INFORMATION THEORY & CODING:

CONTENTS

- Quick recap
- Waveform channel
- Capacity of a Bandlimited Gaussian channel

Waveform channel

Ex: a physical communication system, the input and output of a channel is taken as real value.

A waveform channel is one which takes transmission in continuous time. eg: Gaussian channel.

Gaussian channel

- it is the most commonly used model for a noisy channel with real input and output because →
  - it is highly analytically tractable.
  - most intense kind of additive noise subject to a constraint on the noise power.
- Gaussian channel with noise energy N is a continuous channel with the following specification

1) $f(y|x) = \frac{1}{\sqrt{2\pi N}} e^{-\frac{(y-x)^2}{2N}}$

2) $Z \sim o\ (0, N)$ and $\alpha(X, Z) = X + Z$.

* model for some common communication channels such as wired and wireless telephone channels and satellite links

* Without further conditions, capacity of this channel may be infinite if the noise variance is zero.

    - the receiver receives the transmitted symbol perfectly.

    - Since x can take on any real value, the channel can transmit an arbitrary real number with no error.

if the noise variance is non-zero

- There is no constraint on the input, we can choose an infinite subset of inputs arbitrarily far apart, so that they are distinguishable at the output with arbitrarily small probability of error. Such a scheme has infinite capacity.

Capacity of bandlimited Gaussian channel

Shannon Hartley Theorem (or) Shannon Information capacity theorem.

This theorem is complementary to Shannons second theorem and applies to a channel in which the noise is Gaussian.

The channel capacity of a white bandlimited gaussian channel is given by

$$C = B \log_2 \left(1 + \frac{S}{N}\right) \text{ bits/sec} \quad \text{--- (1)}$$

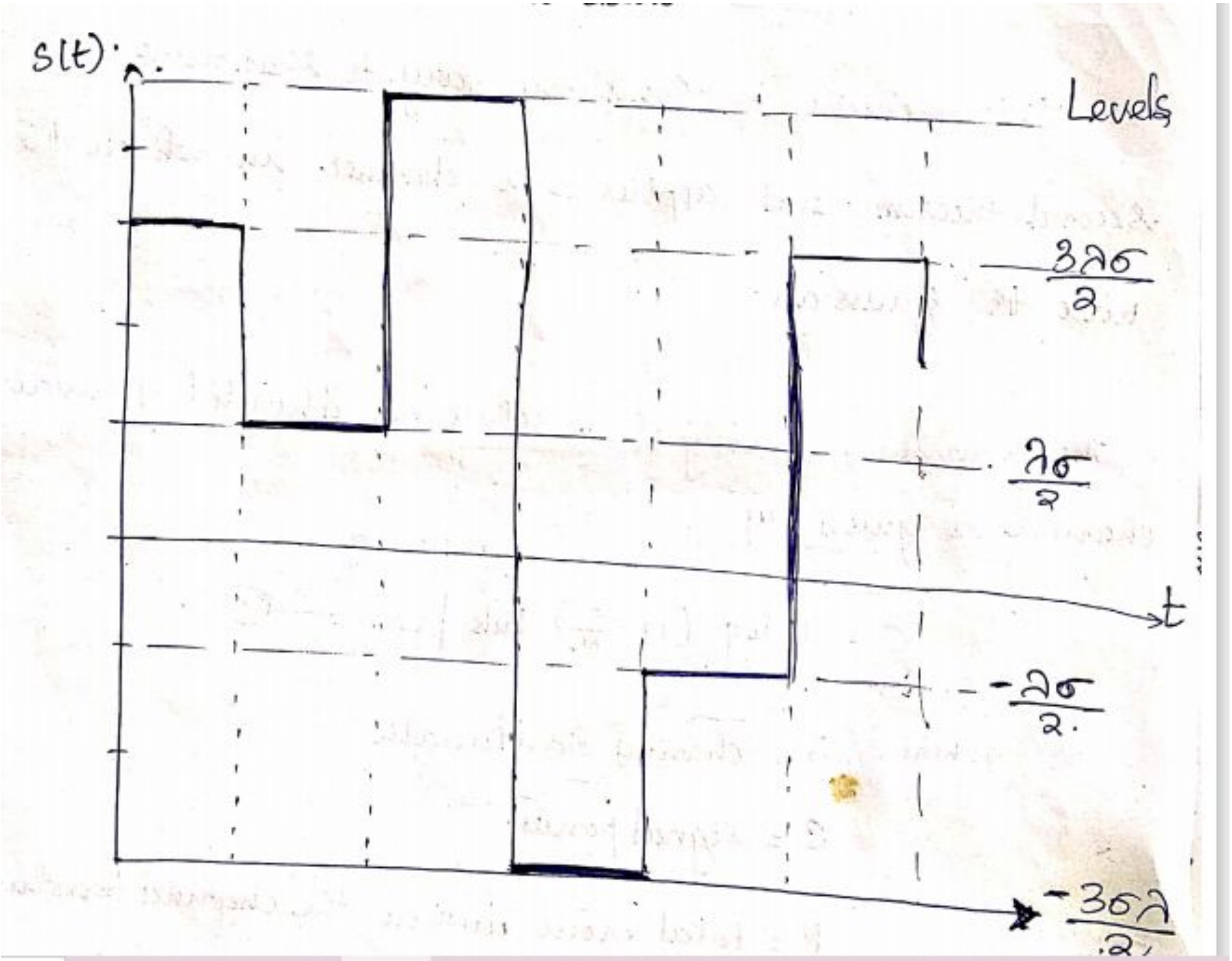
where B = channel Bandwidth

S = signal power

N = total noise within the channel Bandwidth

ie $N = N_0 B$ with $\frac{N_0}{2}$ being the 2sided PSD

Suppose that for the purpose of transmission over the channel, the messages are represented by fixed voltage levels. As the source generates one message after other in sequence, the transmitted signal s(t) takes one waveform similar to the one shown below.


s(t)
Levels
$\frac{3\lambda\sigma}{2}$
$\frac{\lambda\sigma}{2}$
t
$-\frac{\lambda\sigma}{2}$
$-\frac{3\sigma\lambda}{2}$

The received signal is accompanied by noise, when root mean square voltage is σ.

The levels have been separated by an interval $\lambda\sigma$

where λ is the number presumed large enough to allow recognition of individual levels with an acceptable probability of error.

Assuming an even no: of levels, the levels are located at voltage

$\pm \frac{\lambda\sigma}{2}$, $\pm \frac{3\lambda\sigma}{2}$ etc.

If there are M possible message, then there must be M levels. we assume that the message and hence the levels occur with equal likelihood.

Then the avg signal power is

$$S = \frac{2}{M}\left[\left(\frac{\lambda\sigma}{2}\right)^2 + \left(\frac{3\lambda\sigma}{2}\right)^2 + \dots + \left[\frac{(M-1)\sigma\lambda}{2}\right]^2\right].$$

$$= \frac{2}{M}\left(\frac{\lambda\sigma}{2}\right)^2\left[1^2 + 3^2 + 5^2 + \dots (M-1)^2\right].$$

$$= \frac{2}{M}\,\frac{(\lambda\sigma)^2}{4}\left[\frac{M(M^2-1)}{6}\right]$$

$$= \frac{(M^2-1)}{12}\,\lambda^2\sigma^2.$$

eg: $M=2$
levels
$\frac{\lambda\sigma}{2}, -\frac{\lambda\sigma}{2}$

eg $M = 6$.
$\frac{\lambda\sigma}{2}, -\frac{\lambda\sigma}{2}$
$\frac{3\lambda\sigma}{2}, -\frac{3\lambda\sigma}{2}$
$\frac{5\lambda\sigma}{2}, -\frac{5\lambda\sigma}{2}$.

M;
$\frac{(M-1)\lambda\sigma}{2}$

Power $= V^2$

avg m levels $= \frac{\quad}{M}$.

The no: of levels for a given avg signal power is

$$M^2 - 1 = \frac{12S}{\lambda^2 \sigma^2}$$

$$M^2 = 1 + \frac{12S}{(\lambda\sigma)^2}$$

$$M^2 = (\lambda\sigma)^2 + 12S.$$

$$M = \sqrt{1 + \frac{12S}{(\lambda\sigma)^2}}$$

$$M = \left[1 + \frac{12}{\lambda^2}\frac{S}{N}\right]^{1/2}.$$

$$\frac{S}{\sigma^2} = \frac{S}{N}.$$

Each message is equally likely and therefore it conveys an amount of information

$$H = \log_2 M$$

$$= \log_2\left(1 + \frac{12}{\lambda^2}\frac{S}{N}\right)^{1/2}.$$

$$= \frac{1}{2}\log_2\left(1 + \frac{12}{\lambda^2}\frac{S}{N}\right) \text{ bits/message}$$

To find the <u>information rate</u> of the signal waveform $s(t)$.

we need to estimate how many message per unit time may be carried by this signal. ie we need to estimate the interval T which should be assigned to each message to allow the transmitted levels to be recognized individually at the receiver, eventhough the bandwidth B is limited.

Now the effect of limited BW on $s(t)$ will be ~~sac~~ rounding of the initially abrupt transitions from one level to another.

when an abrupt step is applied to an ideal LPF of BW B, the response has 10% to 90% rise time τ given by.

$$\tau = \frac{0.44}{B}$$

if we set $T = \tau$, we shall distinguish levels reliably.

Let $T = \tau = \frac{0.5}{B}$ (Heuristic Approach)

Message rate $r = \frac{1}{T} = 2B$ (Nyquist sampling rate).

Since the reception of any of the M messages is equally likely

$$H = \log_2 M$$

$\boxed{\text{Information Rate } R = rH.}$

we already assumed that the channel is just able to allow the transmission with acceptable probability of error.

$\therefore \quad R \simeq C.$

Channel capacity

$$C \cong R = rH.$$

$$= 2B \times \frac{1}{2} \log_2 \left(1 + \frac{12}{\lambda^2} \frac{S}{N}\right).$$

$$= B \log_2 \left(1 + \frac{12}{\lambda^2} \frac{S}{N}\right). \quad \text{——— } (2)$$

Comparing I and II

we observe that the results would be identical if

$$\frac{12}{\lambda^2} = 1 \Rightarrow \lambda^2 = 12$$

$$\lambda = 3.5$$

Equation ② contemplates that with a sufficiently sophisticated transmission technique.

Transmission at channel capacity is possible with arbitrarily small error.

if $\eta/2$ is 2-sided PSD of noise in watts/Hz

$\eta/2 = N_o/2$.

$N = N_o B$.

$$\boxed{C = B \log_2 \left[1 + \frac{S}{N_o B}\right] \text{ bits/se.}}$$

NOTE

1) We find that channels encountered in physical s/ms are atleast approx. Gaussian.

2) Results obtained for a Gaussian channel often provide a lowerbound on the performance of a system operating over a non-gaussian channel.

3) if a particular encoder -decoder is used with a Gaussian channel and an error probability Pe results, then with a non-gaussian channel another encoder-decoder can be designed so that Pe will be smaller.

CONCLUSION

- Gaussian channel bandwidth

THANK YOU