# MAT 202, MAT 204

## Module. 5

## Numerical Methods. II

Text: Erwin kreyszig, Advanced Engineering Mathematics, 10th Edi: John Wiley & Sons 2016

[Sections 20.3, 20.5, 21.1]

Solution of linear systems - Gauss Siedal and Jacobi iteration methods. Curve fitting - method of least squares, fitting straight lines and parabolas. Solution of ordinary differential equations - Euler and classical Runge-kutta method of second and fourth order, Adams-Moulton predictor-correction method (Proof or derivation of the formulae not required for any of the methods in this module)

# Jacobi iteration method

[arrange the given system of equation in diagonaly dominant form

$$3x_1 - x_2 + x_3 = 1$$

$$3x_1 + 3x_2 + 7x_3 = 4$$

$$3x_1 + 6x_2 + 2x_3 = 0$$

$3+3=6<7$

$$3x_1 - x_2 + x_3 = 1$$

$$3x_1 + 6x_2 + 2x_3 = 0$$

$$3x_1 + 3x_2 + 7x_3 = 4$$

initial value is

$$x_1 = 0, \; x_2 = 0, \; x_3 = 0$$

Solve the following system of equations by Gauss-Jacobi method.

$x+y+54z=110,\ 6x+15y+2z=72,\ 27x+6y-z=85$

$$\left.\begin{aligned} 27x+6y-z&=85\\ 6x+15y+2z&=72\\ x+y+54z&=110 \end{aligned}\right\}$$

$$\boxed{\begin{aligned} x&=\frac{1}{27}\left[85-6y+z\right]\\ y&=\frac{1}{15}\left[72-6x-2z\right]\\ z&=\frac{1}{54}\left[110-x-y\right] \end{aligned}}$$

Let initial values be $x_0=0,\ y_0=0,\ z_0=0$

**1st iteration**

$x_1=\frac{1}{27}\left[85-6y_0+z_0\right]=\frac{1}{27}\left[85-0+0\right]=3.1481=3.148$

$y_1=\frac{1}{15}\left[72-6x_0-2z_0\right]=\frac{1}{15}\left[72-0-0\right]=4.8$

2.037

$z_1=\frac{1}{54}\left[110-x_0-y_0\right]=\frac{1}{54}\left[110-0-0\right]=2.0370=\cancel{2.037}$

**2nd iteration**

2.1569

$x_2=\frac{1}{27}\left[85-6y_1+z_1\right]=\frac{1}{27}\left[85-6(4.8)+(2.037)\right]=\underline{2.157}$

$y_2=\frac{1}{15}\left[72-6x_1-2z_1\right]=\frac{1}{15}\left[72-6(3.148)-2(2.037)\right]=\underline{3.269}$

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$z_2=\frac{1}{54}\left[110-x_1-y_1\right]=\frac{1}{54}\left[110-3.148-4.8\right]=1.8898$

$=1.890$

## 3rd iteration

$$x_3 = \frac{1}{27}\left[85 - 6y_2 + z_2\right] = \frac{1}{27}\left[85 - 6(3.269) + (1.890)\right] = 2.49170 = 2.492$$

$$y_3 = \frac{1}{15}\left[72 - 6x_2 - 2z_2\right] = \frac{1}{15}\left[72 - 6(2.157) - 2(1.890)\right] = 3.6852 = 3.685$$

$$z_3 = \frac{1}{54}\left[110 - x_2 - y_2\right] = \frac{1}{54}\left[110 - 2.157 - 3.269\right] = 1.937$$

## 4th iteration

$$x_4 = \frac{1}{27}\left[85 - 6y_3 + z_3\right] = \frac{1}{27}\left[85 - 6(3.685) + 1.937\right] = 2.401$$

$$y_4 = \frac{1}{15}\left[72 - 6x_3 - 2z_3\right] = \frac{1}{15}\left[72 - 6(2.492) - 2(1.937)\right] = 3.545$$

$$z_4 = \frac{1}{54}\left[110 - x_3 - y_3\right] = \frac{1}{54}\left[110 - 2.492 - 3.685\right] = 1.923$$

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## $5^{th}$ iteration

$$x_5 = \frac{1}{27}[85 - 6y_4 + z_4] = \frac{1}{27}[85 - 6(3.545) + 1.923] = 2.432$$

$$y_5 = \frac{1}{15}[72 - 6x_4 - 2z_4] = \frac{1}{15}[72 - 6(2.401) - 2(1.923)] = 3.583$$

$$z_5 = \frac{1}{54}[110 - x_4 - y_4] = \frac{1}{54}[110 - (2.401) - (3.545)] = 1.927$$

## $6^{th}$ iteration

$$x_6 = \frac{1}{27}[85 - 6y_5 + z_5] = \frac{1}{27}[85 - 6(3.583) + 1.927] = 2.423$$

$$y_6 = \frac{1}{15}[72 - 6x_5 - 2z_5] = \frac{1}{15}[72 - 6(2.432) - 2(1.927)] = 3.570$$

$$z_6 = \frac{1}{54}[110 - x_5 - y_5] = \frac{1}{54}[110 - 2.432 - 3.583] = 1.926.$$

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## 7th Iteration

$$x_7 = \frac{1}{27}\left[85 - 6y_6 - z_6\right] = \frac{1}{27}\left[85 - 6(3.570) + 1.926\right] = 2.426$$

$$y_7 = \frac{1}{15}\left[72 - 6x_6 - 2z_6\right] = \frac{1}{15}\left[72 - 6(2.423) - 2(1.926)\right] = 3.574$$

$$z_7 = \frac{1}{54}\left[110 - x_6 - y_6\right] = \frac{1}{54}\left[110 - 2.423 - 3.570\right] = 1.926$$

## 8th Iteration

$$x_8 = \frac{1}{27}\left[85 - 6y_7 - z_7\right] = \frac{1}{27}\left[85 - 6(3.574) - 1.926\right] = 2.425$$

$$y_8 = \frac{1}{15}\left[72 - 6x_7 - 2z_7\right] = \frac{1}{15}\left[72 - 6(2.426) - 2(1.926)\right] = 3.573.$$

$$z_8 < \frac{1}{54}\left(110 - x_7 - y_7\right] = \frac{1}{54}\left(110 - 2.426 - 3.574\right] = 1.926.$$

## 9th Iteration

$$x_9 = \frac{1}{27}\left[85 - 6y_8 - z_8\right] = \frac{1}{27}\left[85 - 6(3.573) - 1.926\right] = 2.426$$

$$y_9 = \frac{1}{15}\left[72 - 6x_8 - 2z_8\right] = \frac{1}{15}\left[72 - 6(2.425) - 2(1.926)\right] = 3.573$$

$$z_9 = \frac{1}{54}\left[110 - x_8 - y_8\right] = \frac{1}{54}\left(110 - 2.425 - 3.573\right] = 1.926$$

## $10^{th}$ Iteration

$$x_{10} = \frac{1}{27}[85 - 6y_9 + z_9] = \frac{1}{27}[85 - 6(3.573) + 1.926] = 2.426$$

$$y_{10} = \frac{1}{15}[72 - 6x_9 - 2z_9] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z_{10} = \frac{1}{54}[110 - x_9 - y_9] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926.$$

$\therefore$ Soln is $x = 2.426, y = 3.573, z = 1.926$

(3D)

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$x_0 = 0$

$y_0 = 0$

$z_0 = 0$

HW

$3x_1 - x_2 + x_3 = 1$

$3x_1 + 6x_2 + 2x_3 = 0$

$3x_1 + 3x_2 + 7x_3 = 4$

$x_1 = 0.0354$ $x_2 = -0.2367$

$x_3 = 0.6577$

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## Gauss-Siedel Method

First arrange the system of equations as diagonally dominant

Initial values are. $y_0=0, z_0=0$

Solve the following system of equations by Gauss - Seidal method.

$6x + 15y + 2z = 72, \quad 27x + 6y - z = 85, \quad x + y + 54z = 110.$

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Diagonally dominant form:

$$27x + 6y - z = 85 \quad \text{—(1)}$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

$$x = \frac{1}{27}[85 - 6y + z]$$

$$y = \frac{1}{15}[72 - 6x - 2z]$$

$$z = \frac{1}{54}[110 - x - y]$$

initial value $y_0 = 0, z_0 = 0$

1st Iteration

$$x_1 = \frac{1}{27}[85 - 6y_0 + z_0] = 3.148$$

$$y_1 = \frac{1}{15}[72 - 6x_1 - 2z_0] = \frac{1}{15}[72 - 6(3.148) - 0] = 3.541$$

$$z_1 = \frac{1}{54}[110 - x_1 - y_1] = \frac{1}{54}[110 - 3.148 - 3.541] = 1.913$$

2nd iteration.

$$x_2 = \frac{1}{27}[85 - 6y_1 + z_1] = \frac{1}{27}[85 - 6(3.541) + (1.913)] = 2.432$$

$$y_2 = \frac{1}{15}[72 - 6x_2 - 2z_1] = \frac{1}{15}[72 - 6(2.432) - 2(1.913)] = 3.57$$

$$z_2 = \frac{1}{54}[110 - x_2 - y_2] = \frac{1}{54}[110 - 2.432 - 3.572] = 1.926$$

$x_2 = 2.432$ , $y_2 = 3.572$ , $z_2 = 1.926$

## 3rd Iteration

$x_3 = \frac{1}{27}[85 - 6y_2 + z_2] = \frac{1}{27}[85 - 6(3.572) + (1.926)] = 2.426.$

$y_3 = \frac{1}{15}[72 - 6x_3 - 2z_2] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$

$z_3 = \frac{1}{54}[110 - x_3 - y_3] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$

## 4th Iteration

$x_4 = \frac{1}{27}[85 - 6y_3 + z_3] = \frac{1}{27}[85 - 6(3.573) + (1.926)] = 2.426$

$y_4 = \frac{1}{15}[72 - 6x_4 - 2z_3] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573.$

$z_4 = \frac{1}{54}[110 - x_4 - y_4] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926.$

$\therefore$ Soln is $x = 2.426$, $y = 3.573$, $z = 1.926$ Correct to 3.D.

$x_2 = 2.432,\ y_2 = 3.572,\ z_2 = 1.926$

Seidel.

Gauss Seidel.

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$4x + 2y + z = 14$

$x + y + 8z = 20$

$x + 5y - z = 10$

8th $x = 2,\ y = 2,\ z = 2$

## 3rd iteration

$$x_3 = \frac{1}{27}[85 - 6y_2 + z_2] = \frac{1}{27}[85 - 6(3.572) + (1.926)] = 2.426.$$

$$y_3 = \frac{1}{15}[72 - 6x_3 - 2z_2] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z_3 = \frac{1}{54}[110 - x_3 - y_3] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$$

## 4th iteration

$$x_4 = \frac{1}{27}[85 - 6y_3 + z_3] = \frac{1}{27}[85 - 6(3.573) + (1.926)] = 2.426$$

$$y_4 = \frac{1}{15}[72 - 6x_4 - 2z_3] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573.$$

$$z_4 = \frac{1}{54}[110 - x_4 - y_4] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926.$$

$\therefore$ Soln is $x = 2.426,\ y = 3.573,\ z = 1.926$ Correct to 3.D.

## $3^{rd}$ Iteration

$$x_3 = \frac{1}{20}\left[17 - y_2 + 2z_2\right] = \frac{1}{20}\left[17 - (-0.9998) + 2(0.9998)\right] = 1.00001 = 1.0000$$

$$y_3 = \frac{1}{20}\left[-18 - 3x_3 + z_2\right] = \frac{1}{20}\left[-18 - 3(1.0000) + (0.9998)\right] = -1.0000$$

$$z_3 = \frac{1}{20}\left[25 - 2x_3 + 3y_3\right] = \frac{1}{20}\left[25 - 2(1.0000) + 3(-1.0000)\right] = 1.0000$$

## $4^{th}$ Iteration

$$x_4 = \frac{1}{20}\left[17 - y_3 + 2z_3\right] = \frac{1}{20}\left[17 - (-1.0000) + 2(1.0000)\right] = 1.0000$$

$$y_4 = \frac{1}{20}\left[-18 - 3x_4 + z_3\right] = \frac{1}{20}\left[-18 - 3(1.0000) + 1.0000\right] = -1.0000$$

$$z_4 = \frac{1}{20}\left[25 - 2x_4 + 3y_4\right] = \frac{1}{20}\left[25 - 2(1.0000) + 3(-1.0000)\right] = 1.0000.$$

Soln is $x = 1.0000$, $y = -1.0000$, $z = 1.0000$ (4.D)

## Principle of Least Squares:

The principle of least squares states that the sum of squares of difference between the actual value and the approximated value should be minimum ie $E = \sum_{i=1}^{n} (y_i - f(x_i))^2$ is minimum.

### Fitting of straight line

$y = a + bx.$

y - dep:
x - indep

**Normal Equations:**

$\sum_{i=1}^{n} y = na + b\sum x$

$\sum xy = a\sum x + b\sum x^2$

$y = ax + b$

Normal equ:

$\sum y = a\sum x + nb$

$\sum xy = a\sum x^2 + b\sum x$

## ting of parabola.

$$y = a + bx + cx^2$$

### rmal equation

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

**Note:**

Parabola $y = ax^2 + bx + c.$

$$y = ax^2 + b$$

$$y = a + bx^2$$

### Normal equation.

$$\Sigma y = na + b\Sigma x^2$$

$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^4$$

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Fit a straight line by method of least squares of the following data.

| $x$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $y$ | 15 | 19 | 23 | 26 | 30 |

(5,15) (10,19)

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equ: of St: line $\boxed{y = a + bx}$

Normal equation.

$$\Sigma y = na + b\Sigma x$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 5 | 15 | 75 | 25 |
| 10 | 19 | 190 | 100 |
| 15 | 23 | 345 | 225 |
| 20 | 26 | 520 | 400 |
| 25 | 30 | 750 | 625 |
| $\Sigma x = 75$ | $\Sigma y = 113$ | $\Sigma xy = 1880$ | $\Sigma x^2 = 1375$ |

$$\Sigma y = na + b\Sigma x$$

$$113 = 5a + b(75)$$

$$113 = 5a + 75b \quad \text{--- } (1)$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

$$1880 = a(75) + b(1375)$$

$$1880 = 75a + 1375b \quad \text{--- } (2)$$

$$(1) \times 15 \Rightarrow \quad 1695 = 75a + 1125b.$$

$$(2) \Rightarrow \quad 1880 = 75a + 1375b$$

$$-185 = 0 - 250b.$$

$$b = \frac{-185}{-250} = 0.74$$

$$\boxed{b = 0.74}$$

$$(1) \Rightarrow 113 = 5a + 75(0.74)$$

$$5a = 113 - 55.5$$

$$5a = 57.5$$

$$\boxed{a = 11.5}$$

$\therefore$ equ. of St. line $y = a + bx$

$$\boxed{y = 11.5 + 0.74x}$$

HW Use the method of least squares to fit an equation of the form $y = ax + b$ to the following data.

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| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 6 | 7 | 9 | 10 | 12 |

$a = 1.5$

$b = \cancel{15}\ 4.3$

$y = a + bx$   3.66

$b = 1.07$

Fit a St: line to the following data.

a) regarding $x$ as the independent variable and $y$ as dependent variable.

b) regarding $y$ as the independent variable and $x$ as the dependent variable.

$x = c + dy$

| $x$ | 9 | 15 | 19 | 26 | 29 |
|---|---|---|---|---|---|
| $y$ | 14 | 18 | 25 | 31 | 35 |

$\Sigma x = nc + d\Sigma y$

$\Sigma xy = c\Sigma y + d\Sigma y^2$.

| $x$ | $y$ | $xy$ | $x^2$ | $y^2$ |
|---|---|---|---|---|

$c = -3.10$

$d = 0.92$.

The following table gives the tensile force $x$ (in thousands of pounds) applied to a steel specimen and the resulting elongation (in thousands of an inch)

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 14 | 33 | 40 | 63 | 76 | 85 |

find the equation of the least square line that fits the above data. Hence predict the elongation when the tensile force is 3.5 thousand pounds.

equ. of str. line $y = a + bx$.

normal equation is

$$\Sigma y = na + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 1 | 14 | 14 | 1 |
| 2 | 38 | 76 | 4 |
| 3 | 40 | 120 | 9 |
| 4 | 63 | 252 | 16 |
| 5 | 76 | 380 | 25 |
| 6 | 85 | 510 | 36 |
| $\Sigma x = 21$ | $\Sigma y = 311$ | $\Sigma xy = 1352$ | $\Sigma x^2 = 91$ |

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$$311 = 6a + 21b \quad \text{—(1)}$$

$$1342 = 21a + 91b \quad \text{—(2)}$$

$$(1) \times 21 \Rightarrow 6531 = 126a + 441b.$$

$$(2) \times 6 \Rightarrow 8052 = 126a + 546b$$

$$-1521 = 0 - 105b$$

$$b = \frac{-1521}{-105} = 14.4857$$

$$\boxed{b = 14.4857}$$

$$(1) \Rightarrow 311 = 6a + 21 \times 14.4857$$

$$a = \frac{311 - 21 \times 14.4857}{6}$$

$$\boxed{a = 1.1333}$$

$\therefore$ equ: of st: line

$$y = a + bx$$

$$\boxed{y = 1.1333 + 14.4857x}$$

when $x = 3.5$

$$y = 1.1333 + 14.4857(3.5)$$

$$= 51.8333$$

ii: Use method of least squares to fit a curve of the ... the following data by converting the equation $y=ax+bx^2$ to linear form.

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| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 2.1 | 4.4 | 7.2 | 10.4 | 15.7 | 18.3 |

$y = ax + bx^2$

$\div$ by $x$.

$\dfrac{y}{x} = a + bx$ $\qquad Y = \dfrac{y}{x}$

$Y = a + bx$

normal equ:

$\Sigma Y = na + b\Sigma x$

$\Sigma xY = a\Sigma x + b\Sigma x^2$

| $x$ | $y$ | $Y = \frac{y}{x}$ | $xY$ | $x^2$ |
|---|---|---|---|---|
| 1 | 2.1 | 2.1 | 2.1 | 1 |
| 2 | 4.4 | 2.2 | 4.4 | 4 |
| 3 | 7.2 | 2.4 | 7.2 | 9 |
| 4 | 10.4 | 2.6 | 10.4 | 16 |
| 5 | 15.7 | 3.14 | 15.7 | 25 |
| 6 | 18.3 | 3.05 | 18.3 | 36 |
| $\Sigma x = 21$ | $\Sigma y = 58.1$ | $\Sigma Y = 15.49$ | $\Sigma xY = 58.1$ | $\Sigma x^2 = 91$ |

$$15.49 = 6a + 21b \quad \text{---}(1)$$

$$58.1 = 21a + 91b \quad \text{---}(2)$$

$$(1) \times 21 \Rightarrow \quad 325.29 = 126a + 441b$$

$$(2) \times 6 \Rightarrow \quad 348.6 = 126a + 546b$$

$$-23.31 = 0 - 105b$$

$$b = \frac{-23.31}{-105} = 0.222$$

$$(1) \Rightarrow 15.49 = 6a + 21(0.222)$$

$$6a = 15.49 - 4.662.$$

$$a = \frac{10.828}{6} = 1.8046$$

$$= \underline{1.805}$$

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$$\therefore y = ax + bx^2$$

$$\boxed{y = 1.805x + 0.222x^2}$$

Fit a curve of the form $y = a + bx^2$ to the following data.

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 0.56 | 0.89 | 1.04 | 1.63 | 2.95 | 4.5 |

$y = a + bx^2$

normal form:

$\Sigma y = na + b\Sigma x^2$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^4$

| $x$ | $y$ | $x^2$ | $x^2y$ | $x^4$ |
|---|---|---|---|---|
| 1 | 0.56 | 1 | 0.56 | 1 |
| 2 | 0.89 | 4 | 3.56 | 16 |
| 3 | 1.04 | 9 | 9.36 | 81 |
| 4 | 1.63 | 16 | 26.08 | 256 |
| 5 | 2.95 | 25 | 73.75 | 625 |
| 6 | 4.5 | 36 | 162 | 1296 |
| | $\Sigma y = 11.57$ | $\Sigma x^2 = 91$ | $\Sigma x^2y = 275.31$ | 2275 |

$$11.57 = 6a + 91b \quad \text{—— (1)}$$

$$275.31 = 91a + 2275b \quad \text{—— (2)}$$

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$$\text{(1)} \times 91 \Rightarrow 1052.87 = 546a + 8281b$$

$$\text{(2)} \times 6 \Rightarrow 1651.86 = 546a + 13650b$$

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$$-598.99 = 0 - 5369b$$

$$b = \frac{-598.99}{-5.369}$$

$$\boxed{b = 0.1116}$$

$a = 1.4257$ $\quad b = -0.5214$ $\quad c = 0.3929$

(1) $\rightarrow$

$$11.57 = 6a + 91(0.1116)$$

$$6a = 11.57 - 10.1574$$

$$\boxed{a = 0.2363}$$

$$\therefore \quad y = a + bx^2$$

$$\boxed{y = 0.2363 + 0.1116x^2}$$

Fit a second degree parabola of the form $y = a + bx + cx^2$

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 1.2 | 1.7 | 2.1 | 2.8 | 5.9 |

5/ Fit a curve of the form $y = ae^{bx}$ to the following data.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 2.02 | 6.03 | 10.11 | 17.89 | 29.23 |

$y = ae^{bx}$ $\quad \log_{10}$ $\ln_e$

$\ln y = \ln(ae^{bx})$ $\quad \log(ab) = \log a + \log b$

$= \ln a + \ln e^{bx}$ $\quad \log e^{x} = x$

$= \ln a + bx$ $\quad$ Put $\ln y = Y$

$\ln a = A$

$$\boxed{Y = A + bx}$$

normal eqn: $\sum Y = nA + b\sum x$

$\sum xY = A\sum x + b\sum x^2$

| $x$ | $y$ | $Y = \ln y$ | $xY$ | $x^2$ |
|---|---|---|---|---|
| 0 | 2.02 | 0.7031 | 0 | 0 |
| 1 | 6.03 | 1.7967 | 1.7967 | 1 |
| 2 | 10.11 | 2.3135 | 4.6270 | 4 |
| 3 | 17.89 | 2.8842 | 8.6526 | 9 |
| 4 | 29.23 | 3.3752 | 13.5008 | 16 |
| $\sum x = 10$ | | 11.0727 | 28.5771 | 30 |

$11.0727 = 5A + 10b$ —(1)

$28.5771 = 10A + 30b$ —(2)

$$(1)\times 2 \Rightarrow 22.1454 = 10A + 20b$$

$$25.5771 = 10A + 30b$$

$$-6.4317 = 0 - 10b$$

$$b = \frac{-6.4317}{-10}$$

$$b = 0.64317$$

$$\boxed{b = 0.6432}$$

$$(1) \Rightarrow 11.0727 = 5A + 10 \times 0.6432$$

$$5A = 11.0727 - 6.4317$$

$$A = \frac{4.641}{5} = 0.9282.$$

$$A = \ln_e a = 0.9282$$

$$\ln_e a = 0.9282.$$

$$a = (e)^{0.9282}$$

$$= 2.53$$

$$y = a e^{bx}$$

$$\boxed{y = 2.53\, e^{0.6432x}}$$

$$\log_b a = c$$

$$a = (b)^c$$

$$\log_{10} a = 0.9282.$$

$$a = (10)^{0.9282}$$

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# Numerical Solution of ordinary differential equation (ODE)

$$\frac{dy}{dx} + y = e^{x}$$

$$y = CF + PI$$

$$\frac{dy}{dx} = e^{x} - y$$

$$f(x,y) = e^{x} - y$$

## 1) Eulers Method

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

**Note:-**

$$y_1 = y_0 + h f(x_0, y_0) \qquad h = x_1 - x_0$$

$$(x_0, y_0)$$

$$y_2 = y_1 + h f(x_1, y_1)$$

Solve numerically the initial value problem $y' - y = e^{x}\cos x$, $y(0) = 0$ using Euler method on the interval $0 \le x \le 1$ taking step size $h = 0.2$.

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

$$y' - y = e^{x}\cos x$$

$$y' = e^{x}\cos x + y$$

$$f(x, y) = e^{x}\cos x + y$$

$$h = 0.2$$

given $y(0) = 0$

$x_0 = 0$ $\quad y_0 = 0$

$x_1 = 0.2$ $\quad y_1 = ?$

$x_2 = 0.4$ $\quad y_2 = ?$

$x_3 = 0.6$ $\quad y_3 = ?$

$x_4 = 0.8$ $\quad y_4 = ?$

$x_5 = 1$ $\quad y_5 = ?$

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 0 + h f(0, 0)$$

$$= 0 + 0.2\left[e^{0}\cos 0 + 0\right]$$

$$\boxed{y_1 = 0.2}$$

$x_1 = 0.2$ $\quad y_1 = 0.2$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 0.2 + 0.2 f(0.2, 0.2)$$

$$= 0.2 + 0.2\left[e^{0.2}\cos 0.2 + 0.2\right]$$

(Radian)

$$= 0.4794112$$

$$\boxed{y_2 = 0.4794}$$

$x_2 = 0.4$ $\quad y_2 = 0.4794$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 0.4794 + 0.2 f(0.4, 0.4794)$$

$$= 0.4794 + 0.2 \left[ e^{0.4} \cos(0.4) + 0.4794 \right]$$

$$= 0.8500923$$

$$\boxed{y_3 = 0.8501}$$

$x_3 = 0.6 \qquad y_3 = 0.8501$

$$y_4 = y_3 + h f(x_3, y_3)$$

$$= 0.8501 + 0.2 f(0.6, 0.8501)$$

$$= 0.8501 + 0.2 \left[ e^{0.6} \cos(0.6) + 0.8501 \right]$$

$$= 1.32089 \qquad \boxed{y_4 = 1.3209}$$

$y_4 = 0.8 \qquad y_4 = 1.3209$

$$y_5 = y_4 + h f(x_4, y_4)$$

$$= 1.3209 + 0.2 f(0.8, 1.3209)$$

$$= 1.3209 + 0.2 \left[ e^{0.8} \cos(0.8) + 1.3209 \right]$$

$$= 1.895189$$

$$= 1.8952 \quad \text{(correct to 4D)}.$$

$x_4 = 1 \qquad \underline{y_5 = 1.3209}$.

| x | 0 | 0.2 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|---|
| y | 0 | ~~0.4794~~ 0.2 | ~~0.8501~~ 0.4794 | ~~0.8~~ 0.8501 | 1.3209 |

| x | 1 |
|---|---|
| y | 1.8952 |

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Use Euler's method to solve $\frac{dy}{dx} = x + xy + y$, $y(0)=1$, Compute $y$ at $x = 0.15$

~~by taking h=0.05.~~ ~~$x = 0.3$~~

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

$f(x,y) = x + xy + y$

given $y(0) = 1$

$x_0 = 0$ $\quad y_0 = 1$

$x_1 = 0.15$ $\quad y_1 = ?$

$h = x_1 - x_0 = 0.15 - 0$
$= \underline{\underline{0.15}}$

$y_1 = y_0 + h f(x_0, y_0)$.
$= 1 + 0.15 f(0, 1)$
$= 1 + 0.15 [0 + 0 + 1]$
$= \underline{\underline{1.15}}$

<u>HW</u>

Use Euler's method to evaluate $y(0.4)$ with $h = 0.1$ from the equation $\frac{dy}{dx} = \frac{y^2 - 2x}{y}$

given $y(0) = 1$

$x_0 = 0 \quad y_0 = 1 \quad f(x,y) = \,$ "

$x_1 = 0.1 \quad y_1 = ?$

$x_2 = 0.2 \quad y_2 = ?$

$x_3 = 0.3 \quad y_3 = ?$

$x_4 = 0.4 \quad y_4 = ?$

$y_4 = y_3 + h f(x_3, y_3)$

# Second order Runge-Kutta Method (R.K)

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f(x_n + h, x_n + k_1)$$

**Note:**

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h f(x_0, y_0)$$

$$k_2 = h f(x_0 + h, y_0 + k_1)$$

Using Modified Euler's method (R.k method of order 2) Solve $\frac{dy}{dx} = y - x^2 + 1$, $y(0) = 0.5$ for $0 \le x \le 0.4$ taking Step Size $h = 0.2$

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + h, y_n + k_1)$$

$f(x, y) = y - x^2 + 1$

$h = 0.2$

given $y(0) = 0.5$

$x_0 = 0 \qquad y_0 = 0.5$

$x_1 = 0.2 \qquad y_1 = ?$

$x_2 = 0.4 \qquad y_2 = ?$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$k_1 = hf(x_0, y_0)$

$= 0.2 f(0, 0.5)$

$= 0.2[0.5 - 0 + 1]$

$= 0.3$

$k_2 = hf(x_0 + h, y_0 + k_1)$ $\quad 0.5 + 0.3$

$= 0.2 f(0.2, 0.8)$

$= 0.2[0.8 - (0.2)^2 + 1]$

$= 0.352$

$y_1 = 0.5 + \frac{1}{2}(0.3 + 0.352)$

$= 0.826$

$\boxed{x_1 = 0.2 \qquad y_1 = 0.826}$

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2)$$

$k_1 = hf(x_1, y_1)$

$= 0.2 f(0.2, 0.826)$

$= 0.2[0.826 - (0.2)^2 + 1]$

$= 0.3572$

$k_2 = hf(x_1 + h, y_1 + k_1)$

$= 0.2 f(0.4, 1.1832)$

$= 0.2[1.1832 - (0.4)^2 + 1]$

$= 0.4046$

$y_2 = 0.826 + \frac{1}{2}[0.3572 + 0.4046]$

$= 1.2069$

Use Runge-kutta second order find the value of $y$ at $x=0.25$ and $x=0.5$ given $\frac{dy}{dx}=2xy$, $y(0)=1$

$$\boxed{\begin{aligned} y_{n+1} &= y_n+\frac{1}{2}(k_1+k_2)\\ k_1 &= hf(x_n,y_n)\\ k_2 &= hf(x_n+h,y_n+k_1)\end{aligned}}$$

$f(x,y)=2xy$

given $y(0)=1$

$x_0=0 \qquad y_0=1$

$x_1=0.25 \qquad y_1=?$

$x_2=0.5 \qquad y_2=?$

$$\begin{aligned} h &= x_1-x_0\\ &=0.25-0\\ &=0.25\end{aligned}$$

$$y_1=y_0+\frac{1}{2}(k_1+k_2)$$

$$\begin{aligned} k_1 &= hf(x_0,y_0)\\ &=0.25f(0,1)\\ &=0.25[0]\\ &=0\end{aligned}$$

0.125

$$\begin{aligned} k_2 &= hf(x_0+h,y_0+k_1)\\ &=0.25f(0.25,1)\\ &=0.25[2\times0.25\times1]\\ &=0.125\end{aligned}$$

$$\begin{aligned} y_1 &= 1+\frac{1}{2}(0+0.125)\\ &=1.0625\end{aligned}$$

$$\boxed{x_1=0.25 \qquad y_1=1.0625}$$

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$$y_2=y_1+\frac{1}{2}(k_1+k_2)$$

$$\begin{aligned} k_1 &= hf(x_1,y_1)\\ &=0.25f(0.25,1.0625)\\ &=0.25[2\times0.25\times1.0625]\\ &=0.1328\end{aligned}$$

$$\begin{aligned} k_2 &= hf(x_1+h,y_1+k_1)\\ &=0.25f(0.5,1.1953)\\ &=0.25[2\times0.5\times1.1953]\\ &=0.2988\end{aligned}$$

$$\begin{aligned} y_2 &= 1.0625+\frac{1}{2}[0.1328+0.2988]\\ &=\underline{1.2783}\end{aligned}$$

$x_1=0.5 \qquad y_2=1.2783$

① Solve by improved Euler's method $\frac{dy}{dx} = y + e^x$, $y(0) = 0$ for

$x = 0.2, 0.4$

$x_0 = 0 \quad y = 0$

$x_1 = 0.2 \quad y = ?$

$x_2 = 0.4 \quad y = ?$

$h = x_1 - x_0$

$= 0.2$

$f(x, y)$

$y_1 = 0.24214$

$y_2 = 0.59116$

| $x$ | 0 | 0.2 | 0.4 |
|---|---|---|---|
| $y$ | 0 | 0.24214 | 0.59116 |

② . Solve by R.k method of order 2. $y' = x^2 - y$, $y(0) = 1$. find correct to 4.D the value of $y(0.1)$

$x_0 = 0 \quad y_0 = 1$

$x_1 = 0.1 \quad y_1 = ?$

$h = 0.1$

$y_1 = 0.9055$

# Fourth order Runge-Kutta method [Classical Runge-Kutta Method]

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$$y_{n+1} = y_n + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

u.Q Given the initial value problem $\frac{dy}{dx} = \sqrt{x+y}$, $y(0)=1$, use Runge-Kutta method of fourth order to find $y(0.2)$ with $h=0.1$

$$y_{n+1} = y_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4].$$

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

$$f(x,y) = \sqrt{x+y}$$

$$h = 0.1$$

$$y(0) = 1$$

$x_0 = 0$ $\quad y_0 = 1$

$x_1 = 0.1$ $\quad y_1 = ?$

$x_2 = 0.2$ $\quad y_2 = ?$

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1\, f(0, 1)$$

$$= 0.1\sqrt{0+1}$$

$$= 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\, f(0.05, 1.05)$$

$$= 0.1\sqrt{0.05 + 1.05}$$

$$= 0.1048808$$

$$= 0.104881$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\, f(0.05, 1.05244)$$

$$= 0.1\sqrt{0.05 + 1.05244}$$

$$= 0.104997$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1\, f(0.1\ ,\ 1.104997)$$

$$= 0.1 \sqrt{0.1 + 1.104997}$$

$$= \underline{\underline{0.109772}}$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}\big(0.1 + 2(0.104881) + 2(0.104997) + 0.109772\big)$$

$$= \underline{\underline{1.104921}}$$

$$\boxed{x_1 = 0.1 \qquad y_1 = 1.104921}$$

$$y_2 = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$k_1 = hf(x_1, y_1)$$

$$= 0.1\, f(0.1, 1.104921)$$

$$= 0.1 \sqrt{0.1 + 1.104921}$$

$$= \underline{0.109769}$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1\, f(0.15, 1.1598055)$$

$$= 0.1 \sqrt{0.15 + 1.1598055}$$

$$= 0.114447$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1\, f\left(0.15, 1.104921 + \frac{0.114447}{2}\right)$$

$$= 0.1\, f(0.15, 1.162156)$$

$$= 0.1 \left[0.15 + 1.162156\right]$$

$$= \cancel{0.119142} \quad 0.114549$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1\, f\left(0.2, 1.104921 + \overset{0.114549}{\cancel{0.119142}}\right)$$

$$= 0.119142$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1.104921 + \frac{1}{6}\big(0.109769 + 2(0.114447) + 2(0.114549) + 0.119142\big)$$

$$= 1.219405$$

4. Using R.K method of order 4, evaluate $y(0.4)$ given $\cancel{f(x,y)}\ \frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$

$x_0 = 0 \quad y_0 = 1$

$x_1 = 0.4 \quad y_1 = ?$

$h = x_1 - x_0$

$= 0.4$

# Adams - Moulton Method [Predictor Corrector Method]

$y_1 = y_0 + h f(x_0, y_0)$

$y_1 = y_0 + h[k_1 + k_2]$

## Adams - Bashforth Predictor Method

$$y_{n+1} = y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right] \quad n = 3, 4, \ldots$$

$y_0, y_1, y_2, y_3$

$y_4$

$y_1, y_2, y_3, y_4$

$$f^*_{n+1} = f(x_{n+1}, y_{n+1})$$

$y_5$

$x_0$

$x_1$

$h = x_1 - x_0$

## Adams - Moulton Corrector

$$y_{n+1} = y_n + \frac{h}{24}\left[9 f^*_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right] \quad n = 3, 4, \ldots$$

$$y_4 = y_3 + \frac{h}{24}\left[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right]$$

$$y_4 = y_3 + \frac{h}{24}\left[9 f^*_4 + 19 f_3 - 5 f_2 + f_1\right]$$

$f(x, y)$

$f_0 = f(x_0, y_0)$

$f_0 = f(x_0, y_0)$

$f^*_4(x_4, y_4)$

$f_3 = f(x_3, y_3)$

$f(x_1, y_1)$

Using Adam's method find $\underline{y(0.4)}$ given $y' = \frac{xy}{2}$, $y(0)=1$, $y(0.1)=1.01$, $y(0.2)=1.022$, $y(0.3)=1.023$.

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$f(x,y) = \frac{xy}{2}$

$x_0 = 0 \qquad y_0 = 1$

$x_1 = 0.1 \qquad y_1 = 1.01$

$x_2 = 0.2 \qquad y_2 = 1.022$

$x_3 = 0.3 \qquad y_3 = 1.023$

$x_4 = 0.4 \qquad y_4 = ?$

$h = x_1 - x_0$

$= 0.1 - 0$

$\boxed{h = 0.1}$

Adams Bashforth predictor formula.

$$y_{n+1} = y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right]$$

$$y_4 = y_3 + \frac{h}{24}\left[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0\right]$$

$f_0 = f(x_0, y_0) = f(0,1) = 0$

$f_1 = f(x_1, y_1) = f(0.1, 1.01) = \frac{0.1 \times 1.01}{2} = 0.0505$

$f_2 = f(x_2, y_2) = f(0.2, 1.022) = \frac{0.2 \times 1.022}{2} = 0.1022$.

$f_3 = f(x_3, y_3) = f(0.3, 1.023) = \frac{0.3 \times 1.023}{2} = 0.1534$

$$y_4 = 1.023 + \frac{0.1}{24}\left[55(0.1534) - 59(0.1022) + 37(0.0505) - 9 \times 0\right]$$

$$\boxed{y_4 = 1.0408}$$

$$f_4 = f(x_4, y_4) = f(0.4, 1.0408) = \frac{0.4 \times 1.0408}{2} = 0.20816$$

**Adams moulton Corrector ~~corrector~~ formula:**

$$y_{n+1} = y_n + \frac{h}{24}\left[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right]$$

$$y_4 = y_3 + \frac{h}{24}\left[9f_4 + 19f_3 - 5f_2 + f_1\right]$$

$$= 1.023 + \frac{0.1}{24}\left[9(0.20816) + 19\left(\overset{0.1584}{\cancel{0.1022}}\right) - 5(0.1022) + (0.0505)\right]$$

$$= 1.0410$$

$y_4^P = 1.0408$

$y_4^C = 1.0410$

Solve the initial value problem $\frac{dy}{dx} = x - y^2$, $y(0)=1$ to find $y(0.4)$ by Adam's method. Starting solution required are to be obtained using Runge kutta method of order 4. using step size $h=0.1$

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$f(x,y) = x - y^2$

$h = 0.1$

$$\begin{array}{ll} x_0 = 0 & y_0 = 1 \\ x_1 = 0.1 & \left.\begin{array}{l} y_1 = ? \\ y_2 = ? \\ y_3 = ? \end{array}\right\} \text{R.K(4)} \\ x_2 = 0.2 & \\ x_3 = 0.3 & \\ x_4 = 0.4 & \left. y_4 = ? \right\} \text{Adams.} \end{array}$$

To find $y_1, y_2, y_3$ By R.K method.

$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$k_1 = hf(x_0, y_0) = 0.1\, f(0,1) = 0.1\,[0 - (1)^2] = -0.1$

$k_2 = \cancel{hf(x_1,y_1) = 0.1\, f(0.1,}\; hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= 0.1\, f(0.05 - 0.9025^2]$

$= -0.08525$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1\, f(0.05, 0.9165)$

$= -0.08665$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1\, f(0.1, 0.8341) = -0.07341$

$$\left.\begin{array}{ll} x_1 = 0.1 & y_1 = 0.9138 \\ x_2 = 0.2 & y_2 = 0.8512 \\ x_3 = 0.3 & y_3 = 0.8076 \end{array}\right\} \text{R.K method } 4$$

$x_4 = 0.4 \qquad y_4 = ?$

y.

$$y_4^p = y_3 + \frac{h}{24}\left[55f_3 - 59f_2 + 37f_1 - 9f_0\right]$$

$$= \underline{0.7799}$$

$$f_4 = f(x_4, y_4) = f(0.4, 0.7799)$$

$$= \underline{-0.2082}$$

$$y_4^c = y_3 + \frac{h}{24}\left[9f_4 + 19f_3 - 5f_2 + f_1\right]$$

$$= \underline{0.7797}$$

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