3.6.1 *Periodicity*

If $X(k)$ is N-point DFT of a finite duration sequence $x(n)$ then

$$\begin{aligned} x(n+N) &= x(n) \quad \text{for all} \quad n \\ X(k+N) &= X(k) \quad \text{for all} \quad k \end{aligned} \tag{3.30}$$

proof

we have $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \, e^{+j\frac{2\pi}{N}kn}$.

then $x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \, e^{+j\frac{2\pi}{N}k(n+N)}$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \, e^{+j\frac{2\pi}{N}kn} \, \underbrace{e^{+j2\pi k}}_{1}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k) \, e^{+j\frac{2\pi}{N}kn} = x(n)$$

we have $X(k) = \sum_{n=0}^{N-1} x(n) \, e^{-j\frac{2\pi}{N}kn}$.

then $X(k+N) = \sum_{n=0}^{N-1} x(n) \, e^{-j\frac{2\pi}{N}(k+N)n}$.

$$= \sum_{n=0}^{N-1} x(n) \, e^{-j\frac{2\pi}{N}kn} \cdot e^{-j2\pi}.$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = X(k).$$

Hence the proof.

2. Linearity Property.

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k).$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k).$

then for real or complex coefficients

a_1 and a_2

then if the two sequences are linearly

combined as

$$x_3(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(k) + a_2 X_2(k)$$

If $x_1(n)$ has a length N_1 and

$x_2(n)$ has a length N_2 then

maximum length of $x_3(n)$ is

$N_3 = \max(N_1, N_2).$

proof.

$$a_1 x_1(n) + a_2 x_2(n) \xrightarrow[N]{DFT} \sum_{n=0}^{N-1} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} a_2 x_2(n) e^{-j\frac{2\pi}{N}kn}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn} + a_2 \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi}{N}kn}$$

$$= a_1 X_1(k) + a_2 X_2(k)$$

Hence the proof.

3.6.3 *Circular shift of a sequence*

From Eq. (3.15) the periodic extension of the sequence $x(n)$ can be written as

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

One way of visualizing the periodic sequence $x_p(n)$ is wrapping the finite duration sequence $x(n)$ around a circle in counterclockwise direction, which is selected as

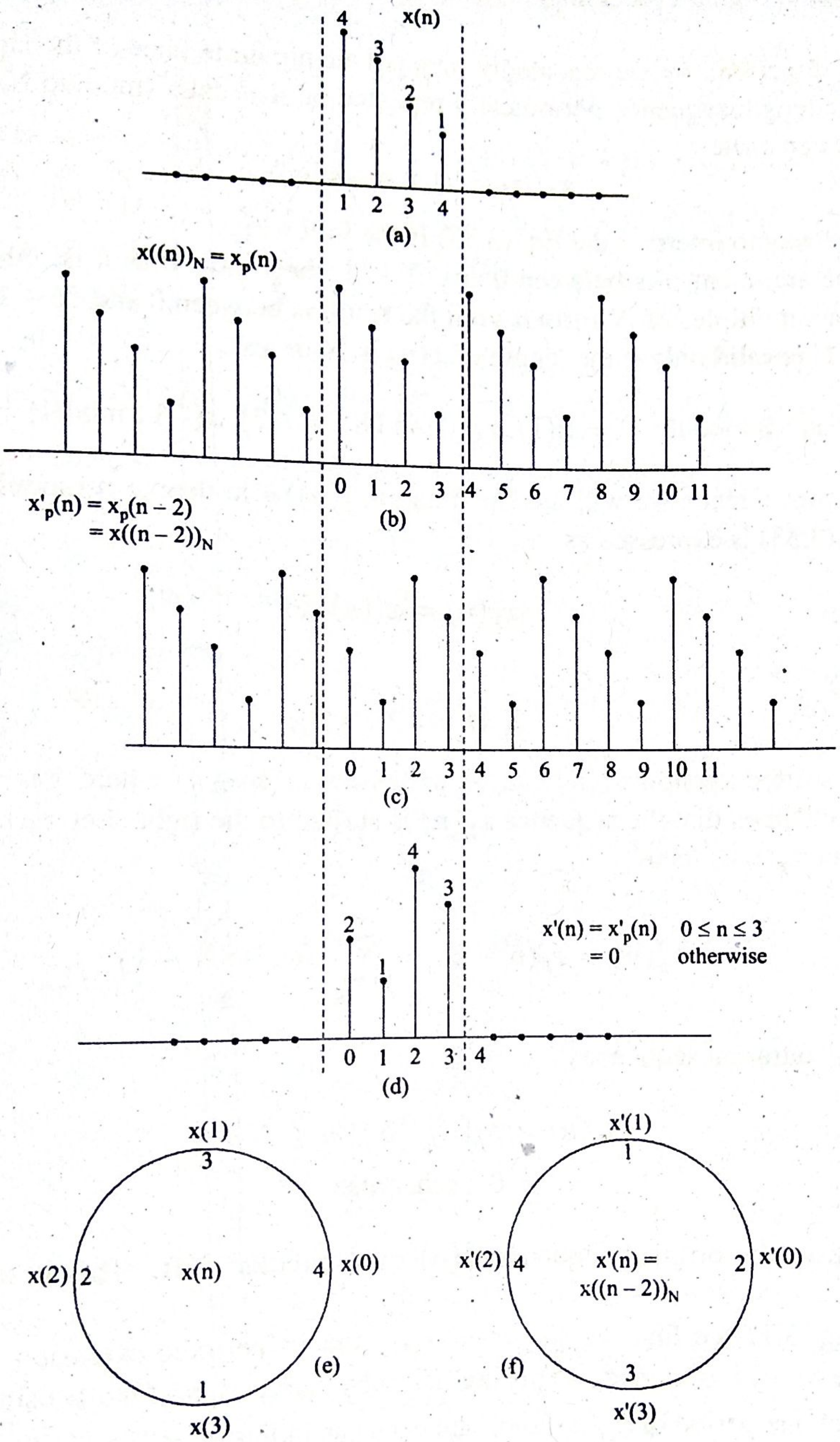


Fig. 3.11 Circular shift of a sequence

positive direction. As we repeatedly traverse the circumference of the circle we see the finite length sequence periodically repeated on a circular (modulo N) time axis. Now we can write

$$x_p(n) = x[(n \text{ modulo } N)] \tag{3.33}$$

A simple way to interpret the Eq. (3.33) is the following :

If the argument n is between 0 and $N - 1$, then leave it as it is; otherwise add or subtract multiples of N from n until the result is between 0 and $N - 1$. Note that Eq. (3.33) is valid only if the length of $x(n)$ is N or less.

$$\text{ex: } x(-3 \text{ modulo } 4) = x(1), x(10 \text{ mod } 8) = x(2), x(-11 \text{ mod } 4) = x(1)$$

For convenience we will use the notation $((n))_N$ to denote n modulo N. With this Eq. (3.33) is expressed as

$$x_p(n) = x((n))_N \tag{3.34a}$$

Similarly

$$X_p(k) = X((k))_N \tag{3.34b}$$

Circular shift of a sequence Property.

If $x(n) \xrightarrow[N]{DFT} X(k)$

then $x((n-m))_N \xrightarrow[N]{DFT} e^{-j\frac{2\pi}{N}km} X(k)$.

proof

$$DFT\left[x((n-m))_N\right] = \sum_{n=0}^{N-1} x((n-m))_N\, e^{-j\frac{2\pi}{N}kn}$$

$$= \sum_{n=0}^{m-1} \cdots + \sum_{n=m}^{N-1} \cdots$$

we have $x((n-m))_N = x(n-m+N)$

Ist term.

$$\sum_{n=0}^{m-1} x(n-m+N)\, e^{-j\frac{2\pi}{N}kn}.$$

put $n-m+N = l$.

$n=0 \Rightarrow l = N-m$

$n=m-1 \Rightarrow l = N-1$

$$= \sum_{l=N-m}^{N-1} x(l)\, e^{-j\frac{2\pi}{N}k(l+m-N)}$$

$$= \sum_{l=N-m}^{N-1} x(l)\, e^{-j\frac{2\pi}{N}k(l+m)}.$$

$$= e^{-j\frac{2\pi}{N}km} \sum_{l=N-m}^{N-1} x(l)\, e^{-j\frac{2\pi}{N}kl}.$$

II^{nd} Term.

$$\sum_{n=m}^{N-1} x(n-m) \cancel{\ } e^{-j\frac{2\pi}{N}kn}.$$

put $n-m \cancel{\ } = l$

$n=m \Rightarrow l = \cancel{\ } 0$

$n = N-1 \Rightarrow l = N-1-m$

$$\sum_{l=0}^{N-m-1} x(l)\, e^{-j\frac{2\pi}{N}k(l+m)}.$$

$$= e^{-j\frac{2\pi}{N}km} \sum_{l=0}^{N-m-1} x(l)\, e^{-j\frac{2\pi}{N}kl}.$$

combining I^{st} and II^{nd} term

$$\text{RHS} = e^{-j\frac{2\pi}{N}km} \sum_{l=0}^{N-1} x(l)\, e^{-j\frac{2\pi}{N}kl}.$$

$$= e^{-j\frac{2\pi}{N}km} X(k)$$

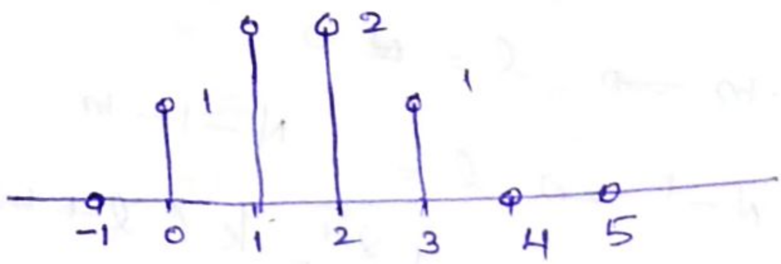
Hence the proof.

Q) Consider the finite length sequence x(n) as in fig and fine point

DFT of $x(n)$ is denoted by $X(k)$

Plot the sequence $y(n)$ whose DFT is

$$Y(k) = e^{-j\frac{4\pi}{5}k} X(k)$$



A) we have the circular shift pply

$$x(n) \xrightarrow{DFT} X(k).$$

$$x((n-m))_N \xrightarrow{DFT} e^{-j\frac{2\pi}{N}km} X(k).$$

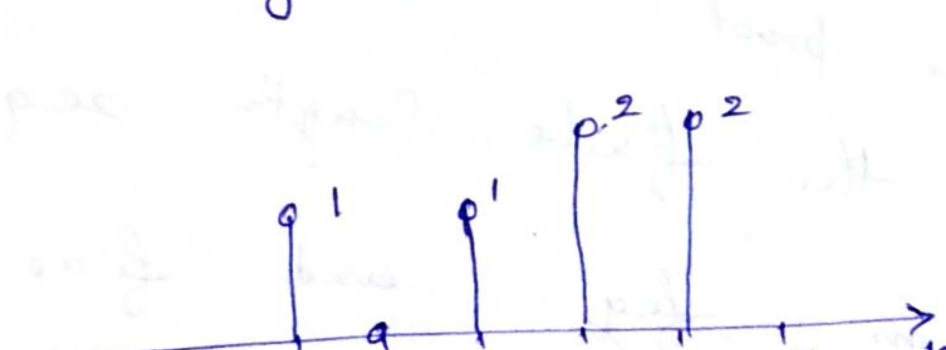
$$Y(k) = e^{-j\frac{4\pi}{5}k} X(k) = e^{-j\frac{2\pi}{5}k \cdot 2} X(k)$$

$$\Rightarrow m = 2.$$

$$\therefore y(n) = x((n-2))_5.$$

$y(0) = x((-2))_5$
$= x(3) = 1$

$y(1) = x((1-2))_5$
$= x(4) = 0$



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$$y(2) = x((2-2))_5$$
$$= x((0))_5 = x(0) = 1$$
$$y(3) = x((3-2))_5$$
$$= x((1))_5 = x(1) = 2$$
$$y(4) = x((4-2))_5 = x(2) = 2$$
$$y(n) = \{1\ 0\ 1\ 2\ 2\}$$

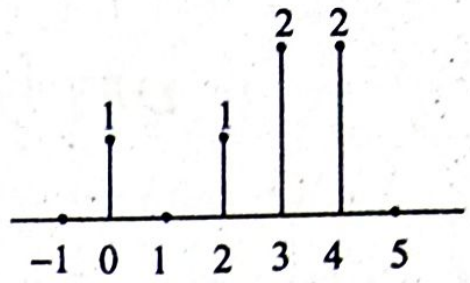


Fig. 3.13

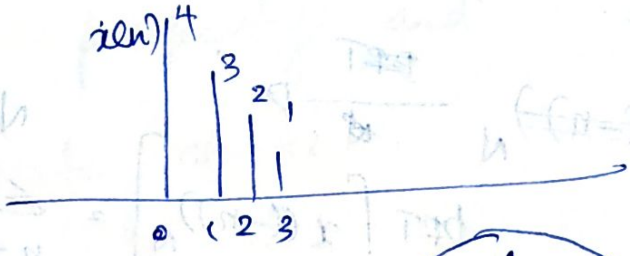
Practice Problem 3.3 If the DFT of the sequence $x(n) = \{1, 2, 1, 1, 2, -1\}$ is $X(k)$. Plot the sequence whose DFT is

$$Y(k) = e^{-j\pi k} X(k)$$

Ans: $\{1, 2, -1, 1, 2, 1\}$

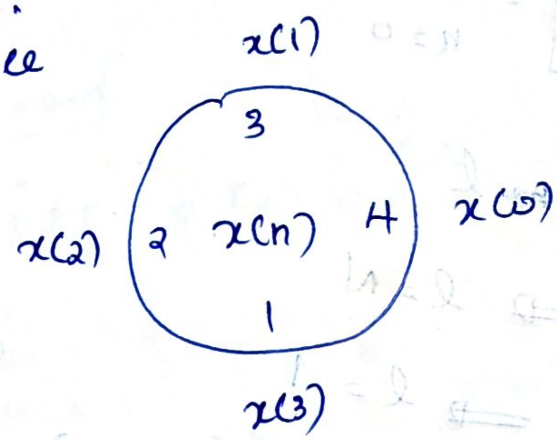
4) Time Reversal of Sequence

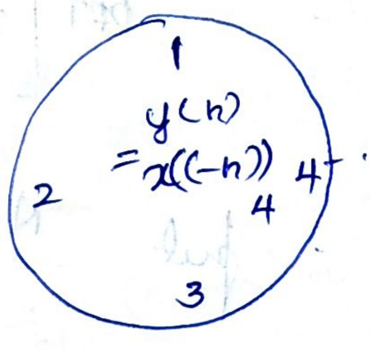
Time reversal of an N-point sequence is attained by wrapping the sequence $x(n)$ around the circle in clockwise direction. It is denoted as

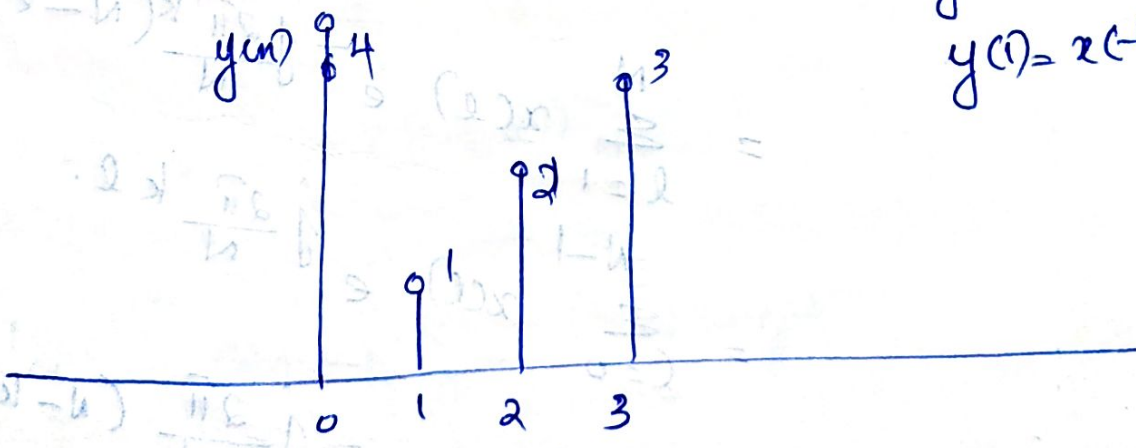


$x((-n))_N$.

ie







$y(0) = x(0)$.

$y(1) = x(-1) = x(3)$

Property

if $x(n) \xleftrightarrow[N]{DFT} X(k)$.

then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} X((-k))_N = X(N-k)$

proof

$\cancel{x((-n))_N} \xrightarrow[\cancel{N}]{\cancel{DFT}}$

$$DFT\left[x((-n))_N\right] = \sum_{n=0}^{N-1} x(-n+N)\, e^{-j\frac{2\pi}{N}kn}.$$

put $N - n = l$

$n = 0 \Rightarrow l = N$.

$n = N-1 \Rightarrow l = 1$

$$= \sum_{l=1}^{N} x(l)\, e^{-j\frac{2\pi}{N}k(N-l)}$$

$$= \sum_{l=0}^{N-1} x(l)\, e^{j\frac{2\pi}{N}kl}.$$

$$= \sum_{l=0}^{N-1} x(l)\, e^{-j\frac{2\pi}{N}(N-k)l}.$$

$$= X(N-k)$$

⑤ Circular frequency shift

If $x(n) \xleftrightarrow[N]{DFT} X(k)$.

then $x(n)\, e^{j\frac{2\pi}{N} ln} \xleftrightarrow[N]{DFT} X((k-l))_N$.

— This is the dual to the circular time shifting property and its proof is similar to above.

proof.

$$DFT\left[x(n)\, e^{j\frac{2\pi}{N} ln}\right] = \sum_{n=0}^{N-1} x(n) e^{j\frac{2\pi}{N} ln} e^{-j\frac{2\pi}{N} kn}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}(k-l)n}.$$

$$= X((k-l))_N$$

(6). Complex conjugate Properties.

If $x(n) \xleftrightarrow[N]{DFT} X(k)$.

then $x^*(n) \xleftrightarrow[N]{DFT} X^*((-k))_N = X^*(N-k)$

proof:

$$\text{DFT}\left[x^*(n)\right] = \sum_{n=0}^{N-1} x^*(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}(-k)n}\right]^*$$

$$= X^*((-k))_N = X^*(N-k).$$

(b) $x^*(N-n) \xleftrightarrow[N]{\text{DFT}} X^*(k).$

proof.

$$\text{IDFT}\left[X^*(k)\right] = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}k(-n)}\right]^*$$

$$= x^*((-n))_N$$

$$= x^*(N-n)$$

Q) Let $X(k)$ be a 14 pt DFT of a length 14 real sequence $x(n)$. The first 8 samples of $X(k)$ are $X(0) = 12$, $X(1) = -1+3j$, $X(2) = 3+4j$, $X(3) = 1-5j$, $X(4) = -2+2j$, $X(5) = 6+3j$, $X(6) = -2-3j$, $X(7) = 10$. Determine the remaining samples.

A): $DFT[x(n)] \longrightarrow X(k)$.

where $DFT[x^*(n)] = X^*(N-k)$.

for a real valued sequence

$$x(n) = x^*(n).$$

$$X(k) = X^*(N-k).$$

$$\therefore X(8) = X^*(14-8) = X^*(6)$$

$$= -2+3j$$

$$X(9) = X^*(14-9) = X^*(5) = 6-3j$$

$x(10) = x^*(14-10) = x^*(4) = -2-2j$

$x(11) = x^*(14-11) = x^*(3) = 1+5j$

$x(12) = x^*(14-12) = x^*(2) = 3-4j$

$x(13) = x^*(14-13) = x^*(1) = -1-3j$

H.W ~~x114~~ First five points of the 8 pt DFT of a real valued sequence are $\{28, -4+j\,9.565, -4+j4, -4+j\,1.656, -4\}$. Determine the remaining three points.

Ans: $\{-4-j\,1.656, -4-4j, -4-9.56j\}$