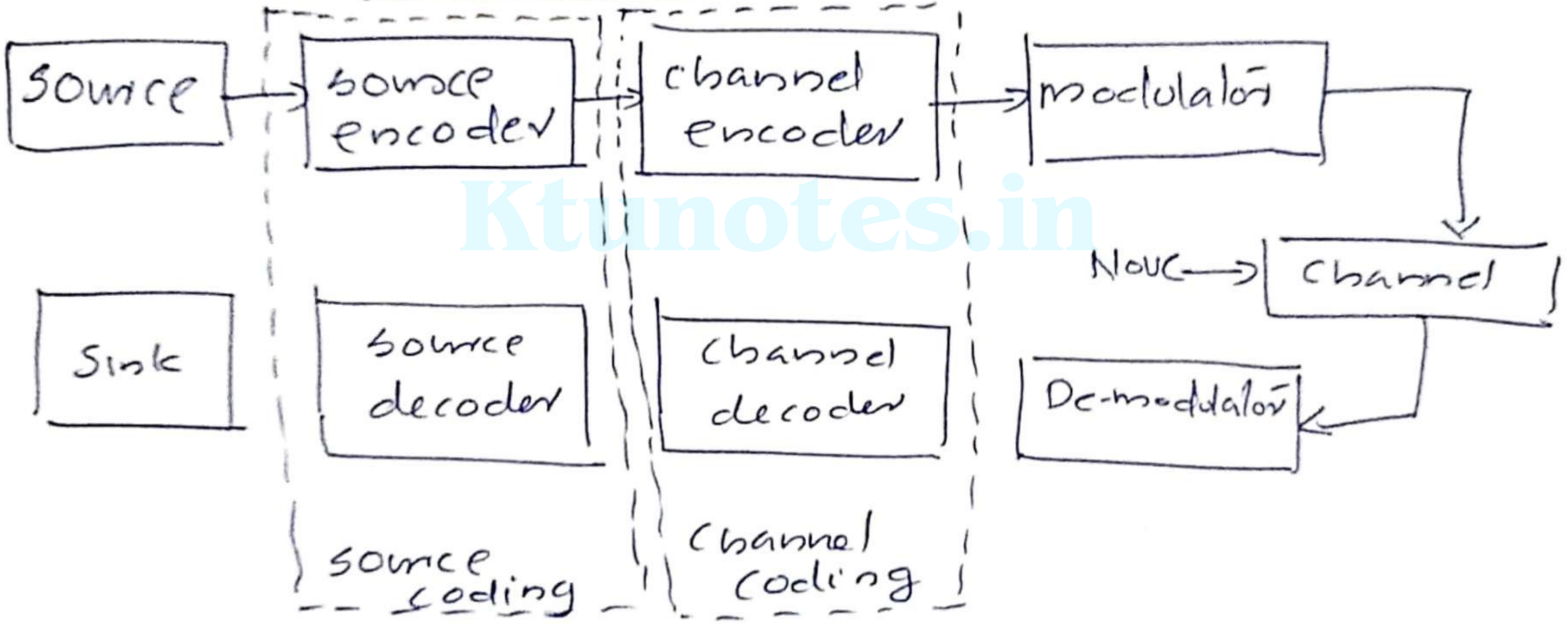
Communication System



Source :-
- analog (microphone, TV camera, scanner etc)
- Digital (computer)

Source encoding :- converts source outputs to bits
source i/p can be voice, video
text, sensor output etc.

Channel coding :- adds extra bits (redundancy) to data which is to be transmitted over the channel

This redundancy helps reduce the errors which can be introduced in transmitted bits due to channel noise

Channel :- provides connection between the source and the destination. It may be a pair of wire, cable, or free space etc. Communication channel have only finite bandwidth. Information bearing signal suffers amplitude and phase distortion as it travels over the channel. Due to attenuation of the channel, signal power also reduces. and signal is affected by noise in the channel.

1.2 Concept and Measure of Information

Consider an information source produces independent sequence of symbol $S = \{S_1, S_2 \ldots S_q\}$ with probability $P = \{P_1, P_2 \ldots P_q\}$ respectively.

Let S_K: symbol chosen for transmission at any instant of time with probability P_K

Self information (Amount of information) of message S_K,

$$I_K = \log \frac{1}{P_K}$$

Ex: Sun rises in the east on every day in Kerala.

Probability of Sun rising in East ;

$$P_K = 1$$

So $I_K = 0$

So this message contains absolutely no information

Ex: There will be snowfall on a particular day in Kollam.

Since the occurrence of snowfall in Kollam is almost an impossible event. The probability of snowfall in Kollam is very very small. So information content is very very large.

Q. The binary symbols '0' and '1' are transmitted with probabilities 1/4 and 3/4 respectively. Find corresponding self information.

Self information in a '0', $I_0 = \log_2 \frac{1}{P_0}$

$$= \log_2 \frac{1}{1/4}$$

$$= \log_2 4$$

$$= 2 \text{ bits}$$

Self information in a '1', $I_1 = \log_2 1/P_1$

$$= \log_2 4/3$$

$$= 0.415 \text{ bits.}$$

Note: 1

If base of logarithm is 2, unit is BITS

$I = \log_2 \frac{1}{p}$ bits (short form of Binary Unit)

If base is 10, unit is HARTLEY or DECITS

$I = \log_{10} \frac{1}{p}$ Hartleys

If base is e, unit is NATS.

$I = \log_e \frac{1}{p}$ nats

Note: 2

Why is logarithmic expression chosen for measuring information.

1. Self information of any message cannot be negative.
2. Each message must contain certain amount of information.
3 Lowest possible self information is zero which occurs for a sure event since probability of sure event is One.
4 More information is carried by a less likely message
5 When independent symbols are transmitted, total self information is equal to the sum of individual self information.

1.3 Units of Information

$$I = \log_{10} \frac{1}{p} \text{ Hartleys}$$

$$I = \log_e \frac{1}{p} \text{ nats}$$

$$I = \log_2 \frac{1}{p} \text{ bits}$$

Relationship between Hartley & nats

$$1 \text{ Hartley} = \frac{I}{\log_{10}\frac{1}{p}} = \frac{\log_e \frac{1}{p}}{\log_{10}\frac{1}{p}}$$

$$= \frac{-\log_e p}{-\log_{10} p} \text{ nats} \quad \left[\because \log \frac{a}{b} = -\log \frac{b}{a}\right]$$

$$= \frac{\log_e p}{\log_{10} p} \text{ nats}$$

$$= \frac{\log p}{\log e} \Big/ \frac{\log p}{\log 10} \quad \left[\because \log_n m = \frac{\log m}{\log n}\right]$$

$$= \frac{\log p}{\log e} \times \frac{\log 10}{\log p} \text{ nats}$$

$$= \log_e 10 \text{ nats}$$

$$1 \text{ Hartley} = 2.303 \text{ nats}$$

Relationship between Hartley and ~~nat~~ bits

$$1 \text{ Hartley} = \frac{I}{\log_{10} \frac{1}{p}} \text{ bits}$$

$$= \frac{\log_2 \frac{1}{p}}{\log_{10} \frac{1}{p}} \text{ bits}$$

$$= \log_2 10 \text{ bits}$$

$$1 \text{ Hartley} = 3.32 \text{ bits}$$

Relationship between nats and bits

$$1 \text{ nat} = \frac{I}{\log_e \frac{1}{p}} \text{ bits}$$

$$= \frac{\log_2 \frac{1}{p}}{\log_e \frac{1}{p}} \text{ bits}$$

$$= \log_2 e \text{ bits}$$

$$= \frac{1}{\log_e 2} \text{ bits}$$

$$= \frac{1}{\ln 2} \text{ bits}$$

$$1 \text{ nat} = 1.443 \text{ bits}$$

1.4 ENTROPY (Average Information Content)

Consider source alphabet $S = \{ s_1, s_2 \ldots s_q \}$ with probabilities $P = \{ P_1, P_2 \ldots P_q \}$ respectively

Consider a long independent sequence of length L symbols. This long sequence contains:

$P_1 L$ number of messages of type s_1

$P_2 L$ " " " s_2

$\vdots$

$P_q L$ " " " s_q

We know, self information of $S_1 = \log \frac{1}{P_1}$ bits

$P_1 L$ number of messages of type S_1 Contains

$P_1 L \log \frac{1}{P_1}$ bits of information

$P_2 L$ number of messages of type S_2 contains

$$P_2 L \log \frac{1}{P_2} \text{ bits of information}$$

⋮

$P_q L$ number of messages of type S_q contains

$$P_q L \log \frac{1}{P_q} \text{ bits of information}$$

Total self information $I_{tot} = P_1 L \log \frac{1}{P_1} +$

$$P_2 L \log \frac{1}{P_2} + \ldots + P_q L \log \frac{1}{P_q} \text{ bits}$$

$$= L \sum_{i=1}^{q} P_i \log \frac{1}{P_i} \text{ bits}$$

Average self information $= \frac{I_{tot}}{L}$

$$H(s) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} \text{ bits/message symbol}$$

Q Consider a source alphabet $S = \{S_1, S_2\}$ with probabilities $P = \left\{\frac{1}{256}, \frac{255}{256}\right\}$. Find entropy

$$H(s) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i}$$

$$H(s) = \sum_{i=1}^{2} p_i \log \frac{1}{p_i}$$

$$= \frac{1}{256} \log 256 + \frac{255}{256} \log \frac{256}{255}$$

$$H(s) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i}$$

$$= \frac{1}{256} \log 256 + \frac{255}{256} \log \frac{256}{255}$$

$$= 0.037 \text{ bits/message symbol}$$

Note: The average information is very very small and it is very easy to guess whether S_1 or S_2 will occur.

Q. Let $S = \{S_1, S_2\}$ with $P = \{\frac{7}{16}, \frac{9}{16}\}$ Find entropy.

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$$H(S) = \frac{7}{16}\log\frac{16}{7} + \frac{9}{16}\log\frac{16}{9}$$

$$= 0.989 \text{ bits/message symbol}$$

Q. Let $S = \{S_1, S_2\}$ with $P = \{1/2, 1/2\}$. Find entropy.

Q. Let $S = \{S_1, S_2\}$ with $P = \{1/2, 1/2\}$. Find entropy.

$H(S) = \frac{1}{2}\log 2 + \frac{1}{2}\log 2 = 1$ bit/message symbol

Note:

Here it is impossible to guess which symbol is transmitted because uncertainity is maximum.

1.5 Information Rate

Average source of information Rate R_s is defined as the product of the average information content per symbol and message symbol rate r_s.

$$R_s = r_s \cdot H(s) \quad \text{bits/sec}$$

Q. Consider a source $S = \{S_1, S_2, S_3\}$ with $P = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\}$

find (a) self information of each message

(b) entropy of source S.

(a) Self information of S_1, $I_1 = \log \frac{1}{P_1} = \log 2 = 1$ bits

" S_2, $I_2 = \log \frac{1}{P_2} = \log 4 = 2$ bits

" S_3, $I_3 = \log \frac{1}{P_3} = \log 4 = 2$ bits

(b) Entropy $H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i}$

Here $q = 3$

$$H(S) = \sum_{i=1}^{3} P_i I_i \qquad \left[I = \log \frac{1}{P} \right]$$

$$= \frac{1}{2} \times 1 + \frac{1}{4} \times 2 + \frac{1}{4} \times 2$$

$= 1.5$ bits / message symbol.

Q. The collector voltage of a certain circuit is to lie between −5 and −12 volts. The voltage reading with respective probability is shown below

$V(i)$	V_1	V_2	V_3	V_4	V_5	V_6
$P(i)$	$1/6$	$1/3$	$1/12$	$1/12$	$1/6$	$1/6$

$$H(S) = \sum_{i=1}^{6} P_i \log \frac{1}{P_i} \text{ bits/level}$$

$$= \frac{1}{6}\log 6 + \frac{1}{3}\log 3 + \frac{1}{12}\log 12 + \frac{1}{12}\log 12$$

$$+ \frac{1}{6}\log 6 + \frac{1}{6}\log 6$$

$$= 2.418 \text{ bits/level.}$$

Q. A discrete source emits one of six symbols once every m-sec. The symbol probabilities are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$ and $\frac{1}{32}$ respectively.

Find source entropy and information rate.

$$H(s) = \sum_{i=1}^{6} p_i \log \frac{1}{p_i}$$

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{16}\log 16$$

$$+ \frac{1}{32}\log 32 + \frac{1}{32}\log 32$$

$$= \frac{1}{2} + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 +$$

$$\frac{1}{32}\times 5 + \frac{1}{32}\times 5$$

$= 1.9375$ bits / message symbol.

Information rate $R_s = r_s H(s)$

Given $r_s = 1$ message symbol / m.sec

$= 10^3$ message symbol / sec

$R_s = (10^3$ message symbol / sec$)$ $(1.9375$ bits / message symbol$)$

$= 1937.5$ bits/sec

Q. A source emits one of 4 possible symbols x_0 to x_3 during each signalling interval. The symbol occur with probabilities as shown below

Symbol	x_0	x_1	x_2	x_3
Probability	$P_0 = 0.4$	$P_1 = 0.3$	$P_2 = 0.2$	$P_3 = 0.1$

Find the amount of information gained by observing the source emitting each of these symbols and also the entropy of the source.

Self information $I_k = \log_2 \frac{1}{P_k}$ bits

$$I_{x_0} = \log \frac{1}{P_0} = \log \frac{1}{0.4} = 1.322 \text{ bits}$$

$$I_{x_1} = \log \frac{1}{P_1} = \log \frac{1}{0.3} = 1.737 \text{ bits}$$

$I_{x_2} = \log \frac{1}{P_2} = \log \frac{1}{0.2} = 2.322$ bits

$I_{x_3} = \log \frac{1}{P_3} = \log \frac{1}{0.1} = 3.322$ bits

Entropy $H(x) = \sum_{i=0}^{3} P_i \log \frac{1}{P_i}$ bits / message symbol

$$= P_0 \log \frac{1}{P_0} + P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2} + P_3 \log \frac{1}{P_3}$$

$$= (0.4)(1.322) + (0.3)(1.737) + (0.2)(2.322) + (0.1)(3.322)$$

$= 1.8465$ bits / message symbol.

Q. If x represents the outcome of a single roll of a fair die. What is the entropy of x?

We know, a die is having six faces. So probability of getting any face is 1/6.

$$P_x = 1/6$$

Q. A binary source is emitting an independent source of 0's and 1's with probabilities p and 1-p respectively. Plot the entropy of the source versus p.

OR

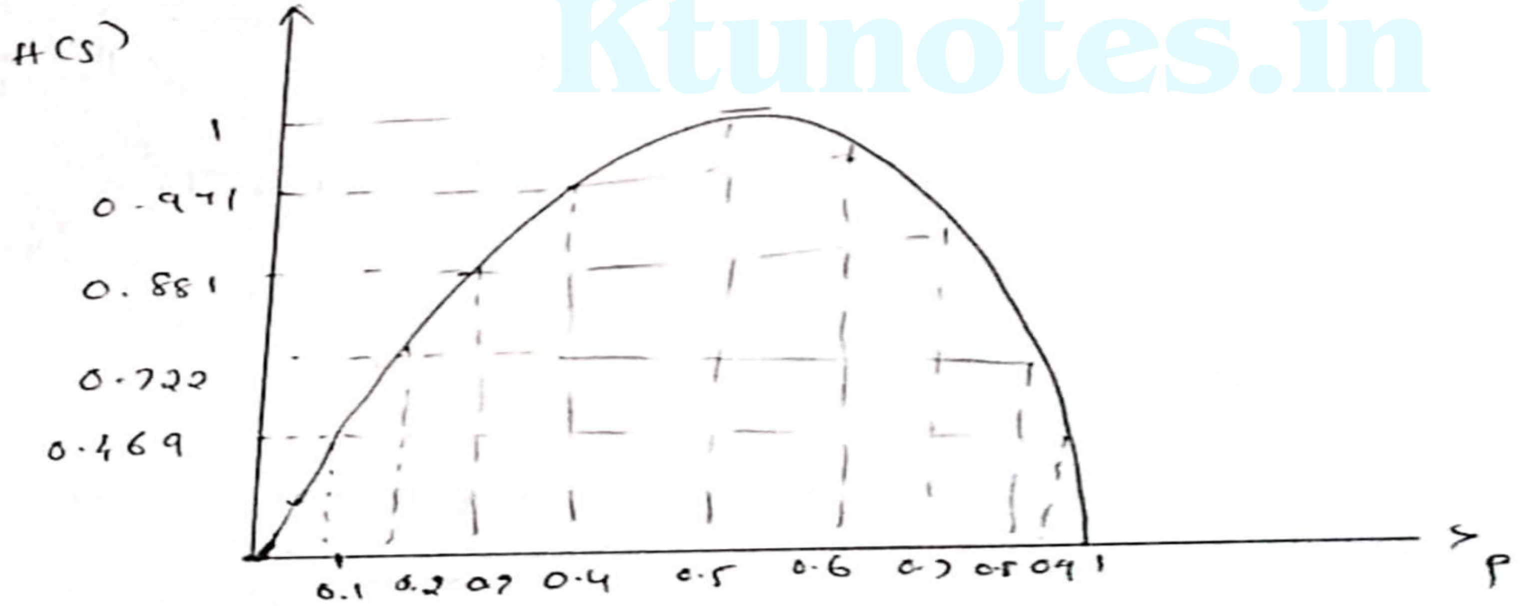
Plot the entropy of binary symmetric channel.

$$H(S) = \sum_{i=1}^{2} P_i \log \frac{1}{P_i}$$

$$= P_1 \log \frac{1}{P_1} + P_2 \log \frac{1}{P_2}$$

$$H(S) = p \log \frac{1}{p} + (1-p) \log \frac{1}{(1-p)}$$

P	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
H(S)	0	0.469	0.722	0.881	0.971	1	0.971	0.881	0.722	0.469	0



Q. A card is drawn from a deck.

(a) How much information did you receive if you are told it is a spade.

(b) Repeat (a) if it is an ace.

(c) Repeat (a) if it is an ace of spade

(d) Verify that information obtain in (c) is the sum of information obtained in (a) and (b)

(a) There are 13 spade cards in a deck of 52 cards

$$P_{spade} = \frac{13}{52} = 0.25$$

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Self information $I_{spade} = \log \frac{1}{P_{spade}} = \log 4 = 2$ bits.

(b) There are 4 Aces in a deck of 52 cards,

$$P_{ace} = 4/52 = 1/13$$

Self information $I_{ace} = \log \frac{1}{P_{ace}}$

$$= \log 13$$

$$= \log 13$$

$$= 3.7 \text{ bits}$$

(c) There is only 1 ace of spade in a deck of 52 cards.

$$P_{\text{ace of spade}} = 1/52$$

Self information $I_{\text{ace of spade}} = \log \dfrac{1}{P_{\text{ace of spade}}}$

$= 5.7$ bits

(h) $I_{ace} + I_{spade} = I_{ace\ of\ spade}$

$3.7 + 2 = 5.7$ bits.

An analog signal is bandlimited to 500Hz and is sampled at Nyquist rate. The samples are quantized into 4 levels. The quantization levels are assumed to be independent and occur with probability $P_1 = P_4 = \frac{1}{8}$, $P_2 = P_3 = 3/8$. Find information rate of the source.

$$H(s) = \sum_{i=1}^{4} P_i \log \frac{1}{P_i}$$

$$= \left(\frac{1}{8} \log 8\right) \times 2 + \left(\frac{3}{8} \log \frac{8}{3}\right) \times 2$$

$$= 1.8113 \text{ bits/level}$$

Signal is sampled at nyquist rate.

Signal is sampled at nyquist rate.

So $r_s = 2 \times BW$

$= 2 \times 500$

$= 1000$ symbols/sec

Information rate $R_s = r_s \cdot H(s)$

$= 1000 \times 1.81$

$= 1810$ bits/sec

Q. A zero memory source has a source alphabet $S = \{s_1, s_2, s_3\}$ with $P = \{1/2, 1/4, 1/4\}$. Find the entropy of this source. Also determine the entropy of its 2^{nd} extension and verify that $H(S^2) = 2H(S)$.

For the basic source with 3 symbols,

$$H(s) = \sum_{i=1}^{3} P_i \log \frac{1}{P_i}$$

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{4}\log 4$$

$= 1.5$ bits / message symbol.

Second extension

$s_1 s_1$ with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$s_1 s_2$ " $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$s_1 s_3$ " $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$s_2 s_1$ " $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

$s_2 s_2$ " $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$s_2 s_3$ " $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$S_3 S_1$ " " $= \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

$S_3 S_2$ " " $= \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$S_3 S_3$ " " $= \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$$H(S^2) = \sum_{i=1}^{9} P_i \log \frac{1}{P_i}$$

$$= \frac{1}{4} \log 4 + 4 \times \frac{1}{8} \log 8 + 4 \times \frac{1}{16} \log 16$$

$= 3$ bits / message symbol

$= 2 \times 1.5$ bits/message symbol

$H(S^2) = 2\, H(S)$

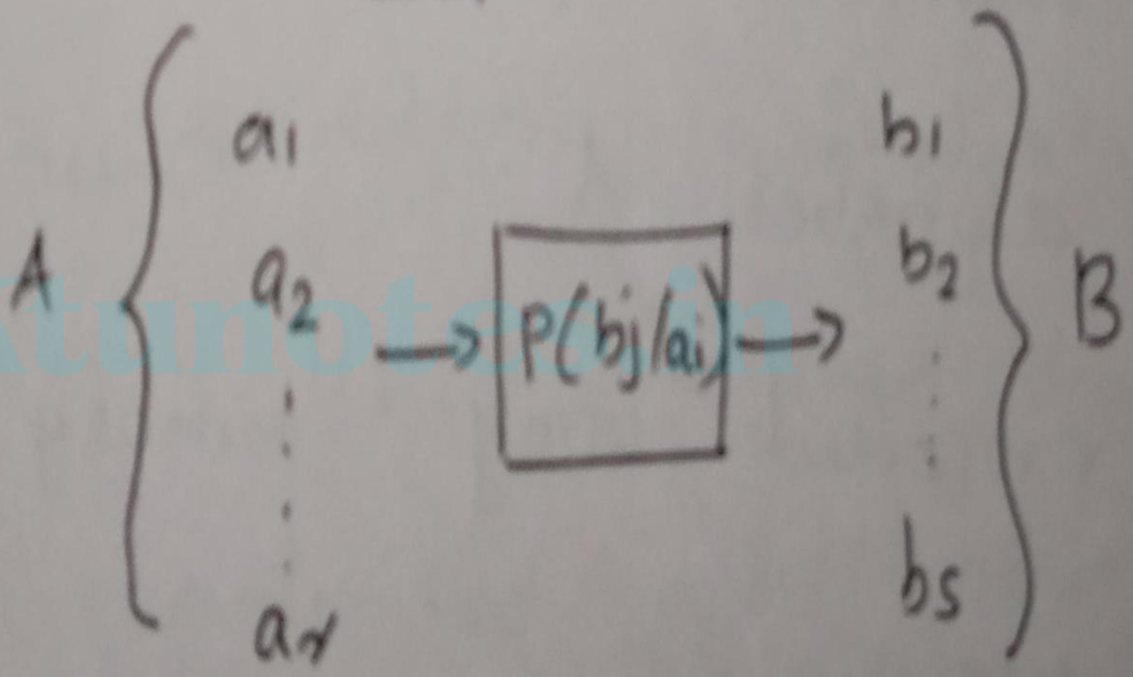
Communication Channel

A channel is defined as the medium through which the coded signals are transmitted. The maximum rate at which data is transferred across the channel with an arbitrarily small probability of error is called channel capacity.

Representation of a channel

Channel may be represented by a set of input alphabet $A = \{a_1, a_2 \cdots a_r\}$ consisting of r symbols, a set of output alphabet $B = \{b_1, b_2 \cdots b_s\}$ consisting of s symbols and a set of conditional probabilities $P(b_j / a_i)$ with $i = 1, 2 \cdots r$ and $j = 1, 2 \cdots s$.

channel representation



Channel Matrix or Noise matrix

Totally we have $r \times s$ number of conditional probabilities, which are represented in a matrix form. This matrix is called channel matrix or Noise Matrix.

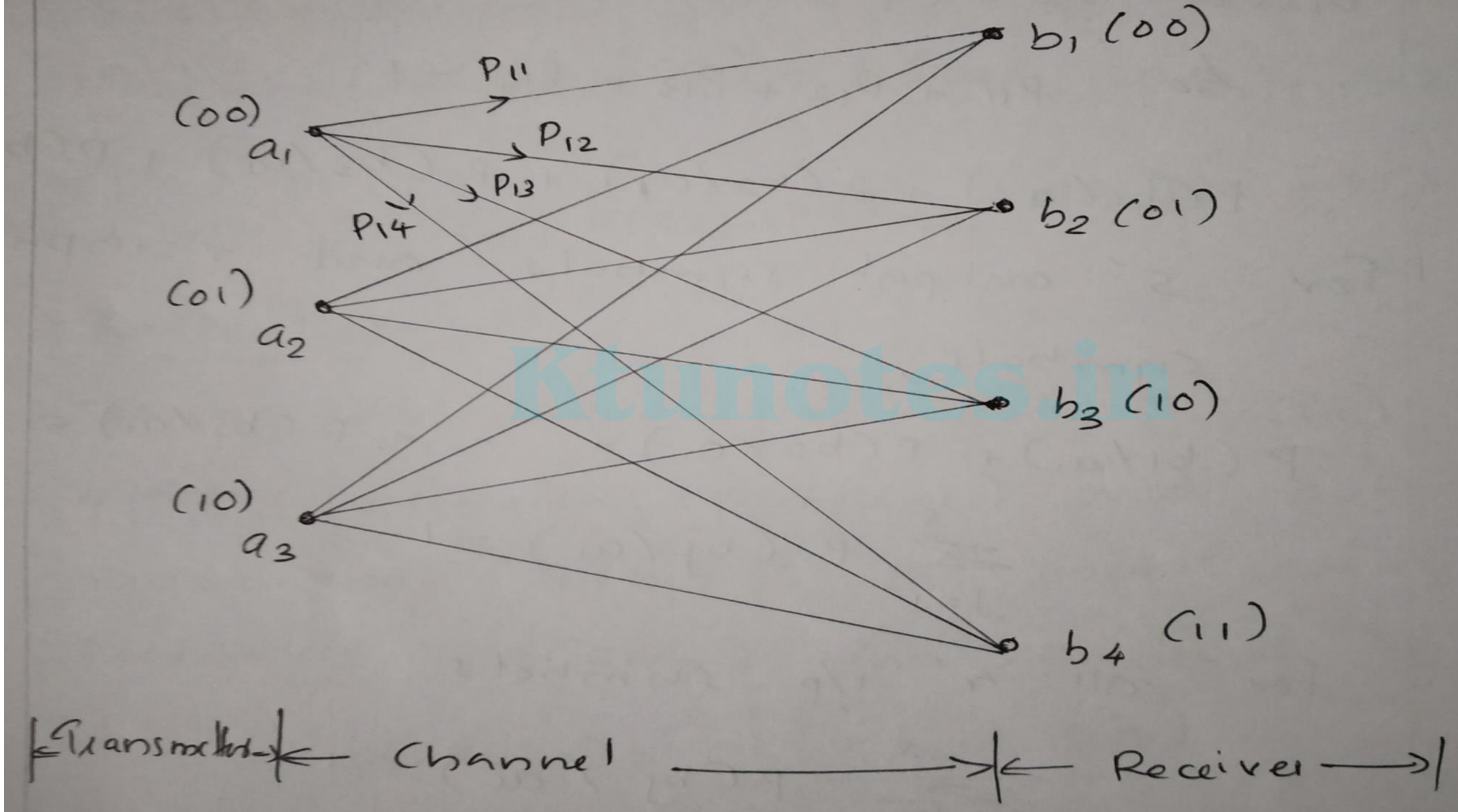
$P(b_j/a_i)$ or $P(B/A)$

$p(b_j/a_i)$ or $p(B/A)$

$$
= \begin{array}{c c}
 & \begin{array}{cccc} b_1 & b_2 & \cdots & b_s \end{array} \\
\begin{array}{c} a_1 \\ a_2 \\ \vdots \\ a_r \end{array} &
\begin{bmatrix}
p(b_1/a_1) & p(b_2/a_2) & \cdots & p(b_s/a_1) \\
p(b_1/a_2) & p(b_2/a_2) & & p(b_s/a_2) \\
\vdots & & & \\
p(b_1/a_r) & p(b_2/a_r) & \cdots & p(b_s/a_r)
\end{bmatrix}
\end{array}
$$

Here all the input symbols are represented row wise and output symbols column wise.

Channel diagram



$P_{11} = P(b_1/a_1)$ = conditional probability of receiving $b_1(00)$ given that $a_1(00)$ is transmitted with no noise.

$P_{12} = p(b_2/a_1)$ = conditional probability of receiving $b_2(01)$ given that $a_1(00)$ is transmitted with noise affecting the ~~f~~ second '0'.

$P_{13} = P(b_3/a_1)$ = conditional probability of receiving b_3 (10) given that a_1 is transmitted with noise affecting the first 0.

$P_{14} = P(b_4/a_1)$ = conditional probability of receiving b_4(11) given that a_1 is transmitted with noise affecting both the symbols

When a_1 (00) is transmitted, we have to receive only one of the four symbols b_1 (00) or b_2 (01) or b_3 (10) or b_4 (11)

So $P_{11} + P_{12} + P_{13} + P_{14} = 1$

$$P(b_1/a_1) + P(b_2/a_1) + P(b_3/a_1) + P(b_4/a_1)$$

For 's' output symbols and r input symbols,

$$P(b_1/a_1) + P(b_2/a_1) + \cdots + P(b_s/a_1) = 1$$

$$\sum_{j=1}^{s} P(b_j/a_1) = 1$$

For all 'r' i/p symbols

$$\sum_{j=1}^{s} P(b_j / a_i) = 1$$

Note:

* The sum of all the elements in any row of the channel matrix is equal to unity.

* The sum of the input probability is equal to unity.

$$P(a_1) + P(a_2) + \cdots + P(a_r) = 1$$

$$\sum_{i=1}^{r} P(a_i) = 1$$

$$p(b_1) = p(b_1/a_1)\cdot p(a_1) + p(b_1/a_2)\cdot p(a_2) + \cdots + p\left(\frac{b_1}{a_r}\right)p(a_r)$$

$$p(b_2) = p(b_2/a_1)\cdot p(a_1) + p(b_2/a_2)\cdot p(a_2) + \cdots + p\left(\frac{b_2}{a_r}\right)p(a_r)$$

$$\vdots \quad = \quad \vdots$$

$$p(b_s) = p(b_s/a_1)\cdot p(a_1) + p(b_s/a_2)\cdot p(a_2) + \cdots + p\left(\frac{b_s}{a_r}\right)\cdot p(a_r)$$

$$p(a_i/b_j) = \frac{p(b_j/a_i) \cdot p(a_i)}{p(b_j)}$$

Joint probability matrix (JPM)

The joint probability between input symbol a_i and output symbol b_j is given by

$$P(a_i, b_j) = P(a_i \cap b_j) = P(b_j / a_i) \cdot P(a_i) = P(a_i / b_j) \cdot P(b_j)$$

Multiply all the elements of the I row of channel matrix by $P(a_1)$, 2^{nd} row by $P(a_2)$ . . . r^{th} row by $P(a_r)$ to get $P(a_i, b_j)$.

$$P(a_i, b_j) = P(b_j / a_i) \cdot P(a_i)$$

$$=$$

$$
= \begin{array}{c|cccc}
 & b_1 & b_2 & \cdots & b_s \\
\hline
a_1 & P(b_1/a_1)\cdot P(a_1) & P(b_2/a_2)\,P(a_1) & \cdots & P(b_s/a_1)\cdot P(a_1) \\
a_2 & P(b_1/a_2)\,P(a_2) & P(b_2/a_2)\,P(a_2) & & P(b_s/a_2)\,P(a_2) \\
\vdots & \vdots & & & \\
a_r & P(b_1/a_r)\,P(a_r) & P(b_2/a_r)\,P(a_r) & & P(b_s/a_r)\,P(a_r)
\end{array}
$$

$$P(a_i, b_j) = P(A, B) = \begin{array}{c|cccc} & b_1 & b_2 & & b_s \\ \hline a_1 & & & & \\ a_2 & & & & \\ \vdots & & & & \\ a_r & & & & \end{array}$$

$$\begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ a_r \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & & b_s \end{array} \\ \begin{bmatrix} p(a_1, b_1) & p(a_1, b_2) & \cdots & p(a_1, b_s) \\ p(a_2, b_1) & p(a_2, b_2) & \cdots & p(a_2, b_s) \\ \vdots & & & \vdots \\ p(a_r, b_1) & p(a_r, b_2) & \cdots & p(a_r, b_s) \end{bmatrix} \end{array}$$

~~Rules~~ Property

1\. We know

$$P(b_1) = P(b_1/a_1) \cdot P(a_1) + P(b_1/a_2) \cdot P(a_2) + \cdots + P(b_1/a_r) \cdot P(a_r)$$

$$P(b_1) = P(a_1, b_1) + P(a_2, b_1) + \cdots + P(a_r, b_1)$$

This is the I column of JPM.

Hence we can conclude that by adding all the elements of I column of JPM, we get the probability of first o/p symbol b_1

Similarly $$P(b_2) = P(a_1, b_2) + P(a_2, b_2) + \cdots P(a_r, b_2)$$

$$\vdots$$

$$P(b_s) = P(a_1, b_s) + P(a_2, b_s) + \cdots + P(a_r, b_s)$$

By adding the elements of JPM columnwise we can obtain the probability of output symbols.

$$\sum_{i=1}^{r} P(a_i, b_j) = P(b_j)$$

Property 2

We know

$$p(a_1) = p(a_1/b_1) \cdot p(b_1) + p(a_1/b_2) \cdot p(b_2) + \dots + p(a_1/b_s) \cdot p(b_s)$$

$$p(a_1) = p(a_1, b_1) + p(a_1, b_2) + \dots + p(a_1, b_s)$$

$$p(a_2) = p(a_2, b_1) + p(a_2, b_2) + \dots + p(a_2, b_s)$$

$$\vdots$$

$$p(a_r) = p(a_r, b_1) + p(a_r, b_2) + \dots + p(a_r, b_s)$$

ie By adding the elements of JPM row wise we can obtain the probability of input symbols

$$\sum_{j=1}^{s} p(a_i, b_j) = p(a_i)$$

Proposition 3

The sum of all the elements of the JPM is unity.

$$\sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) = 1$$

PRIORI ENTROPY

The entropy of input symbols $a_1, a_2 \ldots a_r$ before their transmission is defined as priori entropy, $H(A)$.

$$H(A) = \sum_{i=1}^{r} P(a_i) \log \frac{1}{P(a_i)} \quad \text{bits/msg symbol}$$

Marginal entropy is the average information provided by observing a single variable.

Conditional entropy, $H(A/b_j)$

The entropy of the input symbols $a_1, a_2 \ldots a_r$ after ~~before~~ their transmission ~~is defined as~~ and reception of a particular output symbol b_j is defined as conditional entropy, $H(A/b_j)$

$$H(A/b_j) = \sum_{i=1}^{r} p(a_i/b_j) \log \frac{1}{p(a_i/b_j)} \text{ bits/msg symbol}$$

Since $j = 1$ to s, there are 's' no. of conditional entropy.

Equivocation $H(A/B)$

We know when $j = 1$ to s, there are s no. of conditional entropies. Average value of all the conditional entropies is called equivocation

$$H(A/B) = E\left[H(A/b_j)\right]$$

$$= \sum_{j=1}^{s} p(b_j) \cdot H(A/b_j)$$

Substitute the value of $H(A/b_j)$

$$H(A/B) = \sum_{j=1}^{s} p(b_j) \sum_{i=1}^{r} p(a_i/b_j) \log \frac{1}{p(a_i/b_j)}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} p(b_j)\, p(a_i/b_j) \log \frac{1}{p(a_i/b_j)}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i/b_j)}$$

$$H(A/B) = \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i/b_j)}$$

bits / message symbol.

This $H(A/B)$ is a measure of uncertainity when symbols are transmitted over the channel and hence represents the amount of information lost due to noise w.r.t any of the output symbols.

Interchanging A & B in H(A/B), we get

$$H(B/A) = \sum_{j=1}^{s} \sum_{i=1}^{r} p(b_j, a_i) \log \frac{1}{p(b_j/a_i)}$$

We know $p(b_j, a_i) = p(a_i, b_j)$

$$H(B/A) = \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(b_j/a_i)}$$

bits/message symbol.

H(B/A) is a measure of information about the receiver.

Mutual Information

We know

$$H(A) = \sum_{i=1}^{r} p(a_i) \log \frac{1}{p(a_i)} \quad \text{bits / message symbol}$$

$$H(A/B) = \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i/b_j)} \quad \text{bits / msg symbol}$$

Mutual information $I(A,B) = H(A) - H(A/B)$

ie when an average information of $H(A)$ is transmitted over the channel, an average amount of information $H(A/B)$ is lost in the channel due to noise. The balance information received at the receiver with respect to an observed output symbol is the mutual information.

$$I(A,B) = H(A) - H(A/B)$$

Substitute the values

$$I(A,B) = \sum_{i=1}^{r} p(a_i) \log \frac{1}{p(a_i)} - \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i/b_j)}$$

We know $\sum_{j=1}^{s} p(b_j/a_i) = 1$

$$I(A,B) = \sum_{i=1}^{r} p(a_i) \left[\sum_{j=1}^{s} p(b_j/a_i) \right] \log \frac{1}{p(a_i)} - \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i/b_j)}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i)\, p(b_j/a_i) \log \frac{1}{p(a_i)} - \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i/b_j)}$$

We know, joint Probability $p(a_i, b_j) = p(a_i)\, p(b_j/a_i)$

$$I(A,B) = \sum_{i=1}^{r}\sum_{j=1}^{s} p(a_i, b_j)\log\frac{1}{p(a_i)} + \sum_{i=1}^{r}\sum_{j=1}^{s} p(a_i, b_j)\log p(a_i/b_j)$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{s} p(a_i, b_j)\left[\log\frac{1}{p(a_i)} + \log p\left(\frac{a_i}{b_j}\right)\right]$$

We know, $p(a_i/b_j) = \dfrac{p(a_i, b_j)}{p(b_j)}$

$$I(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \left[\log \frac{1}{p(a_i)} + \log \frac{p(a_i, b_j)}{p(b_j)} \right]$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{p(a_i, b_j)}{p(a_i)\, p(b_j)} \text{ bits/msg symbol}$$

Interchanging A and B

$$I(B,A) = \sum_{j=1}^{s} \sum_{i=1}^{r} p(b_j, a_i) \log \frac{p(b_j, a_i)}{p(b_j)\, p(a_i)}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{p(a_i, b_j)}{p(a_i)\, p(b_j)}$$

ie $I(A,B) = I(B,A)$

ie mutual information is symmetrical

We have $I(A,B) = H(A) - H(A/B)$

Interchanging A and B,

$$I(B,A) = H(B) - H(B/A)$$

$H(B)$ = entropy of output symbols

$$H(B) = \sum_{j=1}^{s} p(b_j) \log \frac{1}{P(b_j)} \text{ bits/msg symbol}$$

$$H(A) - H(A/B) = H(B) - H(B/A) \quad [\because I(A,B) = I(B,A)]$$

$$H(B) + H(A/B) = H(A) + H(B/A)$$

$$H(B) + H(A/B) = \cancel{H(A,B)}\ H(B,A) \quad \text{LHS}$$

$$H(A) + H(B/A) = H(A,B) \quad \text{RHS}$$

$H(A,B) = H(B,A)$ is called joint entropy

Joint Entropy

$$H(A,B) = H(A) + H(B/A)$$

$$= \sum_{i=1}^{r} p(a_i) \log \frac{1}{p(a_i)} + \sum_{i=1}^{r}\sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(b_j/a_i)}$$

$$= \sum_{i=1}^{r} p(a_i) \left[\sum_{j=1}^{s} p(b_j/a_i)\right] \log \frac{1}{p(a_i)} + \sum_{i=1}^{r}\sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(b_j/a_i)}$$

We know $p(a_i) \cdot p(b_j/a_i) = p(a_i, b_j)$

$$H(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i)} +$$

$$\sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(b_j/a_i)}$$

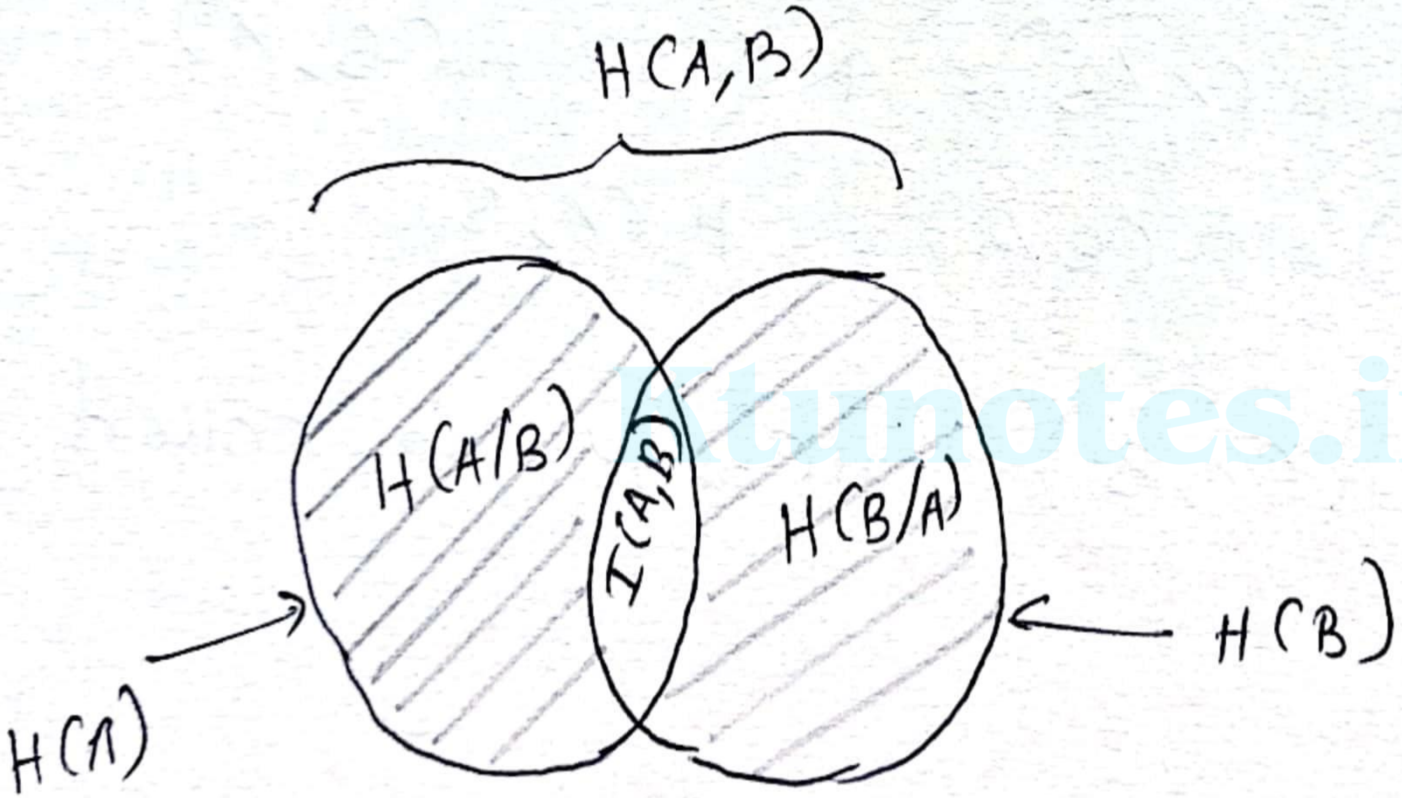
$$= \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \left[\log \frac{1}{p(a_i)} + \log \frac{1}{p(b_j/a_i)} \right]$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i)\, p(b_j/a_i)}$$

$$H(A, B) = \sum_{i=1}^{r} \sum_{j=1}^{s} p(a_i, b_j) \log \frac{1}{p(a_i, b_j)}$$

bits/message symbol

Graphical representation of Entropies



$$I(A,B) = I(B,A)$$

$$I(A,B) = H(B) - H(B/A) \quad \text{--- } (1)$$

$$(1) \Rightarrow H(B/A) = H(B) - I(A,B) \quad \text{--- } (2)$$

we have $H(A,B) = H(A) + H(B/A)$ — (3)

(2) in (3)

$$H(A,B) = H(A) + H(B) - I(A,B)$$

$$\boxed{I(A,B) = H(A) + H(B) - H(A,B)}$$

Properties of Mutual information

1. Mutual information of a channel is symmetric

$$I(A,B) = I(B,A)$$

2. Mutual information is always positive. (nonnegative)

$$I(A,B) \geq 0$$

Mutual information is related to joint entropy by

$$I(A,B) = H(A) + H(B) - H(A,B)$$

Rate of Information

We have entropy of input symbol given by

$$H(A) = \sum_{i=1}^{r} p(a_i) \log \frac{1}{p(a_i)} \quad \text{bits / message symbol}$$

The average rate at which information is passed in to the channel is given by

$$R_{in} = H(A) \cdot r_s \quad \text{bits/sec}$$

r_s = message symbols /sec.

At the receiver, it is not possible to reconstruct the input symbol due to errors introduced when the signal pass through the channel. These errors are introduced due to the noise present in the channel. Thus some amount of information is lost in the channel due to noise. This information which lost in the channel is called equivocation $H(A/B)$.

Net amount of information $I(A,B)$

$$= H(A) - H(A/B) \text{ bits/message symbol}$$

Then the average rate of information

transmission $R_t = I(A,B)\, r_s$ bits/sec

$$= [H(A) - H(A/B)]\, r_s \text{ bits/sec}$$

We know $I(A,B) = I(B,A)$

ie $H(A) - H(A/B) - H(B) - H(B/A)$

So R_t ~~also~~ can also be written as

$$R_t = \left[H(B) - H(B/A) \right] r_s \text{ bits/sec.}$$

Q. A transmitter has an alphabet consisting of 5 letters $\{a_1, a_2, a_3, a_4, a_5\}$ and the receiver has an alphabet of four letters $\{b_1, b_2, b_3, b_4\}$ The joint probabilities of the system are given below

$$P(A,B) = \begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & 0.25 & 0 & 0 & 0 \\ a_2 & 0.10 & 0.30 & 0 & 0 \\ a_3 & 0 & 0.05 & 0.10 & 0 \\ a_4 & 0 & 0 & 0.05 & 0.1 \\ a_5 & 0 & 0 & 0.05 & 0 \end{array}$$

Compute different entropies of the channel.

We know, addition of elements of JPM columnwise results in probability of output symbols.

$$p(b_1) = 0.25 + 0.10 + 0 + 0 + 0 = 0.35$$

$$p(b_2) = 0 + .30 + 0.05 + 0 + 0 = 0.35$$

$$p(b_3) = 0 + 0 + 0.10 + 0.05 + 0.05 = 0.2$$

$$p(b_4) = 0 + 0 + 0 + 0.1 + 0 = 0.1$$

Addition of elements of JPM rowwise results in probability of input symbols.

$$p(a_1) = 0.25 + 0 + 0 + 0 = 0.25$$

$$p(a_2) = 0.10 + 0.30 + 0 + 0 = 0.4$$

$$p(a_3) = 0 + 0.05 + 0.10 + 0 = 0.15$$

$$p(a_4) = 0 + 0 + 0.05 + 0.1 = 0.15$$

$$p(a_5) = 0 + 0 + 0.05 + 0 = 0.05$$

Entropy of input symbol. $H(A) = \sum_{i=1}^{5} p(a_i) \log \frac{1}{p(a_i)}$

$$= 0.25 \log \frac{1}{0.25} + 0.4 \log \frac{1}{0.4} + 0.15 \log \frac{1}{0.15} + 0.15 \log \frac{1}{0.15} + 0.05 \log \frac{1}{0.05}$$

$= 2.066$ bits / message symbol

Entropy of output symbol $H(B) = \sum_{j=1}^{4} p(b_j) \log \frac{1}{p(b_j)}$

$$= 0.35 \log \frac{1}{0.35} + 0.35 \log \frac{1}{0.35} + 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1}$$

$= 1.857$ bits / message symbol.

Joint entropy, $H(A,B) = \sum_{i=1}^{5} \sum_{j=1}^{4} p(a_i, b_j) \log \frac{1}{p(a_i, b_j)}$

$$= 0.25 \log \frac{1}{0.25} + 0.1 \log \frac{1}{0.1} +$$

$$0.3 \log \frac{1}{0.3} + 0.05 \log \frac{1}{0.05} +$$

$$0.1 \log \frac{1}{0.1} + 0.05 \log \frac{1}{0.05} +$$

$$0.05 \log \frac{1}{0.05} + 0.1 \log \frac{1}{0.1}$$

$= 2.666$ bits/message symbol.

Equivocation $H(B/A) = H(A,B) - H(A)$

$= 2.666 - 2.066$

$= 0.6$ bits/message symbol.

Equivocation $H(A/B) = H(A,B) - H(B)$

$= 2.666 - 1.857$

$= 0.809$ bits/message symbol.

Mutual information $I(A,B) = H(A) - H(A/B)$

$= 2.066 - 0.809$

$= 1.257$ bits/message symbol.

OR

$$I(A,B) = H(B) - H(B/A)$$

$$= 1.857 - 0.6$$

$$= 1.257 \text{ bits/message symbol.}$$

Q. A transmitter transmits 5 symbols with probabilities 0.2, 0.3, 0.2, 0.1 and 0.2. Given the channel matrix $P(B/A)$. Calculate (i) $H(B)$ (ii) $H(A,B)$.

$$P(B/A) : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$p(a_i, b_j) = p(a_i) \cdot p(b_j / a_i)$$

The JPM is obtained by multiplying I row elements by $p(a_1) = 0.2 = 1/5$, 2^{nd} row by $p(a_2) = 0.3 = 3/10$, 3^{rd} row by $p(a_2) = 0.2 = 1/5$, 4^{th} row by $p(a_4) = 0.1 = 1/10$ and 5^{th} row by $p(a_5) = 0.2 = 1/5$

$$p(A,B) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 3/40 & 9/40 & 0 & 0 \\ 0 & 1/15 & 2/15 & 0 \\ 0 & 0 & 1/30 & 1/15 \\ 0 & 0 & 1/5 & 0 \end{bmatrix} \end{array}$$

Adding the elements colomnwise,

$p(b_1) = 1/5 + 3/40 = 11/40$

$p(b_2) = 9/40 + 1/15 = 7/24$

$p(b_3) = 2/15 + 1/30 + 1/5 = 11/30$

$p(b_4) = 1/15$

(i)

We know $H(B) = \sum_{j=1}^{4} p(b_j) \log \frac{1}{p(b_j)}$

$$= \frac{11}{40} \log \frac{40}{11} + \frac{7}{24} \log \frac{24}{7} + \frac{11}{30} \log \frac{30}{11} + \frac{1}{15} \log 15$$

$= 1.822$ bits / message symbol.

(ii)

$$H(A,B) = \sum_{i=1}^{5} \sum_{j=1}^{4} p(a_i, b_j) \log \frac{1}{p(a_i, b_j)}$$

$$= \frac{1}{5} \log 5 + \frac{3}{40} \log \frac{40}{3} + \frac{9}{40} \log \frac{40}{9} + \frac{1}{15} \log 15 + \frac{2}{15} \log \frac{15}{2} + \frac{1}{30} \log 30 + \frac{1}{5} \log 5 + \frac{1}{15} \log 15$$

$= 2.7653$ bits / message symbol.

Q. For the JPM given below, compute individually $H(x)$, $H(y)$, $H(x,y)$, $H(x/y)$, $H(y/x)$ and $I(x,y)$. Verify the relationship among these entropies.

$$p(x,y) = \begin{bmatrix} 0.05 & 0 & 0.20 & 0.05 \\ 0 & 0.10 & 0.10 & 0 \\ 0 & 0 & 0.20 & 0.10 \\ 0.05 & 0.05 & 0 & 0.10 \end{bmatrix}$$

Given JPM,

$$
p(x,y) = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.05 & 0 & 0.20 & 0.05 \\ 0 & 0.10 & 0.10 & 0 \\ 0 & 0 & 0.20 & 0.10 \\ 0.05 & 0.05 & 0 & 0.10 \end{bmatrix} \end{array}
$$

Addition of elements of JPM columnwise results in probability of output symbols.

$P(y_1) = 0.05 + 0.05 = 0.10$

$P(y_2) = 0.10 + 0.05 = 0.15$

$P(y_3) = 0.2 + 0.1 + 0.2 = 0.50$

$P(y_4) = 0.05 + 0.1 + 0.1 = 0.25$

$$= \left(0.05 \log \frac{1}{0.05}\right)(4) + \left(0.10 \log \frac{1}{0.10}\right)(4) + \left(0.2 \log \frac{1}{0.2}\right)(2)$$

$$= 3.122 \text{ bits / message symbol.}$$

Equivocation $H(x/y) = \sum_{i=1}^{4} \sum_{j=1}^{4} p(x_i, y_j) \log \frac{1}{p(x_i/y_j)}$

We know $p(x_i/y_j) = \frac{p(x_i, y_j)}{p(y_j)}$ bits /message symbol

So $p(x/y)$ can be constructed as

$$p(x/y) = \begin{bmatrix} \frac{0.05}{0.10} & 0 & \frac{0.2}{0.5} & \frac{0.05}{0.25} \\ 0 & \frac{0.10}{0.15} & \frac{0.1}{0.5} & 0 \\ 0 & 0 & \frac{0.2}{0.5} & \frac{0.10}{0.25} \\ \frac{0.05}{0.10} & \frac{0.05}{0.15} & 0 & \frac{0.10}{0.25} \end{bmatrix}$$

$$p(x/y) = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 1/2 & 0 & 2/5 & 1/5 \\ 0 & 2/3 & 1/5 & 0 \\ 0 & 0 & 2/5 & 2/5 \\ 1/2 & 1/3 & 0 & 2/5 \end{bmatrix} \end{array}$$

$$H(x/y) = 0.05 \log 2 + 0.05 \log 2 + 0.1 \log 3/2$$

$$+ 0.05 \log 3 + 0.2 \log 5/2 + 0.1 \log 5$$

$$+ 0.2 \log 5/2 + 0.05 \log 5 + 0.1$$

$$\log 5/2 + 0.1 \log 5/2$$

$$H(x/y) = 1.379 \text{ bits/message symbol}$$

Equivocation $H(y/x) = \sum_{i=1}^{4} \sum_{j=1}^{4} p(x_i, y_j) \log \frac{1}{p(y_j/x_i)}$

bits / message symbol.

We know $p(y_j/x_i) = \frac{p(x_i, y_j)}{p(x_i)}$.

So channel matrix $p(y/x)$ can be defined as follows.

$$P(y/x) = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 1/6 & 0 & 2/3 & 1/6 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 0 & 1/2 \end{bmatrix} \end{array}$$

$$H(y/x) = 0.05 \log 6 + 0.2 \log 3/2 + 0.05 \log 6$$
$$+ 0.1 \log 2 + 0.1 \log 2 + 0.2 \log 3/2$$
$$+ 0.1 \log 3 + 0.05 \log 4 +$$
$$0.05 \log 4 + 0.1 \log 2$$
$$= 1.151 \text{ bits/message symbol}$$

Verification

$$H(y/x) = H(x,y) - H(x)$$

$$= 3.122 - 1.971$$

$$= 1.151 \text{ bits/message symbol}$$

$$H(x/y) = H(x,y) - H(y)$$

$$= 3.122 - 1.743$$

$$= 1.379 \text{ bits/message symbol}$$

$$I(x,y) = H(x) - H(x/y)$$

$$= 1.971 - 1.379$$

$$= 0.592 \text{ bits / message symbol}$$

OR

$$I(x,y) = H(y) - H(y/x)$$

$$= 1.743 - 1.151$$

$$I(x,y) = 0.592 \text{ bits/message symbol.}$$

$$P(x_i, y_j) = P\left(\frac{x_i}{y_j}\right) \cdot P(y_j)$$

$$= P\left(\frac{y_j}{x_i}\right) \cdot P(x_i)$$

Source Coding

It is the process by which the output of an information source is converted in to r-ary sequence.

r = no. of different symbols used.

Symbol sequence from source $\longrightarrow$ $\boxed{\text{Source encoder}}$ $\longrightarrow$ r-ary sequence.

If $r = 2$, output is a binary sequence

If $r = 3$, " ternary sequence

If $r = 4$ " quaternary sequence

Let S = source alphabet consist of q number of source message

$$S = \{ s_1, s_2 \dots s_q \}$$

X = code alphabet consist of r number of coding symbols

$$X = \{ x_1, x_2 \dots x_r \}$$

Coding is defined as transformation of the source symbols in to some sequence of symbols from code alphabet X.

Properties

1. Block code :- It maps each of the symbols of the source alphabet S in to some finite sequence of code symbols from the code alphabet X. These finite sequence is called code word

Ex:

Source S emitting four symbols

$$S = \{ S_1, S_2, S_3, S_4 \}$$

This is encoded with binary coding.

So $r = 2$ (binary)

Code ~~alph~~ alphabet $X = \{0, 1\}$

Block code

Table 1

Source symbol	Codewords
S_1	00
S_2	01
S_3	10
S_4	11

Nonsingular code

In a block code, if all the codewords are distinct and easily distinguishable from one another, it is nonsingular

The above block code is nonsingular

ex: $S = \{S_1, S_2, S_3, S_4\}$ with $X = \{0, 1\}$

Source symbol	code words
S_1	0
S_2	00
S_3	01
S_4	11

Table 2.

This appears to be nonsingular, but it is not so. Because when you take second extension of these code words $S_1 S_2 = 000$ and $S_2 S_1 = 000$ are not destinct. So it become singular.

3. Uniquely decodable codes

A non singular code is said to be 'uniquely decodable' or 'uniquely decipherable' if every code word present in a long received sequence can be uniquely identified.

Consider the second extension of code in Table 1, we can see that it is nonsingular ~~su su~~. Like 3rd extension will have symbols from $S_1 S_1 S_1$ to $S_4 S_4 S_4$ with respective code words from 000000 to 111111, which is also nonsingular. Similar way 4th, 5th ..... extensions are also nonsingular.

A block code is ~~said~~ said to be uniquely decodable if and only if the n^{th} extension code words of the code is non-singular for every finite value of n.

ex:

Let received sequence = 0 01100

If table 1 is used, received sequence is decodable as $S_1 S_4 S_1$. It is the only possible way of decoding. So this is uniquely decodable code.

If table 2 is used, received code is decoded in several ways as $S_2 S_4 S_2$ or $S_1 S_1 S_4 S_2$ or $S_2 S_4 S_1 S_1$ or $S_1 S_1 S_4 S_1 S_1$.

Since there ~~are~~ is more than one way of decoding the received sequence, code in table 2 is not uniquely decodable code.

4. Instantaneous code.

A uniquely decodable code is said to be instantaneous, if it is possible to recognize the end of any code word in any received sequence, without reference to the succeeding symbols. ie there is no time delay in decoding process, decoding is done instantaneously as and when the symbols arrives at the receiver ex

Source symbol	Code C	Code D	Code E
S_1	00	0	0
S_2	01	10	01
S_3	10	110	011
S_4	11	1110	0111

Let recieved sequence = 001100

If code C is used, then it is decoded $S_1 S_4 S_1$

If code D " " $S_1 S_1 S_3 S_1$

If code E " " $S_1 S_3 S_1 S_1$

When code D is used for decoding & when the symbol '0' (from left) arrives at the receive, it can be immediately decoded as s_1. This is Because.

no other code words S_2, S_3 or S_4 starts with a '0'. Similarly second '0' is decoded as S_1 again.

So code C and D are instantaneous code.

When code E is used for decoding and when symbol '0' arrives at the receiver, we have to wait for the succeeding symbol to arrive at the receiver. Because all the code words in E starts with '0'.

If the second symbol is 1, still we are unable to decode it as S_2 since code words for S_3 and S_4 also start with 01. Thus at every stage, we have to wait for the succeeding symbols. So code E is not an instantaneous code

Test for instantaneous Property (Prefix property)

"No complete word of a code be a prefix of any other code word"

ie If prefixes are present, not instantaneous code

If prefixes are absent, instantaneous code

In the code E given above, prefixes of code for S_4 are '0', '01', '011' which are all present as code words for S_1, S_2 and S_3 respectively.

So code E is not instantaneous.

Note.

Even if one prefix is present, then the code will not be instantaneous.

In code D, prefixes of the code for S_4 are 1, 11, 111 which are not present as code words for any source symbol.

Since all prefixes are absent, code D is instantaneous.

KRAFT INEQUALITY

A necessary and sufficient condition for the existence of an instantaneous code with word length $l_1, l_2 \ldots l_q$ is that

$$\sum_{i=1}^{q} r^{-l_i} \leq 1$$

r = number of different symbols used in the code alphabet x

l_i = word length in binary digits of the code word corresponding to i^{th} source symbol

q = number of source symbols

Proof

The wordlengths $l_1, l_2 \ldots l_q$ are arranged in the ascending order so that

$$l_1 \leq l_2 \leq \ldots \leq l_q$$

If we have to choose one - length code-words for all "q" number of source symbols satisfying prefix property, we can do so only when $q \le r$.

If $q > r$, combination of r symbols must be used to form instantaneous code words.

Let n_i = number of messages encoded into code words of length i, then

for $i = 1$ $\quad n_i \le r$

$\therefore$ for $i = 2$ $\quad n_2 \leq (r - n_1) \times r$

$$n_2 \leq r^2 - n_1 r$$

$$n_3 \leq \left[(r^2 - n_1 r) - n_2 \right] \times r$$

$$n_3 \leq r^3 - n_1 r^2 - n_2 r$$

Generally

$$n_i \leq r^i - n_1 r^{(i-1)} - n_2 r^{(i-2)} \cdots - n_{(i-1)} r$$

$$n_i + n_1 r^{(i-1)} + n_2 r^{(i-2)} + \cdots\cdots + n_{(i-1)} r \leq r^i$$

multiplying throughout by r^{-i}

$$n_i + n_1 r^{(i-1)} + n_2 r^{(i-2)} + \cdots\cdots + n_{(i-1)} r \leq r^i$$

$$n_i + n_1 r^i \cdot r^{-1} + n_2 r^i \cdot r^{-2} + \cdots\cdots + n_{(i-1)} r^i \cdot r \leq r^i$$

multiplying throughout by r^{-i}, we get

$$n_i \cdot r^{-i} + n_1 r^{-1} + n_2 r^{-2} + \dots + n_{(i-1)} \cdot r \cdot r^{-i}$$

$$n_1 r^{-1} + n_2 r^{-2} + \dots + n_{i-1} r^{-(i-1)} + n_i r^{-i} \leq 1 \qquad \text{---} \; (A)$$

$$n_1 r^{-1} \Rightarrow \underbrace{r^{-1} + r^{-1} + r^{-1} + \dots + r^{-1}}_{\text{Total } n_1 \text{ terms}}$$

$n_1, n_2 \dots n_i = \text{integers}$.

$$n_2 r^{-2} \Rightarrow \underbrace{r^{-2} + r^{-2} + \dots + r^{-2}}_{\text{Total } n_2 \text{ terms}}$$

$\vdots$

$$n_i r^{-i} \Rightarrow \underbrace{r^{-i} + r^{-i} + \dots + r^{-i}}_{\text{Total } n_i \text{ terms}}$$

(A) $\Rightarrow$ $\underbrace{r^{-1} + r^{-1} + \dots + r^{-1}}_{n_1 \text{ terms}} + \underbrace{r^{-2} + r^{-2} + \dots r^{-2}}_{n_2 \text{ terms}} + \dots\dots + \underbrace{r^{-i} + r^{-i} + \dots r^{-i}}_{n_i \text{ terms}} \le 1$

$$\underbrace{\sum_{j=1}^{n_1} r^{-1}}_{\text{length } l_1 \text{ group}} + \underbrace{\sum_{j=1}^{n_2} r^{-2}}_{\text{length } l_2 \text{ group}} + \dots + \underbrace{\sum_{j=1}^{n_i} r^{-i}}_{\text{length } l_q \text{ group}} \le 1$$

Since $n_1 + n_2 + \dots + n_i = q$

Combining all the groups; we can write

$$\boxed{\sum_{i=1}^{q} r^{-l_i} \le 1}$$; Hence Prouped.

Unit of wordlength is binits (short form of binary digits)

For binary code, $r = 2$

So equation becomes

$$\sum_{i=1}^{q} 2^{-l_i} \leq 1$$

Ex:

Source symbol	Code A	Code B	Code C	Code D	Code E
S_1	00	0	0	0	0
S_2	01	100	10	100	10
S_3	10	110	110	110	110
S_4	11	111	111	11	11

Code A

$l_1 = 2$ binits ; code word lengths of S_i

$l_2 = l_3 = l_4 = 2$ binits

$$\sum_{i=1}^{q} 2^{-l_i} = 2^{-l_1} + 2^{-l_2} + 2^{-l_3} + 2^{-l_4}$$

$$= 2^{-2} + 2^{-2} + 2^{-2} + 2^{-2}$$

$$= 1$$

Kraft inequality is satisfied. So there exist an instantaneous code with four code-words each of length 2.

Note: Kraft inequality does not tell us that given code is instantaneous; only check the possibility to construct an instantaneous code satisfying the prefix property.

Code B

$l_1 = 1 \quad l_2 = l_3 = l_4 = 3$ binits

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-3} + 2^{-3} + 2^{-3}$$

$$= \frac{7}{8} < 1 \quad \therefore \text{ satisfied.}$$

It is possible to construct an instantaneous code with wordlengths 1 binits for l_1, 3 binits for l_2, 3 binits for l_3, 3 binits for l_4.

Code C $\quad \sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-3} = 1$; satisfied

Code D

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-3} + 2^{-3} + 2^{-2}$$

$= 1$; satisfied

But code C is not instantaneous, since code $S_4 \rightarrow 11$ is a prefix of the code word S_3. But we can easily design an instantaneous code with wordlength 1, 3, 3 and 2.

Code E

$l_1 = 1 \quad l_2 = 2 \quad l_3 = 3 \quad l_4 = 2$ binary

$$\sum_{i=1}^{4} 2^{-l_i} = 2^{-1} + 2^{-2} + 2^{-3} + 2^{-2}$$

$$= 1\frac{1}{8} > 1 \quad \therefore \text{ not satisfied.}$$

ie Given the wordlengths 1, 2, 3, 2, it is not possible to construct an instantaneous code with prefix property.

Q. Which of the following sets of word-lengths are acceptable for the existence of an instantaneous code given

$$X = \{0, 1, 2\}$$

Number of word lengths l_i			Word lengths
Code A	Code B	Code C	
2	2	1	1
1	2	4	2
2	2	6	3
4	3	0	4
1	1	0	5

Code A

There are 2 code-words of length 1 trinit each, one code-word of length 2 trinits 2 code-words of length 3 trinits, 4 code-words of length 4 trinits and one code-word of length 5 trinits

Given $X : \{0, 1, 2\}$ so $r = 3$ unit is trinit.

So $$\sum_{i=1}^{q} r^{-l_i} = 2\times 3^{-1} + 1\times 3^{-2} + 2\times 3^{-3} + 4\times 3^{-4} + 1\times 3^{-5}$$

$$= 0.90535 < 1 \text{ ; satisfied}$$

condition

So Code A is acceptable for construction of instantaneous code

Code B

$$\sum_{i=1}^{q} r^{-l_i} = 2\times 3^{-1} + 2\times 3^{-2} + 2\times 3^{-3} + 3\times 3^{-4} + 1\times 3^{-5}$$

$$= 1.0041 > 1 ; \text{ not satisfied}$$

It is impossible to construct using code B condition

Code C

$$\sum_{i=1}^{q} r^{-l_i} = 1\times 3^{-1} + 4\times 3^{-2} + 6\times 3^{-3}$$

$$= 1 ; \text{ satisfied}$$

It is possible to construct instantaneous code using word lengths specified in Code C.

Construction of instantaneous code with Prefix Property

Q Construct an instantaneous binary code for a source producing 5 symbols S_1 to S_5.

Let $S_1 \longrightarrow 0$

All other code words should start with 1 according to prefix property

$S_2 \longrightarrow 10$ [we can't assign single 1 for S_2, because there would be no symbol left for S_3]

$S_3 \longrightarrow 110$ [we can't assign 11 for S_3 since we have no other combination of binary symbols left for S_4 and S_5

$S_4 \longrightarrow 1110$

$S_5 \longrightarrow 1111$

Source symbols	Code	Another way of construction
S_1	0	00
S_2	10	01
S_3	110	10
S_4	1110	110
S_5	1111	111

Q. Find the smallest number of letters in the alphabet for devising a code with prefix property such that $[W] = [0, 3, 0, 5]$ where W is the set of no. of words with word length 1, 2, 3 ...

No. of word of length l_i	word length l_i
0	1
3	2
0	3
5	4

ii There are three code words of length 2 digits $(l_2 = 2)$ and five code words of length 4 digits $(l_4 = 4)$

ie total 8 symbols $q = 8$

$$\sum_{i=1}^{8} r^{-l_i} = \underbrace{\left(r^{-2} + r^{-2} + r^{-2}\right)}_{\text{3 messages with word length 2}} + \underbrace{\left(r^{-4} + r^{-4} + r^{-4} + r^{-4}\right)}_{\text{5 messages with word length 4}} \leq 1$$

$$3\,r^{-2} + 5\,r^{-4} \leq 1$$

Let $r = 2$; $3\,(2)^{-2} + 5\,(2)^{-4}$

$= 1.0625 > 1$; Impossible

Let $r = 3$; $3\,(3)^{-2} + 5\,(3)^{-4}$

$= 0.395 < 1$; possible.

So it is possible to construct instantaneous code using $r = 3$

Source	code	Alternative code
S_1	22	10
S_2	21	11
S_3	20	12
S_4	1222	2222
S_5	1122	2221
S_6	1112	2211
S_7	1111	2111
S_8	1000	0210

Q. Consider a binary block code with 2^n code-words of same length n. S.T Kraft inequality is satisfied for such a code.

Given $l_i = n$ [Since all the code-words have same length n]

$q = 2^n$ with $r = 2$ (binary block code)

$$\sum_{i=1}^{q} r^{-l_i} = \sum_{i=1}^{2^n} 2^{-n}$$

$$= \underbrace{2^{-n} + 2^{-n} + \cdots + 2^{-n}}_{2^n \text{ terms}}$$

$$= 2^{-n} \cdot 2^n$$

$$= 2^0$$

$$= 1 \; ; \text{ satisfied.}$$

Let $n = 2$

$q = 4$

$l_i = n = 2$ for $i = 1, 2, 3, 4$

Source symbol	code
S_1	00
S_2	01
S_3	10
S_4	11

Let $n = 3$ $\quad q = 2^3 = 8$ $\quad l_i = n = 3$

Source symbol	Code	Source symbol	Code
S_1	000	S_5	100
S_2	001	S_6	101
S_3	010	S_7	110
S_4	011	S_8	111

Code efficiency and Redundancy

The average length L of the code is given by

$$L = \sum_{i=1}^{q} P_i l_i \text{ binits / message symbol}$$

$P_i \Rightarrow P_1, P_2 \ldots P_q$ = probabilities of the source symbol $S_1, S_2 \ldots S_q$

$l_i \Rightarrow l_1, l_2 \ldots l_q$ = word length in binits of $S_1, S_2 \ldots S_q$

We know

$$\text{entropy, } H(S) = \sum_{i=1}^{q} P_i \log \frac{1}{P_i} \text{ bits / message symbol}$$

$L > H(s)$ for binary codes

$L \geq H_r(s)$ for r-ary codes

$H_r(s)$ = entropy in r-ary units/message symbol

$$H_r(s) = \frac{H(s)}{\log_2 r}$$

r = no. of different symbol used in code alphabet

Coding efficiency $\eta_c = \frac{H(s)}{L}$; for binary code

$$\eta_c = \frac{H_r(s)}{L} \rightarrow \text{for r-ary code}$$

Since $L \geq H_r(s)$

$$\eta_c \leq 100\%$$

Coding redundancy $R_{\eta c} = 1 - \eta_c$

Q. A source having an alphabet $S = \{S_1, S_2, S_3, S_4, S_5\}$ produces these symbols with respective probabilities of $\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{9}, \frac{1}{18}$. When these symbols are coded as shown below, find code-efficiency and redundancy.

Source symbol	Code
S_1	0
S_2	10
S_3	110
S_4	1110
S_5	1111

We have $l_1 = 1$ binits, $l_2 = 2$ binits,

$l_3 = 3$ binits $l_4 = l_5 = 4$ binits

$$L = \sum_{i=1}^{5} P_i l_i$$

$$= P_1 l_1 + P_2 l_2 + P_3 l_3 + P_4 l_4 + P_5 l_5$$

$$= \left(\frac{1}{2}\right) \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{9} \times 4 + \frac{1}{18} \times 4$$

$L = 2.0$ binits / message symbol

$$H(S) = \sum_{i=1}^{5} P_i \log \frac{1}{P_i}$$

$$= \frac{1}{2} \log 2 + 2 \times \frac{1}{6} \log 6 + \frac{1}{9} \log 9 + \frac{1}{18} \log 18$$

$H(S) = 1.94553$ bits/message symbol

$$\eta_c = \frac{H(S)}{L} = \frac{1.94553}{2} \times 100\%$$

$$= 97.28\%$$

$$R_{\eta c} = (1 - 0.9728) \times 100\%$$

$$= 2.72\%$$