MODULE – IV
Frequency Response Analysis

Introduction:

The response of a system can be partitioned into both the transient response and the steady state response. We can find the transient response by using Fourier integrals. The steady state response of a system for an input sinusoidal signal is known as the **frequency response**. In this chapter, we will focus only on the steady state response.

If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signals have the same frequency, but different amplitudes and phase angles.

Let the input and output signal be −

$$r(t)=A\sin(\omega t) \tag{1}$$

$$c(t)=B\sin(\omega t+\phi) \tag{2}$$

The open loop transfer function will be −

$$G(s)=G(j\omega)$$

We can represent $G(j\omega)$ in terms of magnitude and phase as shown below.

$$G(j\omega)=|G(j\omega)|\angle G(j\omega)$$

The output signal is

$$c(t)=A|G(j\omega)|\sin(\omega t+\angle G(j\omega)) \tag{3}$$

- The **amplitude** of the output sinusoidal signal is obtained by multiplying the amplitude of the input sinusoidal signal and the magnitude of $G(j\omega)$ at ω.
- The **phase** of the output sinusoidal signal is obtained by adding the phase of the input sinusoidal signal and the phase of $G(j\omega)$ at ω

Where,

- **A** is the amplitude of the input sinusoidal signal.
- **Ω** is angular frequency of the input sinusoidal signal.

We can write, angular frequency ω as shown below.

$$\omega=2\pi f$$

Here, f is the frequency of the input sinusoidal signal. Similarly, you can follow the same procedure for closed loop control system.

Correlation between time and frequency response:

The frequency domain specifications are resonant peak, resonant frequency and bandwidth.

Consider the transfer function of the second order closed loop control system as,

$$T(s) = C(s)/R(s)=\omega_n^2/(s2+2\zeta\omega_n s+\omega_n^2)$$

Substitute, s= jω in the above equation.

$$T(j\omega)= \omega_n^2 /(j\omega)^2+2\zeta\omega_n(j\omega)+ \omega_n^2$$

$$\Rightarrow T(j\omega)= \frac{\omega_n^2}{-\omega^2+2j\zeta\omega\omega_n+\omega_n^2} = \frac{\omega_n^2}{\omega_n^2}\frac{1}{\left(-\frac{\omega^2}{\omega_n^2}\right)+2j\zeta\left(\frac{\omega}{\omega_n}\right)+1}$$

$$\Rightarrow T(j\omega)= \frac{1}{\left(1-\frac{\omega^2}{\omega_n^2}\right)+j2\zeta\left(\frac{\omega}{\omega_n}\right)} \qquad (4)$$

Let, $\frac{\boldsymbol{\omega}}{\boldsymbol{\omega_n}}$ **=u** Substitute this value in the above equation.

$$T(j\omega)= \frac{1}{(1-u^2)+j2\zeta u} \qquad (5)$$

Magnitude of T(jω) is -

$$M=|T(j\omega)|= \frac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}} \qquad (6)$$

Phase of T(jω) is -

$$\angle T(j\omega)=-\tan-1\frac{2\zeta u}{(1-u^2)} \qquad (7)$$

The steady-state output of the system for a sinusoidal input of unit magnitude and variable frequency ω is given by

$$C(t) = \frac{1}{\sqrt{(1-u^2)^2+(2\zeta u)^2}}\text{Sin}(\omega t- \tan^{-1}\frac{2\zeta u}{(1-u^2)}) \qquad (8)$$

Resonant Frequency:

It is the frequency at which the magnitude of the frequency response has peak value for the first time. It is denoted by ω_r. At $\omega=\omega_r$ the first derivate of the magnitude of T(jω) is zero.

Differentiate M with respect to u.

$$\frac{dM}{du}\Big|_{u=u_r} = -\frac{1}{2}\frac{-4(1-u_r^2)u_r+8\zeta^2u_r}{\left[(1-u_r^2)^2+(2\zeta u_r)^2\right]^{3/2}} = 0$$

$$\Rightarrow 4u_r^3 - 4u_r +8\zeta^2u_r =0$$

$$\Rightarrow u_r = \sqrt{1-2\zeta^2} \qquad (9)$$

$$\text{i.e, } \omega_r = \omega_n\sqrt{1-2\zeta^2} \qquad (10)$$

Resonant Peak:

It is the peak (maximum) value of the magnitude of T(jω). It is denoted by Mr.

At $u=u_r$, the Magnitude of T(jω) is -

$$M=|T(j\omega)|=\frac{1}{\sqrt{\left(1-u^2\right)^2+(2\zeta u)^2}}$$

Substitute, $u_r = \sqrt{1-2\zeta^2}$ in the above equation, we get

$$M_r=\frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{11}$$

The phase angle of $T(j\omega)$ at resonant frequency u_r obtained from equation 7 is

$$\Phi_r = -\tan^{-1}\left[\sqrt{1-2\zeta^2} / \zeta\right] \tag{12}$$

Resonant peak in frequency response corresponds to the peak overshoot in the time domain transient response for certain values of damping ratio ζ. So, the resonant peak and peak overshoot are correlated to each other.

Bandwidth:

It is the range of frequencies over which, the magnitude of $T(j\omega)$ drops to 70.7% (0.707) from its zero frequency value.

At $\omega=0$, the value of u will be zero.

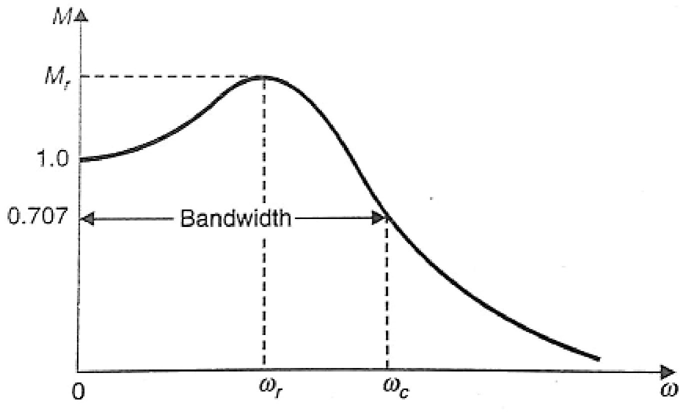
Substitute, u=0 in M, frm equation 6

$$M=\frac{1}{\sqrt{\left(1-0^2\right)^2+(2\zeta 0)^2}}=1$$

Therefore, the magnitude of $T(j\omega)$ is one at $\omega=0$.

At 3-dB frequency, the magnitude of $T(j\omega)$ will be 70.7% of magnitude of $T(j\omega)$ at $\omega=0$.

i.e., at $\omega=\omega_b$, $M=0.707(1)=1/\sqrt{2}$



Typical magnification curve of a feedback control system.

From Equation 6: $$M=\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{(1-u_b^2)^2+(2\zeta u_b)^2}}$$

$$\Rightarrow 2=(1-u_b^2)^2+(2\zeta)^2 u_b^2$$

Let, $u_b^2=x$

$$\Rightarrow 2=(1-x)^2+(2\zeta)^2 x \Rightarrow x^2+(4\zeta^2-2)x-1=0$$

$$\Rightarrow x = \frac{-(4\zeta^2-2)\pm\sqrt{(4\zeta^2-2)^2-4}}{2}$$

Consider only the positive value of x.

$$x = \frac{-(4\zeta^2-2)+\sqrt{(4\zeta^2-2)^2-4}}{2}$$

or

$$X = 1 - 2\zeta^2\sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

Substitute, $x = u_b^2 = \omega_b^2/\omega_n^2$

$$\omega_b^2/\omega_n^2 = 1 - 2\zeta^2\sqrt{2 - 4\zeta^2 + 4\zeta^4}$$

$$\Rightarrow \omega_b = \omega_n\sqrt{1 - 2\zeta^2\sqrt{2 - 4\zeta^2 + 4\zeta^4}} \qquad (13)$$

Bandwidth ωb in the frequency response is inversely proportional to the rise time tr in the time domain transient response.

Bode Plots

Sinusoidal transfer function is graphically represented by Bode plot for determining the stability of the control system. Bode plot is a logarithmic plot and consists of two plots.

- A plot of the logarithmic (base 10) of magnitude (in decibel) Vs frequency in logarithmic scale i.e. log ω.
- A plot of Phase plot (ϕ) Vs frequency in logarithmic scale i.e. log ω.

In both the plots, x-axis represents angular frequency (logarithmic scale). Whereas, y-axis represents the magnitude (linear scale) of open loop transfer function in the magnitude plot and the phase angle (linear scale) of the open loop transfer function in the phase plot.

The **magnitude** of the open loop transfer function in dB (decibel) is -

$$M = 20\log|G(j\omega)H(j\omega)| \qquad (1)$$

The **phase angle** of the open loop transfer function in degrees is -

$$\phi = \angle G(j\omega)H(j\omega) \qquad (2)$$

Note − the base of logarithm is 10.

Basic of Bode Plots

Let the generalised expression for open-loop transfer function of a system be given by:

$$G(s)H(s) = \frac{K\left[(1+sT_1)(1+sT_2)\ldots\ldots\right]\omega_n^2}{s^n\left[(1+sT_a)(1+sT_b)\ldots\ldots\right]\left[s^2+2\xi\omega_n s+\omega_n^2\right]} \qquad (3)$$

where $K, T_1, T_2, \ldots, T_a, T_b, \ldots, \xi, \omega_n$ are all real coefficients

Put s= jω in equation 3, we get

$$G(j\omega)H(j\omega) = \frac{K\left[(1+j\omega T_1)(1+j\omega T_2)\dots\right]}{(j\omega)^n\left[(1+j\omega T_a)(1+j\omega T_b)\dots\right]\left[1+j2\xi\left(\frac{\omega}{\omega_n}\right)+\left(\frac{j\omega}{\omega_n}\right)^2\right]} \quad (4)$$

$$= \frac{K\left[(1+j\omega T_1)(1+j\omega T_2)\dots\right]}{(j\omega)^n\left[(1+j\omega T_a)(1+j\omega T_b)\dots\right]\left[1+j2\xi u-u^2\right]} \quad (5)$$

Where u= ω/ω_n

From equation 5, the magnitude of G(jω)H(jω) in decibels is given by

From equation (11.9), the magnitude of G (jω) H (jω) in decibels is given by

$$20\log_{10}|G(j\omega)H(j\omega)| = 20\log_{10}K + (20\log_{10}|1+j\omega T_1| + 20\log_{10}|1+j\omega T_2|\dots)$$
$$-[20n\log\omega + 20\log|1+j\omega T_a| + 20\log|1+j\omega T_b| + \dots] - 20\log_{10}|(1-u^2)+j2\xi u| \quad (6)$$

or $$20\log_{10}|G(j\omega)H(j\omega)| = [20\log K + 20\log\sqrt{1+\omega^2T_1^2} + 20\log\sqrt{1+\omega^2T_2^2}+\dots]$$
$$-20N\log\omega - 20\log\sqrt{1+\omega^2T_a^2} - 20\log\sqrt{1+\omega^2T_b^2}\dots$$
$$\dots - 20\log\sqrt{(1-u^2)^2+4\xi^2u^2} \quad (7)$$

and phase angle of G (jω) H (jω) is given by

$$\angle G(j\omega)H(j\omega) = \angle K + \angle(1+j\omega T_1) + \angle(1+j\omega T_2) + \dots$$
$$-\angle(j\omega)^n - \angle(1+j\omega T_a) - \angle(1+j\omega T_b)\dots$$
$$-\angle(1-u^2+j2\xi u) \quad (8)$$

$$\angle G(j\omega)H(j\omega) = 0^\circ + \tan^{-1}\omega T_1 + \tan^{-1}\omega T_2 + \dots$$
$$-(90\times n) - \tan^{-1}\omega T_a - \tan^{-1}\omega T_b\dots$$
$$-\tan^{-1}\left(\frac{2\xi u}{1-u^2}\right) \quad (9)$$

Generally there are the following seven simple types of factors in G (jω) H (jω) :

(i) Constant K

(ii) Zeros at origin $(j\omega)^{+n}$

(iii) Poles at the origin $(j\omega)^{-n}$

(iv) Simple zero on real axis $(1 + j\omega T)$

(v) Simple pole on real axis $\frac{1}{(1+j\omega T)}$

(vi) Complex conjugate pole $\frac{1}{(1+j2\xi u-u^2)}$

(vii) Complex conjugate zero $(1 + j2\xi u - u^2)$

Procedure for plotting Bode plot:

Step 1: Rewrite the open loop transfer function in the time constant form as given in equation 4.

Step 2: Identify the corner frequencies associated with each factor of the transfer function.

Step 3: After knowing the corner frequencies, draw the asymptotic magnitude plot. This plot consists of straight line segments with line slope changing at each corner frequency as follows.

(i) + 20 db / decade for a zero and + 20n db/decade for a zero of multiplicity n.

(ii) -20db/decade for a pole and - 20n db/decade for a pole of multiplicity n.

(iii) + 40db/decade for a complex conjugate zero and + 40n db/decade for a complex Conjugate zero of multiplicity n.

(iv)-40db/decade for a complex conjugate pole and - 40n db/decade for a complex Conjugate pole of multiplicity n.

Step 3: Initial slope of Bode plot are calculated as follows.

(i) For type zero system draw a line up to first (lowest) corner frequency having 0 db/decade slope.

(ii) For type one system dra w a line having slope of -20db/decade up to ω=K. Mark first (lowest) corner frequency.

(ii) For type two system draw a line having slope of -40db/decade up to ω=√K and so on. Mark first (lowest) corner frequency.

Step 4: Draw a line up to second corner frequency by adding the slope of next pole or zero to the previous slope and so on.

Step 5: Calculate phase angle for different values of ω from the equation 9 and join all points.

Note − The corner frequency (ω=1/K) is the frequency at which there is a change in the slope of the magnitude plot.

Example 1: Draw the bode plot for unity feedback control system having G(s)= $\frac{1000}{(s+100)}$.

Solution:

Step1: Open-loop transfer function in time constant form is given by

$$G(s)H(s)= \frac{1000}{(s+100)}$$

$$= \frac{1000}{100\frac{(s+100)}{100}} = \frac{10}{(1+0.01s)} \text{ (Time constant form)}$$

Put s= jω $$G(j\omega)H(j\omega)= \frac{10}{(1+j0.01\omega)}$$

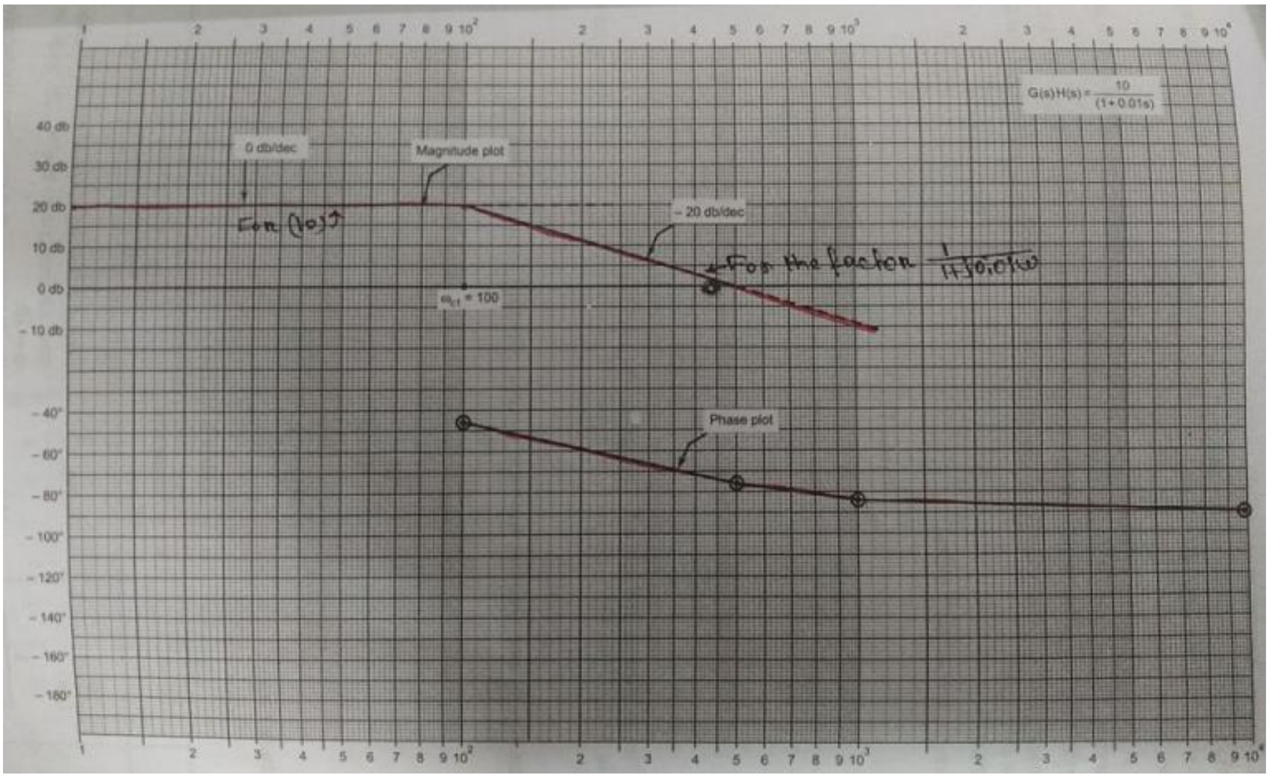
Step 2: Corner frequency ω = 1/0.001 = 100

Step 3: There is one pole on the real axis hence magnitude plot is a straight line having slope of -20 db/decade.

Step 4: As the system is type zero system so magnitude plot is a straight line parallel to o db axis and having magnitude $20\log_{10}K= 20\log_{10}10= 20$db.

Step 5: phase angle $\phi = -\tan^{-1} 0.01\omega$. The table shows value of ϕ when ω varies from 0 to ∞.

S. No.	ω	φ
1.	0	0°
2.	10	– 5.7°
3.	100	– 45°
4.	500	– 78.7°
5.	1000	– 84.29°
6.	10,000	– 90°



Example 2: Draw the bode plot for unity feedback control system having

G(s) = 5(s+2)/s(s+10)

Put s= jω

$$G(j\omega) = \frac{5(j\omega+2)}{j\omega(j\omega+10)} = \frac{5\times 2\left(1+\frac{j\omega}{2}\right)}{10j\omega\left(1+\frac{j\omega}{10}\right)} = \frac{1+\frac{j\omega}{2}}{j\omega\left(1+\frac{j\omega}{10}\right)}$$

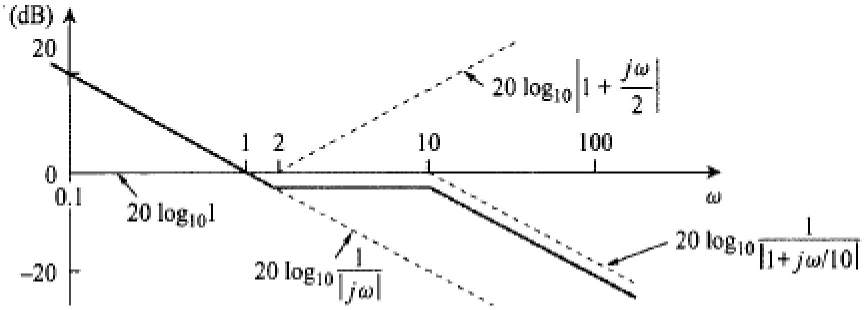
$$|G(j\omega)|\angle G(j\omega) = \frac{\left|1+\frac{j\omega}{2}\right|}{|j\omega|\left|1+\frac{j\omega}{10}\right|}\angle\left[90^\circ+\tan^{-1}\left(\frac{\omega}{2}\right)-\tan^{-1}\left(\frac{\omega}{10}\right)\right]$$

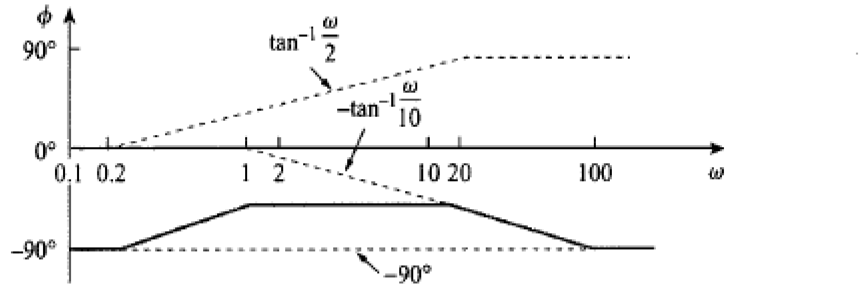
Magnitude plot:

Sl. No	Factor	Corner Frequency	Slope	Asymptotic log magnitude
1	$\frac{1}{j\omega}$	None	-20 db/decade	Straight line of slope -20 db/decade and intersecting the 0 dB axis at ω=K=1and extend upto first corner frequency 2.
2	1+0.5 $j\omega$	2	+20 db/decade	Draw a net slope (-20) + (+20)= 0 db/decade from corner frequency 2 to the next corner frequency 10.
3	$\frac{1}{1+0.1\,j\omega}$	10	-20 db/decade	Draw the net slope of 0+(-20)= -20 db/decade from corner frequency 10 to ∞.

Note: Arrange the table in increasing order of corner frequency.

For different value of ω calculate phase angle ∠G(jω) and join all the points by free hand.





Computation of Gain Margin and Phase Margin

From the Bode plots, we can say whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

Phase Cross over Frequency

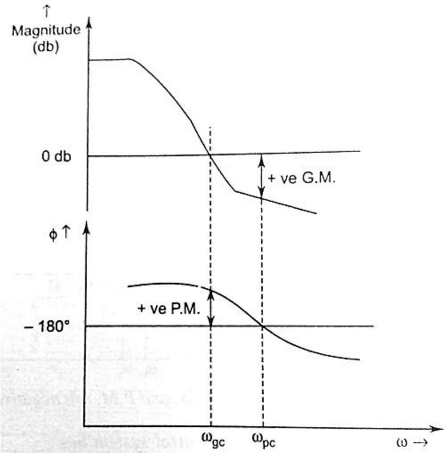
The frequency at which the phase plot is having the phase of -180^0 is known as **phase cross over frequency**. It is denoted by ω_{pc}. The unit of phase cross over frequency is **rad/sec**.

Gain Cross over Frequency

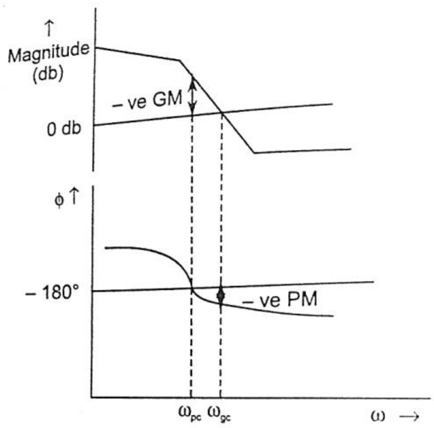
The frequency at which the magnitude plot is having the magnitude of zero dB is known as **gain cross over frequency**. It is denoted by ω_{gc}. The unit of gain cross over frequency is **rad/sec**.

The stability of the control system based on the relation between the phase cross over frequency and the gain cross over frequency is listed below.

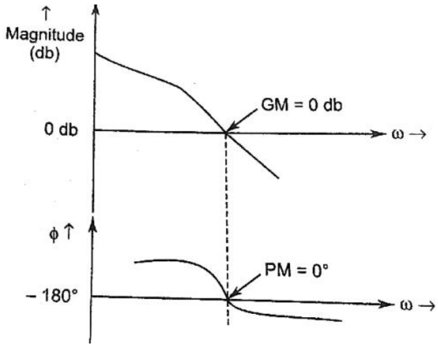
- If the phase cross over frequency ω_{pc} is greater than the gain cross over frequency ω_{gc}, then the control system is **stable**.
- If the phase cross over frequency ω_{pc} is equal to the gain cross over frequency ω_{gc}, then the control system is **marginally stable**.
- If the phase cross over frequency ω_{pc} is less than the gain cross over frequency ω_{gc}, then the control system is **unstable**.



$\omega_{pc} > \omega_{gc}$, GM & PM are +ve
Stable System



$\omega_{pc} < \omega_{gc}$, GM & PM are –ve
Un-stable System



$\omega_{pc} = \omega_{gc}$, GM= PM=0
marginally stable system

Gain Margin

Gain margin GM is defined as the margin in gain allowable by which gain can be increased till system reaches on the verge of instability. It is equal to negative of the magnitude in dB at phase cross over frequency. Mathematically

$$GM=20\log_{10}\left(\frac{1}{|G(j\omega)|_{\omega=\omega_{pc}}}\right)= - 20\log_{10} |G(j\omega)|_{\omega=\omega_{pc}}$$

The unit of gain margin (GM) is dB.

Phase Margin

Phase margin can be defined as the amount of additional phase lag which can be introduced in the system till the system reaches on the verge of instability. The formula for phase margin PM is

$$PM=\left[\angle G(j\omega)|_{\omega=\omega_{gc}}\right] - (-180^{o})$$

$$= 180^{o} + \left[\angle G(j\omega)|_{\omega=\omega_{gc}}\right]$$

The unit of phase margin is degrees.

The stability of the control system based on the relation between gain margin and phase margin is listed below.

- If both the gain margin GM and the phase margin PM are positive, then the control system is **stable**.
- If both the gain margin GM and the phase margin PM are equal to zero, then the control system is **marginally stable**.
- If the gain margin GM and / or the phase margin PM are/is negative, then the control system is **unstable**.

Example3: A unity feedback control system has

$$G(s)= \frac{20}{s(1+0.1s)(1+0.01s)}$$

Draw the bode plot. Find Gain crossover frequency, phase crossover frequency, gain margin and phase margin.

Solution: Put s= jω in open loop transfer function

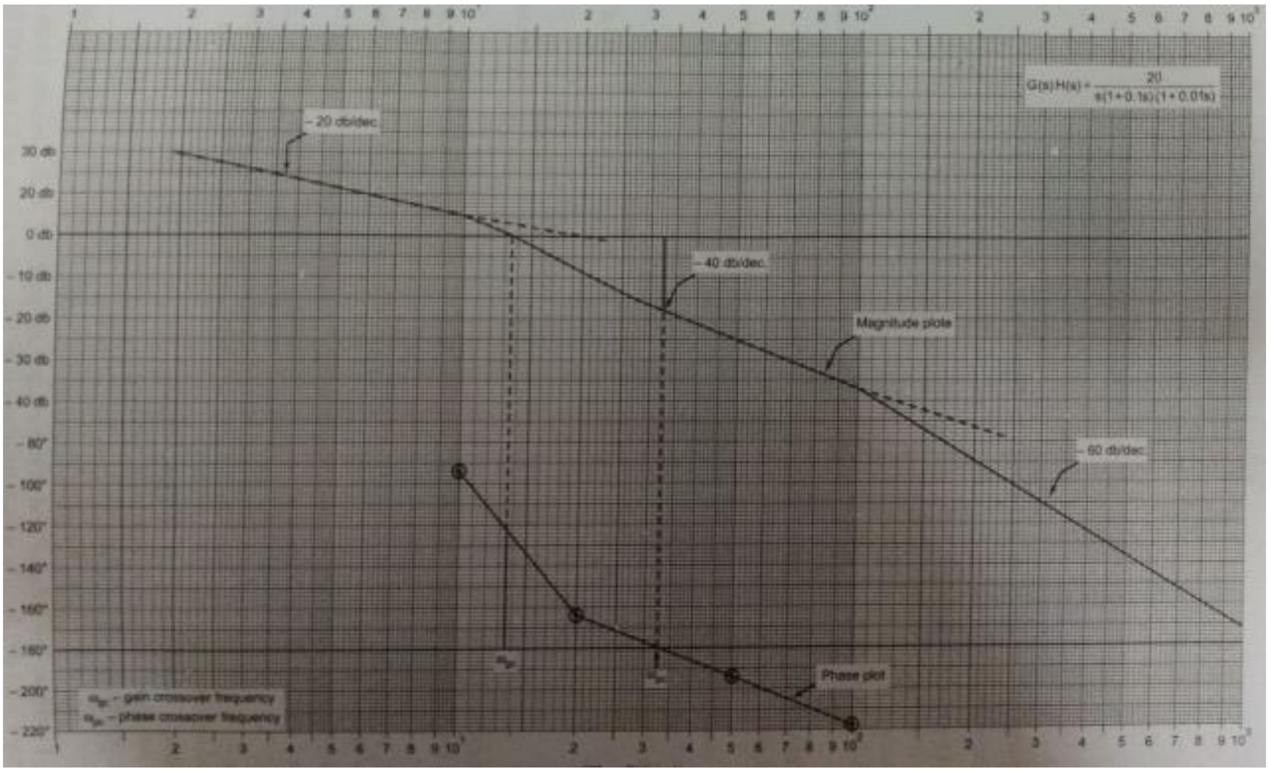
$$G(s)= \frac{20}{j\omega(1+0.1j\omega)(1+0.01j\omega)}$$

$$|G(j\omega)|\angle G(j\omega)= \frac{20}{-\omega^2\sqrt{1+(0.1\omega)^2}\sqrt{1+(0.01\omega)^2}}\angle -90^\circ - tan^{-1}0.1\omega - tan^{-1}0.01\omega$$

Sl. No	Factor	Corner Frequency	Slope	Asymptotic log magnitude
1	$\frac{20}{j\omega}$	None	-20 db/decade	Straight line of slope -20 db/decade and intersecting the 0 dB axis at ω=K=20 and extend upto first corner frequency 10.
2	$\frac{1}{1+0.1\,j\omega}$	10	20 db/decade	Draw a net slope (-20) + (-20)= -40 db/decade from corner frequency 10 to the next corner frequency 100.
3	$\frac{1}{1+0.01\,j\omega}$	100	-20 db/decade	Draw the net slope of (- 40)+(-20)= -60 db/decade from corner frequency 100 to ∞.

The table shown below shows phase angle for the different value of ω.

S.No.	ω	ϕ
1.	0	– 90°
2.	10	– 95.71°
3.	20	– 164.74°
4.	50	– 195.25°
5.	100	– 219.3°



From the plots

1. Gain crossover frequency $\boldsymbol{\omega_{gc}}$ = 13.5
2. Phase crossover frequency $\boldsymbol{\omega_{pc}}$ = 33
3. $\omega_{pc} > \omega_{gc}$, GM & PM are +ve, hence the system become stable.
4. Gain Margin = + 15 db
5. Phase Margin= 180^o-($+124^o$)= $+56^o$

All pass and minimum phase system

If all the poles and zeros of any transfer function lie o the left half of s-plane, such type of transfer function is known as ***minimum phase transfer function.***

The transfer function having a pole-zero pattern which is antisymmetric about the imaginary axis i.e for every pole in the left half plane, there is a zero in the mirror image position. This type of transfer function is known as ***all pass transfer function.***

A common example of such transfer function is

$$G(j\omega)= \frac{1- j\omega T}{1+ j\omega T} \tag{1}$$

Pole zero configuration of equation 1is shown below:

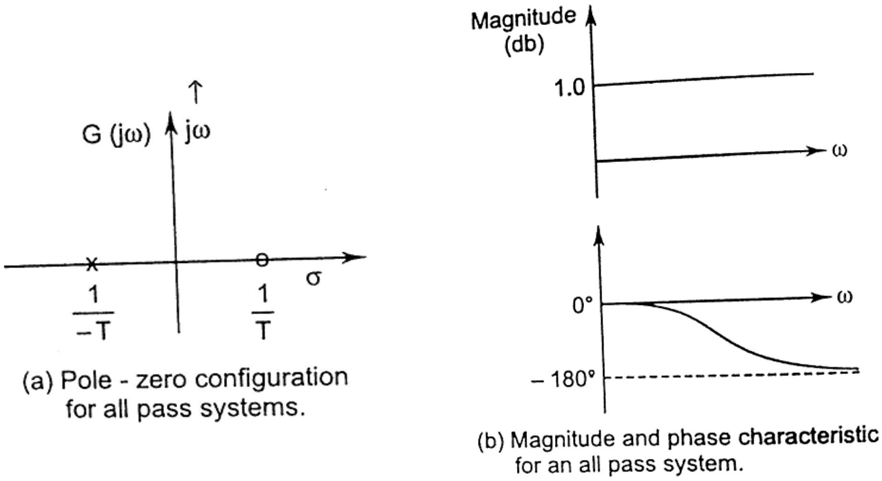


Figure 1. All pass system

All pass transfer function has a magnitude of unity at all frequency and a phase angle of (- 2 $\tan^{-1}\omega T$)which varies from 0^o to -180^o as ω increases from 0 to ∞. *The property of unit magnitude at all frequencies applies to all transfer function with antisymmetric pole-zero pattern.* Physical systems with this property are called all-pass system.

Now consider the case where the transfer function has poles in the left half s-plane and zero in both left and right half s-plane. Poles are not permitted to lie in the right half s-plane because such a system would be unstable. Consider the following transfer function

$$G_1(j\omega)= \frac{1-j\omega T}{(1+j\omega T_1)(1+j\omega T_2)} \quad (2)$$

Whose pole zero pattern is shown in figure . This transfer function may be rewritten as

$$G_1(j\omega)= \left[\frac{1+j\omega T}{(1+j\omega T_1)(1+j\omega T_2)}\right]\left[\frac{1-j\omega T}{1+j\omega T}\right] = G_2(j\omega)\, G(j\omega) \quad (3)$$

Which is now become the product of two transfer function $G_2(j\omega)$ i.e minimum phase transfer function shown in figure (2b) and $G(j\omega)$ i.e all pass transfer function shown in figure (2c). It is clear that $G_1(j\omega)$ and $G_2(j\omega)$ have identical curve of magnitude Vs frequency but their phase Vs frequency curve are different as shown in figure(3). $G_2(j\omega)$ having a smaller range of phase angle than $G_1(j\omega)$. A transfer function which has one or more zeros and no pole in the right half s-plane is known as non- minimum phase transfer function.

In general if the transfer function has any zeros in the right half s-plane, it is possible to extract them one by one by associating them with all-pass transfer function as shown in figure(2a).

A common example of a non-minimum phase element is transportation lag which has transfer function

$$G(j\omega)= e^{-j\omega T}= 1\angle -\omega T \text{ rad} = 1\angle -57.3\omega T \text{ degree}$$

Other possible non-minimum transfer function are:

1. where more than one possible signal paths are available between input and output as in lattice network.
2. When there is inductive coupling between input and output in addition to conduction.

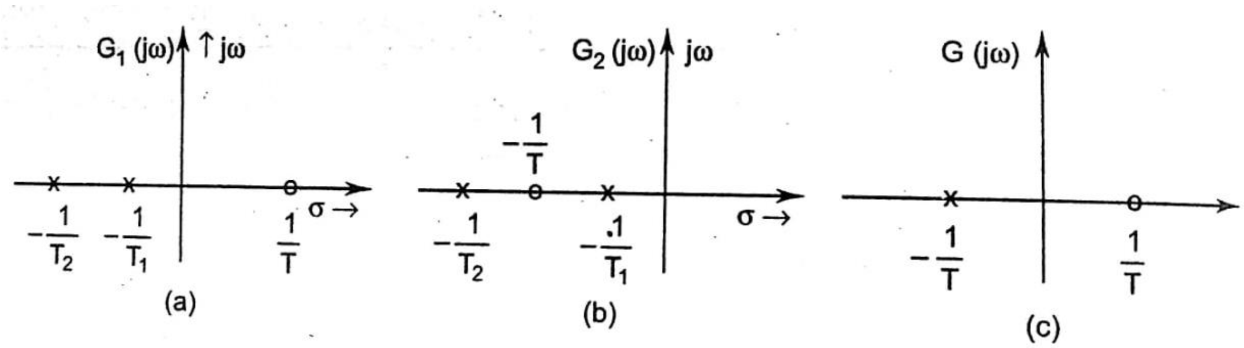


Figure 2

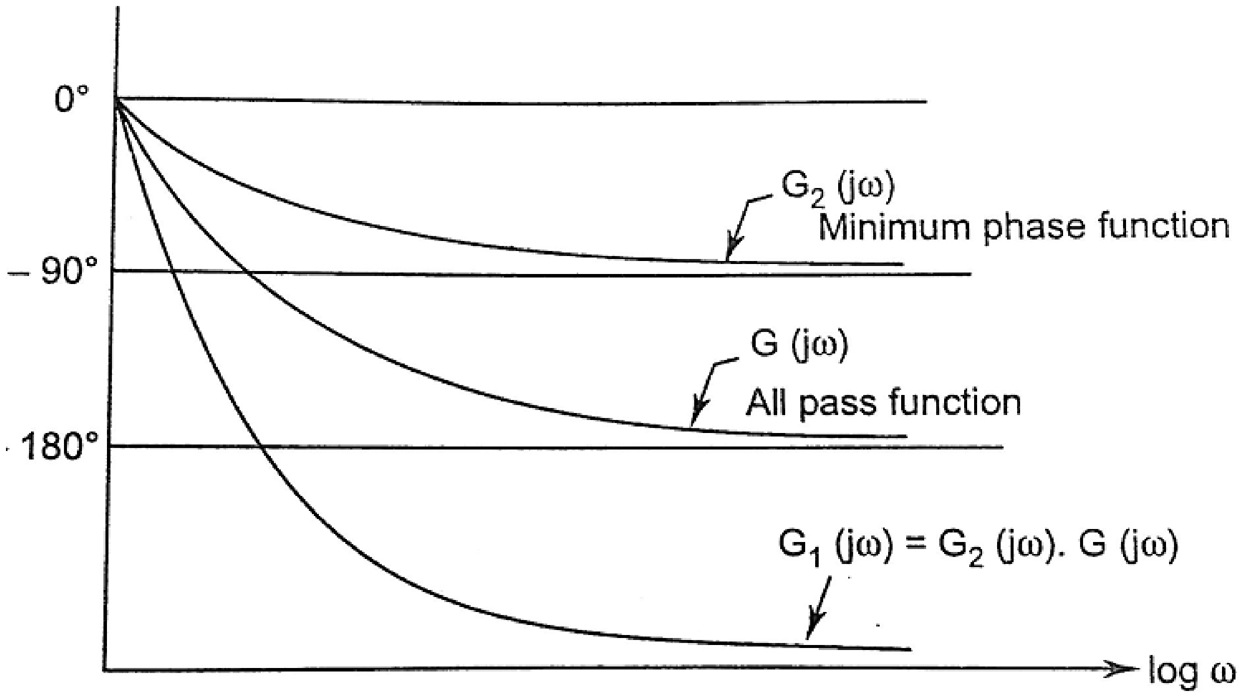


Figure 3 Phase Vs frequency graph

Polar Plots

Polar plot is a plot which can be drawn between magnitude and phase. It is a plot of magnitude $|G(j\omega)|$ versus phase angle$\angle G(j\omega)$on polar co-ordinates as input frequency (ω) is varied from 0 to ∞. Here, the magnitudes are represented by normal values only.

The polar form of G(jω) is

$$G(j\omega)= |G(j\omega)| \angle G(j\omega)$$

Rules for Drawing Polar Plots

Follow these rules for plotting the polar plots.

Step1. Substitute, s=jω in the open loop transfer function.
Step2. Write the expressions for magnitude and the phase of G(jω)

Step3. Find the starting magnitude and the phase of G(jω) by substituting ω=0. So, the polar plot starts with this magnitude and the phase angle.
Step4. Find the ending magnitude and the phase of G(jω) by substituting ω=∞. So, the polar plot ends with this magnitude and the phase angle.
Step5. Check whether the polar plot intersects the real axis, by making the imaginary term of G(jω) equal to zero and find the value(s) of ω.
Step6. Determine the intersection of polar plot with real axis and imaginary axis, as follows:

i. Rationalise the function G(jω) and separate the real and imaginary parts.

ii. Intersection with imaginary axis: equate the real term of |G(jω)| to zero and find the value of frequency (ω) at which the polar plot intersects the imaginary axis. Now put this value of ω into |G(jω)|. Which gives |G(jω)| at this point of intersection.

iii. Intersection with real axis: equate the imaginary term of |G(jω)| to zero and find the value of frequency (ω) at which the polar plot intersects the real axis. Now put this value of ω into |G(jω)|. Which gives |G(jω)| at this point of intersection.

Step7. By using this information, plot the points on the complex plane. Make the arrow on the plot for increasing frequency from 0 to ∞.

Example1: Consider the open loop transfer function of a closed loop control system.

$$G(s)=\frac{1}{(1+sT_1)(1+sT_2)}$$

Draw the polar plot.

Step 1 − Substitute, s=jω in the open loop transfer function.

$$G(j\omega)=\frac{1}{(1+j\omega T_1)(1+j\omega T_2)}$$

The magnitude of the open loop transfer function is

$$|G(j\omega)|=\frac{1}{\sqrt{1+(\omega T_1)^2}\sqrt{1+(\omega T_2)^2}}$$

The phase angle of the open loop transfer function is

$$\angle G(j\omega)=-\tan^{-1}\omega T_1-\tan^{-1}\omega T_2$$

Step 2 − The following table shows the magnitude and the phase angle of the open loop transfer function at ω=0 rad/sec and ω=∞ rad/sec.

Frequency (rad/sec)	Magnitude	Phase angle(degrees)
0	1	0
∞	0	-180°

So, the polar plot starts at $(1,0^0)$ and ends at $(0,-180^0)$. The first and the second terms within the brackets indicate the magnitude and phase angle respectively.

Step 3 − This polar plot will intersect the negative imaginary axis. The phase angle corresponding to the negative imaginary axis is -90^0 or 270^0. So, by equating the phase angle of the open loop transfer function to either -90^0 or 270^0, we will get the ω value as

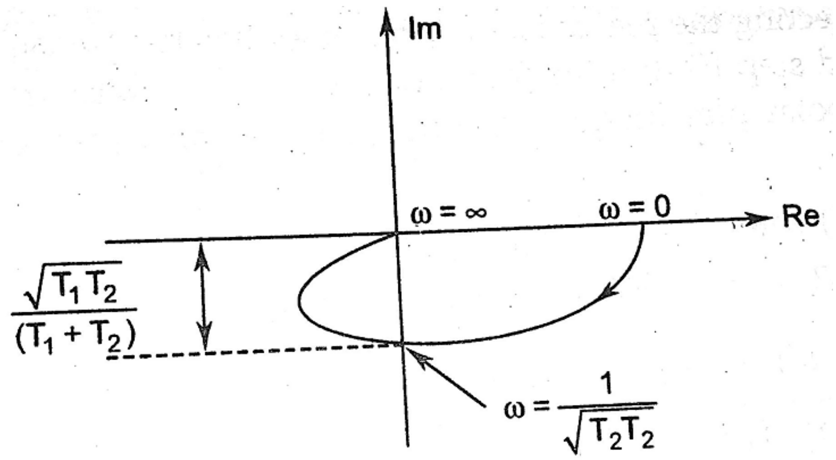
$$\angle G(j\omega) = -\tan^{-1} \omega T_1 - \tan^{-1} \omega T_2 = -90^0$$

$$\Rightarrow \frac{\omega T_1 + \omega T_2}{1-\omega^2 T_1 T_2} = \infty \Rightarrow \omega = \frac{1}{\sqrt{T_1 T_2}}$$

By substituting $\omega = \frac{1}{\sqrt{T_1 T_2}}$ in the magnitude of the open loop transfer function, we will get

$$|G(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{1}{\sqrt{T_1 T_2}} T_1\right)^2}\sqrt{1+\left(\frac{1}{\sqrt{T_1 T_2}} T_2\right)^2}} = \frac{\sqrt{T_1 T_2}}{T_1 + T_2}$$

So, we can draw the polar plot with the above information on the polar graph sheet.



The following table shows polat plot for different type of control system: