PROPAGATION OF EM WAVE IN A CONDUCTING MEDIUM (LOSSY DIELECTRIC)

Consider a linear (J=0), isotropic (ρ=0), homogeneous(σ=0), lossy dielectric medium (ρ_v=0). Electric filed and magnetic fields are assumed to be varying sinusoidally with time.

The wave equation in phaser form for conducting medium

$$\nabla^2 \boldsymbol{E_s} = \boldsymbol{j\omega\mu}\,(\boldsymbol{\sigma} + \boldsymbol{j\omega}\varepsilon)\mathbf{E_s}$$

Lets substitute $\gamma^2 = \boldsymbol{j\omega\mu}\,(\boldsymbol{\sigma} + \boldsymbol{j\omega}\varepsilon)$

$$\boxed{\begin{aligned}\nabla^2 \boldsymbol{E_s} &= \gamma^2 \mathbf{E_s} \\ \nabla^2 \boldsymbol{H_s} &= \gamma^2 \boldsymbol{H_s}\end{aligned}}$$

Propagation constant is a complex value so,

$$\gamma = \alpha + j\beta$$

$$\gamma^2 = (\alpha + j\beta)^2$$

$$\gamma^2 = \alpha^2 - \beta^2 + 2j\alpha\beta \quad \text{----------- (1)}$$

$$\gamma^2 = \boldsymbol{j\sigma\omega\mu} - \boldsymbol{\omega}^2\boldsymbol{\mu}\varepsilon \quad \text{----------- (2)}$$

Comparing real & imaginary values of (1) & (2)

$$\alpha^2 - \beta^2 = -\boldsymbol{\omega}^2\boldsymbol{\mu}\varepsilon \quad \text{------ (3)}$$

$$2\alpha\beta = \boldsymbol{\sigma\omega\mu} \quad \text{------ (4)}$$

From (3) & (4) we get

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\boldsymbol{\omega}\varepsilon}\right]^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\boldsymbol{\omega}\varepsilon}\right]^2} + 1\right]}$$

$$\gamma = \pm\sqrt{\boldsymbol{j\omega\mu}\,(\boldsymbol{\sigma} + \boldsymbol{j\omega}\varepsilon)}$$

PROPAGATION OF EM WAVE IN A CONDUCTING MEDIUM (LOSSY DIELECTRIC)

If we assume that the wave propagates along z direction & it has only x components

$$E_s = E_{xs}(z)\hat{a}_x$$

We know

$$\nabla^2 \boldsymbol{E_s} = \gamma^2 \mathbf{E_s}$$

$$\nabla^2 E_{xs}(z) - \gamma^2 E_{xs}(z) = 0$$

$$\frac{\partial^2}{\partial x^2} E_{xs}(z) + \frac{\partial^2}{\partial y^2} E_{xs}(z) + \frac{\partial^2}{\partial z^2} E_{xs}(z) - \gamma^2 E_{xs}(z) = 0$$

$$\frac{\partial^2}{\partial z^2} E_{xs}(z) - \gamma^2 E_{xs}(z) = 0$$

$$\left[\frac{\partial^2}{\partial z^2} - \gamma^2\right] E_{xs}(z) = 0$$

This a scalar wave equation

PROPAGATION OF EM WAVE IN A CONDUCTING MEDIUM (LOSSY DIELECTRIC)

Let

$$E_{xs}(z) = E_0 e^{-\gamma z} + E_0' e^{\gamma z}$$

Assuming E_0'=0, and including the time factor $e^{j\omega t}$ in the above equation

$$E(z,t) = Re[E_{xs}(z)e^{j\omega t}\hat{a}_x]$$

$$= Re[E_0 e^{-\gamma z} e^{j\omega t}\hat{a}_x]$$

$$= Re[E_0 e^{-(\alpha+j\beta)z} e^{j\omega t}\hat{a}_x]$$

$$= Re[E_0 e^{-\alpha z} e^{-j\beta z} e^{j\omega t}\hat{a}_x]$$

$$= Re[E_0 e^{-\alpha z} e^{-j(\omega t-\beta z)}\hat{a}_x]$$

$$\boxed{\begin{aligned} E(z,t) &= E_0 e^{-\alpha z} cos(\omega t - \beta z)\hat{a}_x \\ H(z,t) &= H_0 e^{-\alpha z} cos(\omega t - \beta z - \theta_\eta)\hat{a}_y \end{aligned}}$$

Where

$$\frac{E_0}{H_0} = \eta$$ *(intrinsic impedance of medium)*

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = |\eta| e^{j\theta_n}$$

For perfect dielectric (σ=0) $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

$$\boxed{|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\left[1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]^{\frac{1}{4}}}, \qquad \tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon}}$$

SUMMERY	
Propagation constant	$\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$
Attenuation constant	$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$
Phase shift constant	$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$
Intrinsic impedance	$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$
Wave length	$\lambda = \frac{2\pi}{\beta}$
Velocity	$V = \frac{\omega}{\beta}$

PROPAGATION OF EM WAVE IN PERFECT DIELECTRIC (LOSSLESS MEDIUM)

Consider an isotropic homogeneous perfect dielectric of permittivity ε and permeability μ

Wave equation

$$\nabla^2 \boldsymbol{E_s} = \gamma^2 \mathbf{E_s}$$

Where

$$\gamma^2 = j\boldsymbol{\sigma\omega\mu} - \boldsymbol{\omega}^2\boldsymbol{\mu}\varepsilon$$

For a homogeneous medium (σ=0)

$$\gamma^2 = -\boldsymbol{\omega}^2\boldsymbol{\mu}\varepsilon$$

$$\gamma = \boldsymbol{j\omega}\sqrt{\boldsymbol{\mu\varepsilon}}$$

$$\boxed{\gamma = \alpha + j\beta}$$

Since there is no real part

(i) Attenuation constant $\alpha = 0$

(ii) Phase shift constant $\beta = \omega\sqrt{\mu\varepsilon}$

(iii) Intrinsic impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

(iv) Wave length $\lambda = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{f\sqrt{\mu\varepsilon}}$

(v) Velocity $V = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu\varepsilon}}$

SUMMERY	
Propagation constant	$\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$
Attenuation constant	$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$
Phase shift constant	$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$
Intrinsic impedance	$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$
Wave length	$\lambda = \frac{2\pi}{\beta}$
Velocity	$V = \frac{\omega}{\beta}$

PROPAGATION OF PLANE EM WAVE IN GOOD CONDUCTOR

In a good conductor $\sigma \approx \infty$, $\varepsilon = \varepsilon_0$ $\mu = \mu_0\mu_r$

$$\sigma \gg \omega\varepsilon$$

$$\frac{\sigma}{\omega\varepsilon} \gg 1$$

$$\alpha, \beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} \pm 1\right]}$$

$$= \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{\left[\frac{\sigma}{\omega\varepsilon}\right]^2} \pm 1\right]}$$

$$= \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\frac{\sigma}{\omega\varepsilon}\right]}$$

$$\alpha, \beta = \sqrt{\frac{\omega\mu\sigma}{2}}$$

(i) Attenuation constant $\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$

(ii) Phase shift constant $\beta = \sqrt{\frac{\omega\mu\sigma}{2}}$

(iii) Intrinsic impedance $\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^0$

(iv) Wave length $\lambda = \frac{2\pi}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \frac{2\pi}{\sqrt{\pi f \mu\sigma}}$

(v) Velocity $V = \frac{\omega}{\sqrt{\frac{\omega\mu\sigma}{2}}} = \sqrt{\frac{2\omega}{\mu\sigma}}$

SUMMERY	
Propagation constant	$\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)}$
Attenuation constant	$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}$
Phase shift constant	$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\omega\varepsilon}\right]^2} + 1\right]}$
Intrinsic impedance	$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$
Wave length	$\lambda = \frac{2\pi}{\beta}$
Velocity	$V = \frac{\omega}{\beta}$

Q) Derive the expression for attenuation constant and phase shift constant in a lossy dielectric medium?

Solution:

Propagation constant is a complex value so,

$$\gamma = \alpha + j\beta \quad \text{------ (1)}$$

$$\gamma^2 = \alpha^2 - \beta^2 + 2j\alpha\beta$$

$$\gamma^2 = j\omega\mu\,(\sigma + j\omega\varepsilon) \quad \text{------ (2)}$$

$$\gamma^2 = j\sigma\omega\mu - \omega^2\mu\varepsilon$$

Comparing real & imaginary values of (1) & (2)

$$\alpha^2 - \beta^2 = -\omega^2\mu\varepsilon \quad \text{------ (3)}$$

$$2\alpha\beta = \sigma\omega\mu$$

$$\beta = \frac{\sigma\omega\mu}{2\alpha} \quad \text{------ (4)}$$

$$\alpha^2 - \left(\frac{\sigma\omega\mu}{2\alpha}\right)^2 = -\omega^2\mu\varepsilon$$

Multiply by α^2 in both sides

$$(\alpha^2)^2 - \left(\frac{\sigma\omega\mu}{2\alpha}\right)^2 \alpha^2 = -\omega^2\mu\varepsilon\,\alpha^2$$

$$(\alpha^2)^2 + \omega^2\mu\varepsilon\,\alpha^2 = \left(\frac{\sigma\omega\mu}{2}\right)^2$$

IMPLearn $(\alpha^2)^2 + \boldsymbol{\alpha^2\omega^2\mu\varepsilon} = \left(\frac{\boldsymbol{\sigma\omega\mu}}{2}\right)^2$

Converting LHS into (a + b)² format $\left(a = \alpha^2, b = \frac{\omega^2\mu\varepsilon}{2}\right)$

a^2 2ab b^2

$$(\alpha^2)^2 + \frac{2\alpha^2\omega^2\mu\varepsilon}{2} + \left(\frac{\omega^2\mu\varepsilon}{2}\right)^2 - \left(\frac{\omega^2\mu\varepsilon}{2}\right)^2 = \left(\frac{\sigma\omega\mu}{2}\right)^2$$

$$\left(\alpha^2 + \frac{\omega^2\mu\varepsilon}{2}\right)^2 = \left(\frac{\omega^2\mu\varepsilon}{2}\right)^2 + \left(\frac{\sigma\omega\mu}{2}\right)^2$$

$$= \frac{\omega^4\mu^2\varepsilon^2}{4} + \frac{\sigma^2\omega^2\mu^2}{4}$$

$$= \frac{\omega^4\mu^2\varepsilon^2}{4}\left[1 + \frac{\sigma^2}{\omega^2\varepsilon^2}\right]$$

Take square root on both sides

$$\alpha^2 + \frac{\omega^2\mu\varepsilon}{2} = \frac{\omega^2\mu\varepsilon}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - \frac{\omega^2\mu\varepsilon}{2} \quad \text{------ (5)}$$

$$= \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}} - 1\right]$$

Taking square root

$$\boxed{\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left[\frac{\sigma}{\omega\varepsilon}\right]^2} - 1\right]}}$$

Attenuation constant

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From equation (3)

$$\alpha^2 - \beta^2 = -\boldsymbol{\omega^2\mu}\varepsilon \quad \text{------ (3)}$$

$$\beta^2 = \alpha^2 + \boldsymbol{\omega^2\mu}\varepsilon$$

$$\alpha^2 = \frac{\omega^2\mu\varepsilon}{2}\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - \frac{\omega^2\mu\varepsilon}{2} \quad \text{------ (5)}$$

From equation (5)

$$\beta^2 = \frac{\omega^2\mu\varepsilon}{2}\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} - \frac{\omega^2\mu\varepsilon}{2} + \boldsymbol{\omega^2\mu}\varepsilon$$

$$= \frac{\omega^2\mu\varepsilon}{2}\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + \frac{\omega^2\mu\varepsilon}{2}$$

$$\beta^2 = \frac{\omega^2\mu\varepsilon}{2}\left[\sqrt{1+\frac{\sigma^2}{\omega^2\varepsilon^2}} + 1\right]$$

Taking square root

$$\boxed{\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left[\frac{\sigma}{\boldsymbol{\omega}\varepsilon}\right]^2} + 1\right]}}$$

Phase shift constant

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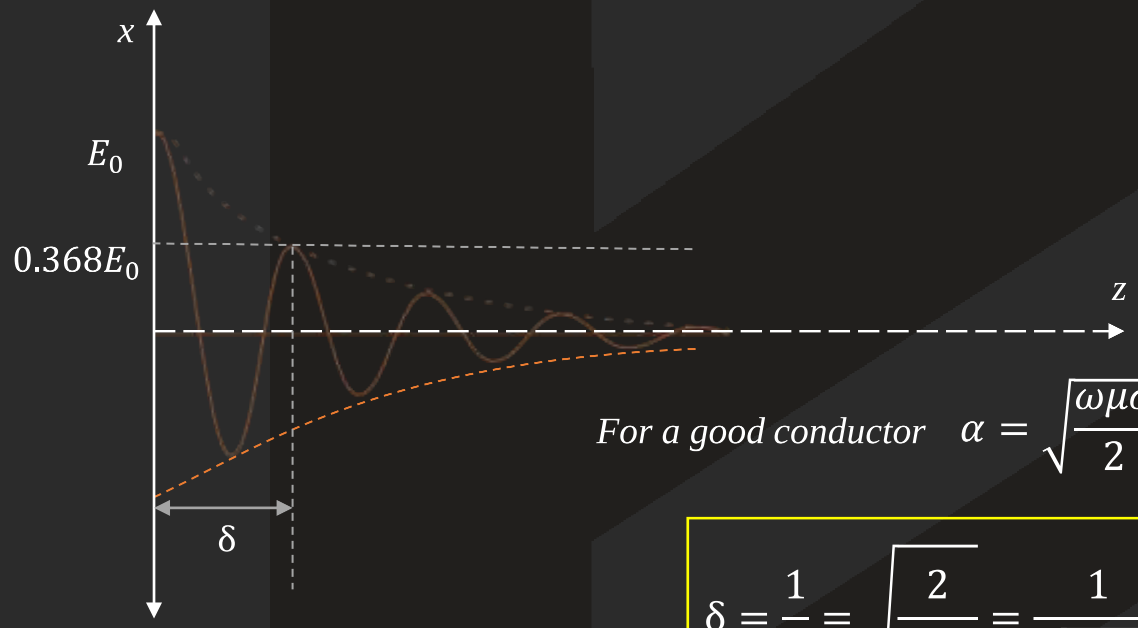
SKIN DEPTH & SKIN EFFECT

We saw the equation

$$E = E_0 e^{-\alpha z} cos(\omega t - \beta z)\hat{a}_x$$

$$H = \frac{E_0}{\eta} e^{-\alpha z} cos(\omega t - \beta z)\hat{a}_y$$

An E (or H) was travels in a conducting medium, its amplitude is attenuated by a factor $e^{-\alpha z}$



For a good conductor $\alpha = \sqrt{\frac{\omega\mu\sigma}{2}}$

$$\boxed{\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega\mu\sigma}} = \frac{1}{\sqrt{\pi f \mu\sigma}}}$$

intrinsic impedance $\eta = \frac{E_0}{H_0}$

$$\boxed{\begin{aligned} E(z,t) &= E_0 e^{-\alpha z} cos(\omega t - \beta z)\hat{a}_x \\ H(z,t) &= H_0 e^{-\alpha z} cos(\omega t - \beta z)\hat{a}_y \end{aligned}}$$

*The distance δ through which the wave amplitude decrease to a factor e^{-1} (about 37% of the original value) is called **skin depth** or penetration depth of the medium (δ). The **skin depth** is the measure of the depth to which an EM wave can penetrate the medium*

$$E_0 e^{-\alpha\delta} = E_0 e^{-1} \quad \rightarrow \quad \alpha\delta = 1 \qquad \delta = \frac{1}{\alpha}$$

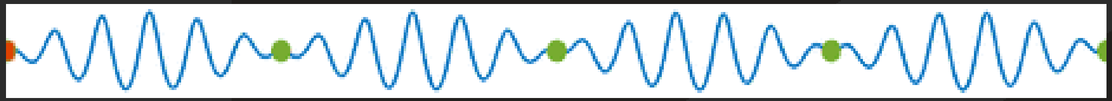
*The phenomenon when by field in a conductor rapidly decrease is known as **skin effect**. It is the tendency of charge to migrate from the bulk of the conducting material to the surface, resulting in higher resistance, As frequency increases skin depth decreases*

PHASE VELOCITY & GROUP VELOCITY

Phase velocity

- *Phase velocity is the rate at which the phase and the wave propagates in space*
- *This is the velocity at which the phase of any one frequency component of the wave travels.*
- *It is represented in terms of wave length (λ) & time period (T)(or angular frequency (ω) and phase constant (β))*

$$\boxed{v_p = \frac{\lambda}{T} \text{ or } v_p = \frac{\omega}{\beta}}$$



The ■ red square moves with the phase velocity, and the ● green circles propagate with the group velocity.

Group velocity

- *Group velocity is the velocity with the overall shape of the wave amplitude also known as modulation or envelop of wave propagates through space*
- *It is represented in terms of angular frequency (ω) and angular wave number (k)*

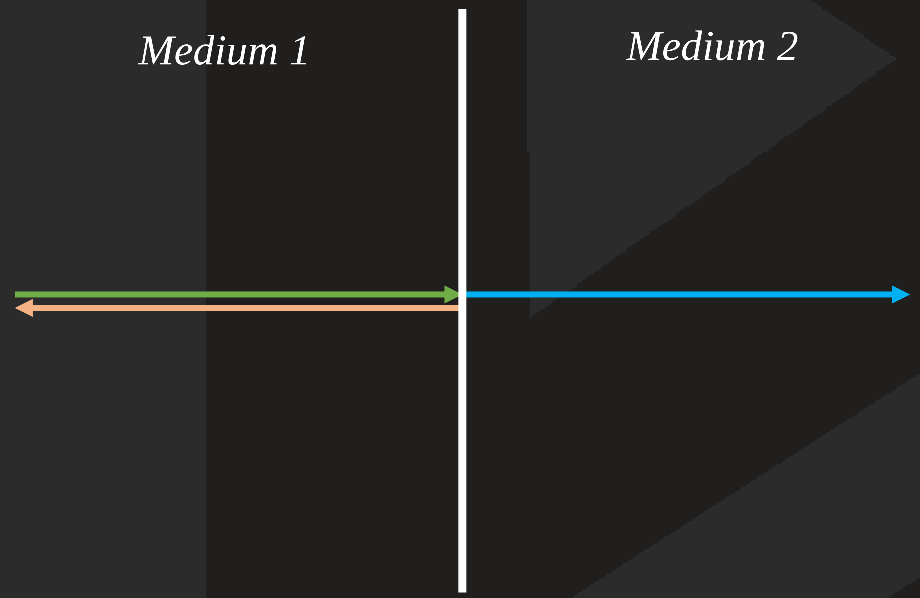
$$\boxed{v_g = \frac{\partial \omega}{\partial k}}$$

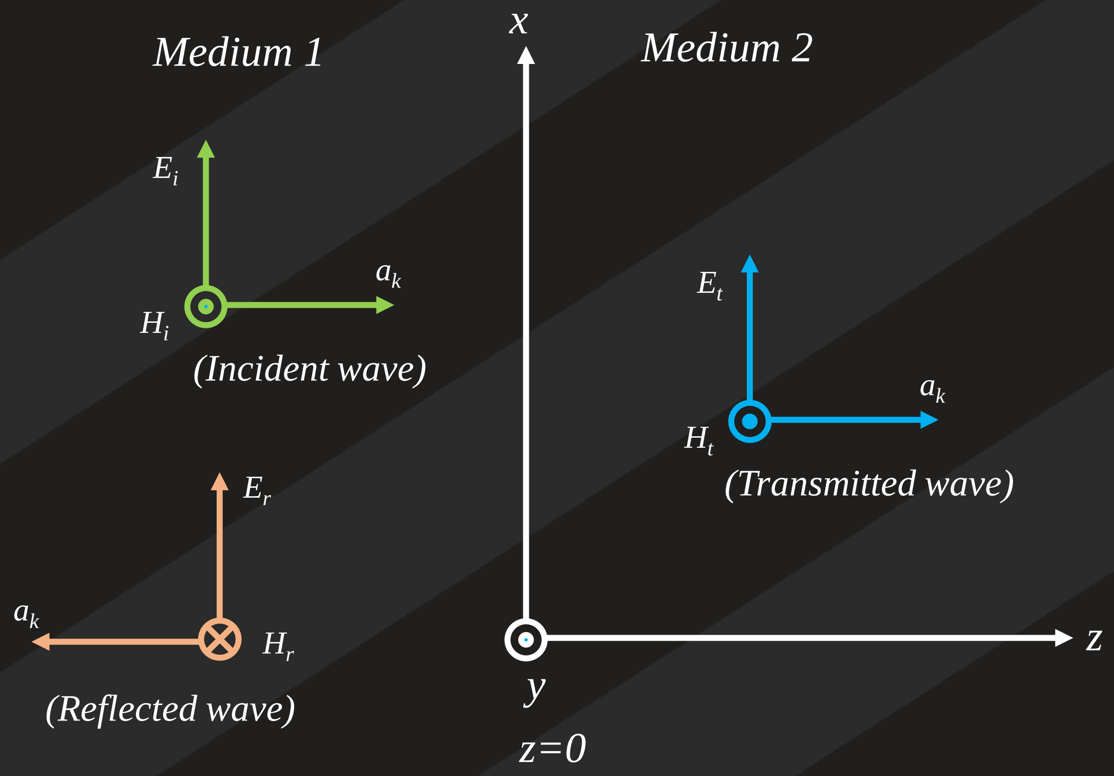
- *If ω is directly proportional to k then group velocity is equal to phase velocity (wave traves undistorted)*
- *If ω is a linear function of K but not directly proportional (ω=ak+b), then both velocities are different*
- *If ω is not a linear function of k, then the envelop of the wave packet will distorted as it travels which is related to group velocity*
- *Since wave packet contains a range of different frequencies / values the envelop does not move at a single velocity, so the envelop distorted*

REFLECTION OF A PLANE WAVE AT NORMAL INCIDENCE

When a plane wave from one medium meets a different medium it is partly reflected and partly transmitted. The proportion of the incident wave that is reflected or transmitted depends on the constitutive parameters off the two medium involved

here we will assume that the incident wave plane is normal to the boundary between the media





$$E(z,t) = E_{xs}(z)e^{j\omega t}\hat{a}_x$$

Incident wave (E_i, H_i)

If we suppress the time factor $e^{j\omega t}$ and wave us traveling along $+a_z$

$$E_{is}(z) = E_{i0}e^{-\gamma_1 z}\hat{a}_x$$

$$H_{is}(z) = H_{i0}e^{-\gamma_1 z}\hat{a}_y$$

$$H_{is}(z) = \frac{E_{i0}}{\eta_1}e^{-\gamma_1 z}\hat{a}_y$$

Reflected wave (E_r, H_r)

Travels along $-\hat{a}_z$ in medium 1

$$E_{rs}(z) = E_{r0}e^{\gamma_1 z}\hat{a}_x$$

$$H_{rs}(z) = H_{r0}e^{\gamma_1 z} - \hat{a}_y$$

$$H_{rs}(z) = -\frac{E_{r0}}{\eta_1}e^{\gamma_1 z}\hat{a}_y$$

Transmitted wave (E_t, H_t)

Travels along $+\hat{a}_z$ in medium 2

$$E_{ts}(z) = E_{t0}e^{-\gamma_2 z}\hat{a}_x$$

$$H_{ts}(z) = H_{t0}e^{-\gamma_2 z}\hat{a}_y$$

$$H_{ts}(z) = \frac{E_{t0}}{\eta_2}e^{-\gamma_2 z}\hat{a}_y$$

Also at z=0

$$E_{is}(0) = E_{i0}$$

$$E_{rs}(0) = E_{r0}$$

$$E_{ts}(0) = E_{t0}$$

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$$E_1 = E_i + E_r \qquad E_2 = E_t$$

$$H_1 = H_i + H_r \qquad H_2 = H_t$$

Applying boundary conditions the tangential components of E & H are continuous at z=0

$$E_i(0) + E_r(0) = E_t(0) \qquad H_i(0) + H_r(0) = H_t(0)$$

$$E_i + E_r = E_t \qquad H_i - H_r = H_t$$

Substitute the value of E_t

$$\frac{1}{\eta_1}(E_i - E_r) = \frac{E_t}{\eta_2}$$

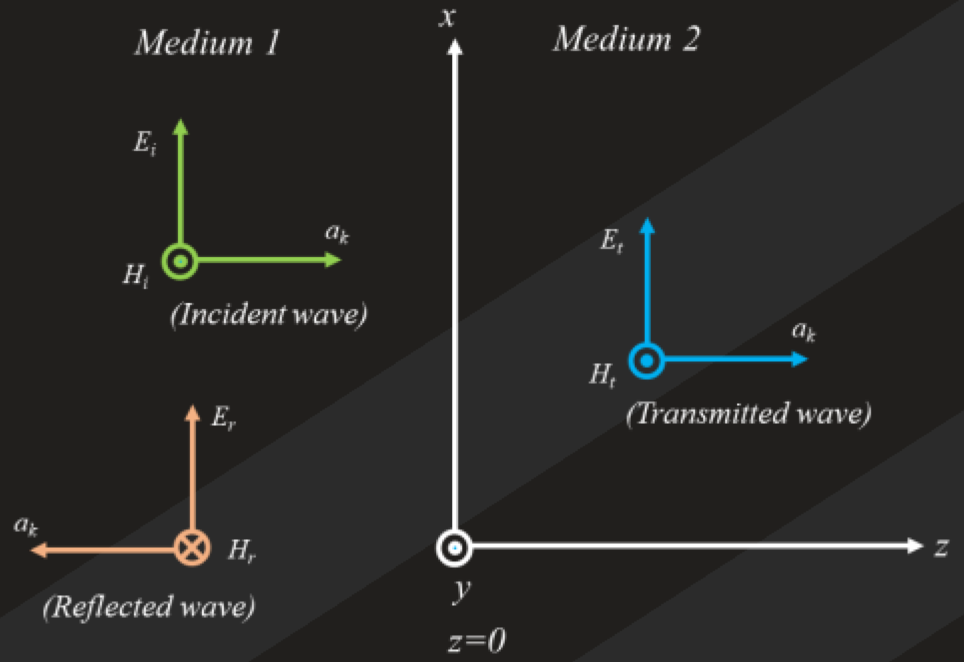
$$\frac{E_i - E_r}{\eta_1} = \frac{E_i + E_r}{\eta_2}$$

$$\eta_2(E_i - E_r) = \eta_1(E_i + E_r)$$

$$E_r(\eta_2 + \eta_1) = E_i(\eta_2 - \eta_1)$$

$$\boxed{\frac{E_r}{E_i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma}$$

*Where **Γ** is the reflection coefficient*



$$\boxed{E_{1t} = E_{2t}}$$

Thus the tangential component of E are same on the two sides of the boundaries

$$\boxed{H_{1t} = H_{2t}}$$

Thus the tangential component of H are continuous on the two sides of the boundaries

$$\boxed{\frac{E_0}{H_0} = \eta}$$

$\Gamma \rightarrow$ ***Capital Gamma***

(intrinsic impedance of medium)

$$\frac{E_i - E_r}{\eta_1} = \frac{E_t}{\eta_2}$$

$$E_i + E_r = E_t$$

$$\eta_2 E_i - \eta_2 E_r = \eta_1 E_t$$

$$E_r = E_t - E_i$$

$$\eta_2 E_i - \eta_2 (E_t - E_i) = \eta_1 E_t$$

$$\eta_2 E_i - \eta_2 E_t + \eta_2 E_i = \eta_1 E_t$$

$$2\eta_2 E_i - \eta_2 E_t = \eta_1 E_t$$

$$2\eta_2 E_i = (\eta_2 + \eta_1) E_t$$

$$\boxed{\frac{E_t}{E_i} = \frac{2\eta_2}{\eta_1 + \eta_2} = \tau}$$

Where $\boldsymbol{\tau}$ is the transmission coefficient

Note :

1. $1+\Gamma=\tau$
2. *Both $\boldsymbol{\Gamma}$ and $\boldsymbol{\tau}$ are dimensionless and may be complex*
3. $0 \leq |\Gamma| \leq 1$

REFLECTION OF A PLANE WAVE AT OBLIQUE INCIDENCE

To simplify the analysis the analysis we will assume that we are dealing with lossless media. We may extend our analysis to that of lossy media. It can be shown that a uniform plane wave taken the general form of

Assume a time dependent field $E = Re[E_0 e^{j(k.r-\omega t)}]$ *and* $H = Re[H_0 e^{j(k.r-\omega t)}]$

$$E = E_0 \cos(k.r - \omega t)$$

$\boldsymbol{r} = \boldsymbol{x}a_x + \boldsymbol{y}a_y + \boldsymbol{z}a_z$ ***Position vector***

$\boldsymbol{k} = k_x a_x + k_y a_y + k_z a_z$ ***Propagation vector*** *(in the direction of wave propagation)*

The magnitude of $\boldsymbol{k}$ *is related to* $\boldsymbol{\omega}$

$$\boldsymbol{k^2} = k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon$$

and

$$k \times E = \omega \mu H$$

$$k \times H = -\omega \varepsilon E$$

$$k.E = 0$$

$$k.H = 0$$

$$E \perp H \perp k$$

$$k.r = k_x x + k_y y + k_z z = constant$$

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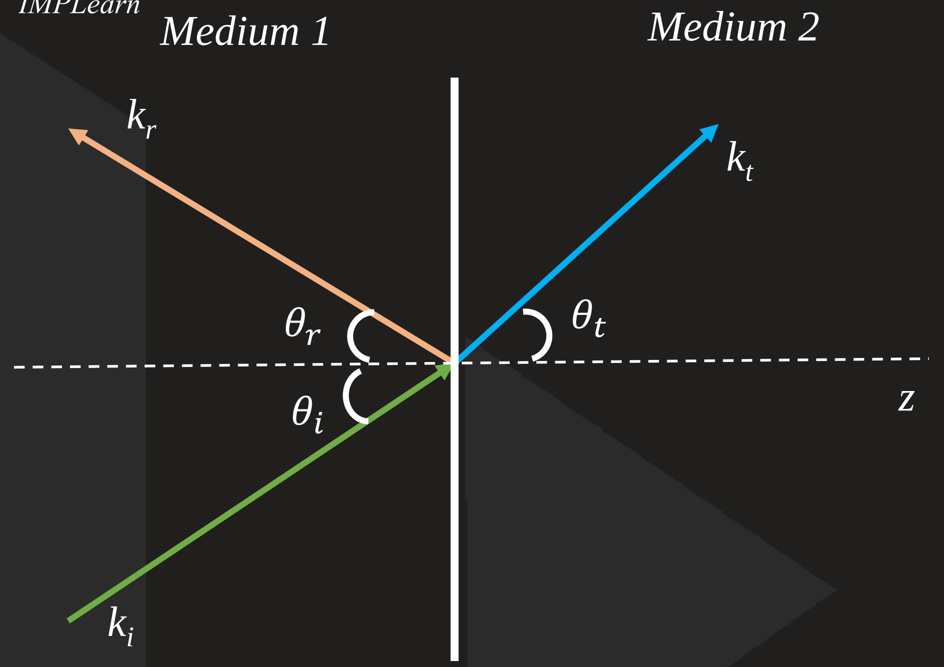


Incident wave

$$E_i = E_{i0}\cos\left(k_{ix}x + k_{iy}y + k_{iz}z - \omega_i t\right)$$

Reflected wave

$$E_r = E_{r0}\cos\left(k_{rx}x + k_{ry}y + k_{rz}z - \omega_r t\right)$$

Transmitted wave

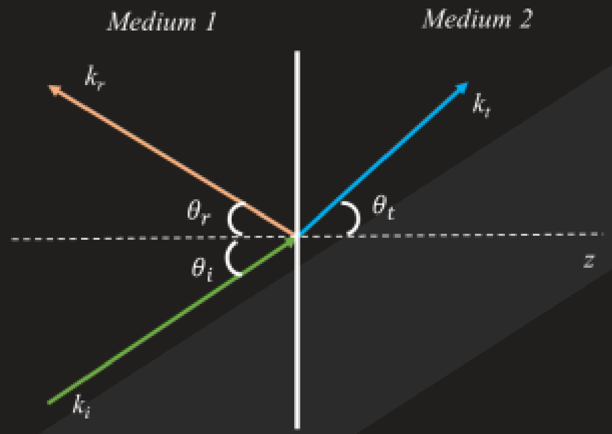
$$E_t = E_{t0}\cos\left(k_{tx}x + k_{ty}y + k_{tz}z - \omega_t t\right)$$

Since the tangential component of E must be continuous across the boundary z=0

$$E_i(z=0) + E_r(z=0) = E_t(z=0)$$

This boundary condition can be satisfied by the waves for all x and y only if

1. $\omega_i = \omega_r = \omega_t = \omega$ *Frequency is unchanged*
2. $k_{ix} = k_{rx} = k_{tx} = k_x$
3. $k_{iy} = k_{ry} = k_{ty} = k_y$ *Phase-matching conditions*



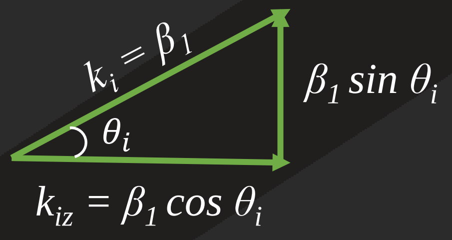
Under these conditions

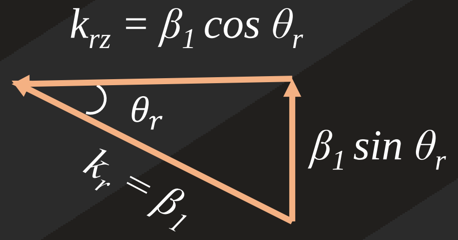
$$k_i \sin\theta_i = k_r \sin\theta_r \qquad k_i \sin\theta_i = k_t \sin\theta_t$$

But for lossless medium

$$k_i = k_r = \beta_1 = \omega\sqrt{\mu_1\varepsilon_1} \qquad k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

$$\boxed{\beta = \omega\sqrt{\mu\varepsilon}}$$

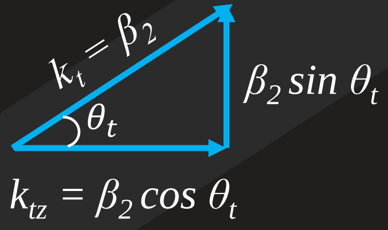




$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{k_i}{k_t} = \sqrt{\frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$$

Snell's law



$$n_1 = c\sqrt{\mu_1\varepsilon_1}$$
$$n_2 = c\sqrt{\mu_2\varepsilon_2}$$

Refractive indices

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PLANE WAVE AT OBLIQUE INCIDENCE - CASE 1: PARALLEL POLARIZATION

E field lies in the xz – plane, the plane of incident, illustrates the case of parallel polarization,

In medium 1

$$E_{is} = E_{i0}(\cos\theta_i a_x - \sin\theta_i\, a_z)e^{-j\beta_1(x\, sin\theta_i + z\cos\theta_i)}$$

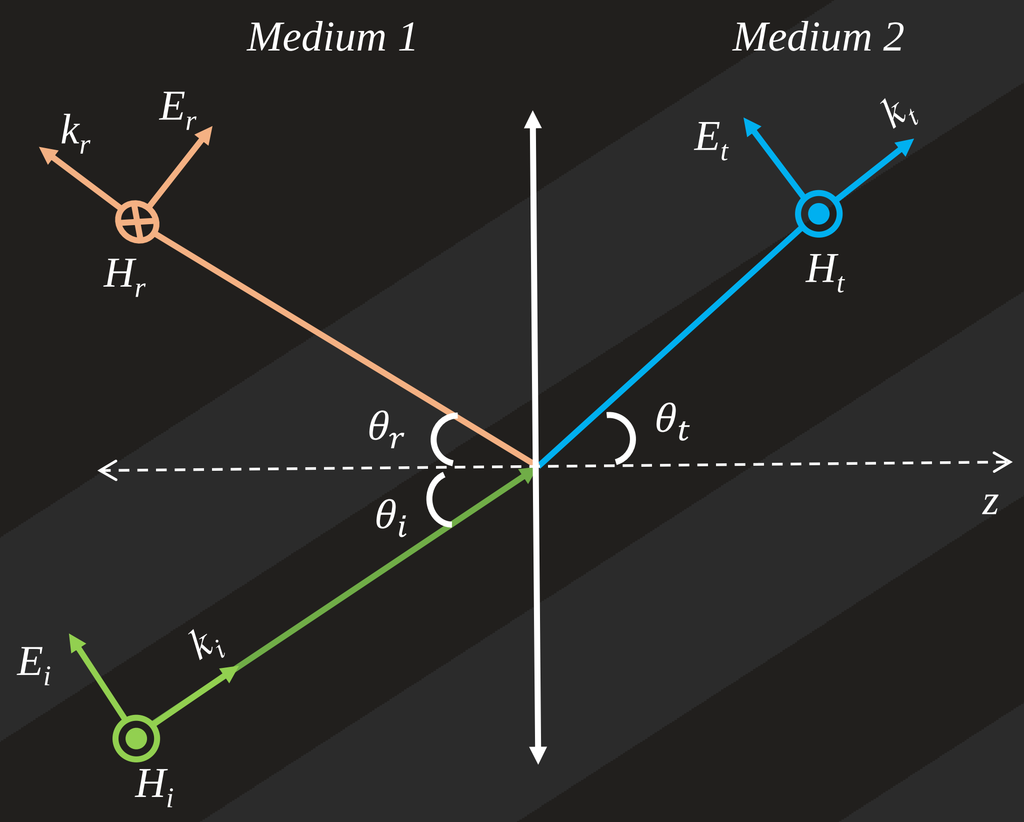
$$H_{is} = \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\, sin\theta_i + z\cos\theta_i)} a_y$$

$$E_{rs} = E_{r0}(\cos\theta_r a_x + \sin\theta_r\, a_z)e^{-j\beta_1(x\, sin\theta_r - z\cos\theta_r)}$$

$$H_{rs} = -\frac{E_{r0}}{\eta_1} e^{-j\beta_1(x\, sin\theta_r - z\cos\theta_r)} a_y$$

Where

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$$



In medium 2

$$E_{ts} = E_{t0}(\cos\theta_t a_x - \sin\theta_t\, a_z)e^{-j\beta_2(x\, sin\theta_t + z\cos\theta_t)}$$

$$H_{ts} = \frac{E_{t0}}{\eta_2} e^{-j\beta_2(x\, sin\theta_t + z\cos\theta_t)} a_y$$

Where

$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$$

Requiring that $\theta_r = \theta_i$ and that the tangential components of E & H be continuous at the boundaries z=0, we get

$$E_{is} + E_{rs} = E_{ts}$$

$$H_{is} + H_{rs} = H_{ts}$$

$$(E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{1}{\eta_2}E_{t0}$$

Expressing E_{r0} and E_{t0} in terms of E_{i0}

$$\boxed{\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} \qquad E_{r0} = \Gamma_{||}E_{i0}}$$

and

$$\boxed{\tau_{||} = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} \qquad E_{t0} = \boldsymbol{\tau}_{||}E_{i0}}$$

These equations are called Fresnel's equations for parallel polarization

In medium 1

$$E_{is} = E_{i0}(\cos\theta_i a_x - \sin\theta_i\, a_z)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$H_{is} = \frac{E_{i0}}{\eta_1}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}a_y$$

$$E_{rs} = E_{r0}(\cos\theta_r a_x + \sin\theta_r\, a_z)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$H_{rs} = -\frac{E_{r0}}{\eta_1}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}a_y$$

In medium 2

$$E_{ts} = E_{t0}(\cos\theta_t a_x - \sin\theta_t\, a_z)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$H_{ts} = \frac{E_{t0}}{\eta_2}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}a_y$$

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$$\Gamma_{||} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} \qquad E_{r0} = \Gamma_{||} E_{i0}$$

In this equation, it is possible that $\Gamma_{||}$ =0, under this condition there is no reflection (E_{r0}=0), and the incident angle at which this takes place is called ***<u>Brewster angle</u>*** ***$\theta_{B||}$ (polarizing angle)***

The Brewster angle is obtained by settling $\theta_i = \theta_{B||}$ when $\Gamma_{||}$ =0

$$\eta_2 \cos\theta_t = \eta_1 \cos\theta_{B||}$$

$$\eta_2^2(1 - \sin^2\theta_t) = \eta_1^2(1 - \sin^2\theta_{B||})$$

$$\boxed{\sin^2\theta_{B||} = \frac{1 - \frac{\mu_2\varepsilon_1}{\mu_1\varepsilon_2}}{1 - \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2}}$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$$ *Snell's law*

$$n_1 = c\sqrt{\mu_1\varepsilon_1}$$
$$n_2 = c\sqrt{\mu_2\varepsilon_2}$$
Refractive indices

When $\mu_1 = \mu_2 = \mu_0$

$$\boxed{\tan\theta_{B||} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}}$$

PLANE WAVE AT OBLIQUE INCIDENCE - CASE 2 : PERPENDICULAR POLARIZATION

When the E field is perpendicular to the plane of incidence which is perpendicular polarization

In medium 1

$$E_{is} = E_{i0} e^{-j\beta_1 (x\, sin\theta_i + z\cos\theta_i)} a_y$$

$$H_{is} = \frac{E_{i0}}{\eta_1} (-\cos\theta_i a_x + \sin\theta_i\, a_z) e^{-j\beta_1 (x\, sin\theta_i + z\cos\theta_i)}$$

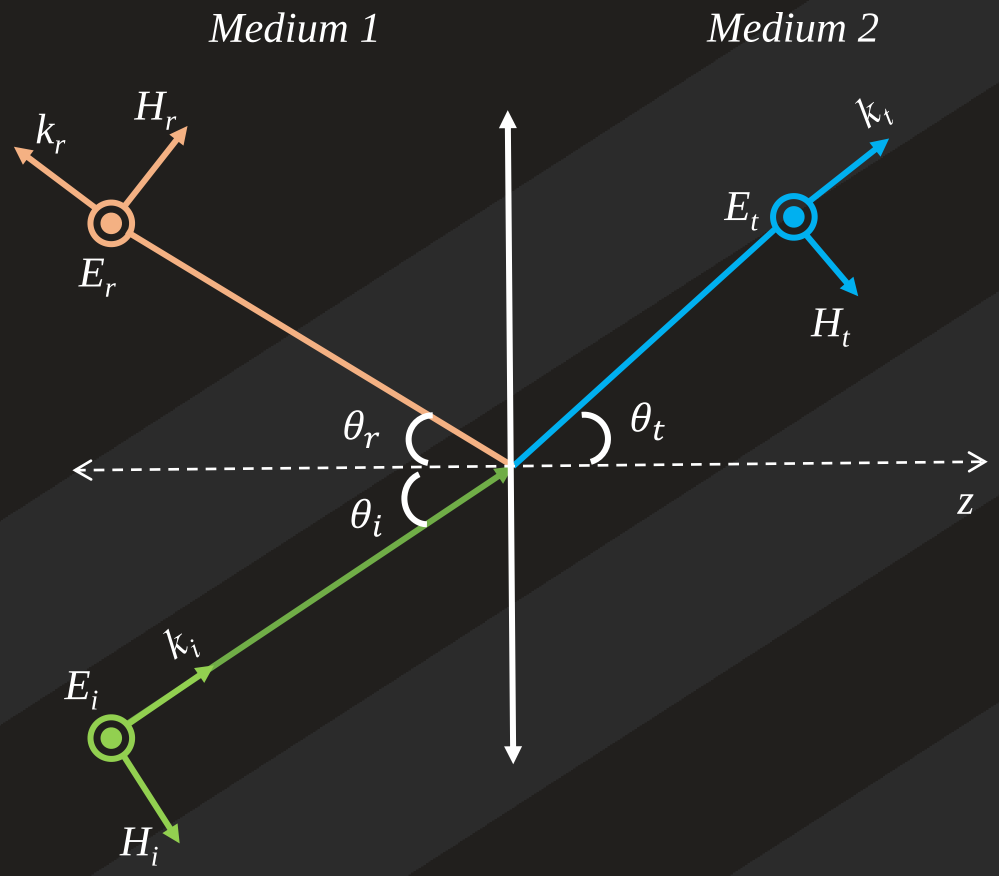
$$E_{rs} = E_{r0} e^{-j\beta_1 (x\, sin\theta_r - z\cos\theta_r)} a_y$$

$$H_{rs} = \frac{E_{r0}}{\eta_1} (\cos\theta_r a_x + \sin\theta_r\, a_z) e^{-j\beta_1 (x\, sin\theta_r - z\cos\theta_r)}$$

In medium 2

$$E_{ts} = E_{t0} e^{-j\beta_2 (x\, sin\theta_t + z\cos\theta_t)} a_y$$

$$H_{ts} = \frac{E_{t0}}{\eta_2} (-\cos\theta_t a_x + \sin\theta_t\, a_z) e^{-j\beta_2 (x\, sin\theta_t + z\cos\theta_t)}$$



Requiring that $\theta_r = \theta_i$ and that the tangential components of E & H be continuous at the boundaries z=0, we get

$$E_{is} + E_{rs} = E_{ts} \qquad H_{is} + H_{rs} = H_{ts}$$

$$E_{i0} + E_{r0} = E_{t0} \qquad \frac{1}{\eta_1}(E_{i0} - E_{r0})\cos\theta_i = \frac{1}{\eta_2}E_{t0}\cos\theta_t$$

Expressing E_{r0} and E_{t0} in terms of E_{i0}

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \qquad E_{r0} = \Gamma_\perp E_{i0}$$

and

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} \qquad E_{t0} = \boldsymbol{\tau}_\perp E_{i0}$$

These equations are called Fresnel's equations for perpendicular polarization

In medium 1

$$E_{is} = E_{i0}e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}a_y$$

$$H_{is} = \frac{E_{i0}}{\eta_1}(-\cos\theta_i a_x + \sin\theta_i\, a_z)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$E_{rs} = E_{r0}e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}a_y$$

$$H_{rs} = \frac{E_{r0}}{\eta_1}(\cos\theta_r a_x + \sin\theta_r\, a_z)e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

In medium 2

$$E_{ts} = E_{t0}e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}a_y$$

$$H_{ts} = \frac{E_{t0}}{\eta_2}(-\cos\theta_t a_x + \sin\theta_t\, a_z)e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

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$$\Gamma_{\perp} = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t} \qquad E_{r0} = \Gamma_{\perp} E_{i0}$$

In this equation, it is possible that $\Gamma_{\perp}$ =0, under this condition there is no reflection (E_r=0), and the incident angle at which this takes place is called ***Brewster angle** $\theta_{B\perp}$ **(polarizing angle)***

The Brewster angle is obtained by settling $\theta_i = \theta_{B\perp}$ when $\Gamma_{\perp}$ =0

$$\eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t$$

$$\eta_2^2(1 - \sin^2\theta_{B\perp}) = \eta_1^2(1 - \sin^2\theta_t)$$

$$\boxed{\sin^2\theta_{B\perp} = \frac{1 - \frac{\mu_1 \varepsilon_2}{\mu_2 \varepsilon_1}}{1 - \left(\frac{\mu_1}{\mu_2}\right)^2}}$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$$

Snell's law

$$n_1 = c\sqrt{\mu_1 \varepsilon_1}$$

$$n_2 = c\sqrt{\mu_2 \varepsilon_2}$$

Refractive indices

When $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$

$$\boxed{\tan\theta_{B\perp} = \sqrt{\frac{\mu_2}{\mu_1}}}$$