{LC}}$

$$= \frac{1}{\sqrt{0.25\times10^{-6}\times100\times10^{-12}}}$$

$$= \underline{\underline{2\times10^{8}\,m/s}}.$$

4) A telephone line has $R = 30\Omega/km$, $L = 100mH/km$ $G=0$ and $C = 20\mu F/km$. At $f = 1kHz$, obtain

a) Characteristic impedance of the line

b) Propagation Constant

c) Phase Velocity.

a)
$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{0.03+j2\pi\times1\times10^{3}\times100\times10^{-3}}{0+j2\pi\times1\times10^{3}\times20\times10^{-6}/10^{3}}}$$

$$= 70.73 - j1.688$$

$$= \underline{\underline{70.75\angle-1.367^{\circ}\,\Omega}}$$

b)
$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = \sqrt{(0.03+j0.2\pi)(j0.4\times10^{-4}\pi)}$$

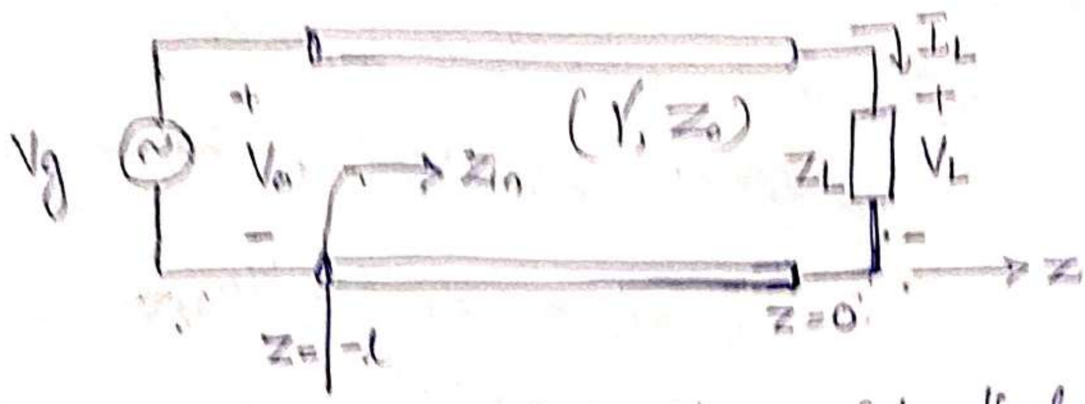
$8.7495\times10^{-6}\angle154.477$

$$= \underline{\underline{2.12\times10^{-4} + j8.888\times10^{-3}/m}}$$

$2.95795\times10^{-3}\angle77.2385$

c)
$$u = \frac{\omega}{\beta} = \frac{2\pi\times10^{3}}{8.888\times10^{-3}} = \underline{\underline{7.069\times10^{5}\,m/s}}$$

Terminated Transmission Line



Consider a transmission line of length l, characterized by γ and Z_0 Connected to a load Z_L

Let the transmission line extend from $Z=0$ at the load to $z=-l$ at the generator.

We have,

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \quad \text{—(7)}$$

$$I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{-\gamma z} \quad \text{—(8)}$$

$$= \frac{V_0^+}{Z_0} e^{-\gamma z} + \frac{-V_0^-}{Z_0} e^{-\gamma z} \qquad \because -\frac{V_0^-}{I_0^-} = Z_0$$

$$I_s(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{-\gamma z}$$

At $Z=0$,

$$Z_L = \frac{V_s(z=0)}{I_s(z=0)} = \frac{V_0^+ e^{-\gamma 0} + V_0^- e^{0}}{\frac{V_0^+}{Z_0} e^{0} - \frac{V_0^-}{Z_0} e^{0}}$$

$$= \frac{V_0^+ + V_0^-}{\frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}}$$

or $Z_L = \left(\frac{V_0^+ + V_0^-}{V_0^+ - V_0^-}\right) Z_0$

Cross multiplying,

$$V_0^+ Z_L - V_0^- Z_L = V_0^+ Z_0 + V_0^- Z_0$$

$$V_0^+ (Z_L - Z_0) = V_0^- (Z_L + Z_0)$$

$$\therefore \quad V_0^- = \frac{(Z_L - Z_0)}{(Z_L + Z_0)} V_0^+$$

Reflection co-efficient Γ_L

The Voltage Reflection Co-efficient Γ_L is defined as the ratio of Voltage of reflected wave to that of incident wave

$$\therefore \boxed{\Gamma_L = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$

Current reflection Co-efficient at any point on the line is negative of the Voltage reflection Co-efficient at that point

In terms of Γ_L (7) & (8) can be written as,

$$V_s = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \qquad \because V_0^- = \Gamma_L V_0^+$$

$$= V_0^+ e^{-\gamma z} + V_0^+ \Gamma_L e^{\gamma z}$$

$$V_0^+ \left[e^{-\gamma z} + \Gamma_L e^{\gamma z} \right]$$

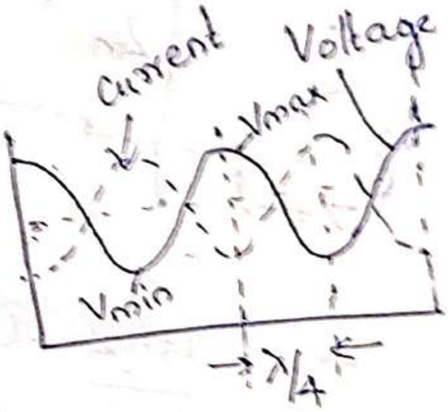
$$I_s = \frac{V_0^+}{Z_0} \left[e^{-\gamma z} - \Gamma_L e^{\gamma z} \right]$$

Thus the Voltage and current on the line are a Superposition of incident and reflected waves.

Such waves are called standing waves. Only when $\Gamma_L = 0$, there is no reflected wave. This happens when $Z_L = Z_0$ ie the load impedance is equal to transmission line characteristic impedance. Such a load is said to be matched to the line

Note The range of values of Γ_L is $-1 \leq \Gamma_L \leq 1$.

Standing Wave Ratio



when reflections occur in an incorrectly terminated line, the voltage and current vary in magnitude along the line.

when transmission line is not correctly terminated the travelling electromagnetic wave from generator at the sending end is reflected completely or partially at the termination.

The combination of incident and reflected wave give rise to interference phenomenon and thus standing waves of current and voltage along the line, with definite minima and maxima of current and voltage along the line.

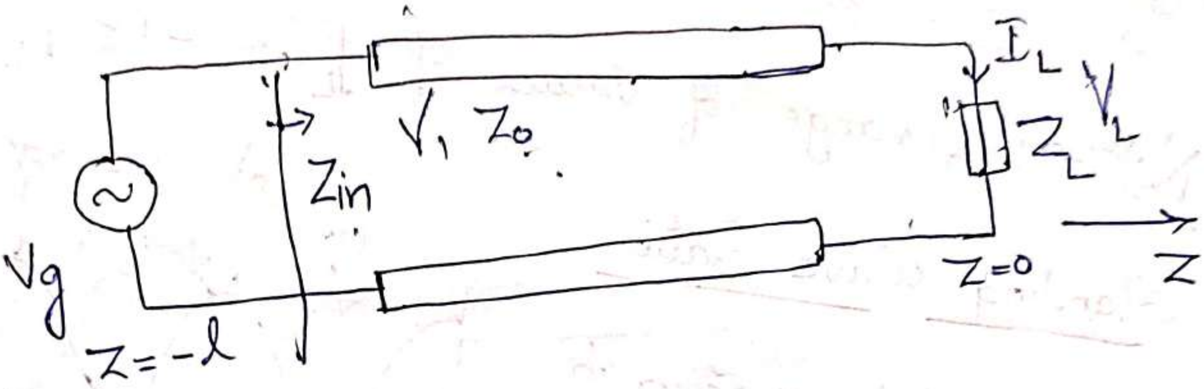
Standing wave ratio is defined as the ratio of maximum to minimum current or voltage on a transmission line having standing waves

$$\boxed{\text{VSWR [Voltage standing Wave ratio]} = \left|\frac{V_{max}}{V_{min}}\right|}$$

$$S = \frac{|V_{max}|}{|V_{min}|} = \frac{|I_{max}|}{|I_{min}|} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|}$$ (20)

The range of Values of VSWR is $1 \leq VSWR \leq \infty$

Input Impedance (Z_{in}) of transmission line



$$Z_{in} = \frac{V_s(z=-l)}{I_s(z=-l)} = \frac{V_0^+\left[e^{\gamma l} + \Gamma_L e^{-\gamma l}\right]}{V_0^+\left[e^{\gamma l} - \Gamma_L e^{-\gamma l}\right]} Z_0$$

Substituting $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$

$$\therefore Z_{in} = \frac{(Z_L+Z_0)e^{\gamma l} + (Z_L - Z_0)e^{-\gamma l}}{(Z_L+Z_0)e^{\gamma l} - (Z_L-Z_0)e^{-\gamma l}} \cdot Z_0$$

$$= \frac{Z_L\left[e^{\gamma l} + e^{-\gamma l}\right) + Z_0\left(e^{\gamma l} - e^{-\gamma l}\right]}{Z_0\left(e^{\gamma l} + e^{-\gamma l}\right) + Z_L\left(e^{\gamma l} - e^{-\gamma l}\right)} \cdot Z_0$$

$$= \left(\frac{2Z_L \cosh\gamma l + 2Z_0 \sinh\gamma l}{2Z_0\cosh\gamma l + 2Z_L \sinh\gamma l}\right) \cdot Z_0 .$$

Th $\therefore Z_{in} = \dfrac{2\cosh\gamma l}{2\cosh\gamma l}\left[\dfrac{Z_L + Z_0 \dfrac{\sinh\gamma l}{\cosh\gamma l}}{Z_0 + Z_L \dfrac{\sinh\gamma l}{\cosh\gamma l}}\right] \cdot Z_0$ (31)

$$\boxed{Z_{in} = \left[\frac{Z_L + Z_0 \tanh\gamma l}{Z_0 + Z_L \tanh\gamma l}\right] \cdot Z_0}$$

This is for a lossy line. for a lossless line, $\alpha = 0$ and so $\gamma = \alpha + j\beta = j\beta$.

$\tanh j\beta l = j\tan\beta l$

$$\boxed{\therefore Z_{in} = \left[\frac{Z_L + jZ_0\tan\beta l}{Z_0 + jZ_L\tan\beta l}\right] \cdot Z_0}$$ for lossless line.

Now consider 3 special cases when the transmission line is connected to load, ~~load~~

$Z_L = 0$ – shorted line

$Z_L = \infty$ – Open circuited line.

$Z_L = Z_0$ – matched line

Case (1) shorted line $(Z_L = 0)$

The input impedance for a shorted line can be expressed as

$$Z_{sc} = Z_{in}\big|_{Z_L = 0}$$

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right]$$

$$Z_{sc} = Z_0 \frac{[0 + jZ_0 \tan\beta l]}{[Z_0 + 0]}$$

$$= \frac{\cancel{Z_0}\, jZ_0 \tan\beta l}{\cancel{Z_0}}$$

$$\boxed{Z_{sc} = jZ_0 \tan\beta l}$$

Voltage reflection Co-efficient,

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

For a shorted line $Z_L = 0$

$$\boxed{\Gamma_L = -1}$$

Standing wave ratio (SWR) $= \dfrac{1 + |\Gamma_L|}{1 - |\Gamma_L|}$

For shorted line

$$SWR = \frac{1+1}{1-1} = 2/0 = \infty$$

$$\boxed{SWR = \infty}$$

Case 2: Open Circuited line $(Z_L = \infty)$

input impedance,

$$Z_{oc} = Z_{in}\big|_{Z_L = \infty}$$

$$Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right]$$

$$Z_{oc} = Z_0 \frac{\cancel{Z_L}\left[1 + j\frac{Z_0}{Z_L}\tan\beta l\right]}{\cancel{Z_L}\left[\frac{Z_0}{Z_L} + j\tan\beta l\right]}$$

$$\therefore Z_{oc} = Z_{L\to\infty}$$

$$= Z_0\left[\frac{1}{j\tan\beta l}\right] = \frac{Z_0}{j\tan\beta l} = -jZ_0\cot\beta l.$$

$$\boxed{Z_{oc} = -jZ_0\cot\beta l}$$

Voltage reflection Co-efficient $\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$

for open circuited line

$$\Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L = \infty}$$

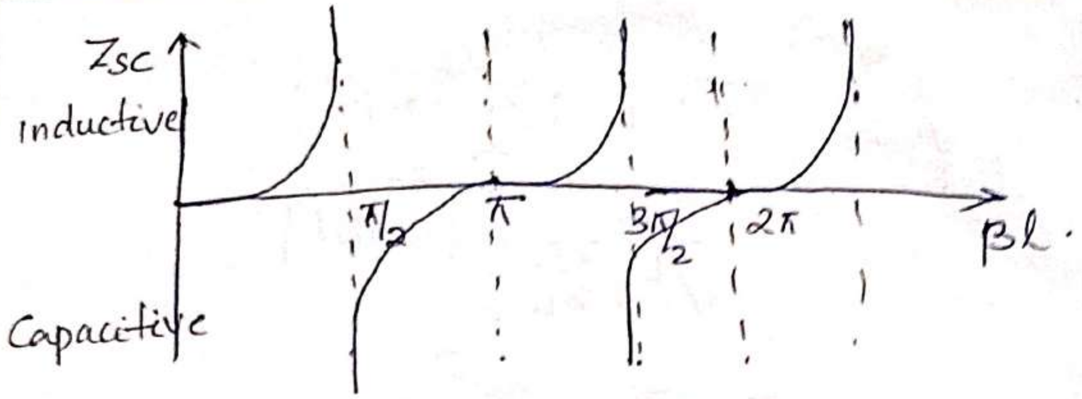
$$= \frac{\cancel{Z_L}\left[1 - Z_0/Z_L\right]}{\cancel{Z_L}\left(1 + Z_0/Z_L\right)}$$

$$\boxed{\Gamma_L = 1/1 = 1}$$

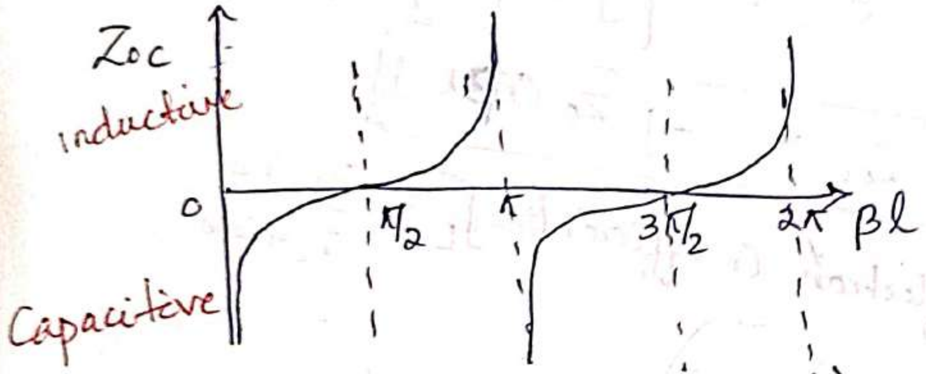
Standing wave ratio $\boxed{SWR = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \infty}$.

Note:

(1) For a shorted line, the input impedance is a pure reactance which will be capacitive or inductive depending on l. The variation of Z_{in} with l is shown below.



Similarly for an Open Circuited line



Case-III Matched line $(Z_L = Z_0)$

$$Z_{in} = Z_0 \frac{(Z_L + jZ_0 \tan\beta l)}{(Z_L + jZ_0 \tan\beta l)}$$

$$= Z_0 \left[\frac{Z_0 + jZ_0 \tan\beta l}{(Z_0 + jZ_0 \tan\beta l)}\right]$$

$$\boxed{Z_{in} = Z_0}$$

Voltage reflection Co-efficient

$$\boxed{\Gamma_L = 0}$$

$\boxed{\text{Standing wave ratio, VSWR} = 1}$

This is the most desired case from a practical point of view. The whole wave is transmitted and there is no reflection.

A lossless transmission line with $Z_0 = 50\Omega$ is 30m long and operates at 2MHz. The line is terminated with a load $Z_L = 60+j40\Omega$. If $u = 0.6C$ on the line (26)

Find a) reflection - Co-efficient

b) SWR

c) input impedance.

a) Reflection Co-efficient

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{60+j40-50}{60+j40+50} = \underline{\underline{0.3522\angle 55.98^\circ}}$$

b) $SWR = \dfrac{1+|\Gamma_L|}{1-|\Gamma_L|} = \dfrac{1+0.3522}{1-0.3522} = \underline{\underline{2.087}}$

c) Input impedance.

$$u = \omega/\beta$$

$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right]$$

$$\beta = \frac{\omega}{u} = \frac{\omega}{0.6C} \qquad \therefore \beta l = \frac{\omega}{0.6C} \times l$$

$$= \frac{2\pi \times 2\times 10^6}{0.6\times 3\times 10^8} \times 30$$

$$\beta l = \underline{\underline{2\pi/3}}$$

$$\therefore \tan\beta l = -\sqrt{3}$$

$$Z_{in} = 50\left[\frac{60+j40+j50\times\sqrt{3}}{50+j(60+j40)\times\sqrt{3}}\right]$$

$$= \cancel{63.58+j40.20\,\Omega}\quad 23.972+1.35j$$

Average Power at a distance l from the load.

$$P_{av} = \frac{1}{2}\,Re\left[V_s(l)\cdot I_s^*(l)\right]$$

For lossless line $\alpha = 0$,

$$P_{av} = \frac{1}{2}\,Re\left[V_0^+\left(e^{j\beta l} + \Gamma_L e^{-j\beta l}\right)\frac{V_0^{*}}{Z_0}\left[e^{-j\beta l} - \Gamma_L^* e^{j\beta l}\right]\right]$$

$$= \frac{1}{2}\,Re\left[\frac{|V_0|^2}{Z_0}\left(1 - \Gamma_L^* e^{j2\beta l} + \Gamma_L e^{-j2\beta l} - |\Gamma_L|^2\right)\right]$$

$$\frac{1}{2}\,\frac{|V_0|^2}{Z_0}\left[1-|\Gamma_L|^2\right]$$

$$P_{av} = \frac{|V_0|^2}{2Z_0}\left(1-|\Gamma_L|^2\right) = P_i - P_r$$

P_{av} is maximum when $\Gamma_L = 0$.

1. ① A lossless transmission line is 80cm long and operates at a frequency of 600MHz. The line parameters are $L = 0.25\,\mu H/m$ and $C = 100\,pF/m$. Find the characteristics impedance, the phase constant, the Velocity on the line

② At an operating radian frequency of 500M rad/s, typical Circuit Values for a Certain transmission line are $R = 0.2\,\Omega/m$, $L = 0.25\,\mu H/m$, $G = 10\,\mu S/m$ and $C = 100\,pF/m$. Find (a) α, (b) β, (c) λ, (d) v_p

Reflection and Refraction of Electromagnetic Wave.

Poynting Vector.

Poynting vector, denoted by $\vec{S}$, is given by the vector product of $\vec{E}$ & $\vec{H}$.

$$\vec{S} = \vec{E} \times \vec{H}$$

$$|\vec{S}| = |\vec{E}||\vec{H}| \sin(\vec{E}, \vec{H})$$

Poynting vector measures the rate of flow of energy of the wave as it propagates. The direction of $\vec{S}$ represents the direction of power flow and it is perpendicular to the plane containing $\vec{E}$ and $\vec{H}$.

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Poynting's Theorem:

When electromagnetic wave propagates through space, there will be a transfer of energy. There exist a simple and direct relation between the rate of this energy transfer and the amplitudes of E & H.

Statement:

The vector product of electric field intensity and magnetic field intensity at any point is a measure of the rate of energy flow per unit area at the point.

Derivation:

Consider the field intensities $\vec{E}$ and $\vec{H}$.
From Maxwell's equations,

$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu \frac{\partial \vec{H}}{\partial t}}$$

$$\boxed{\nabla \times \vec{H} = \vec{J} = \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}}$$

~~We know the~~

We know the poynting Vector

$$\vec{S} = \vec{E} \times \vec{H}$$

Taking Dot product on both sides

$$\nabla \cdot \vec{S} = \nabla \cdot (\vec{E} \times \vec{H})$$

By Vector identity

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\therefore \nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H})$$

Substituting the Values,

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot \left(-\mu \frac{\partial \vec{H}}{\partial t}\right) - \vec{E} \cdot \left(\sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}\right)$$

$$= -\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} - \sigma \vec{E}^2 - \varepsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \quad \text{--- (1)}$$

Let, $\frac{\partial (\vec{H} \cdot \vec{H})}{dt} = \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{H}}{\partial t}$ (product rule)

$$= 2 \vec{H} \cdot \frac{\partial \vec{H}}{\partial t}$$

$$\therefore \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{1}{2} \frac{\partial (\vec{H} \cdot \vec{H})}{\partial t} = \frac{1}{2} \frac{\partial \vec{H}^2}{\partial t} \quad \text{--- (A)}$$

Substituting (A) in equation (1).

$$\nabla\cdot(\vec{E}\times\vec{H}) = \frac{-\mu}{2}\frac{\partial H^2}{\partial t} - \sigma E^2 - \frac{\varepsilon\, \partial E^2}{2\partial t} \quad \left(\because \vec{E}\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}\right)$$

(21)

Taking Volume Integral on both sides,

$$\int_V \nabla\cdot(\vec{E}\times\vec{H})\,dv = \frac{-d}{dt}\int_V\left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right]dv - \int_V \sigma E^2\,dv$$

Applying the divergence theorem to L·H·S

$$\oint_S (\vec{E}\times\vec{H})\cdot ds = \frac{-d}{dt}\int_V\left[\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2\right]dv - \int_V \sigma E^2\,dv$$

Total power leaving the Volume = Rate of decrease in energy stored in electric and magnetic fields — ohmic power dissipated.

This equation is referred to as poynting theorem.

The quantity $\vec{E}\times\vec{H}$ is known as poynting Vector.

$$\boxed{P = \vec{E}\times\vec{H}}$$

Poynting's theorem states that the net power flowing out of a given Volume V is equal to the time rate of decrease in the energy stored within V minus the Conduction losses.

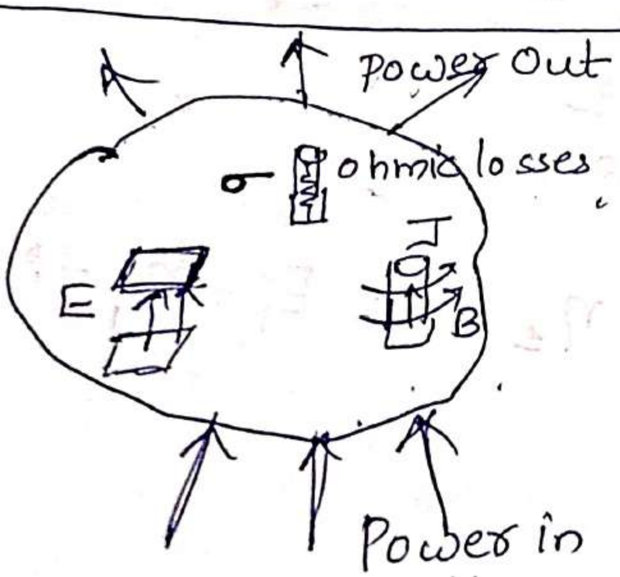


figure: Illustration of power balance for EM fields.

Polarization of Plane Waves:

The polarization of Uniform plane wave is defined as time varying behaviour of the electric field intensity $\vec{E}$ at some fixed point in space, along the direction of propagation

Consider a Uniform plane wave travelling in positive 'z' direction. Then the fields $\vec{E}$ and $\vec{H}$ lie in the x-y plane, which is perpendicular to the direction of propagation.

Being an electromagnetic wave (EM), as it travels in a space, both the fields undergo some variations with respect to time.

There are three different types of polarization

1) Linear polarization
2) Elliptical polarization
3) Circular polarization.

Linear Polarization:

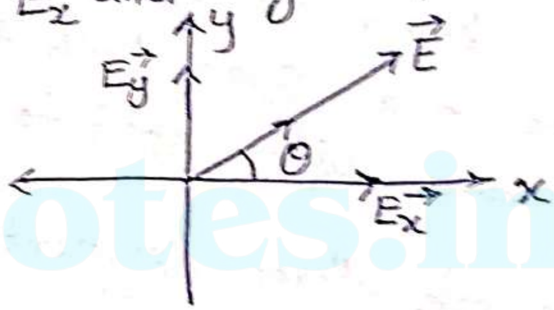
Consider that the electric field $\vec{E}$ has only x component and y component of $\vec{E}$ is zero. Then looking from the direction of propagation, the wave is said to be linearly polarized in x-direction.

Similarly if only the y component in $\vec{E}$ is present and x-component of $\vec{E}$ is zero, then the wave is said to be linearly polarized in y-direction.

Let us assume that both the components of $\vec{E}$ are present denoted by $\vec{E_x}$ and $\vec{E_y}$. The electric field is the resultant of $\vec{E_x}$ and $\vec{E_y}$ and the direction of it depends on the relative magnitude of $\vec{E_x}$ and $\vec{E_y}$

$$\vec{E} = \sqrt{E_x^2 + E_y^2}$$

Angle made by $\vec{E}$ $\quad \theta = \tan^{-1} \frac{E_y}{E_x}$ where E_x and E_y are the magnitudes of $\vec{E_x}$ and $\vec{E_y}$ respectively.

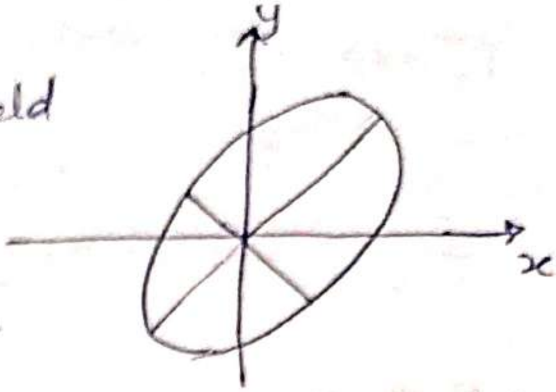


Linear polarization.

This angle 'θ' is constant with respect to time. In other words, the resultant vector $\vec{E}$ is oriented in a direction which is constant with time, thus the wave is said to be linearly polarized.

When both the components have same amplitudes ie $E_x = E_y$, then the polarization is called Linear polarization with constant angle of 45°.

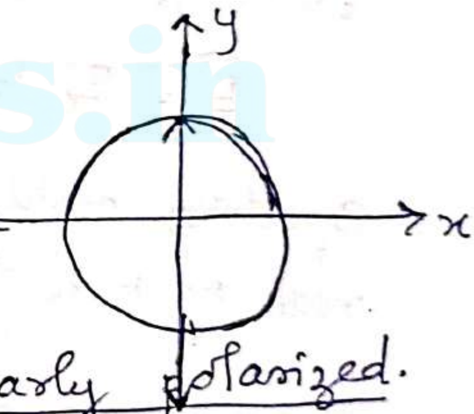
Elliptical Polarization



Elliptical polarization

Consider that the $\vec{E}$ electric field has both the components $\vec{E_x}$ and $\vec{E_y}$ with different amplitudes and are not in phase, the direction of the resultant field $\vec{E}$ varies with time. If the locus of the end points of $\vec{E}$ is traced, it is observed that $\vec{E}$ moves elliptically. Then such a wave is said to be elliptically polarized.

Circular Polarization



Let us consider that $\vec{E}$ has two components, $\vec{E_x}$ and $\vec{E_y}$ of equal amplitudes but the phase difference between them is exactly 90°. Such a wave is said to be circularly polarized.

Condition for the polarization of a sinusoidal wave

Two components of $\vec{E}$ electric field can be expressed in phasor form as,

$$\vec{E_x} = E_1 e^{j(\omega t - \beta z)} \quad (1)$$

$$\vec{E_y} = E_2 e^{j(\omega t - \beta z - \delta)} \quad (2)$$

where δ is the phase difference between the two components.

As the electric field vector $\vec{E}$ is the resultant of $\vec{E_x}$ and $\vec{E_y}$

$$\vec{E} = \vec{E_x} + \vec{E_y}$$

$$\vec{E} = E_1 e^{j(\omega t - \beta z)} \hat{a_x} + E_2 e^{j(\omega t - \beta z - \delta)} \hat{a_y} \quad \text{--- (3)}$$

At $z=0$, above equation becomes,

$$\vec{E} = E_1 e^{j\omega t} \hat{a_x} + E_2 e^{j(\omega t - \delta)} \quad \text{--- (4)}$$

$$\therefore \vec{E} = E_1[\cos\omega t + j\sin\omega t] \hat{a_x} + E_2[\cos(\omega t - \delta) + j\sin(\omega t - \delta)]\hat{a_y} \quad \text{--- (5)}$$

In general, the electric field vector can be represented with its two components as,

$$\vec{E} = E_x \hat{a_x} + E_y \hat{a_y} \quad \text{--- (6)}.$$

Equating right hand side of equation (6) with the real part of the equation (5).

$$E_x \hat{a_x} + E_y \hat{a_y} = E_1 \cos\omega t \, \hat{a_x} + E_2 \cos(\omega t - \delta) \hat{a_y} \quad \text{--- (7)}$$

Comparing x and y components,

$$E_x = E_1 \cos\omega t \quad \text{--- (8)}$$

$$E_y = E_2 \cos(\omega t - \delta) \quad \text{--- (9)}$$

From eq (8),

$$\left(\frac{E_x}{E_1}\right) = \cos\omega t \quad \text{--- (9-a)}.$$

$$\therefore \left(\frac{E_x}{E_1}\right)^2 = \cos^2\omega t = 1 - \sin^2\omega t \quad \text{--- (10)}$$

$$\sin^2\omega t = 1 - \left(\frac{E_x}{E_1}\right)^2$$

$$\therefore \sin\omega t = \sqrt{1 - \left(\frac{E_x}{E_1}\right)^2} \quad \text{——— (10a)}$$

From (9),
$$\frac{E_y}{E_2} = \cos(\omega t - \delta)$$

$$\therefore \frac{E_y}{E_2} = \cos\omega t \cos\delta + \sin\omega t \sin\delta \quad \text{——— (11)}$$

Substituting the values of $\cos\omega t$ and $\sin\omega t$ from equation (9a) and (10a), in equation (11).

$$\frac{E_y}{E_2} = \frac{E_x}{E_1}\cos\delta + \sqrt{1-\left(\frac{E_x}{E_1}\right)^2}\,\sin\delta$$

$$\therefore \frac{E_y}{E_2} - \frac{E_x}{E_1}\cos\delta = \sqrt{1-\left(\frac{E_x}{E_1}\right)^2}\,\sin\delta$$

Squaring both sides,

$$\left(\frac{E_y}{E_2} - \frac{E_x}{E_1}\cos\delta\right)^2 = \left[1-\left(\frac{E_x}{E_1}\right)^2\right]\sin^2\delta \qquad (a-b)^2 = a^2-2ab+b^2$$

$$\left(\frac{E_y}{E_2}\right)^2 - 2\frac{E_y}{E_2}\frac{E_x}{E_1}\cos\delta + \left(\frac{E_x}{E_1}\right)^2\cos^2\delta = \sin^2\delta - \left(\frac{E_x}{E_1}\right)^2\sin^2\delta$$

Simplifying above equation

$$\left(\frac{E_x}{E_1}\right)^2\left[\cos^2\delta + \sin^2\delta\right] + \left(\frac{E_y}{E_2}\right)^2 - 2\frac{E_y}{E_2}\frac{E_x}{E_1}\cos\delta = \sin^2\delta$$

$$\boxed{\therefore \left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right)\cos\delta = \sin^2\delta} \quad \text{— (12)}$$

Above equation is the equation for polarization of Sinusoidal wave. By applying different conditions to equation for the polarization of Sinusoidal wave, we get different types of the polarization.

Condition 1: $\vec{E_x}$ and $\vec{E_y}$ are in phase and $\delta = 0$,

Substituting this condition in equation (12),

$$\left(\frac{E_x}{E_1}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right) + \left(\frac{E_y}{E_2}\right)^2 = 0$$

$$\therefore \left(\frac{E_x}{E_1} - \frac{E_y}{E_2}\right)^2 = 0$$

$$\therefore \frac{E_x}{E_1} - \frac{E_y}{E_2} = 0$$

$$\boxed{\text{or } E_x = \left(\frac{E_1}{E_2}\right) E_y} \quad \text{——— (A)}$$

If the amplitudes of $\vec{E_x}$ and $\vec{E_y}$ are constant Then the ratio $\frac{E_1}{E_2}$ is also constant. The above equation is similar to the equation of a straight line passing through origin $y = mx$. The wave is said to be linearly polarized wave.

Condition 2

$\vec{E_x}$ and $\vec{E_y}$ components of unequal amplitudes with a phase difference $\delta \neq 0$, Let us assume $\delta = \pi/2$

Applying condition to equation (12),

$$\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 - 2\left(\frac{E_x}{E_1}\right)\left(\frac{E_y}{E_2}\right)\cos \pi/2 - \sin^2 \pi/2$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta), \quad \therefore \frac{1}{2}(1-\cos 2\times\pi/2)$$

$$= \frac{1}{2}(1--1) = 1$$

Then equation becomes,

$$\boxed{\left(\frac{E_x}{E_1}\right)^2 + \left(\frac{E_y}{E_2}\right)^2 = 1}$$

This equation represents equation for an ellipse. Hence the wave satisfies above equation is called elliptically polarized.

Condition 3:

Let the amplitude of $\vec{E_x}$ and $\vec{E_y}$ be equal and phase difference between then $\delta = 90°$, Applying conditions to equation (12) gives,

$$\boxed{E_x^2 + E_y^2 = 1}$$

equation represents Circle. The wave satisfies above equation is called circullarly polarized.