Circular Convolution (Multiplication of DFTs)

circular convolution two N point sequences $x_1(n)$ and $x_2(n)$ is

$$x_1(n) \textcircled{N} x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N .$$

Property:

If $x_1(n)$ and $x_2(n)$ are two finite duration sequence of length N, and $X_1(k)$ and $X_2(k)$ are their N point DFTs resply.

i.e. $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$.

then $x_1(n) \circledN x_2(n) \xleftrightarrow[N]{DFT} X_1(k) X_2(k)$.

proof.

given $X_1(k) = \sum_{m=0}^{N-1} x_1(m) e^{-j\frac{2\pi}{N}km}$.

$X_2(k) = \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}kl}$.

Let $X_3(k) = X_1(k) X_2(k)$

$= \sum_{m=0}^{N-1} x_1(m) e^{-j\frac{2\pi}{N}km} \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}kl}$

let $x_3(n) = IDFT\left[X_1(k) X_2(k) \right]$.

$$= \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{N-1} x_1(m) e^{-j\frac{2\pi}{N}km} \cdot \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}kl} \cdot e^{j\frac{2\pi}{N}kn} \right]$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-m-l)k}$$

Take the third summation

$$\sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}(n-m-l)k}$$

if $l = n-m$. summation $= N$.

$l \neq n-m$ summation $= 0$.

$\therefore l = n-m \Rightarrow$ $l=0 \Rightarrow m=n$. $l = N-1 \Rightarrow m = n-$

$$x_3(n) = \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{m=n}^{n-(N-1)} x_2(n-m) \cdot N.$$

$$= \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$x_3(n) = \underline{x_1(n) \circledN x_2(n)}$ Hence proof

Methods to find circular convolution.

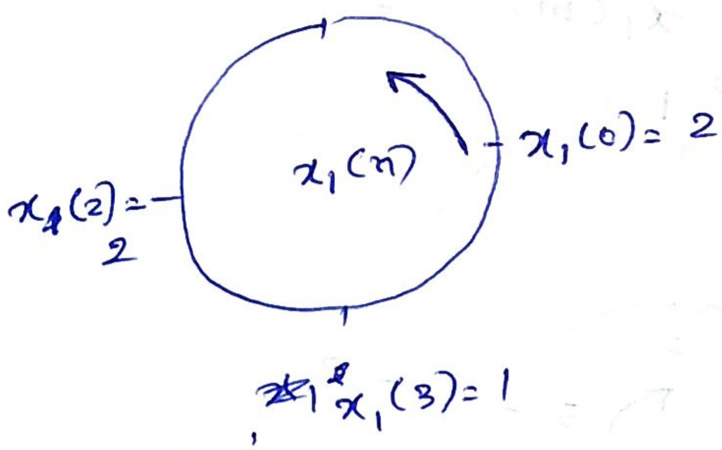
① concentric circle method:

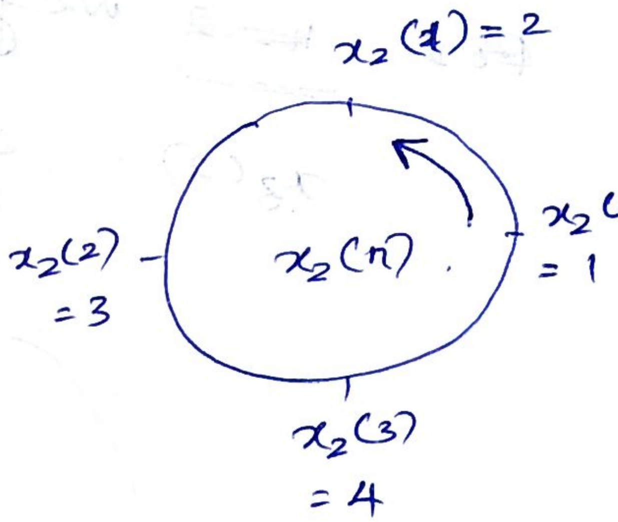
Perform circular convolution of two sequences.

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$x_2(n) = \{1, 2, 3, 4\}.$$

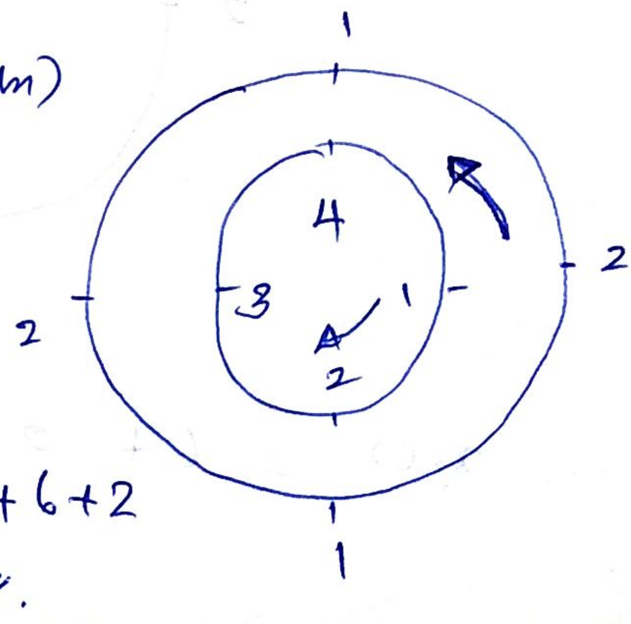
$$\underset{x_3(n)}{\cancel{y(n)}} = x_1(n) \circledast x_2(n) = \underline{\sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)} \quad ①$$





put $n=0$ in eqn ① $\Rightarrow$

$$\underset{x_3(0)}{\cancel{y(0)}} = \sum_{m=0}^{N-1} x_1(m)\, x_2(-m)$$



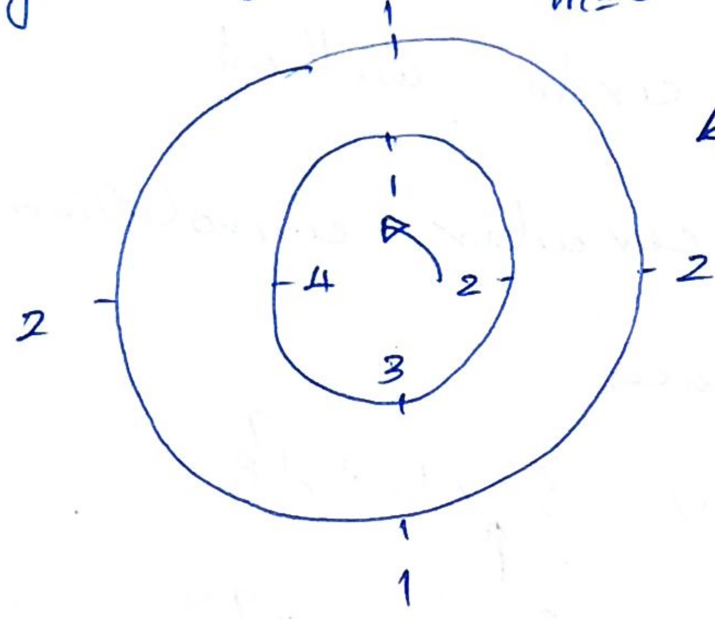
$= 2 \times 1 + 1 \times 4 +$

$2 \times 3 + 1 \times 2 = 2 + 4 + 6 + 2$

$= 14$ //.

put $n=1$ in eqn ①

we ~~to~~ get $x_3(1) = \sum_{m=0}^{N-1} x_1(m)\, x_2(1-m)$



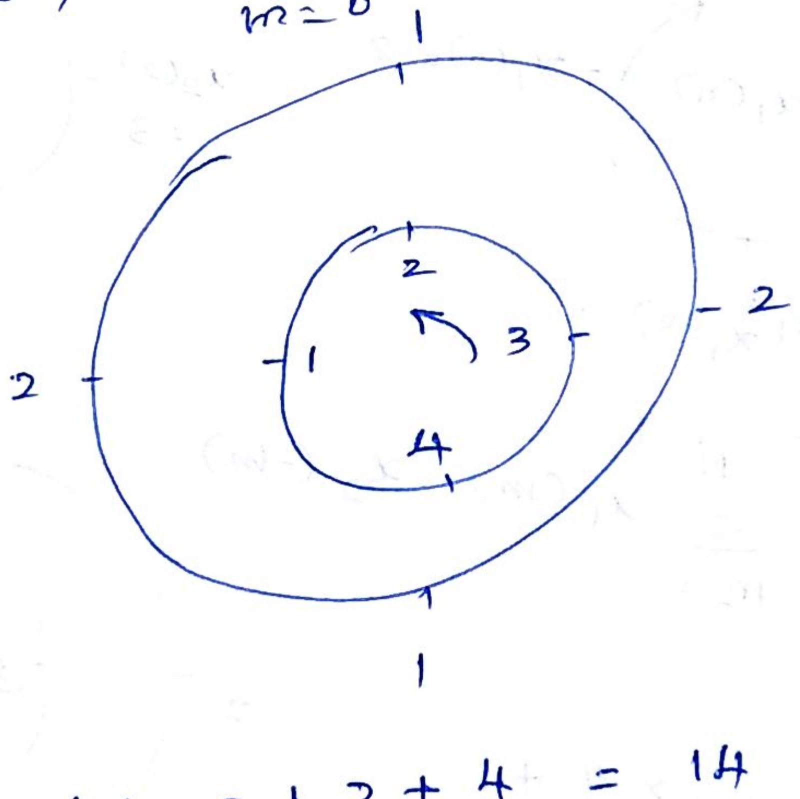
[Rotate inner circle in anti clock wise direction]

$x_3(1) = 4 + 1 + 8 + 3 = 16$

put $n=2$ in eqn ①

we get.

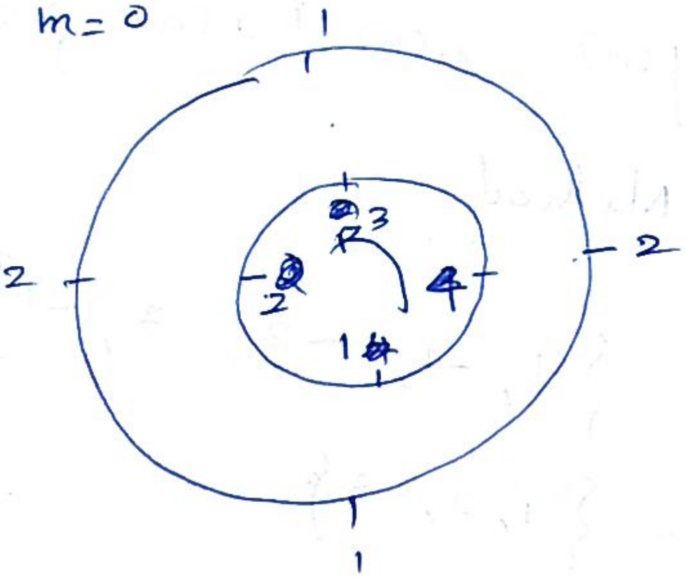
$x_3(2) = \sum_{m=0}^{3} x_1(m)\, x_2(2-m)$.



$x_3(2) = 6 + 2 + 2 + 4 = \underline{\underline{14}}$.

For $n = 3$.

$$x_3(3) = \sum_{m=0}^{3} x_1(m)\, x_2(3-m).$$



$$x_3(3) = \cancel{6+2+2+4} \quad 8 + 3 + 4 + 1 = 16.$$

$$x_3(n) = \{x_3(0),\ x_3(1),\ x_3(2),\ x_3(3)\}.$$

$$x_3(n) = \{14,\ 16,\ 14,\ 16\}.$$

— If we continue the above procedure beyond $m = 3$ same answer is repeated.

HW Find circular convolution of two finite duration sequences ~~$x_1(n)$~~

$x_1(n) = \{1, -1, -2, 3, -1\}$ and ~~$x_2(n)$~~

$x_2(n) = \{1, 2, 3\}.$

Ans: $y(n) = x_1(n) \; ⓝ \; x_2(n) = \{8, -2, -1, -4, -1\}$

② Matrix Method

$x_1(n) = \{1, -1, -2, 3, -1\}$

$x_2(n) = \{1, 2, 3\}$

Find circular convolution $y(n)$ using matrix method.

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+9-2 \\ 2-1-3 \\ 3-2-2 \\ -3-4+3 \\ -6+6-1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

$y(n) = \{8, -2, -1, -4, -1\}$

HW. Find circular convolution of two sequences $x_1(n) = \{1, 2, 3, 1\}$ and $x_2(n) = \{4, 3, 2, 2\}$ using (a) concentric circle method (b) matrix method.

Ans: $y(n) = \{17, 19, 22, 19\}$

③ Circular convolution by DFT – IDFT method:

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

$x_1(n) \; Ⓝ \; x_2(n)$ ~~then~~ $\cancel{x_3(n)}$ $=$ IDFT $[X_1(k) \cdot X_2(k)]$

Q) Perform circular convolution of the following two sequences using DFT – IDFT method.

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4$$

① First compute 4 point DFT of $x_1(n)$

$$X_1(k) = \{6, 0, 2, 0\}.$$

② 4 point DFT of $x_2(n)$.

$$X_2(k) = \{10, -2+2j, -2, -2-2j\}$$

③ Multiply the two DFTs to get

$$X_3(k) = X_1(k)\, X_2(k).$$

$$= \{60, 0, -4, 0\}$$

④ Now compute IDFT of $X_3(k)$ i.e

$$x_3(n) = \frac{1}{4} \sum_{k=0}^{3} X_3(k)\, e^{j\frac{2\pi}{4}nk}.$$

$$x_3(n) = \underline{\{14, 16, 14, 16\}}.$$

which is the result already obtained.

HW

Q) By means of DFT - IDFT determine the circular convolution of two sequences

$x_1(n) = \{\underset{\uparrow}{1}, 2, 3\}$ and

$x_2(n) = \{1, 2, 2, 1\}$.