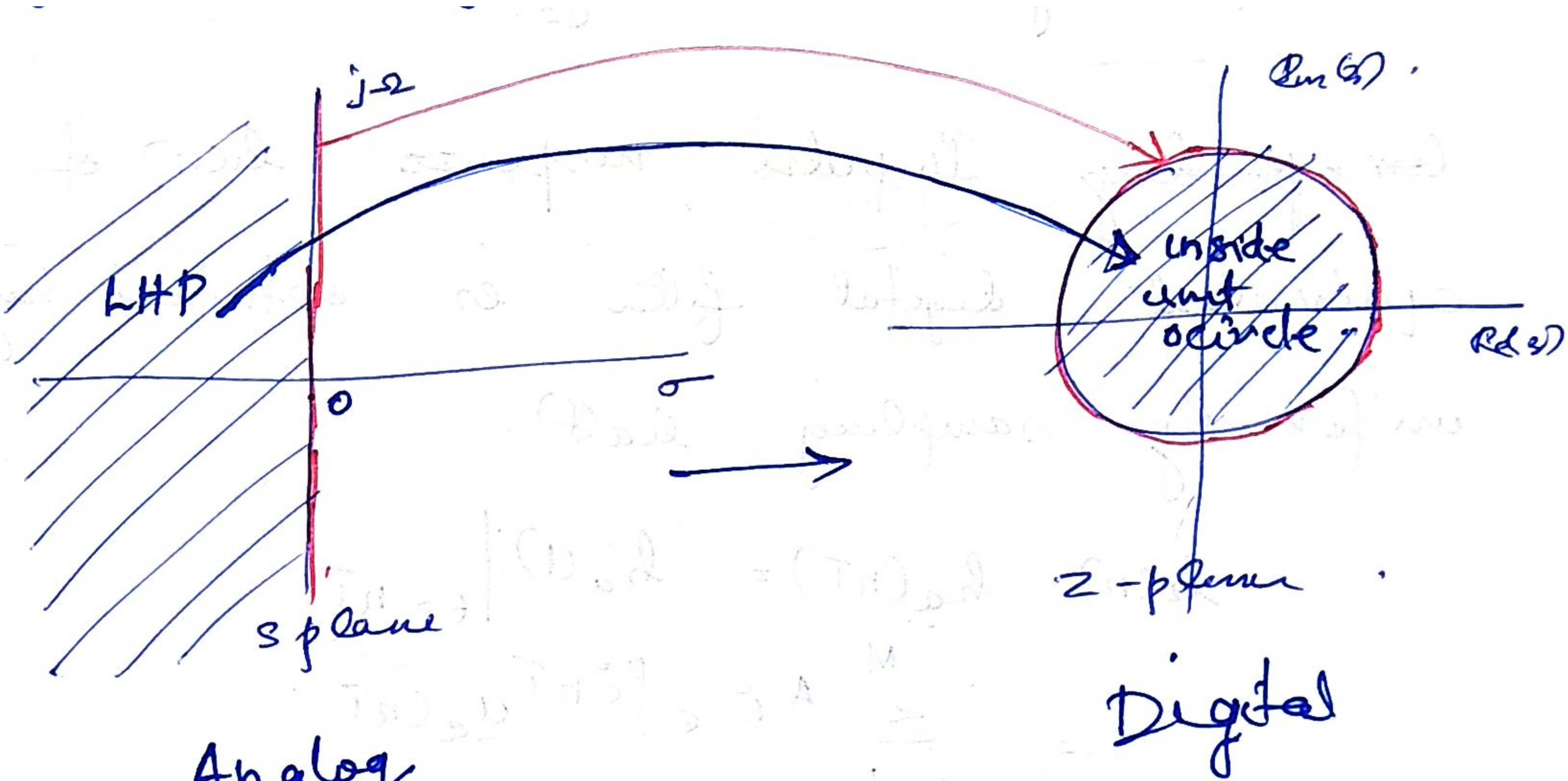
5.12 Design of IIR filters from analog filters.

There are several methods that can be used to design digital filters having an infinite duration unit sample response. The techniques described are all based on converting an analog filter into a digital filter. If the conversion technique is to be effective, it should possess the following desirable properties.

1. The $j\Omega$-axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship between the two frequency variables in the two domains.
2. The left-half plane of the s-plane should map into the inside of the unit circle in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.

The four most widely used methods for digitizing the analog filter into a digital filter include

1. Approximation of derivatives.
2. The impulse invariant transformation.
3. The bilinear transformation.
4. The matched z-transformation technique.



Mapping of s-plane to z-plane.

5.12.2 *Design of IIR Filter using Impulse Invariance Technique*

In impulse invariance method the IIR filter is designed such that the unit impulse response $h(n)$ of digital filter is the sampled version of the impulse response of analog filter.

The z-transform of an infinite impulse response is given by

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{5.71}$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n) e^{-sTn}$$

Let us consider the mapping of points from the s-plane to the z-plane implied by the relation

$$z = e^{sT} \tag{5.72}$$

If we substitute $s = \sigma + j\Omega$ and express the complex variable z in polar form as $z = re^{j\omega}$ we get

$$\begin{aligned} re^{j\omega} &= e^{(\sigma + j\Omega)T} \\ &= e^{\sigma T} e^{j\Omega T} \end{aligned} \tag{5.73}$$

which gives

$$r = e^{\sigma T} \tag{5.74a}$$

and

$$\omega = \Omega T \tag{5.74b}$$

The first term in the product in Eq.(5.73), $e^{\sigma T}$, has a magnitude of $e^{\sigma T}$ and an angle of 0 - a real number. The second term $e^{j\Omega T}$, has unity magnitude and an angle of ΩT. Therefore, our analog pole is mapped to a place in the z-plane of magnitude

$e^{\sigma T}$ and angle ΩT. The real part of the analog pole determines the radius of the z-plane pole and the imaginary part of the analog pole dictates the angle of the digital pole.

Consider any pole on the $j\Omega$-axis, where $\sigma = 0$ as shown in Fig. 5.20. These poles map to the z-plane at a radius $r = e^{0.T} = 1$. Therefore, the impulse invariant mapping map poles from the s-plane's $j\Omega$-axis to the z-plane's unit circle.

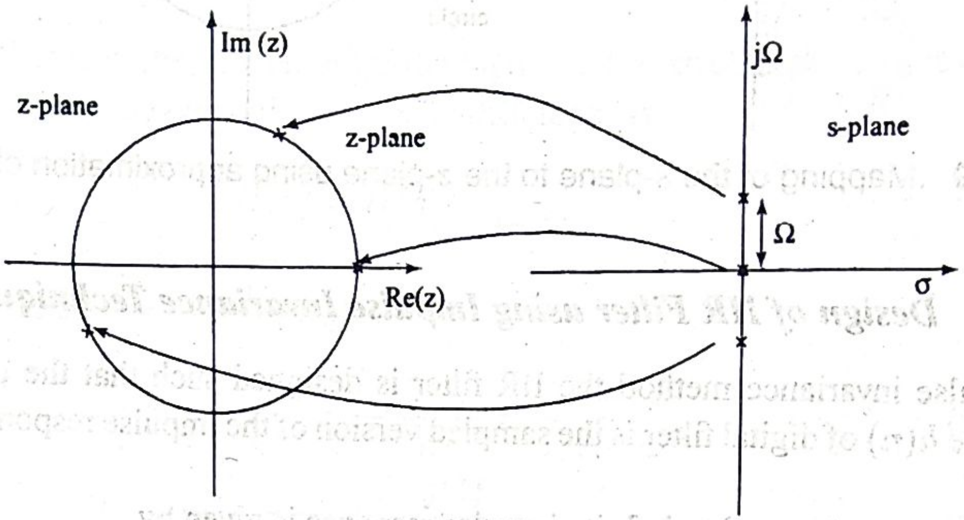


Fig. 5.20 $j\Omega$-axis mapping to the unit circle

Now consider the poles in the left half of s-plane where $\sigma < 0$. These poles map inside the unit circle as shown in Fig. 5.21, because $r = e^{\sigma T} < 1$ for $\sigma < 0$.

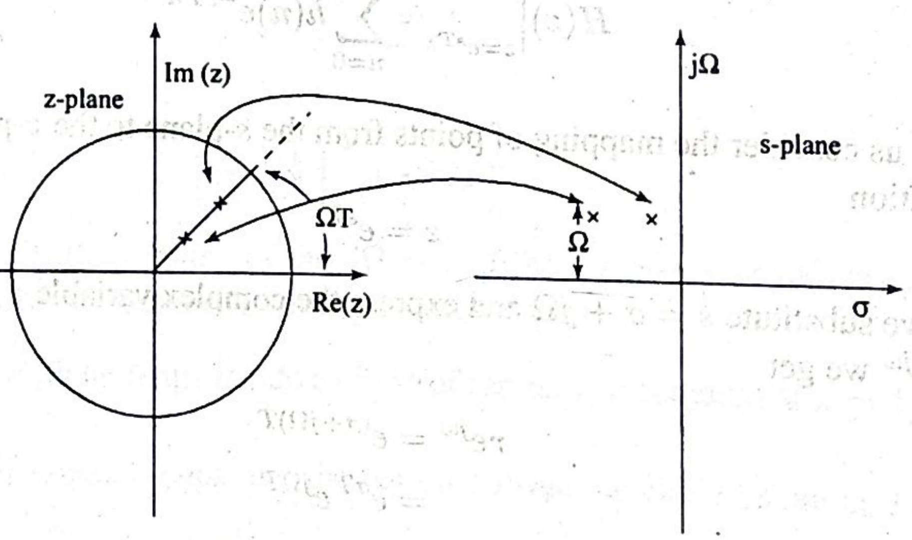


Fig. 5.21 Stable poles mapping inside the unit circle

Therefore, all s-plane poles with negative real parts map to z-plane poles inside the unit circle - stable analog poles are mapped to stable digital poles. The impulse invariant mapping preserves the stability of the filter.

All poles in the right half of the s-plane map to digital poles outside the unit circle.

$$r = e^{\sigma T} > 1 \quad \text{for} \quad \sigma > 0$$

The mapping is shown in Fig. 5.22.

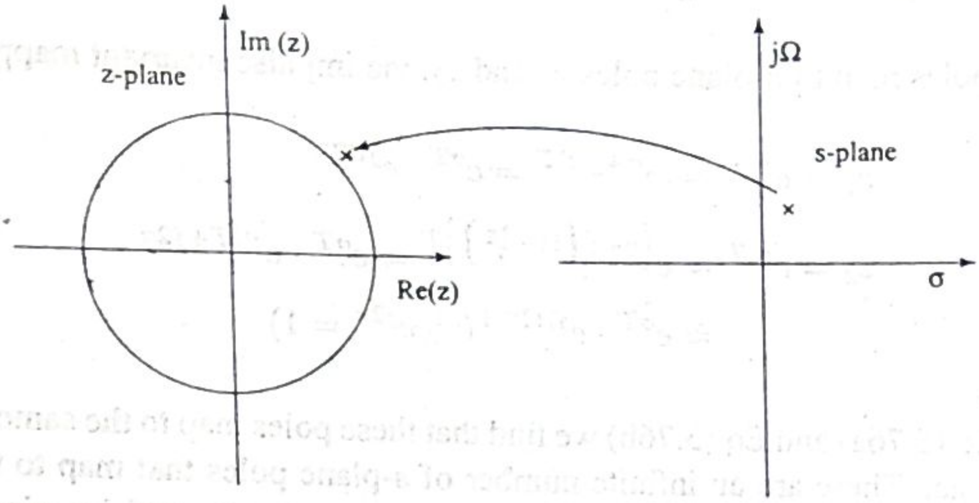


Fig. 5.22 Unstable poles mapping, outside the unit circle

Although the $j\Omega$-axis is mapped into the unit circle, it is not one-to-one mapping rather it is many-to-one mapping, where many points in s-plane are mapped to a single point in the z-plane. The easiest way to explain this is to consider two poles in the s-plane with identical real parts, but with imaginary components differing by $\frac{2\pi}{T}$ as shown in Fig. 5.23.

Let the poles be

$$s_1 = \sigma + j\Omega$$
$$s_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right) \tag{5.75}$$

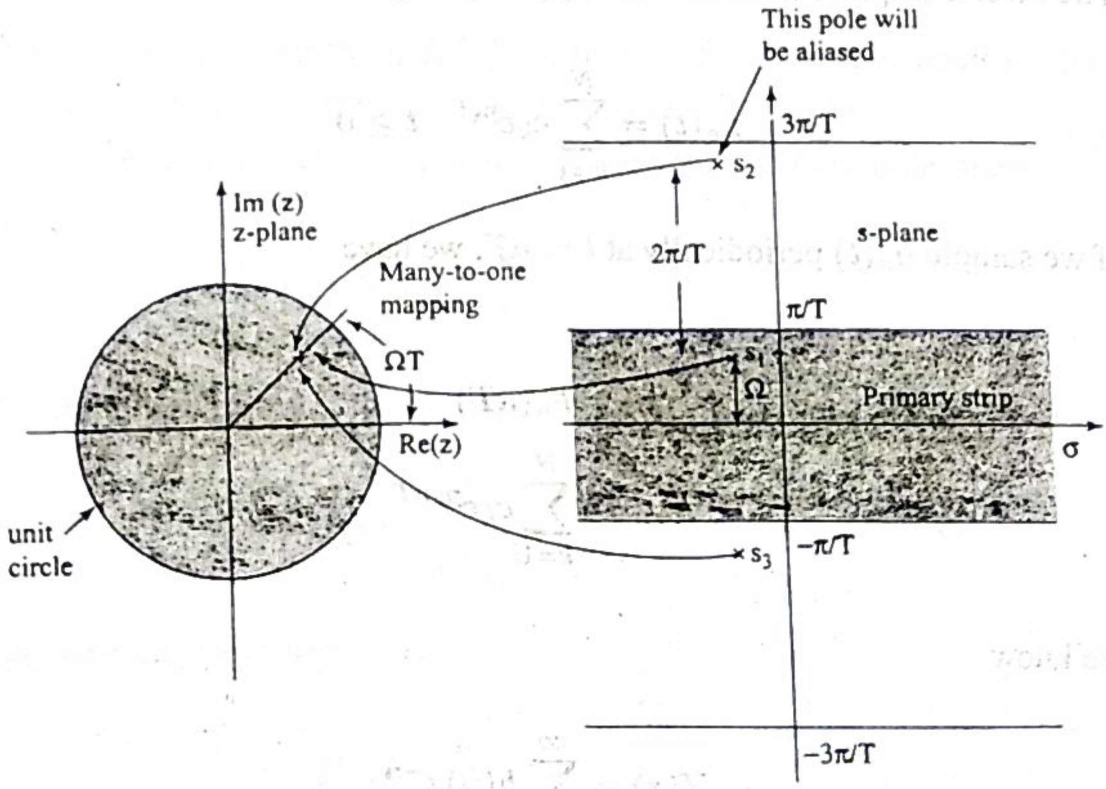


Fig. 5.23 Impulse invariant pole mapping

These poles map to z-plane poles z_1 and z_2, via impulse invariant mapping.

$$z_1 = e^{s_1 T} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T} \tag{5.76a}$$

$$\begin{aligned} z_2 = e^{s_2 T} &= e^{\left[\sigma + j\left(\Omega + \frac{2\pi}{T}\right)\right]T} = e^{\sigma T} \cdot e^{j\Omega T + j2\pi} \\ &= e^{\sigma T} \cdot e^{j\Omega T} \quad (\because\ e^{j2\pi} = 1) \end{aligned} \tag{5.76b}$$

From Eq.(5.76a) and Eq.(5.76b) we find that these poles map to the same location in the z-plane. There are an infinite number of s-plane poles that map to the same location in the z-plane. They must have the same real parts and imaginary parts that differ by some integer multiple of $\frac{2\pi}{T}$. This is the main disadvantage of impulse invariant mapping. The s-plane poles having imaginary parts greater than $\frac{\pi}{T}$ or less than $-\frac{\pi}{T}$ cause aliasing, when sampling analog signals. The analog poles will not be aliased by the impulse invariant mapping if they are confined to the s-plane's "Primary strip" (within π/T of the real axis).

Let $H_a(s)$ is the system function of an analog filter. This can be expressed in partial fraction form as

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \tag{5.77}$$

where $\{p_k\}$ are the poles of the analog filter and $\{c_k\}$ are the coefficients in the partial fraction expansion.

The inverse Laplace transform of Eq.(5.77) is

$$h_a(t) = \sum_{k=1}^{N} c_k e^{p_k t} \quad t \geq 0 \tag{5.78}$$

If we sample $h_a(t)$ periodically at $t = nT$, we have

$$\begin{aligned} h(n) &= h_a(nT) \\ &= \sum_{k=1}^{N} c_k e^{p_k nT} \end{aligned} \tag{5.79}$$

We know

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \tag{5.80}$$

Substituting Eq.(5.79) in Eq.(5.80) we obtain

$$
\begin{aligned}
H(z) &= \sum_{n=0}^{\infty} \sum_{k=1}^{N} c_k e^{p_k n T} z^{-n} \\
&= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1}\right)^n \\
&= \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}} \qquad (5.81a)
\end{aligned}
$$

That is if $H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$ then $H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$.

For high sampling rates (for small T), the digital filter gain is high. Therefore, instead of Eq. (5.81a) we can use

$$
H(z) = \sum_{k=1}^{N} \frac{T c_k}{1 - e^{p_k T} z^{-1}} \qquad (5.81b)
$$

Due to the presence of aliasing, the impulse invariant method is appropriate for the design of lowpass and bandpass filters only. The impulse invariance method is unsuccessful for implementing digital filters such as a highpass filter.

Steps to design a digital filter using Impulse Invariance method

1. For the given specifications, find $H_a(s)$, the transfer function of an analog filter.
2. Select the sampling rate of the digital filter, T seconds per sample.
3. Express the analog filter transfer function as the sum of single-pole filters.

$$
H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}
$$

4. Compute the z-transform of the digital filter by using the formula

$$
H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}.
$$

For high sampling rates use

$$
H(z) = \sum_{k=1}^{N} \frac{T c_k}{1 - e^{p_k T} z^{-1}}
$$

Example 5.11 For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

Solution

Given $H(s) = \dfrac{2}{(s+1)(s+2)}$

Using partial fraction we can write

$$\boxed{\begin{aligned} A &= (s+1)\frac{2}{(s+1)(s+2)}\bigg|_{s=-1} \\ &= 2 \\ B &= (s+2)\frac{2}{(s+1)(s+2)}\bigg|_{s=-2} \\ &= -2 \end{aligned}}$$

$$\begin{aligned} H(s) &= \frac{A}{s+1} + \frac{B}{s+2} \\ H(s) &= \frac{2}{s+1} - \frac{2}{s+2} \\ &= \frac{2}{s-(-1)} - \frac{2}{s-(-2)} \end{aligned}$$

Using impulse invariance technique we have, if

$$H(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

i.e., $(s - p_k)$ is transformed to $1 - e^{p_k T} z^{-1}$.

There are two poles $p_1 = -1$ and $p_2 = -2$. So

$$H(z) = \frac{2}{1 - e^{-T} z^{-1}} - \frac{2}{1 - e^{-2T} z^{-1}}$$

For $T = 1$ sec

$$\begin{aligned} H(z) &= \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}} \\ &= \frac{2}{1 - 0.3678 z^{-1}} - \frac{2}{1 - 0.1353 z^{-1}} \\ H(z) &= \frac{0.465 z^{-1}}{1 - 0.503 z^{-1} + 0.04976 z^{-2}} \end{aligned}$$

Example 5.13 Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period $T = 1$ sec.

Solution

From the table 5.1, for $N = 3$, the transfer function of a normalised Butterworth filter is given by

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$
$$= \frac{A}{s+1} + \frac{B}{s+0.5+j0.866} + \frac{C}{s+0.5-j0.866}$$

$$A = (s+1)\frac{1}{(s+1)(s^2+s+1)}\bigg|_{s=-1} = \frac{1}{(-1)^2-1+1} = 1$$

$$B = (s+0.5+j0.866)\frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)}\bigg|_{s=-0.5-j0.866}$$

$$= \frac{1}{(-0.5-j0.866+1)(-j0.866-j0.866)}$$

$$= \frac{1}{-j1.732(0.5-j0.866)} = \frac{1}{-j0.866-1.5}$$

$$= \frac{-1.5+j0.866}{3} = -0.5+j0.288$$

$$C = B^* = -0.5-j0.288$$

Hence

$$H(s) = \frac{1}{s+1} + \frac{-0.5+0.288j}{s+0.5+j0.866} + \frac{-0.5-0.288j}{s+0.5-j0.866}$$

$$= \frac{1}{s-(-1)} + \frac{-0.5+0.288j}{s-(-0.5-j0.866)} + \frac{-0.5-0.288j}{s-(-0.5+j0.866)}$$

In impulse invariant technique

$$\text{if} \quad H(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}, \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T} z^{-1}}$$

Therefore,

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j0.288}{1-e^{-0.5}e^{-j0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5}e^{j0.866}z^{-1}}$$

$$= \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$