Linear phase FIR filters:

- The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} \longrightarrow \text{Transfer function} \quad (1)$$

where $h(n)$ is the impulse response of the filter.

- The fourier transform of $h(n)$ is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n) e^{-j\omega n} \longrightarrow \text{Frequency response} \quad (2)$$

which is periodic in frequency with a period 2π.

- $H(e^{j\omega})$ can also be represented as.

$$H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j\theta(\omega)} \quad (3)$$

(in magnitude phase form).

where $|H(e^{j\omega})| \longrightarrow$ magnitude response

and $\theta(\omega) \longrightarrow$ phase response.

- The phase delay and group delay of a filter is defined as.

$$\text{phase delay } \tau_p = \frac{-\theta(\omega)}{\omega} \quad \text{——— ④}$$

$$\text{group delay } \tau_g = \frac{-d\,\theta(\omega)}{d\omega} \quad \text{——— ⑤}$$

- For FIR filters with linear phase we can define

$$\theta(\omega) = -\alpha\,\omega \quad ; \; -\pi \leq \omega \leq \pi \quad \text{——— ⑥}$$

where α is a constant phase delay in samples.

- So in the case of linear phase FIR filters, group delay and phase delay is given by.

substitute eqn ⑥ in ④ and ⑤.

$$\left.\begin{aligned} \tau_p &= \frac{+\alpha\omega}{\omega} = +\alpha \\ \tau_g &= \frac{-d}{d\omega}(-\alpha\omega) = +\alpha \end{aligned}\right\} \quad \text{——— ⑦}$$

- From equ ② and ③ we can write.

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm\left|H(e^{j\omega})\right| e^{j\theta(\omega)} \quad \text{——— ⑧}$$

taking the real part.

$$\sum_{n=0}^{N-1} h(n) \cos\omega n = \pm\left|H(e^{j\omega})\right| \cos\theta(\omega) \quad \text{——— ⑨}$$

from imaginary part

$$-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm |H(e^{j\omega})| \sin \theta(\omega) \quad \text{——— (10)}$$

– By taking the ratio of equ (9) and (10)

$$\frac{-\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\sin \theta(\omega)}{\cos \theta(\omega)} \quad \text{——— (11)}$$

for linear phase FIR filters $\theta(\omega) = -\alpha\omega$

$$(11) \Rightarrow \frac{-\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{-\sin(\alpha\omega)}{\cos(\alpha\omega)}$$

cross multiply.

$$\sum_{n=0}^{N-1} h(n) \sin(\omega n) \cos(\alpha\omega) = \sum_{n=0}^{N-1} h(n) \cos(\omega n) \sin(\alpha\omega)$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \left[\sin(\omega n) \cos(\alpha\omega) - \cos(\omega n) \sin(\alpha\omega)\right] = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n) \sin[\alpha - n]\omega = 0$$

This equation will be zero when

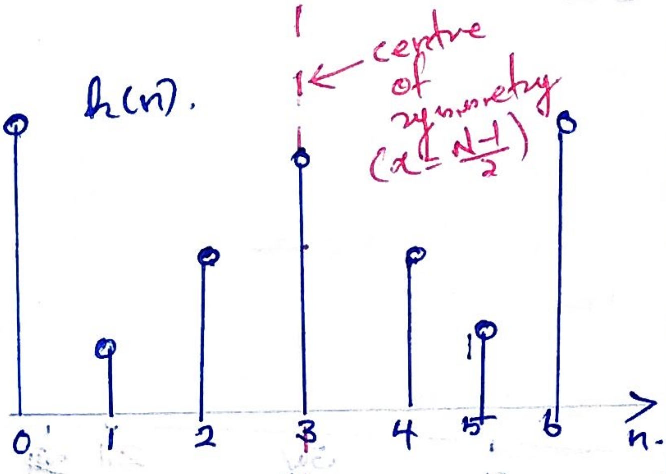
$$\left.\begin{aligned} h(n) &= h(N-1-n) \\ \text{and} \quad \alpha &= \frac{N-1}{2}. \end{aligned}\right\} \quad \text{——— (12)}$$

Therefore FIR filters will have constant phase delay and group delay when impulse response is symmetrical about

$\alpha = \frac{N-1}{2}$.

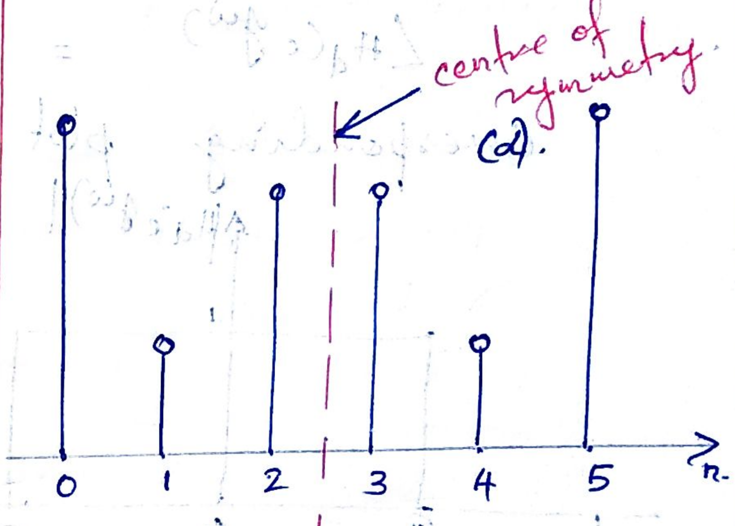
The impulse response satisfying equ (12) for odd and even values of N is shown below.

- when $N=7 \Longrightarrow$ centre of symmetry $\alpha = \frac{N-1}{2} = 3$.
- when $N=6 \Longrightarrow$ centre of symmetry $\alpha = \frac{N-1}{2} = 2\frac{1}{2}$



N – odd (N=7).

$\alpha = \frac{N-1}{2} = 3.$



N → even.

N=6.

$\alpha = \frac{N-1}{2} = 2\frac{1}{2}$ samples.

If only constant group delay is required and not phase delay we can write

$$\theta(\omega) = \beta - \alpha\omega.$$

Now $H(e^{j\omega}) = \pm |H(e^{j\omega})| e^{j(\beta - \alpha\omega)}$ — (13)

From eqn (2) and (13).

$$\sum_{n=0}^{N-1} h(n) e^{-j\omega n} = \pm |H(e^{j\omega})| e^{j(\beta - \alpha\omega)}$$

Considering real part only.

$$\sum_{n=0}^{N-1} h(n) \cos\omega n = \pm |H(e^{j\omega})| \cos(\beta - \alpha\omega) \quad (14)$$

and imaginary part.

$$-\sum_{n=0}^{N-1} h(n) \sin\omega n = \pm |H(e^{j\omega})| \sin(\beta - \alpha\omega) \quad (15)$$

By taking ratio of eqn (15) and (14)

$$\frac{-\sum_{n=0}^{N-1} h(n) \sin\omega n}{\sum_{n=0}^{N-1} h(n) \cos\omega n} = \frac{\sin(\beta - \alpha\omega)}{\cos(\beta - \alpha\omega)}$$

$$\sum_{n=0}^{N-1} h(n) \cos\omega n \sin(\beta - \alpha\omega) + \sum_{n=0}^{N-1} h(n) \sin\omega n \cos(\beta - \alpha\omega) = 0.$$

$$\sum_{n=0}^{N-1} h(n)\left[\sin(\beta - \alpha\omega)\cos\omega n + \cos(\beta - \alpha\omega)\sin n\omega\right] = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} h(n)\sin\left[\beta - (\alpha - n)\omega\right] = 0 \quad \text{--- (16)}$$

if $\beta = \pi/2$ above equ becomes.

$$\sum_{n=0}^{N-1} h(n)\cos(\alpha - n)\omega = 0 \quad \because (\sin(90-\theta) = \cos\theta)$$

The equ will be satisfied when.

$$\left.\begin{aligned} h(n) &= -h(N-1-n) \\ \text{and} \quad \alpha &= \frac{N-1}{2} \end{aligned}\right\} \text{--- (17)}$$

$\therefore$ FIR filters have constant group delay τ_g and not ~~conts~~ constant phase delay when the impulse response is antisymmetrical about $\alpha = \frac{N-1}{2}$.

The impulse response satisfying equ (17) is as shown below.

- when $N = 7$ the centre of ~~ass~~ antisymmetry occurs at $\alpha = \frac{N-1}{2} = 3^{rd}$ sample

when $n = 3 \Rightarrow \therefore h(3) = -h(7-1-3)$

$$= -h(3).$$

Only satisfy if $h(3) = 0$.

i.e. $h\left(\frac{N-1}{2}\right) = 0$ for antisymmetric odd sequence

when $N = 6$ centre of antisymmetry occurs

at $\alpha = \frac{N-1}{2} = 2\frac{1}{2}$ sample.