MODULE - V

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MS-1
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SMITH CHART

The Smith chart is a graphical approach of analysing transmission lines. Using it, we get a graphical indication of the impedance of a transmission line and the corresponding reflection coefficient as we move along the line. The chart consists of a circular plot with lots of interlaced circles on it. Though it takes some practice to read and follow values on the circles, the main advantage of the chart is that transmission line characteristics can be determined without complex computations.

Construction of Smith chart

The chart is constructed based on the relation

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Since different transmission lines have different characteristic impedances (eg $Z_0 = 60, 100, 120\ \Omega$), a different Smith chart would be required for each transmission line. On the other hand, if all impedances are normalized with respect to the characteristic impedance Z_0 of the particular line under consideration, a single Smith chart can be used irrespective of the transmission line used.

Hence, if Z_L is the load impedance, normalized load impedance $z_L = \frac{Z_L}{Z_0}$

$$\therefore \Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= \frac{\frac{Z_L}{Z_0} - 1}{\frac{Z_L}{Z_0} + 1} = \frac{z_L - 1}{z_L + 1}$$

Or, on rearranging

$$z_L = \frac{1 + \Gamma_L}{1 - \Gamma_L}$$

Both z_L & Γ_L are complex and can be expressed in terms of real and imaginary components.

$\therefore$ If $z_L = r + jx$ and $\Gamma_L = u + jv$

$$r + jx = \frac{1 + (u + jv)}{1 - (u + jv)}$$

$$= \frac{(1+u) + jv}{(1-u) - jv}$$

$$= \frac{(1+u) + jv}{(1-u) - jv} \times \frac{(1-u) + jv}{(1-u) + jv}$$

$$= \frac{1 - u^2 + jv + juv + jv - juv - v^2}{(1-u)^2 + v^2}$$

$$= \frac{1 - u^2 - v^2 + j2v}{(1-u)^2 + v^2} \qquad j^2 = -1$$

$$= \frac{1 - u^2 - v^2}{(1-u)^2 + v^2} + \frac{j2v}{(1-u)^2 + v^2}$$

$$\therefore r = \frac{1 - u^2 - v^2}{(1-u)^2 + v^2}$$

$$\& \quad x = \frac{2v}{(1-u)^2 + v^2}$$

Considering the real part,

$$r = \frac{1-u^2-v^2}{(1-u)^2+v^2}$$

$$r(1-u)^2 + rv^2 = 1-u^2-v^2$$

$$r(1-2u+u^2)+rv^2 = 1-u^2-v^2$$

$$r-2ur+ru^2+rv^2 = 1-u^2-v^2$$

$$-2ur+ru^2+rv^2+u^2+v^2 = 1-r$$

$$u^2(1+r)+v^2(1+r)-2ur = 1-r$$

Dividing by $(1+r)$,

$$u^2+v^2-\frac{2ur}{(1+r)} = \frac{(1-r)}{(1+r)}$$

$$u^2+v^2-\frac{2ur}{(1+r)} = \frac{(1-r)}{(1+r)}\cdot\frac{(1+r)}{(1+r)}$$

$$u^2+v^2-\frac{2ur}{(1+r)} = \frac{1-r^2}{(1+r)^2}$$

$$u^2+v^2-\frac{2ur}{(1+r)} = \frac{1}{(1+r)^2}-\frac{r^2}{(1+r)^2}$$

$$u^2-\frac{2ur}{(1+r)}+\frac{r^2}{(1+r)^2}+v^2 = \frac{1}{(1+r)^2}$$

$$\left(u-\frac{r}{(1+r)}\right)^2+(v-0)^2 = \frac{1}{(1+r)^2}$$

MS-3

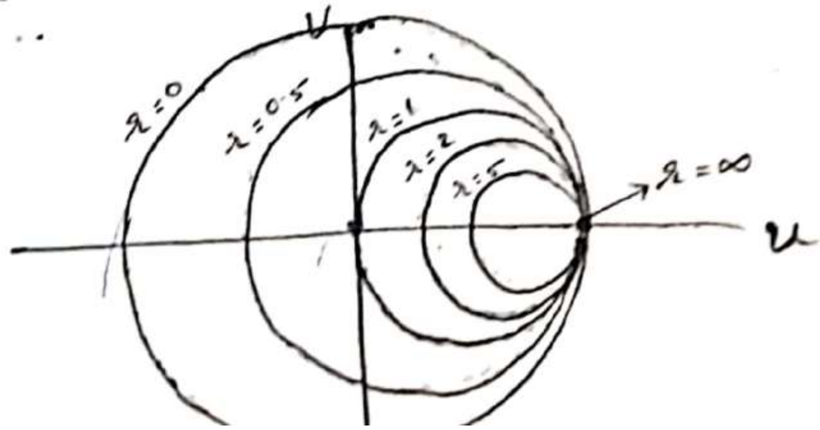
This equation is of the form

$$(x-h)^2+(y-k)^2 = a^2$$

which is the general equation of a circle with centre (h,k) and radius 'a'.

Hence the real part of normalized impedance of the transmission line is represented by a circle of centre $\left(\frac{r}{1+r}, 0\right)$ and radius $\frac{1}{1+r}$. These are called r-circles or resistance circles

Normalized resistance (r)	Radius $\left(\frac{1}{1+r}\right)$	Centre $\left(\frac{r}{1+r}, 0\right)$
0	1	(0,0)
1/2	2/3	(1/3, 0)
1	1/2	(1/2, 0)
2	1/3	(2/3, 0)
5	1/6	(5/6, 0)
∞	0	(1, 0)



Similarly, considering the imaginary

$$x = \frac{2v}{(1-u)^2+v^2}$$

$$= \frac{2v}{(u-1)^2+v^2}$$

$$(u-1)^2+v^2 = \frac{2v}{x}$$

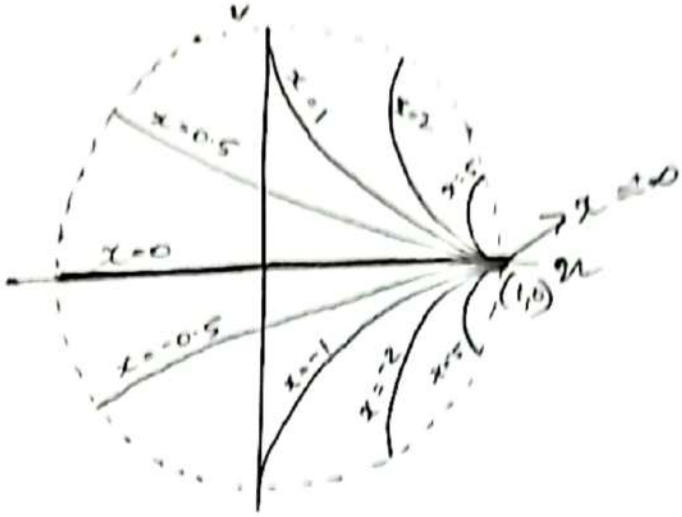
$$(u-1)^2+v^2-\frac{2v}{x} = 0$$

$$(u-1)^2+v^2-\frac{2v}{x}+\frac{1}{x^2} = \frac{1}{x^2}$$

$$(u-1)^2+\left(v-\frac{1}{x}\right)^2 = \frac{1}{x^2}$$

This is also the equation of a circle with centre $\left(1, \frac{1}{x}\right)$ and radius $\frac{1}{x}$. These are called x-circles or reactance circles.

Normalized reactance x	Radius $\left(\frac{1}{x}\right)$	Centre $\left(1, \frac{1}{x}\right)$
0	∞	$(1, \infty)$
$\pm 1/2$	2	$(1, \pm 2)$
± 1	1	$(1, \pm 1)$
± 2	$1/2$	$(1, \pm 1/2)$
± 5	$1/5$	$(1, \pm 1/5)$
$\pm \infty$	0	$(1, 0)$

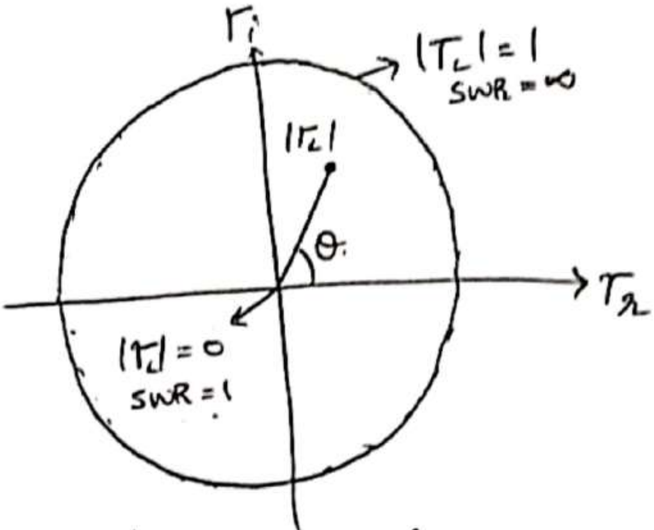


Since x is a reactance, it can be +ve or -ve.

If we superpose the r-circles and the x-circles then we get the Smith chart which lies within a unit circle.

ie $|\Gamma_L| \le 1$.

$$\Gamma_L = |\Gamma_L| e^{j\theta} = \Gamma_r + j\Gamma_i$$ (NOTE: $u+jv = \Gamma_r + j\Gamma_i$



<u>Uses of Smith chart:</u>

* Plotting complex impedance on the Smith chart.

* Finding Γ_L & SWR for a give

* Finding admittance of a given impedance.

* Finding input impedance of a transmission line

* Matching a transmission line to a load with a single series stub.

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Important points about Smith chart

1. The point on the extreme left of the chart corresponds to $r=0$ & $x=0$ ie $Z_L = 0+j0$ which means this corresponds to a short circuit on the transmission line. Similarly, at the extreme right hand side, $r=\infty$ and $x=\infty$ which corresponds to open circuit on the line.

2. A complete revolution (360°) around the Smith chart represents a distance of $\lambda/2$ on the line. Clockwise movement on the chart is regarded as moving towards the generator (or away from load) and counter

clockwise movement on the chart corresponds to moving towards load (or away from generator).

3. There are three scales around the periphery of the Smith chart. They outermost scale is used to determine distance on the line from generator end in terms of λ. The middle scale is used to determine distance on line from load end in terms of λ and the innermost scale is a protractor used to determine θ.

4. The Smith chart can be used both as impedance and admittance chart. Then, r & x-circles correspond to g & b-circle

Method to locate/calculate :

(a) Impedance

calculate the normalized impedance

$$z_L = \frac{Z_L}{Z_0}$$

If for example, $z_L = 1 + j0.5$, identify the point where $r = 1$ circle and

and $x = 0.5$ circle meeting shown in chart. A as point P.

Consider a load impedance
$$Z_L = (300 - j25)\,\Omega.$$

If it is connected to a transmission line of characteristic impedance $50\,\Omega$,

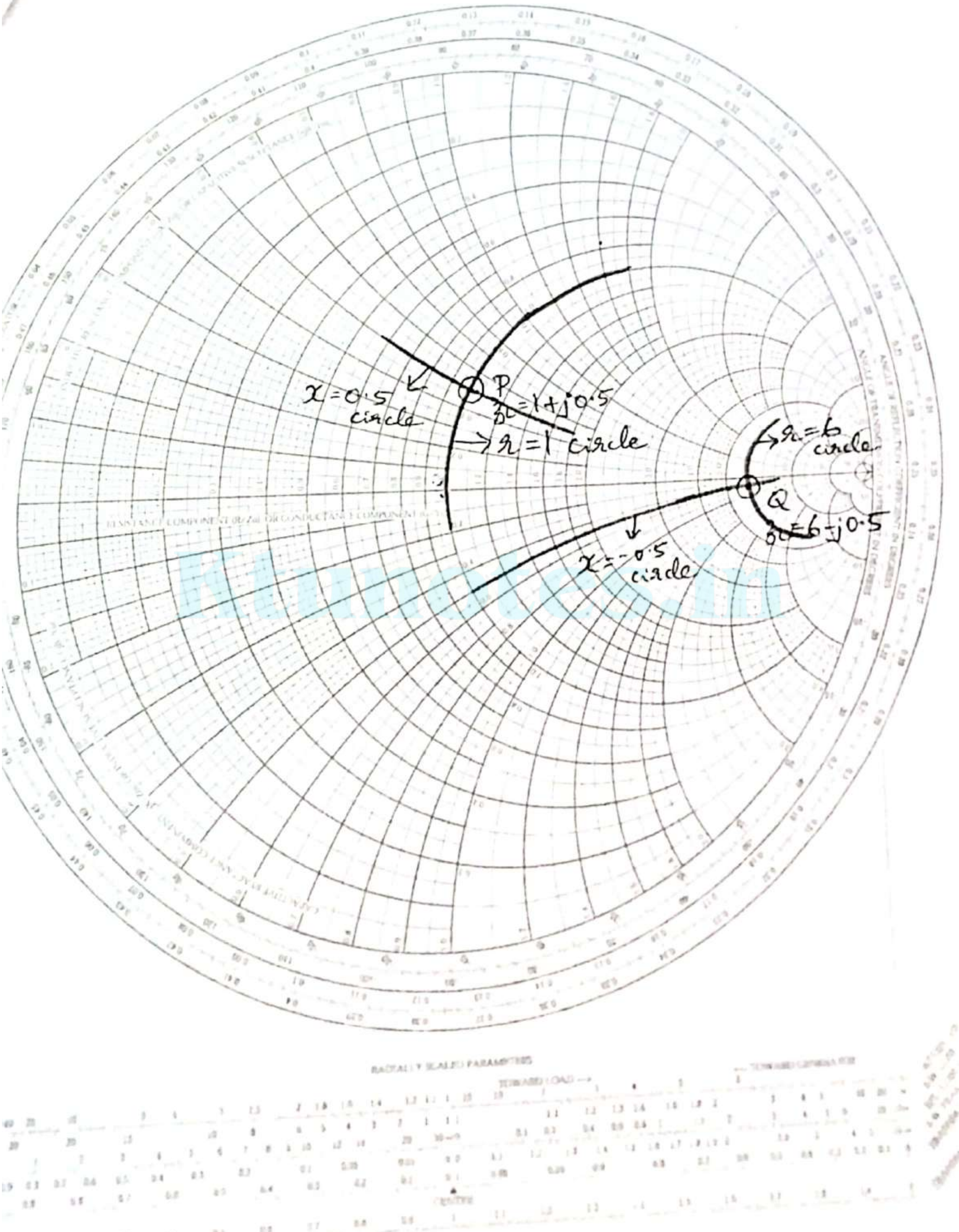
$$z_L = \frac{300 - j25}{50}$$
$$= 6 - j0.5.$$

Since $x = 0.5$, the point will be in the lower half of the chart as shown with point Q.

(b) Reflection coefficient Γ_L.

After normalizing the impedance and locating it on the Smith chart (say point P) draw a line from the centre of the chart ($z_L = 1 + j0$) to P. If the centre point is O, measure

Chart - A



the length OP. Now extend the line to meet $r=0$ circle at point G and measure OG. Since the length OG corresponds to $|\Gamma|=1$, then the $|\Gamma|$ at P is obtained as

$$|\Gamma| = \frac{OP}{OG}$$

$OG = 1$
$\therefore OP = \frac{OP}{OG}$

This gives magnitude of the reflection coefficient. To find angle, we measure the angle line OP makes with horizontal axis. Angle can be read off the innermost scale.

eg:- If $Z_L = 60 + j40\,\Omega$ & $Z_0 = 50\,\Omega$, Γ_L is obtained as shown in chart B.

$$z_L = \frac{60+j40}{50} = 1.2 + j0.8$$

$$|\Gamma_L| = \frac{OP}{OQ} = \frac{2.8\,cm}{8\,cm} = 0.35$$

$$\therefore \Gamma_L = 0.35\angle 56^\circ$$

[NOTE: Irrespective of size of the Smith Chart, OP/OG will be the same]

(c) Standing wave ratio

Once z_L is located, say point P, draw a circle with radius OP. (where O is the centre of the chart). This is called the constant S or $|\Gamma|$ circle. Locate the point S where this meets the Γ_r - axis. The value of r at point S is the standing wave ratio.

For the example shown in chart-B this SWR is 2.1.

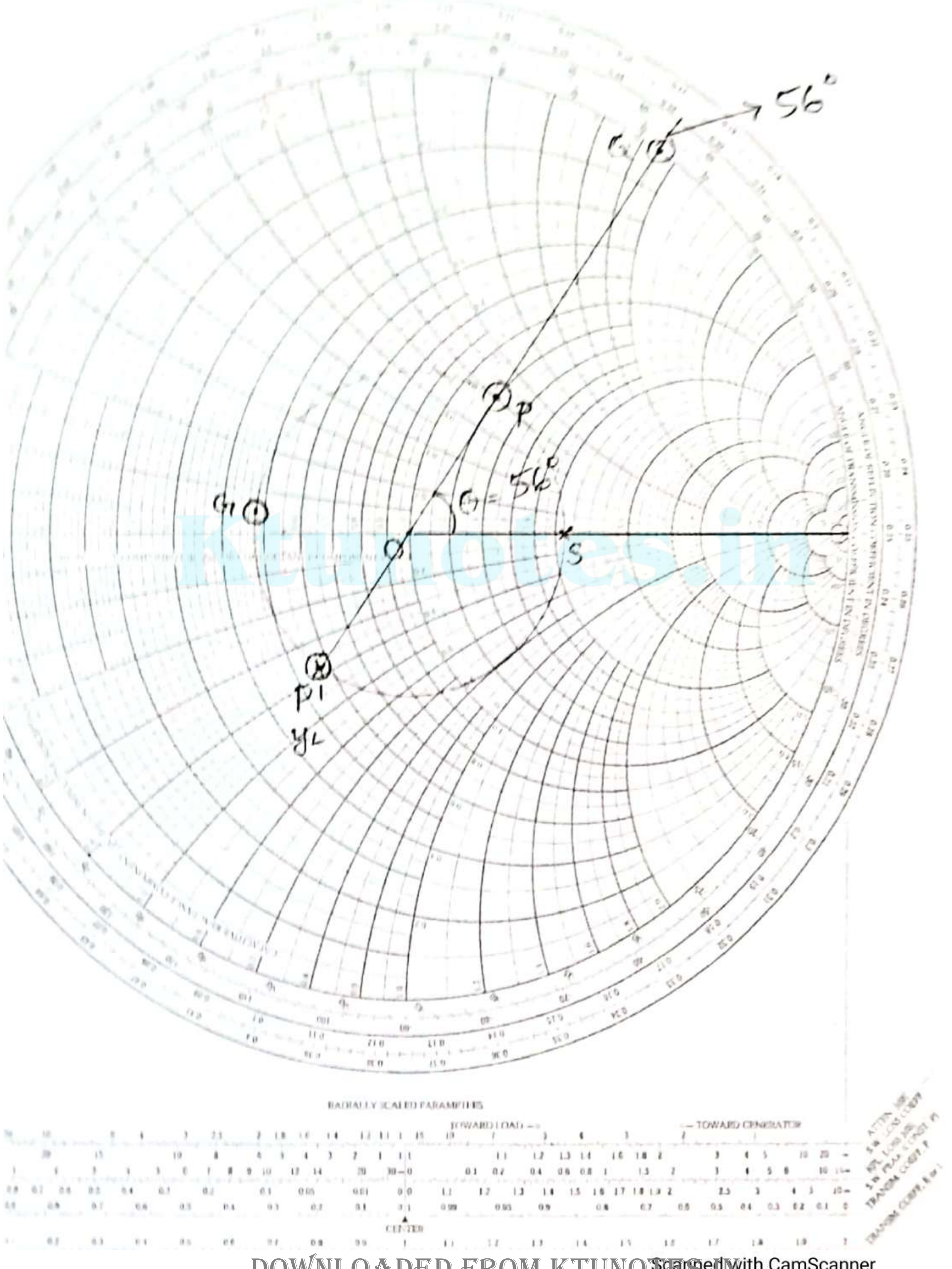
Consider $Z_L = 50\,\Omega$. Then for $Z_0 = 50\,\Omega$

$$z_L = 1 + j0.$$

This will correspond to point O in chart-B. Hence the distance from centre of chart to this point is zero, so $|\Gamma_L| = 0$. And since we cannot draw a circle, SWR=1. This is the value of $|\Gamma_L|$ and SWR for a matched line in which $Z_L = Z_0$.

M5-8

Chart - B


56°
P
θ = 56°
G
O
S
P1
yL
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER

(d) <u>Load admittance</u>

If load impedance z_L is marked at point P, then extend the line OP to P'OP as shown in chart - B. At P',

$$y_L = 0.575 - j0.38$$

Then load admittance

$$Y_L = Y_0\, y_L = \frac{1}{50}(0.575 - j0.38)$$

$$= 0.0115 - j0.0076$$

(e) <u>Input impedance</u>

To obtain the input impedance at any point along the transmission line, first the length of the line at the desired point should be expressed in terms of λ or degrees.

eg:- Consider a line, [illegible]
with $Z_0 = 75\,\Omega$. If $Z_L = 100 + j150$ find Z_{in} at 0.4λ from the loa

Around the chart, 360° corresponds to $\frac{\lambda}{2}$. Therefore λ corresponds to an angular movement of 720°.

0.4 λ corresponds to $0.4 \times 720° = 288°$.

Therefore, identify point P where

$$z_L = \frac{Z_L}{Z_0} = \frac{100 + j150}{75} = 1.33 + j2$$

Draw a line from O to P and move 288° in the clockwise direction (towards generator) on the constant S circle to reach point R as shown in Chart - C ⊛.

At R, $z_{in} = 0.29 + j0.64$

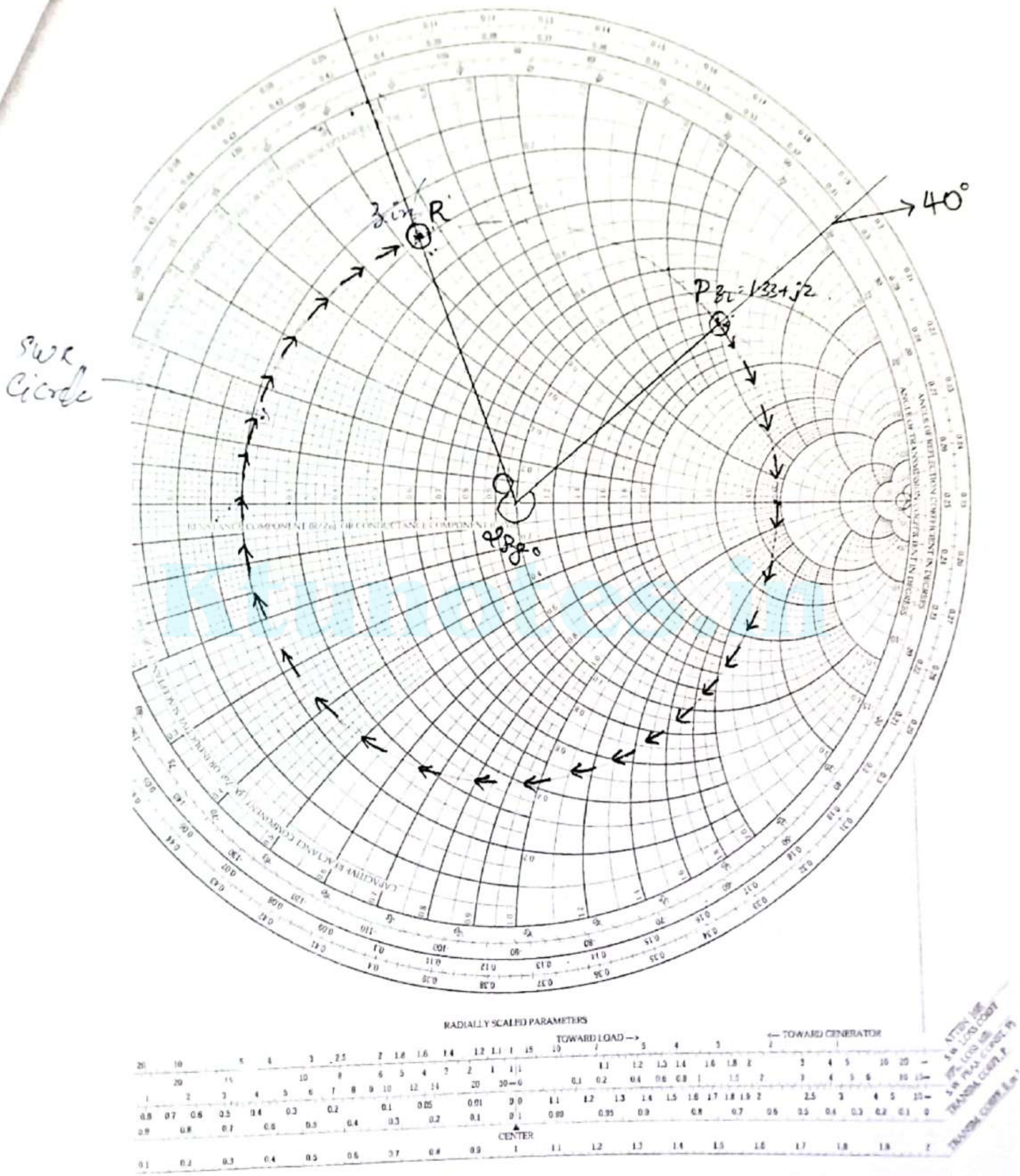
∴ Input impedance at this point is

$$Z_{in} = Z_0 \times z_{in}$$
$$= 75(0.29 + j0.64)$$
$$= (21.75 + j48)\,\Omega$$

⊛ Alternately, without converting to degrees, 0.4 λ distance can be obtained using outer-most scale of the chart.

Chart - C

M5-10



Zin R'
40°
P ZL=1.33+j2
SWR Circle
RADIALLY SCALED PARAMETERS
TOWARD LOAD →
← TOWARD GENERATOR
CENTER

If, for example, a lossless transmission line with $Z_0 = 50\,\Omega$ is 30m long and operates at 2 MHz. For a load impedance $Z_L = 60 + j40\ \Omega$, if $u = 0.6c$ on the line, find input impedance.

$$z_L = \frac{Z_L}{Z_0} = \frac{60 + j40}{50} = 1.2 + j0.8$$

Identify this point (say P on chart B) and draw the constant S circle with OP as radius.

To obtain Z_{in}, first we express l in terms of λ.

$$\lambda = \frac{u}{f} = \frac{0.6 \times 3 \times 10^8}{2 \times 10^6} = 90\text{m}$$

$$l = 30\text{m}$$

$$90 = \lambda$$

$$30 = \frac{30}{90}\lambda = \frac{\lambda}{3}$$

$$\text{ie } \frac{720^\circ}{3} = 240^\circ$$

∴ From point P, we move 240° in clockwise direction on the S-circle. This is shown as point G on chart-B.

At this point,

$$z_{in} = 0.48 + j0.05$$

$$\therefore Z_{in} = 50(0.48 + j0.05)$$
$$= (24 + j2.5)\Omega$$

TRANSMISSION LINE SECTIONS AS CIRCUIT ELEMENTS

From the expression for input impedance $Z_{in} = Z_0 \left[\frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}\right]$

it is seen that if a section of a transmission line is open circuited (ie $Z_L = \infty$), $Z_{in} = -jZ_0 \cot(\beta l)$.

ie Z_{in} is a pure reactance. It is capacitive for $\beta l < \frac{\pi}{2}$ and inductive for $\frac{\pi}{2} < \beta l < \pi$.

Similarly, if the transmission line is short circuited (ie $Z_L = 0$), $Z_{in} = jZ_0 \tan(\beta l)$. Here also Z_{in} is a reactance and will be capacitive for $\frac{\pi}{2} < \beta l < \pi$ and inductive for $\beta l < \frac{\pi}{2}$.

Thus, both an open-circuited transmission line and a short-circuited transmission line can be used as an inductive or capacitive reactance by suitable choice of length of the line.

Two port elements:

Transmission line sections which are half wavelength ($\lambda/2$) and quarter wavelength ($\lambda/4$) long have some interesting characteristics

Half wavelength section

$$l = \frac{\lambda}{2}$$

$$\therefore \beta l = \frac{2\pi}{\lambda} \times \frac{\lambda}{2} = \pi$$

$$\therefore \tan \beta l = 0$$

$$\therefore Z_{in} = Z_0 \left[\frac{Z_L + j0}{Z_0 + j0} \right] = Z_L$$

ie input impedance becomes eq. to load impedance. Such $\lambda/2$ s. can be inserted in transmission & circuits whenever physical spacing between two ports of a circuit is required without changing the electrical behaviour at a particular frequency.

Quarter Wavelength Section

$$l = \frac{\lambda}{4}$$

$$\therefore \beta l = \frac{2\pi}{\lambda} \times \frac{\lambda}{4} = \frac{\pi}{2}$$

$$\therefore \tan\frac{\pi}{2} = \infty$$

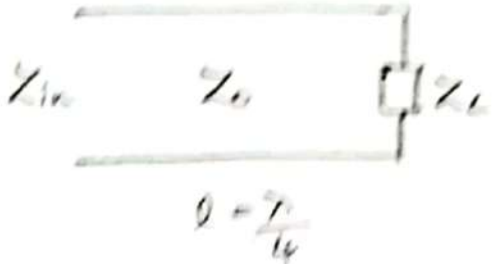
$$Z_{in} = Z_0\left[\frac{Z_L + jZ_0\tan\frac{\pi}{2}}{Z_0 + jZ_L\tan\frac{\pi}{2}}\right]$$

$$= Z_0\left[\frac{\tan\frac{\pi}{2}\left(\frac{Z_L}{\tan\frac{\pi}{2}} + jZ_0\right)}{\tan\frac{\pi}{2}\left(\frac{Z_0}{\tan\frac{\pi}{2}} + jZ_L\right)}\right]$$

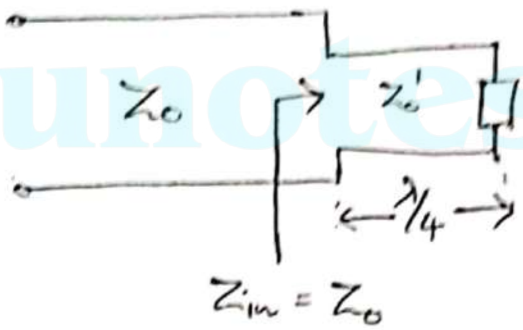
$$= Z_0 \times \frac{jZ_0}{jZ_L}$$

$$= \frac{Z_0^2}{Z_L}$$

or, $Z_0 = \sqrt{Z_{in} Z_L}$



Thus if we have a mismatched load Z_L we can match it to the transmission line (say of impedance Z_0) by inserting a $\lambda/4$ line with characteristic impedance $Z_0' = \sqrt{Z_0 Z_L}$.



Z_0' is selected such that ($Z_{in} = Z_0$)

For this reason, the quarter wave or $\lambda/4$ transmission line is also called quarter wave transformer. It is used for impedance matching like providing transformer.

eg:- if a $120\,\Omega$ load is to be matched to a $75\,\Omega$ line, a $\lambda/4$ section of characteristic impedance $\sqrt{75 \times 120} = 95\,\Omega$ can be used.

Waveguides:

Wave guides are used to transmit EM waves from one point to another. It is a hollow Conducting tube to transmit electromagnetic waves.

Transmission line (TL)	Waveguides (WG)
1. Transmission line can Support only a TEM Wave	Can support many possible field Configurations.
2. At microwave frequency (3-300GHz) TL become inefficient due to Skin effect and dielectric losses	WG are used at microwave frequency to obtain larger bandwidth and lower attenuation.
3. TL operate from (dc, f=0) to Very high frequency	WG can operate only above a Certain frequencies called Cut-off frequency.. It act as high pass-filter. WG Cannot transmit dc.

MODES

The waves that propagate through the waveguides have infinite patterns of electric and magnetic fields these are called Modes.

Commonly seen Modes are.

(1) TEM (Transverse electromagnetic Wave).

(2) TE (Transverse Electric Wave)

(3) TM (Transverse Magnetic Wave).

For a wave propagating in z direction,

TEM are characterized by $E_z = 0$, $H_z = 0$ both are perpendicular to the direction of propagation

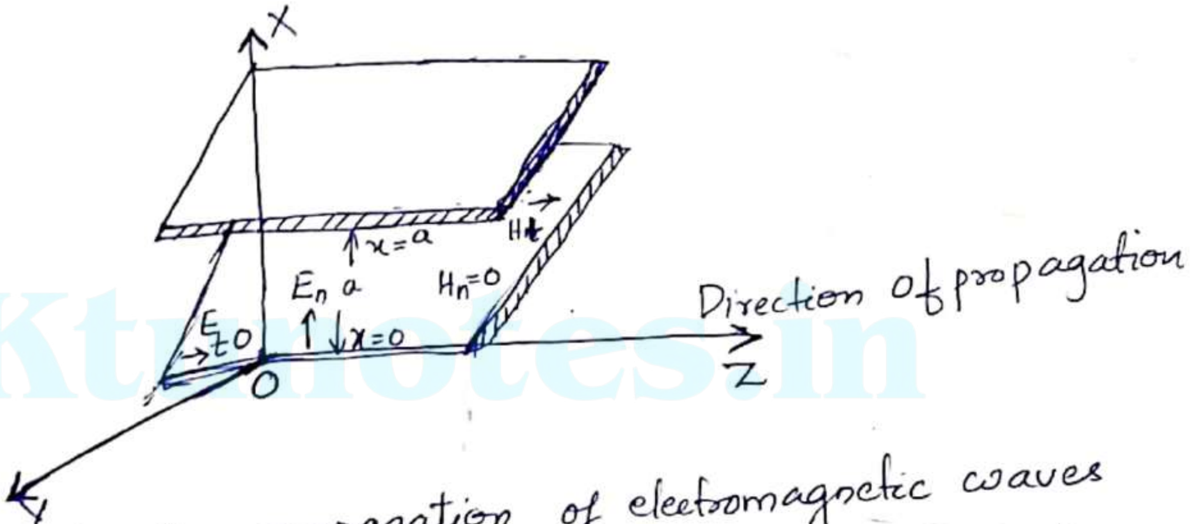
TE (Transverse Electric Wave). In this mode the electric field is perpendicular to the direction of propagation, where as the magnetic field is not. $\boxed{E_z = 0, \text{ and } H_z \neq 0}$

TM (Transverse Magnetic Wave)

In this mode, the magnetic field is perpendicular to the direction of propagation

$\boxed{H_z = 0, \quad E_z \neq 0}$

<u>Parallel Plate Wave-Guide.</u>



Consider the propagation of electromagnetic waves between <u>two parallel perfectly conducting plates</u> of infinite extent in the directions y and z which is separated by a distance 'a'.

The medium between the plates is <u>perfect dielectric of permittivity</u> ε and permeability μ.

The maxwell's equation are solved with proper boundary conditions to determine the E.M field configuration inside the planes of the two plates.

<u>Boundary Conditions.</u>

(1) Tangential component of electric field must be zero $E_t = 0$. ie electric field terminate normally on the Conductor $E_n \neq 0$.

(ii) Normal Component of Magnetic field must be zero (2)

$H_n = 0, \quad H_t \neq 0.$

① Derive the wave equation for guided waves between parallel planes and obtain all the field components.

Maxwell's curl equation in phasor form,

$$\nabla \times \vec{E} = -j\mu\omega \vec{H}$$

$$\nabla \times \vec{H} = (\sigma + j\varepsilon\omega)\vec{E}$$

$$\nabla^2 \vec{E} = \gamma^2 \vec{E}$$

$$\nabla^2 \vec{H} = \gamma^2 \vec{H}$$

where, $\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon)$

For non-conducting region between the planes $(\sigma = 0)$ the above equations reduce to,

$$\nabla \times \vec{E} = -j\mu\omega \vec{H}$$

$$\nabla \times \vec{H} = j\varepsilon\omega \vec{E}$$

$$\nabla^2 \vec{E} = \gamma^2 \vec{E}$$

$$\nabla^2 \vec{H} = \gamma^2 \vec{H}$$

$$\gamma^2 = j\mu\omega(j\omega\varepsilon) = j^2\omega^2\mu\varepsilon = -\omega^2\mu\varepsilon.$$

The above equations can be expanded as,

$$\nabla \times \vec{E} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\mu\omega\left[H_x\hat{a}_x + H_y\hat{a}_y + H_z\hat{a}_z\right]$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\mu\omega H_x \quad \text{——— (1)}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\mu\omega H_y \quad \text{--- (2)}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\mu\omega H_z \quad \text{--- (3)}$$

Similarly,

$$\nabla \times \vec{H} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j\omega\varepsilon(E_x\hat{a_x} + E_y\hat{a_y} + E_z\hat{a_z})$$

$$= \hat{a_x}\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] - \hat{a_y}\left[\frac{\partial H_z}{\partial x} - \frac{\partial H_x}{\partial z}\right] + \hat{a_z}\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]$$

Equating the components,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\varepsilon\omega E_x \quad \text{--- (4)}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\varepsilon\omega E_y \quad \text{--- (5)}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\varepsilon\omega E_z \quad \text{--- (6)}$$

Also, $\nabla^2\vec{E} = \gamma^2\vec{E}$

$$\nabla^2\vec{E} = -\omega^2\mu\varepsilon\vec{E}$$

$$\boxed{\frac{\partial^2\vec{E}}{\partial x^2} + \frac{\partial^2\vec{E}}{\partial y^2} + \frac{\partial^2\vec{E}}{\partial z^2} = -\omega^2\mu\varepsilon\vec{E}} \quad \text{--- (7)}$$

Similarly $\nabla^2 H = \gamma^2\vec{H}$

$$\boxed{\frac{\partial^2\vec{H}}{\partial x^2} + \frac{\partial^2\vec{H}}{\partial y^2} + \frac{\partial^2\vec{H}}{\partial z^2} = -\omega^2\mu\varepsilon\vec{H}} \quad \text{--- (8)}$$

Assumptions:

I Assuming that, the wave is propagating in the positive Z-direction, So the Variation of all the field components w.r.t z direction can be expressed as $e^{-\gamma z}$ where γ is the propagation constant (3)

$$\gamma = \alpha + j\beta.$$

when the time variation factor is combined with direction of Variation factor.

then $e^{j\omega t} \cdot e^{-\gamma z} = e^{(j\omega t - \gamma z)}$.

This represents a wave propagation in the z-direction

II In, Infinite parallel plate guide shown in figure above, there is no boundary condition to be met in the y-direction as the space is infinite in extent in this direction.

So the field component in y direction is assumed to be uniform or Constant, resulting all their derivatives w.r.t y are zero.

Variations of Field components w.r.t z direction can be expressed as,

$$E_x = E_{x0} e^{-\gamma z}$$

$$E_y = E_{y0} e^{-\gamma z}$$

$$E_z = E_{z0} e^{-\gamma z}$$

$$H_x = H_{x0} e^{-\gamma z}$$

$$H_y = H_{y0} e^{-\gamma z}$$

$$H_z = H_{z0} e^{-\gamma z}.$$

$$\therefore \frac{\partial E_y}{\partial z} = E_{y0}(-\gamma)e^{-\gamma z}$$

$$\boxed{\frac{\partial E_y}{\partial z} = -\gamma E_y}$$

$$\boxed{\frac{\partial^2 E_y}{\partial z^2} = \gamma^2 E_y}.$$

$$\because \left[\frac{\partial^2 E_y}{\partial z^2} = E_{y0}(-\gamma)(-\gamma)e^{-\gamma z} = \gamma^2 E_{y0}e^{-\gamma z} \quad \frac{\partial^2 E_y}{\partial z^2} = \gamma^2 E_y \right].$$

Similarly,

$$\frac{\partial H_x}{\partial z} = -\gamma H_x.$$

and, $\frac{\partial E_z}{\partial y} = \frac{\partial E_x}{\partial y} = \frac{\partial H_z}{\partial y} = \frac{\partial H_x}{\partial y} = 0$. [By assumption].

Substituting above results in equations [1 − 8].

(1) becomes, $+\gamma E_y = -j\mu\omega H_x$ —— (9).

(2) becomes,

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\mu\omega H_y \quad \text{—— (10)}$$

(3) becomes, $\boxed{\frac{\partial E_y}{\partial x} = -j\mu\omega H_z}$ —— (11).

(4) becomes, $+\gamma H_y = j\varepsilon\omega E_x$ —— (12).

(5) become, $-\gamma H_x - \frac{\partial H_z}{\partial x} = j\varepsilon\omega E_y$ —— (13).

(6) becomes, $\boxed{\frac{\partial H_y}{\partial x} = j\varepsilon\omega E_z}$ —— (14).

$$\boxed{\begin{aligned} &\frac{\partial^2 \vec{E}}{\partial x^2} + \gamma^2 \vec{E} = -\omega^2 \mu \varepsilon \vec{E} \quad (15) \\ \text{and} \quad &\frac{\partial^2 \vec{H}}{\partial x^2} + \gamma^2 \vec{H} = -\omega^2 \mu \varepsilon \vec{H} \quad (16) \end{aligned}}$$

Simultaneous solving of these equations will give the value of E_x, E_y, H_x & H_y.

From (12)

$$H_y = \frac{j \varepsilon \omega}{\gamma} E_x$$

Putting this in equation (10).

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\mu\omega \left[\frac{j\varepsilon\omega}{\gamma}\right] E_x$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j^2 \frac{\mu \varepsilon \omega^2}{\gamma} E_x$$

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = +\frac{\mu \varepsilon \omega^2}{\gamma} E_x$$

$$-\frac{\partial E_z}{\partial x} = \left[\frac{\mu\varepsilon\omega^2}{\gamma} + \gamma\right] E_x$$

$$= \left[\frac{\mu\varepsilon\omega^2 + \gamma^2}{\gamma}\right] E_x$$

$$E_x = -\frac{\gamma}{(\mu\varepsilon\omega^2 + \gamma^2)} \cdot \frac{\partial E_z}{\partial x}$$

$$\boxed{\begin{aligned} &E_x = -\frac{\gamma}{h^2} \cdot \frac{\partial E_z}{\partial x} \quad (A) \\ &\text{where } h^2 = \gamma^2 + \mu\varepsilon\omega^2 \end{aligned}}$$

From (9),

$$E_y = \frac{-j\mu\omega}{\gamma} H_x .$$

Putting this in equation (13),

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = j\varepsilon\omega \cdot \frac{-j\mu\omega H_x}{\gamma}$$

$$\because j^2$$

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = \frac{+\mu\varepsilon\omega^2}{\gamma} H_x .$$

$$-\frac{\partial H_z}{\partial x} = \left[\frac{\cancel{+}\mu\varepsilon\omega^2}{\gamma} + \gamma\right] H_x$$

$$= \left[\frac{\mu\varepsilon\omega^2 + \gamma^2}{\gamma}\right] H_x .$$

$$\therefore H_x = -\left[\frac{\gamma}{\mu\varepsilon\omega^2 + \gamma^2}\right]\frac{\partial H_z}{\partial x}$$

$$\boxed{H_x = -\left[\frac{\gamma}{h^2}\right]\frac{\partial H_z}{\partial x} .}, \quad h^2 = \gamma^2 + \mu\varepsilon\omega^2 . \longrightarrow (B)$$

Substituting H_x in equation (13)

$$-\gamma H_x - \frac{\partial H_z}{\partial x} = j\varepsilon\omega E_y$$

$$-\gamma \times -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x} - \frac{\partial H_z}{\partial x} = j\varepsilon\omega E_y .$$

$$\left[\frac{\gamma^2}{h^2} - 1\right]\frac{\partial H_z}{\partial x} = j\varepsilon\omega E_y .$$

$$E_y = \left[\frac{\gamma^2 - h^2}{h^2}\right] \cdot \frac{1}{j\varepsilon\omega}\frac{\partial H_z}{\partial x}$$

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$$E_y = \left[\frac{\cancel{\gamma^2} - (\cancel{\gamma^2} + \mu\varepsilon\omega^2)}{h^2}\right] \cdot \frac{1}{j\varepsilon\omega}\frac{\partial H_z}{\partial x}$$

$$= -\frac{\mu\cancel{\varepsilon}\omega^{\cancel{2}}}{h^2}\cdot\frac{1}{j\cancel{\varepsilon\omega}}\cdot\frac{\partial H_z}{\partial x}$$

$$\boxed{E_y = \frac{j\omega\mu}{h^2}\cdot\frac{\partial H_z}{\partial x}} \quad \text{—— (C)}$$

Substituting E_x into equation (10)

$$-\gamma E_x - \frac{\partial E_z}{\partial x} = -j\mu\omega H_y$$

$$-\gamma \times -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} - \frac{\partial E_z}{\partial x} = -j\mu\omega H_y.$$

$$\left[\frac{+\gamma^2}{h^2} - 1\right]\frac{\partial E_z}{\partial x} = -j\mu\omega H_y$$

$$\cancel{\times \frac{-1}{j\mu\omega}\left[\frac{\gamma^2+h^2}{h^2}\right]\frac{\partial E_z}{\partial x} = H_y}$$

$$\cancel{H_y = \frac{1}{j\mu\omega}}$$

$$\frac{\gamma^2 - h^2}{h^2}\frac{\partial E_z}{\partial x} = -j\mu\omega H_y$$

$$\frac{\cancel{\gamma^2} - (\cancel{\gamma^2} + \omega^2\mu\varepsilon)}{h^2}\frac{\partial E_z}{\partial x} = -j\mu\omega H_y$$

$$+\frac{\omega^{\cancel{2}}\cancel{\mu}\varepsilon}{h^2}\cdot\frac{1}{j\cancel{\mu\omega}}\frac{\partial E_z}{\partial x} = H_y$$

$$\boxed{H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}} \longrightarrow (D) \quad \because \frac{1}{j} = -j$$

These are the different components of electric and magnetic fields expressed in terms of E_z component and H_z component.

Electromagnetic wave classified into.

(1) Transverse electric (TE) wave or H-waves.

In this, electric field component lies perpendicular to the direction of propagation (z-direction) where as component of magnetic field $\vec{H}$ lies along the direction of propagation.

$$E_z = 0, \quad H_z \neq 0.$$

(2) Transverse - Magnetic (TM) wave or E-waves.

In this, the component of magnetic field lies perpendicular to the direction of propagation whereas component of electric field lies along the direction of propagation

$$H_z = 0, \quad E_z \neq 0.$$

Q. Derive the solution for TE waves and hence find the expressions for their field strength between parallel planes.

Transverse Electric Wave (TE) or H-waves:

For TE wave $E_z = 0$, but $H_z \neq 0$ when this conditions are substituted in equations A, B, C, and D General solutions,

$$E_x = \frac{-\cancel{\gamma}}{h^2}\frac{\partial E_x}{\partial x} = 0$$

$$H_y = \frac{-j\omega\varepsilon}{h_2}\frac{\partial E_z}{\partial x} = 0.$$

$$H_x = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

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$$E_y = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

Consider the wave equations, and re-write in its Component form,

$$\frac{\partial^2 \vec{E}}{\partial x^2} + \gamma^2 \vec{E} = -\omega^2\mu\varepsilon \vec{E}$$

Component form,

$$\frac{\partial^2 E_x}{\partial x^2} + \gamma^2 E_x = -\omega^2\mu\varepsilon E_x$$

$$\frac{\partial^2 E_y}{\partial x^2} + \gamma^2 E_y = -\omega^2\mu\varepsilon E_y$$

$$\frac{\partial^2 E_z}{\partial x^2} + \gamma^2 E_z = -\omega^2\mu\varepsilon E_z$$

Similarly

$$\frac{\partial^2 H_x}{\partial x^2} + \gamma^2 H_x = -\omega^2\mu\varepsilon H_x$$

$$\frac{\partial^2 H_y}{\partial x^2} + \gamma^2 H_y = -\omega^2\mu\varepsilon H_y$$

$$\frac{\partial^2 H_z}{\partial x^2} + \gamma^2 H_z = -\omega^2\mu\varepsilon H_z.$$

Now the wave equation in y (say E_y) Can be written as

$$\frac{\partial^2 E_y}{\partial x^2} = -\gamma^2 E_y - \omega^2\mu\varepsilon E_y$$

$$= -(\gamma^2+\omega^2\mu\varepsilon) E_y$$

$$\boxed{\frac{\partial^2 E_y}{\partial x} = -h^2 E_y}$$

$$E_y = E_{y0}\, e^{-\gamma z}$$

$$\therefore \frac{\partial^2 E_{y0} e^{-\gamma z}}{\partial x^2} = -h^2 E_{y0} e^{-\gamma z}$$

$$\frac{\partial^2 E_{y0}}{\partial x^2} \cdot \cancel{e^{-\gamma z}} = -h^2 E_{y0} \cancel{e^{-\gamma z}}$$

$$\boxed{\frac{\partial^2 E_{y0}}{\partial x^2} = -h^2 E_{y0}}$$

This is a standard differential equation of simple harmonic motion and its solution is

$$E_{y0} = A_1 \sin hx + A_2 \cos hx$$

A_1 & A_2 — arbitrary Constants.

$$\therefore E_y = (A_1 \sin hx + A_2 \cos hx) e^{-\gamma z}.$$

Now the Values of arbitrary Constants A_1 and A_2 can be determined on applying the boundary Conditions, ie

(i) tangential Component of $\vec{E}$ is zero at the Surface of perfect conductor.

$$\therefore E_y = 0 \text{ at } x = 0 \quad \text{——— (1)}$$

$$E_y = 0 \text{ at } x = a \quad \text{——— (2)}.$$

Applying boundary conditions, At $x=0$,

$$E_y = (A_1 \sin hx + A_2 \cosh x) e^{-\gamma z}$$

$$0 = (A_1 \sin 0 + A_2 \cos 0) e^{-\gamma z}$$

$$0 = 0 + A_2$$

$$A_2 = 0$$

$$E_y = A_1 \sin hx \, e^{-\gamma z}$$

At

At $x = a$, $E_y = 0$.

(7)

$$0 = (A_1 \sin ha + A_2 \cos ha) e^{-\gamma z}.$$

$$\sin ha = 0$$

$$ha = m\pi$$

Where $m = \pm 1, \pm 2, \pm 3, \pm 4, \ldots$

excluding $m = 0$.

$$\therefore \boxed{h = m\pi / a}$$

Hence $\boxed{E_y = A_1 \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}.}$ ——— (1)

The other expressions for the field strength can by obtained by

from (9), $\gamma E_y = -j\mu\omega H_x$

$$H_x = \frac{-\gamma}{j\mu\omega} E_y.$$

$$H_x = \frac{-\gamma}{j\mu\omega}\left[A_1 \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}\right]$$

$$\boxed{H_x = \frac{-\gamma}{j\mu\omega} A_1 \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}}$$

From (11), $\frac{\partial E_y}{\partial x} = -j\mu\omega H_z$, $H_z = \frac{-1}{j\mu\omega}\frac{\partial E_y}{\partial x}$

$\therefore$ Differentiating (1) w.r.t x.

$$\frac{\partial E_y}{\partial x} = A_1 \cos\left(\frac{m\pi x}{a}\right)\left(\frac{m\pi}{a}\right) \cdot e^{-\gamma z}.$$

$$\therefore \boxed{H_z = \frac{-1}{j\mu\omega} A_1 \frac{m\pi}{a} \cos\left(\frac{m\pi x}{a}\right) e^{-\gamma z}}$$

Summarizing the Components of E and H

$$E_x = 0$$
$$E_y = A_1 \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}$$
$$E_z = 0$$
$$H_x = \frac{-\gamma}{j\mu\omega} A_1 \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}$$
$$H_y = 0$$
$$H_z = -\frac{A_1 m\pi}{j\mu\omega a} \cos\left(\frac{m\pi x}{a}\right) e^{-\gamma z}$$

Each value of m specifies a particular field configuration or mode and the wave associated with integer m is designated as T_{m0} waves or T_{m0} mode. The lowest value of m is 1 and higher. As $m=0$ will lead the values of E_y, H_x and H_z identically zero.

Thus the lowest order mode is TE_{10}.

If $\gamma = \alpha + j\beta$, $\alpha = 0$ $\therefore \gamma = j\beta$.

Hence

$$E_x = 0$$
$$E_y = A_1 \sin\left(\frac{m\pi x}{a}\right) e^{-j\beta z}$$
$$E_z = 0$$
$$H_x = \frac{-\gamma}{j\mu\omega} A_1 \sin\left(\frac{m\pi x}{a}\right) e^{-j\beta z}$$
$$H_y = 0$$
$$H_z = -\frac{A_1 m\pi}{j\mu\omega a} \cos\left(\frac{m\pi x}{a}\right) e^{-j\beta z}$$

Transverse Magnetic (TM) waves or E-Waves.

(2)

For TM waves, $H_z = 0$ but $E_z \neq 0$

Q. Derive the Solution for TM waves and hence find the expressions for their field strength between parallel planes.

We have general Solutions

$$H_x = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_x = \frac{-\gamma}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_y = \frac{+j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

In Transverse Magnetic waves $H_z = 0$, and $E_z \neq 0$ Hence.

$$H_x = 0$$

$$E_y = 0.$$

Solving the wave equation for H_y,

$$\frac{\partial^2 H_y}{\partial x^2} + \gamma^2 H_y = -\omega^2\mu\varepsilon H_y$$

$$\frac{\partial^2 H_y}{\partial x^2} = -h^2 H_y$$

This is a standard differential equation,

$$H_y = [A_3 \sin hx + A_4 \cos hx]e^{-\gamma z}$$

Boundary condition, normal component of magnetic field must be zero, $H_n = 0$ but $H_t \neq 0$ at the Surface of Conductor. Thus the Constants A_3 and A_4 Cannot be evaluated directly, by applying boundary condition to H_y therefore H_y is converted in terms of E_z. Then boundary condition on E_z,

At (i) $x=0$, $E_z=0$

(ii) $x=a$, $E_z=0$.

From eq(14),

$$\frac{\partial H_y}{\partial x} = j\varepsilon\omega E_z$$

$$\therefore E_z = \frac{1}{j\varepsilon\omega}\frac{\partial H_y}{\partial x}$$

$$= \frac{1}{j\omega\varepsilon}\cdot\left[\frac{d}{dx}(A_3 \sin hx + A_4 \cos hx)e^{-\gamma z}\right]$$

$$E_z = \frac{h}{j\omega\varepsilon}\left[A_3 \cos hx - A_4 \sin hx\right]e^{-\gamma z}$$

Applying boundary conditions (i) $x=0$, $E_z=0$

$$0 = \frac{h}{j\omega\varepsilon}\left[A_3 \cos 0 - 0\right]e^{-\gamma z}$$

$$\boxed{A_3 = 0}$$

$$\therefore E_z = \frac{h}{j\omega\varepsilon}\left[-A_4 \sin hx\right]e^{-\gamma z}$$

Applying second boundary condition, $x=a$, $E_z=0$

$$0 = \frac{h}{j\omega\varepsilon}\left[\cancel{A_3 \cos ha} - A_4 \sin ha\right)e^{-\gamma z}$$

$A_3 = 0$

$$\sin ha = 0$$

$$ha = m\pi$$

$$\boxed{h = \frac{m\pi}{a}}$$ where m is any integer

$$\boxed{E_z = \frac{-hA_4}{j\varepsilon\omega} \sin\left(\frac{m\pi x}{a}\right) \cdot e^{-\gamma z}}$$

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General solution, $H_y = \frac{-j\omega\varepsilon}{h^2} \frac{\partial E_z}{dx}$

$$= \frac{-j\omega\varepsilon}{h^2} \frac{d}{dz}\left\{-\frac{hA_4}{j\varepsilon\omega} \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}\right\}$$

$$= \frac{-j\omega\varepsilon}{h^2} \cdot \frac{-hA_4}{j\varepsilon\omega} \cos\left(\frac{m\pi x}{a}\right) \cdot \frac{m\pi}{a} \cdot e^{-\gamma z}$$

$\because h = \frac{m\pi}{a}$

$$= \frac{A_4}{h} \cos\left(\frac{m\pi x}{a}\right) \cdot \frac{m\pi}{a} e^{-\gamma z}$$

$$\boxed{H_y = A_4 \cos\left(\frac{m\pi x}{a}\right) e^{-\gamma z}.}$$

From general solution,

$$E_x = \frac{-\gamma}{h^2} \frac{dE_z}{dx}$$

$$E_x = -\frac{\gamma}{h^2} \frac{d}{dx}\left(-\frac{hA_4}{j\varepsilon\omega} \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}\right)$$

$$= \frac{-\gamma}{h^2} \cdot \frac{-hA_4}{j\varepsilon\omega} \cos\left(\frac{m\pi x}{a}\right) \cdot \frac{m\pi}{a} \cdot e^{-\gamma z}$$

$$\boxed{E_x = \frac{+\gamma A_4}{j\varepsilon\omega} \cos\left(\frac{m\pi x}{a}\right) e^{-\gamma z}}$$

Summarizing we have,

$$E_x = \frac{\gamma A_4}{j\epsilon\omega} \cos\left(\frac{m\pi x}{a}\right) e^{-\gamma z}$$

$$E_y = 0$$

$$E_z = \frac{-hA_4}{j\epsilon\omega} \sin\left(\frac{m\pi x}{a}\right) e^{-\gamma z}$$

$$H_x = 0$$

$$H_y = A_4 \cos\left(\frac{m\pi x}{a}\right) e^{-\gamma z}$$

$$H_z = 0.$$

If $\gamma = j\beta$

$$E_x = \frac{\gamma A_4}{j\epsilon\omega} \cos\left(\frac{m\pi x}{a}\right) e^{-j\beta}$$

$$E_y = 0$$

$$E_z = \frac{-hA_4}{j\epsilon\omega} \sin\left(\frac{m\pi x}{a}\right) e^{-j\beta}$$

$$H_x = 0$$

$$H_y = A_4 \cos\left(\frac{m\pi x}{a}\right) e^{-j\beta}$$

$$H_z = 0.$$

There are infinite numbers of modes possible corresponding to $m = \pm 1$ to ∞. In case of TM waves there is a possibility of $m=0$, as by putting $m=0$ in above eqns. Some of the fields (eg: E_x and H_y) exist. Hence lowest order mode that can exist in TM waves is the TM_0 mode. ~~mode~~.

Transverse Electric (TE) Mode Solutions of Rectangular Wave guides. ①

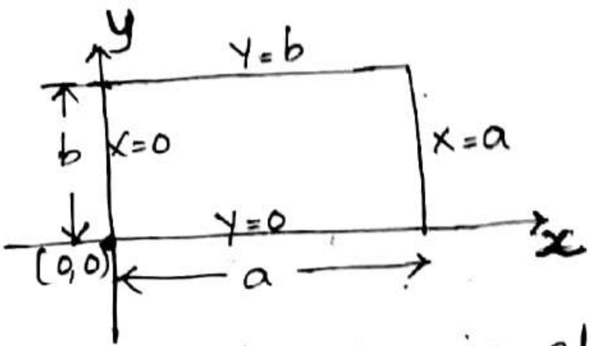


figure : Vertical cut view along x-x' axis.

Assumptions:

a) The hollow, rectangular conducting pipe with dimensions a, b are shown in figure.

b) The waveguide walls assumed ideal perfect conductors.

c) The lossless dielectric material has the parameters μ, ε with $\rho_0 = 0$ and $J = 0$.

d) All field components vary as $e^{j\omega t \pm \beta z}$.

e) For TE mode $E_z = 0$.

Mathematical Analysis,

It is convenient to start with wave equation having field component H_z,

$$\text{ie} \quad \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu \varepsilon H_z \quad \text{--- (1)}$$

By variable separable method, the general solution

(1) can be rewritten as

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z = -\omega^2 \mu \varepsilon H_z$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z + \omega^2 \mu \epsilon H_z = 0$$ (2)

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + (\gamma^2 + \omega^2 \mu \epsilon) H_z = 0$$

$$\boxed{\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + h^2 H_z = 0} \quad \text{--- (2)}$$

or $\boxed{h^2 = \gamma^2 + \omega^2 \mu \epsilon}$

(2) → This partial differential equation can be solved by variable seperable method,

Hence general solution is

$$H_z(x, y) = (A_1 \cos h_x x + A_2 \sin h_x x) \cdot (A_3 \cos h_y \cdot y + A_4 \sin h_y \cdot y) e^{-j\beta z} \quad \text{--- (3)}$$

Now applying boundary conditions to evaluate the complex constants.

$$\boxed{\begin{array}{l} E_x = 0, E_z = 0 \text{ at } y = 0 \text{ and } y = b \\ E_y = 0, E_z = 0 \text{ at } x = 0 \text{ and } x = a. \end{array}}$$

We know, $E_x = \frac{-j\mu\omega}{h^2} \frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2} \frac{\partial E_z}{\partial x}$ --- (a)

$$E_y = \frac{j\omega\mu}{h^2} \frac{\partial H_z}{\partial x} - \frac{\gamma}{h^2} \frac{\partial E_z}{\partial y} \quad \text{--- (b)}$$

$$H_x = \frac{-\gamma}{h^2} \frac{\partial H_z}{\partial x} + \frac{j\epsilon\omega}{h^2} \frac{\partial E_z}{\partial y} \quad \text{--- (c)}$$

$$H_y = \frac{-\gamma}{h^2} \frac{\partial H_z}{\partial y} - \frac{j\epsilon\omega}{h^2} \frac{\partial E_z}{\partial x} \quad \text{--- (d)}$$

For a TE wave $E_z = 0$, substituting in ⓐ, ⓑ, ⓒ and ⓓ results in, ③

$$E_x = \frac{-j\mu\omega}{h^2}\frac{\partial H_z}{\partial y} \quad \text{——— (e)}$$

$$E_y = \frac{+j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} \quad \text{——— (f)}$$

$$H_x = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x} \quad \text{——— (g)}$$

$$H_y = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial y} \quad \text{——— (h)}$$

The boundary conditions are applied to E_x and E_y.

$$\therefore E_x = \frac{-j\mu\omega}{h^2}\frac{\partial H_z}{\partial y}$$

$$= \frac{-j\mu\omega}{h^2}\frac{\partial}{\partial y}\left[(A_1 \cosh_x x + A_2 \sinh_x x)\cdot(A_3 \cosh_y y + A_4 \sinh_y y)\right]e^{-j\beta z}$$

$$E_x = \frac{-j\mu\omega}{h^2} h_y \left(A_1 \cosh_x x + A_2 \sinh_x x\right)\left(-A_3 \sinh_y y + A_4 \cosh_y y\right) e^{-j\beta z} \quad \text{——— (5)}$$

$$E_y = \frac{+j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$= \frac{j\omega\mu}{h^2} h_x \left(-A_1 \sinh_x x + A_2 \cosh_x x\right)\left(A_3 \cosh_y y + A_4 \sinh_y y\right) e^{-j\beta z} \quad \text{——— (6)}$$

From boundary condition

$E_x = 0$ at $y=0$ and $y=b$

∴ Equation 5 becomes,

At $y=0$

$$0 = \frac{-j\omega\mu}{h^2} h_y \left(A_1 \cos h_x x + A_2 \sin h_x x \right)(0 + A_4)$$

$$\therefore \underline{\underline{A_4 = 0}}$$

At $y=b$,

$$0 = \frac{-j\omega\mu}{h^2} h_y \left(A_1 \cos h_x x + A_2 \sin h_x x \right)(-A_3 \sin h_y b + 0)$$

$$\therefore h_y b = n\pi \quad , n = 0, 1, 2 \ldots$$

$$h_y = \frac{n\pi}{b} \quad , n = 0, 1, 2 \ldots$$

Similarly, $E_y = 0$ at $x=0$ and $x=a$

∴ Equation 6 becomes,

At $x=0$

$$0 = \frac{j\omega\mu}{h^2} h_x (0 + A_2)(A_3 \cos h_y y + A_4 \sin h_y y) e^{-j\beta z}$$

$$\therefore \underline{\underline{A_2 = 0}}$$

At $x=a$,

$$0 = \frac{j\omega\mu}{h^2} h_x (-A_1 \sin h_x a + 0)(A_3 \cos h_y y + A_4 \sin h_y y) e^{-j\beta z}$$

$\therefore \quad h_x \cdot a = m\pi$

$$h_x = \frac{m\pi}{a} \qquad m = 0, 1, 2, 3, \ldots$$

$$\therefore H_z(x,y) = A_{m,n} \cos h_x \cdot x \cos h_y \cdot y$$

$$= A_{m,n} \cos \frac{m\pi}{a} \cdot x \cos \frac{n\pi}{b} \cdot y$$

$$\boxed{H_z(x,y,z) = A_{m,n} \cos \frac{m\pi}{a} \cdot x \cos \frac{n\pi y}{b} e^{-j\beta z}.}$$

where $A_{m,n} = A_1 A_3$ = Constant.

Applying to equations (e), (f), (g) and (h) results in

$$E_x = \frac{j\omega\mu n\pi}{h^2 b} A_{m,n} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$$

$$E_y = \frac{-j\omega\mu m\pi}{h^2 a} A_{m,n} \sin \frac{m\pi}{a} x \cos \frac{n\pi}{b} y \; e^{-j\beta z}$$

$$H_x = \frac{+j\beta m\pi}{h^2 a} A_{m,n} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$$

$$H_y = \frac{j\beta n\pi}{h^2 b} A_{m,n} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}.$$

forth between the parallel plates and normal to them.

10.38. RECTANGULAR WAVE GUIDES

So far, the physical picture and techniques for studying the characteristics of guided waves by considering the simple boundary conditions imposed by a pair of infinite parallel conducting planes separated by a dielectric were considered. The techniques so developed will enable us to analyse the characteristics of waves guided along some commonly used waveguides which are more involved. Although such parallel perfectly conducting plate guide system is of hardly convenient in actual practice but it form the basis for analysing the fields of practical waveguides. *e.g.*, rectangular cylindrical etc.

By and large, in case of low frequency transmission line any system of wires may be used as a transmission line but the simplest arrangements like parallel two wire and coaxial lines are preferred. Similarly a pipe with any sort of cross-section could be used as a waveguide, but the simplest cross-section are preferred. Hence, waveguides with constant rectangular or circular cross-sections are usually employed, although other shapes are also possible and are used in some special applications. Like a, b, $f_c =$

transmission line, in waveguides too simplest shapes are those which are easiest to manufacture and most amenable to mathematical treatment. That is why rectangular waveguides are taken up first because they are very common and propagation in them is easy to visualize and calculate.

A rectangular waveguide is shown in Fig. 10.31 in which the walls of the guide are conductors and hence reflections from them takes place. It may be noted that conduction of energy takes place not through wall, whose function is to confine it only, but through the dielectric filling the waveguide, which is usually air.

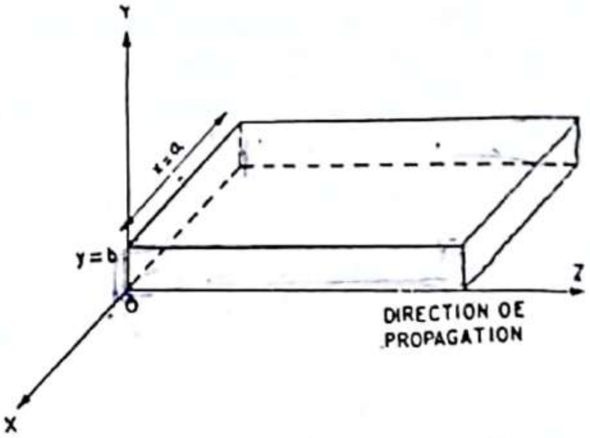


Fig. 10.31. Shows a rectangular waveguide.

Let us consider a rectangular waveguide made of metallic of high conductivity σ with perfect dielectric, such as air, of magnetic permeability μ and permittivity ε inside the guide for the purpose of studying the characteristic and properties (Fig. 10.31). Let the widths of the guide in x and y directions be a and b metres respectively and let the dimensions of the guide be of infinite extent in the z direction

With a view to calculate the electromagnetic field cofiguration inside the rectangular guide, the Maxwell's equations are solved after applying the appropriate boundary conditions at the walls of the guide. As earlier the boundary conditions at the surface of conductor are

(i) the tangential component of electric field must be zero, ie.

$$\boxed{E_t = 0} \qquad \text{... 10.147 (a)}$$

(ii) the normal component of magnetic field must be zero, *i.e.*

$$\boxed{H_n = 0} \qquad \text{... 10.147 (b)}$$

Thus for a rectangular wave guide shown in Fig. 10.27

$$E_x = 0, \; E_z = 0 \text{ at } y = 0 \text{ and } y = b \qquad \text{... 10.147 (c)}$$

$$E_y = 0, \; E_z = 0 \text{ at } x = 0 \text{ and } x = a \qquad \text{... 10.147 (d)}$$

For rectangular waveguides, the Maxwell's equations will be expressed in rectangular coordinate, assuming loss free region where $\sigma = 0$, Thus from eqns. 10.149, 10.150 and 10.160, we have

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\mu\omega H_x \qquad \text{... 10.149 (a)}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\mu\omega H_y \qquad \text{... 10.149 (b)}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\mu\omega H_z \qquad \text{... 10.149 (c)}$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\varepsilon\omega E_x \qquad \text{... 10.150 (a)}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\varepsilon\omega E_y \qquad \text{... 10.150 (b)}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\varepsilon\omega E_z \qquad \text{... 10.150 (c)}$$

and wave equations for E_z and H_z for convenience are

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = -\omega^2 \mu \varepsilon E$$

i.e.
$$\boxed{\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu \varepsilon E_z} \quad \text{... 10.19}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma E_z = -\omega^2 \mu \varepsilon E_z$$

and
$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = -\omega^2 \mu \varepsilon H$$

or
$$\boxed{\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu \varepsilon H_z} \quad \text{... 10.196 (b)}$$

Now with the help of eqns. 10.153 (c) and 10.154 (c), which are showing the variation of field components w.r.t. z coordinate

i.e.
$$E_z = E_{z0}\, e^{-Pz} \quad \text{... (10.197)}$$

or
$$\frac{\partial E_z}{\partial z} = E_{z0}(-P)\, e^{-Pz} = -P E_z \quad \text{... 10.197 (a)}$$

or
$$\frac{\partial^2 E_z}{\partial z^2} = +P^2 E_{z0}\, e^{-Pz}$$

$$\boxed{\frac{\partial^2 E_z}{\partial z^2} = P^2 E_z} \quad \text{... 10.197 (b)}$$

and similarly,
$$\boxed{\frac{\partial^2 H_z}{\partial z^2} = P^2 H_z \text{ etc.}} \quad \text{... 10.197 (c)}$$

Thus, putting derivatives of z as given in eqn. 10.197 into above eqn., we get (eqns. 10.153, 10.149, 10.150)

$$\frac{\partial E_z}{\partial y} + P E_y = -j\mu\omega H_x \quad \text{— 9} \quad \text{... 10.198 (a)}$$

$$-P E_x - \frac{\partial E_z}{\partial x} = -j\mu\omega H_y$$

or
$$P E_x + \frac{\partial E_z}{\partial x} = +j\mu\omega H_y \quad \text{— 10} \quad \text{... 10.198 (b)}$$

and
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\mu\omega H_z \quad \text{— 11} \quad \text{... 10.198 (c)}$$

and
$$\frac{\partial H_z}{\partial y} + P H_y = j\varepsilon\omega E_x \quad \text{— 12} \quad \text{... 10.199 (a)}$$

$$\frac{-\partial H_z}{\partial x} - P H_x = j\varepsilon\omega E_y \text{ or } \frac{\partial H_z}{\partial x} + P H_x = -j\omega\varepsilon E_y \quad \text{— 13} \quad \text{... 10.199 (b)}$$

and
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = +j\varepsilon\omega E_z \quad \text{— 12} \quad \text{... 10.199 (c)}$$

From eqn. 10.199 (a)
$$H_y = \frac{1}{P}\left[j\varepsilon\omega E_x - \frac{\partial H_z}{\partial y}\right] \quad \text{... (10.200)}$$

Substitute this value in eqn. 10.198 (b),

$$P E_x + \frac{\partial E_z}{\partial x} = j\mu\omega\left[\frac{1}{P}\left\{j\varepsilon\omega E_x - \frac{\partial H_z}{\partial y}\right\}\right]$$

$$P E_x - \frac{j^2\omega^2\mu\varepsilon}{P} \times E_x = \frac{j\mu\omega}{P}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}$$

or
$$E_x\left(\frac{P^2 + \omega^2\mu\varepsilon}{P}\right) = -\frac{j\mu\omega}{P}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}$$

or
$$E_x = \frac{P}{K^2}\left[-\frac{j\mu\omega}{P}\frac{\partial H_z}{\partial y} - \frac{\partial E_z}{\partial x}\right]$$

or
$$\boxed{E_x = -\frac{j\mu\omega}{K^2}\cdot\frac{\partial H_z}{\partial y} - \frac{P}{K^2}\cdot\frac{\partial E_z}{\partial x}} \quad \text{... 10.201 (a)}$$

where
$$K^2 = P^2 + \omega^2\mu\varepsilon$$

$h^2 = \gamma^2 + \omega^2\mu\varepsilon$

Now from eqn 10.199 (b),
$$H_x = \frac{1}{P}\left[-\frac{\partial H_z}{\partial x} - j\varepsilon\omega E_y\right] \quad \text{... (10.202)}$$

Putting this into eqn. 10.198 (a), we shall get the value of E_y

$$\frac{\partial E_z}{\partial y} + P E_y = -j\mu\omega H_x$$

or
$$\frac{\partial E_z}{\partial y} + P E_y = -\frac{j\mu\omega}{P}\left[\frac{-\partial H_z}{\partial x} - j\varepsilon\omega E_y\right]$$

or
$$P E_y - \frac{j^2\omega^2\mu\varepsilon}{P} E_y = +\frac{j\mu\omega}{P}\frac{\partial H_z}{\partial x} - \frac{\partial E_z}{\partial y}$$

$$E_y\left(\frac{P^2 + \omega^2\mu\varepsilon}{P}\right) = -\frac{j\omega\mu}{P}\frac{\partial H_z}{\partial x} - \frac{\partial E_z}{\partial y}$$

$$E_y\frac{(P^2 + \omega\mu\varepsilon)}{P} = +\frac{j\omega\mu}{P}\cdot\frac{\partial H_z}{\partial x} - \frac{\partial E_z}{\partial y}$$

or
$$E_y = \frac{P}{K^2}\left[+\frac{j\omega\mu}{P}\cdot\frac{\partial H_z}{\partial x} - \frac{\partial E_z}{\partial y}\right]$$

or
$$\boxed{E_y = +\frac{j\omega\mu}{K^2}\cdot\frac{\partial H_z}{\partial x} - \frac{P}{K^2}\cdot\frac{\partial E_z}{\partial y}} \quad \text{... 10.201 (b)}$$

Substituting the value of E_y into eqn. 10.202 we have the value of H_x as

$$H_x = \frac{1}{P}\left[-\frac{\partial H_z}{\partial x} - j\varepsilon\omega\left\{\frac{j\omega\mu}{K^2}\frac{\partial H_z}{\partial x} - \frac{P}{K^2}\cdot\frac{\partial E_z}{\partial y}\right\}\right]$$

$$= \frac{1}{P}\left[-\frac{\partial H_z}{\partial z}\left(1 + \frac{j^2\omega^2\mu\varepsilon}{K^2}\right) + \frac{j\varepsilon\omega P}{K^2}\cdot\frac{\partial E_z}{\partial y}\right]$$

$$= \frac{1}{P}\left[-\frac{\partial H_z}{\partial x}\left(\frac{K^2 - \omega^2\mu\varepsilon}{K^2}\right) + \frac{j\varepsilon\omega P}{K^2}\frac{\partial E_z}{\partial y}\right]$$

$$= -\frac{P^2}{K^2 P}\frac{\partial H_z}{\partial z} + \frac{j\varepsilon\omega}{K^2}\frac{\partial E_z}{\partial y} \qquad \because K^2 - w^2\mu\varepsilon = P^2$$

or
$$\boxed{H_x = -\frac{P}{K^2}\frac{\partial H_z}{\partial x} + \frac{j\varepsilon\omega}{K^2}\frac{\partial E_z}{\partial y}} \quad \text{... 10.201 (c)}$$

Lastly, the value of H_y component can be obtained from eqn. 10.199 (a) after solving for H_y as

$$PH_y = j\varepsilon\omega E_x - \frac{\partial H_z}{\partial y}$$

$$= j\varepsilon\omega\left\{\frac{-j\mu\omega}{K^2}\frac{\partial H_z}{\partial y} - \frac{P}{K^2}\frac{\partial E_z}{\partial x}\right\} - \frac{\partial H_z}{\partial y} \quad \text{From eqn 10.201 (a)}$$

$$= -\frac{j^2\omega^2\mu\varepsilon}{K^2}\frac{\partial H_z}{\partial y} - \frac{\partial H_z}{\partial y} - \frac{j\varepsilon\omega P}{K^2}\frac{\partial E_z}{\partial x}$$

$$H_y = \frac{1}{P}\left[-\frac{\partial H_z}{\partial y}\left(1 + j^2\frac{\omega^2\mu\varepsilon}{K^2}\right) - \frac{j\varepsilon\omega P}{K^2}\frac{\partial E_z}{\partial x}\right]$$

$$= \frac{1}{P}\left[-\frac{\partial H_z}{\partial y}\left(\frac{K^2 - \omega^2\mu\varepsilon}{K^2}\right) - \frac{j\varepsilon\omega P}{K^2}\frac{\partial E_z}{\partial x}\right]$$

$$\boxed{H_y = -\frac{P}{K^2}\frac{\partial H_z}{\partial y} - \frac{j\varepsilon\omega}{K^2}\frac{\partial E_z}{\partial x}} \quad \text{... 10.201 (d)}$$

Eqns. 10.201 give the relationships among the fields inside the guide and permits one to find out the transverse field components of a rectangular waveguide, whenever the longitudinal components E_z and H_z are known. *It is also observed that in case E_z and H_z are both zero, all the fields within the guide will be zero which means it is imperative that for waveguide transmission, there must exist either an E_z component or H_z component. Further they also serve as a basis for decomposing the field solution into classes known as mode depending on which longitudinal component either E_z or H_z is present.* The modes of the uniform waves guides are defined as

1. *Transverse magnetic (TM) Modes or E-Waves.* For this $H_z = 0$.
2. *Transverse Electric (TE) Modes or H-Mode.* For this $E_z = 0$.
3. *Transverse Electromagnetic (TEM) Modes.* For this both $E_z = 0$ and $H_z = 0$

Here, if the conditions for *TM, TE*, and *TEM* modes (*i.e.* $H_z = 0$, $E_z = 0$ and $H_z = E_z = 0$) respectively are put in eqns. 10.201, then the results derived earlier for these modes (eqn. 10.161), etc. can be obtained.

Now we shall proceed to calculate the solutions for these (eqn. 10.201) for *TM* and *TE* field configurations in a rectangular waveguide.

10.38.1. *Transverse magnetic (TM) mode solutions of rectangular waveguides.* An analysis of the *TM* mode solutions of rectangular hollow waveguides is described as follows. The cross-sectional geometry of the rectangular waveguide is shown in Fig. 10.32 and the following assumptions are made.

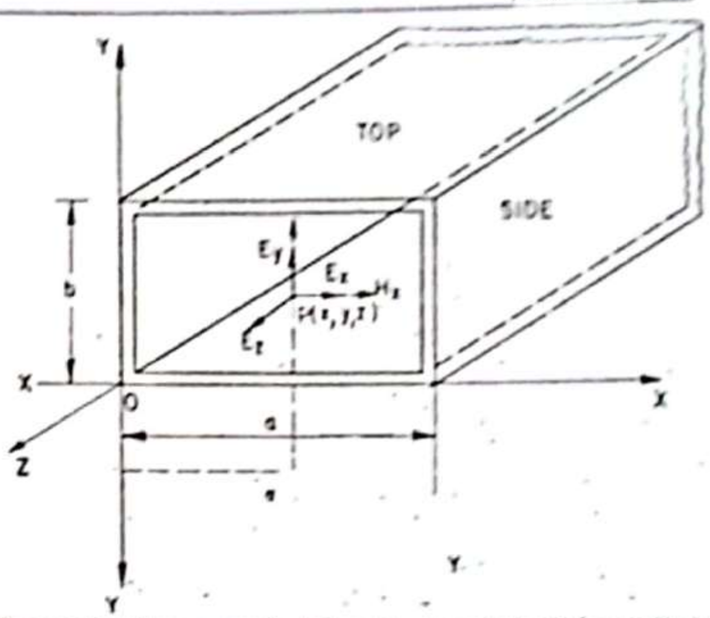
10.38.2. *Assumptions :*

(a) The hollow, rectangular conducting pipe is assumed very long and end effects are avoided. Also uniform transverse dimensions *a, b* are assumed as in figure.

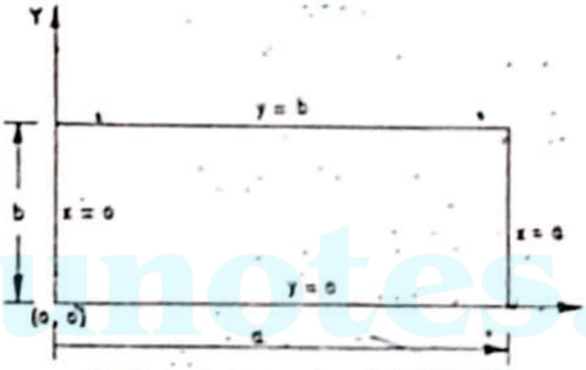
(b) The waveguide walls are assume ideal perfect conductors, simplifying the application of boundary conditions.

(c) The dielectric medium filling the pipe has the constant material parameters μ, ε and is assumed loss-less, so that $\rho_0 = 0$ and $J = 0$ therein.

(d) All the field quantities are assumed to vary with *z* and *t* solely in accordance with the factor $e^{j\omega t \mp \gamma z}$ in which negative sign is associated with positive Z and the positive sign with that of negative Z



(a) Shows geometry of a hollow, rectangular waveguide of dimensions a and b for TM mode filed components.



TM $H_z = 0$ $E_z \neq 0$

(b) Vertical cut view along X-X of filed (a).
Fig. 10.32.

travelling wave solutions. Though the negative wave solution is deleted as we have taken only positive Z direction.

(e) The sinusoidal angular frequency of the fields is ω, determined by the generator frequency.

(f) $H_z = 0$, for *TM* mode under consideration, leaving at most five field components (*e.g.*, E_x, E_y, H_x, H_y and E_z) as shown in figure.

10.38.3. *Mathematical Analysis.* It is convenient to start with the wave eqn. 10.196 (a) having longitudinal field component E_z, *i.e.*,

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu \varepsilon E_z$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 E_z = -\omega^2 \mu_z E_z \qquad \because \frac{\partial^2 E_z}{\partial z^2} = P^2 E_z$$

or
$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + (P^2 + \omega^2 \mu \varepsilon) E_z = 0 \qquad \text{eqn. 10.203 (a)}$$

or
$$\boxed{\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + h^2 E_z = 0} \qquad 10.203 \text{ (a)}$$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right)$

where
$$K^2 = P^2 + \omega^2 \mu \varepsilon$$

This partial differential equation will be solved by the standard method of Variable Separable. Let us assume a solution of the product form as

let $E_z = X(x) Y(y)$.

where $$E_z(x, y, z) = E_{z0}(x, y)\, e^{-\beta z} \quad \text{... (10.204)}$$

$$E_{z0}(x, y) = X(x)\, Y(y) \quad \text{... 10.204 (a)}$$

or simply

$$\boxed{E_{z0} = XY} \quad \text{... 10.204 (b)}$$

where $X(x)$ and $Y(y)$ are functions of x only and y only respectively and in general are complex. Differentiating eqn. 10.204 twice separately once w.r.t. x and next w.r.t. y we have

$$\frac{\partial E_{z0}}{\partial x} = \frac{\partial X}{\partial x} \cdot Y$$

or

$$\frac{\partial^2 E_{z0}}{\partial x^2} = \frac{\partial^2 X}{\partial x^2} \cdot Y \quad \text{... 10.205 (a)}$$

and

$$\frac{\partial E_{z0}}{\partial y} = X \frac{\partial Y}{\partial y}$$

or

$$\frac{\partial^2 E_{z0}}{\partial y^2} = X \frac{\partial^2 Y}{\partial y^2} \quad \text{... 10.205 (b)}$$

k_c^2 / $k_x^2 + k_y^2$

Seperable form /

$\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$

$\frac{\partial^2 Y}{\partial x^2} + k_y^2 Y = 0$

Further putting eqn. 10.204 into eqn. 10.203 the differential eqn. reduces to

$$\frac{\partial^2 E_{z0}}{\partial x^2} + \frac{\partial^2 E_{z0}}{\partial y^2} + K^2 E_{z0} = 0 \quad \text{... 10.203 (b)}$$

$z = 0$

$E_z \neq 0$

Putting eqn. 10.205 in eqn. 10.203 (b) we have

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} + K^2 XY = 0$$

Dividing throughout by XY, we get

$$\boxed{\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + K^2 = 0} \quad \text{... (10.206)}$$

If the two functions of x and y comprising the LHS of eqn. 10.206 are add up to the identical constant for all values of x and y within the cross-section of Fig. 10.30 then both of these functions must equal to constants as well. With $-K_x^2$ and $-K_y^2$ denoting those constants, we have, mathematically from eqn. 10.206.

$$-K_x^2 - K_y^2 = -K^2$$

or

$$\boxed{K_x^2 + K_y^2 = K^2} \quad \text{... 10.207 (a)}$$

Let us assume $h^2 = k_x^2 + k_y^2$.

where

$$-K_x^2 = \frac{1}{X}\frac{\partial^2 X}{\partial x^2} \quad \text{... 10.208 (a)}$$

$$-K_y^2 = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} \quad \text{... 10.208 (b)}$$

Seperating into two different functions

K_x and K_y are called as separation constants and are ascertained from the applications of boundary conditions at the walls.

Since eqns. 10.208 are functions of x and y only so they can be written as ordinary differential equations, *i.e.*,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + K_x^2 = 0$$

$(\ldots + k^2) E_z = 0$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) X(x)\, Y(y)$

Dividing XY $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + h^2 = 0$

$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + h^2 X(x)\, Y(y) = 0$

or

$$\boxed{\frac{\partial^2 X}{\partial x^2} + K_x^2 X = 0} \quad \text{... 10.209 (a)}$$

and similarly,

$$\boxed{\frac{\partial^2 Y}{\partial y^2} + K_y^2 Y = 0} \quad \text{... 10.209 (b)}$$

Ths solutons of these differential equations can be written as

$$X = A_1 \cos K_x x + A_2 \sin K_x x \quad \text{... 10.210 (a)}$$

and

$$Y = A_3 \cos K_y y + A_4 \sin K_y y \quad \text{... 10.210 (b)}$$

where A_1, A_2, A_3, A_4 are constants and are in general complex which will be evaluatedd from the boundary conditions. These separated solutions eqn. 10.210, when inserted into eqn. 10.204 we get the desired particular solution of the wave equation (eqn. 10.203). Thus,

Y

$$\boxed{E_{z0}(x, y) = (A_1 \cos K_x x + A_2 \sin K_x x)\,(A_3 \cos K_y \cdot y + A_4 \sin K_y \cdot y)} \quad \text{... (10.211)}$$

Applying now boundary conditions to evaluate complex constants

$E_z = E_{z0}(x,y)e^{-\gamma z}$

(i) At $x = 0$, $E_{z0} = 0$ this gives

$$E_{z0} = 0 = (A_1 \cos 0 + 0)\,(A_3 \cos K_y \cdot y + A_4 \sin K_y \cdot y)$$

or

$$A_1 = 0$$

$\therefore$

$$E_{z0} = (A_2 \sin K_x \cdot x)\,(A_3 \cos K_y \cdot y + A_4 \sin K_y \cdot y) \quad \text{... 10.212 (a)} \quad \text{... (10.213)}$$

(ii) At $y = 0$, $E_{z0} = 0$ gives from eqn.10.213

$$0 = (A_2 \sin K_x\, x)\,(A_3)$$

This implies either $A_2 = 0$

or $A_3 = 0$

Provided $K_x \neq 0$

First, let us put $A_2 = 0$ in eqn. 10.213. We see that E_{z0} is identically zero. Thus instead of A_2 ; A_3 will be chosen to zero

i.e.

$$A_3 = 0. \quad \text{... 10.212 (b)}$$

Then the general expression reduces to

$$E_{z0} = (A_2 \sin K_x \cdot x\; A_4 \sin K_y \cdot y)$$

$$\boxed{E_{z0} = A_2 A_4 \sin K_x \cdot x \sin K_y \cdot y} \quad \text{... (10.214)}$$

(iii) At $x = a$, $E_{z0} = 0$. Thus gives from eqn. 10.214.

$$0 = A_2 A_4 \sin K_x a \cdot \sin K_y \cdot y$$

or $\quad \sin K_x a = 0 \quad$ if $\sin K_y \neq 0$ $A_2 A_4 \equiv A \neq 0$

or $\quad K_x a = \pm m\pi$

or

$$\boxed{K_x = \pm \frac{m\pi}{a}} \quad \text{... 10.215 (a)}$$

where

$$m = \pm 1, \pm 2, \pm 3, \pm 4, \ldots\ldots \infty \quad \text{... 10.215 (b)}$$

This corresponds to an infinite set of, discrete values for K_x and hence to an infintite number of particular solutions or modes that satisfy the original wave equation. In case m = 0 is substituted so this will produce a null or trivial solution, so deleted. Further, negative values of m add no new solutions to the set, therefore, we restrict and choose the positive value of m, *i.e.*,

$$\boxed{K_x = \frac{m\pi}{a}} \quad \text{... 10.215 (c)}$$

where $\qquad m = 1, 2, 3, 4, \ldots\ldots \infty$ ... 10. 215 (d)

Putting eqn. 10.215 (c) into eqn. 10.214 we get

$$E_{zo} = A \sin \frac{m\pi x}{a} \sin K_y \cdot y \quad \text{... (10.216)}$$

(iv) At $y = b, E_{z0} = 0$. Applying this condition into eqn. 10.216, we obtain,

$$0 = A \sin \frac{m\pi}{a} x \sin K_y \cdot b$$

or $\qquad \sin K_y \cdot b = 0 \qquad$ if $A = A_2 A_4 \neq 0 \quad ; \quad \sin \frac{m\pi x}{a} \neq 0$

or $\qquad K_y\, b = \pm\, n\pi$

or $\qquad K_y = \pm \frac{n\pi}{b}$

or $\qquad \boxed{K_y = \frac{n\pi}{b}}$ ... 10.217 (a)

where $\qquad n = 1, 2, 3, \ldots\ldots \infty$ (as before).

Therefore, the complete solution is,

$$\boxed{E_{zo}(x\,y) = A \sin \frac{m\pi x}{a} \cdot \sin \frac{n\pi y}{b}} \quad \text{... (10.218)}$$

where $\qquad m, n = 1, 2, 3, \ldots$

and $\qquad A = A_2 A_4$ = complex amplitude or any positive of negative travelling E_{zo} component associated with specific values of the mode numbers m and n.

Eqn. 10.218 describe the z directed field component of the transverse-magnetic waves or modes with mode integer m, n assiged, so the field component (eqn. 10.218) is said to belong to the TM_{mn} mode. This is also known as Eigen functions (proper functions) of the boundary value problem.

Now from eqns. 10.201, by putting the condition for TM mode, *i.e.* $H_z = 0$, we get

$$E_x = -\frac{P}{K^2}\frac{\partial E_z}{\partial x} \quad \text{... 10.219 (a)}$$

$$E_y = -\frac{P}{K^2}\frac{\partial E_z}{\partial y} \quad \text{... 10.219 (b)}$$

$$H_x = +\frac{j\varepsilon\omega}{K^2}\frac{\partial E_z}{\partial y} \quad \text{... 10.219 (c)}$$

$$H_y = -\frac{j\varepsilon\omega}{K^2}\frac{\partial E_z}{\partial z} \quad \text{... 10.219 (d)}$$

and $\qquad P = j\beta$ for frequencies above cut–off frequency.

The rest four transverse field components are obtained by substituting the value of eqn. 10.218 into eqn. 10.219 as follows (using eqns. 10.153 and 10.154)

$$E_{zo} = A \sin K_x \cdot x \sin K_y \cdot y$$

$$\frac{\partial E_{z0}}{\partial x} = A\, K_x \cos K_x \cdot x \sin K_y \cdot y \quad \text{... 10.220 (a)}$$

$$\frac{\partial E_{z0}}{\partial y} = A \cdot K_y \sin K_x\, x \cos K_y \cdot y \qquad \text{... 10.220 (b)}$$

$$\therefore \quad E_{x0} = -\frac{j\beta A K_x}{K^2} \cdot \cos K_x \cdot x \cdot \sin K_y \cdot y = \frac{E_x}{e^{-Pz}} \qquad \text{... 10.221 (a)}$$

$$\boxed{E_{x0} = -\frac{j\beta A m\pi}{a K^2} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}} \qquad \text{... 10.221 (b)}$$

$$E_{y0} = -\frac{P}{K^2} A K_y \sin K_x \cdot x \cdot \cos K_y \cdot y \qquad \text{... 10.221 (c)}$$

$$\boxed{E_{y0} = -\frac{j\beta A n\pi}{K^2 b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}} \qquad \text{... 10.221 (d)}$$

$$H_{x0} = +\frac{j\varepsilon\omega A K_y}{K^2} \sin K_x \cdot x \cdot \cos K_y \cdot y \qquad \text{... 10.221 (e)}$$

or

$$\boxed{H_{x0} = +\frac{j\varepsilon\omega A n\pi}{K^2 b} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b}} \qquad \text{... 10.221 (f)}$$

and

$$H_{y0} = -\frac{j\varepsilon\omega A K_x}{K^2} \cos K_x \cdot x \cdot \sin K_y \cdot y \qquad \text{... 10.221 (g)}$$

$$\boxed{H_{y0} = -\frac{j\varepsilon\omega A m\pi}{K^2 a} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b}} \qquad \text{... 10.221 (h)}$$

Eqns. 10.221 expresses how each of the components of electric field and magnetic field varies with x and y. The variation of time along the axis of the guide, *i.e.* z direction can be obtained by using the relations eqn. 10.153 and eqn. 10.153, *i.e.*

$$E_{x0} = \frac{E_x}{e^{-Pz}} \text{ and } H_{x0} = \frac{H_x}{e^{-Pz}}$$

Hence, eqn. 10.221 reduces ($P = j\beta$) for the propagation frequency range as

$$E_x\,(x, y, z) = -\frac{j\beta A m\pi}{K^2 a} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \times e^{-j\beta z} \qquad \text{... 10.222 (a)}$$

$$E_y\,(x, y, z) = -\frac{j\beta A n\pi}{K^2 b} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \times e^{-j\beta z} \qquad \text{... 10.222 (b)}$$

$$H_x\,(xyz) = +\frac{j\varepsilon\omega A n\pi}{K^2 b} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \times e^{-j\beta z} \qquad \text{... 10.222 (c)}$$

$$H_y\,(xyz) = -\frac{j\varepsilon\omega A m\pi}{K^2 a} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \times e^{-j\beta z} \qquad \text{... 10.222 (d)}$$

Each value of m, n specifies a particular field configuration or mode and the wave associated with the integers m, n is designated as the TM_{mn} mode or wave. It can be seen that for m or n equal to zero the fields for TM_{mn} wave will be identically zero and hence, *the lowest order possible TM_{mn} mode or wave is the TM_{11} mode.*

10.38.4. ***Transverse Electric (TE) Mode Solutions of Rectangular waveguides.*** The analysis of the TE mode solutions of the rectangular waveguides is described as follows which is essentially similar to one employed in the preceding article. The cross-sectional geometry of the rectangular waveguide is shown in Fig. 10.33 and the following assumptions are made.

wavelength.

10.37.5. ***Group Velocity and Phase Velocity.*** During discussion with wave propagation between parallel conducting planes (also in waveguide) two different velocities are encountered. This is "**phase velocity**" and "**group velocity**". It happens that any electromagnetic wave has two velocities, the one with which it propagates and the one with which it changes phase. However in free space, these are naturally the same and are called the velocity of light C.

The phase velocity v_p is defined as the velocity of propagation of equiphase surfaces along the guide or in other words phase velocity is the velocity with which the wave changes phase in a direction parallel to the conducting surface. If f is the frequency, then phase velocity is given by the product of wavelength and frequency.

i.e.
$$v_p = f\lambda = \frac{2\pi f\lambda}{2\pi} = 2\pi f \cdot \frac{\lambda}{2\pi}$$

or
$$\boxed{v_p = \frac{\omega}{\beta} \quad \text{m/s}} \qquad \text{eqn 10.185}$$

The group velocity, on the other hand, may be defined under certain condition, as the velocity with which the group of the wave as a whole propagate.. It is usually denoted by v_g as is defined by the equation

$$\boxed{v_g = \frac{d\omega}{d\beta}} \qquad \ldots (10.194)$$

In case of waveguide, *it can be considered as the velocity of energy propagation in the direction of the axis of guide.* It may, however, be noted that group velocity is always less than free space velocity $= \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, where as phase velocity is always more than the

$$\boxed{v_c = \frac{1}{\sqrt{\varepsilon \mu}}}$$

While conveying intelligence, it is always necessary to modulate by some means or the other the carrier frequency being transmitted. On modulation a group of frequencies, usually centred about the carrier has to be propagated along the guide or transmission line. If phase velocity is a function of frequency, the waves of different frequencies, in the group will be transmitted with slightly different velocities. The component waves combine to form a "modulation envelope" which is propagated as a wave having the group velocity v_g given by eqn. 10.194.

Now we shall proceed to show that the product of group velocity and phase velocity of a signal propagating in a waveguide is equal to the square of the velocity of light.

i.e.,
$$\boxed{v_g \; v_p = v_c^{\,2}} \qquad \text{... 10.195 (a)}$$

For calculating the group velocity for a group of TE and TM waves with frequencies centred about the carrier angular ω propagating in a parallel plate waveguide, we have from eqn. 10.181

$$\beta^2 = \omega^2 \mu \varepsilon - \left(\frac{m\pi}{a}\right)^2$$

$$2\beta \, d\beta = 2\omega \, d\omega \mu \varepsilon - 0$$

or
$$\frac{d\omega}{d\beta} = \frac{\beta}{\omega \mu \varepsilon} = \frac{\beta}{\omega} \cdot v_c^{\,2}$$

or
$$v_g = \frac{d\omega}{d\beta} = \frac{v_c^2}{\omega/\beta} \qquad \because v_c = \frac{1}{\sqrt{\mu \varepsilon}}$$

or
$$v_g = \frac{v_c^{\,2}}{v_p} \qquad \text{by eqn. 10.185}$$

or
$$\boxed{v_c^{\,2} = v_p \; v_g = c^2} \qquad \text{... (10.195 b)}$$

or

where v_p = phase velocity.

v_g = group velocity

v_c = velocity of light in free space and equal to c if dielectric is air between plates, otherwise

$$v_c = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = c$$

Thus, the group velocity (v_g) with which the energy is actually transmitted can at most equal to c, the velocity of light in free space. As the angular frequency ω is reduced, the phase velocity v_p becomes very large and group velocity v_g becomes very small. At cut off frequency, v_p becomes infinite and v_g becomes zero. This means there is no propagation of energy along the waveguide. The waves simply bounce back and forth between the parallel plates and normal to them.

... of guided waves by

(Examples 10.76 to 10.105 are based on waveguides)

Example 10.76. *An air filled rectangular waveguide has cross-sectional dimensions of a = 100 mm and b = 50 mm. Calculate the cut-off frequencies for the modes TE_{10}, TE_{20}, TE_{01}, TE_{02}, TE_{11}, TE_{12} and TE_{21} and also the ratio of the guide velocity v_{ps} to the velocity in free space for each of above modes if $f = 2f_c$.*

Solution. Since $f_c = \frac{v_p}{2ab}\sqrt{(mb)^2 + (na)^2}$

Given $$a = \frac{100}{1000} \text{ metres} = \frac{1}{10} \text{ metres}; \ b = \frac{50}{1000} \text{ metres} = \frac{5}{100} \text{ metres}$$

$$v_p = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8 \text{ m/s}$$

Hence $$f_c = \frac{3 \times 10^8 \times 10 \times 100}{2 \times 1 \times 5}\sqrt{\left(m \times \frac{5}{100}\right)^2 + \left(n \times \frac{1}{10}\right)^2}$$

$$= 3 \times 10^{10}\sqrt{\frac{m^2 \times 25}{(100)^2} + \frac{n^2 \times 1 \times 100}{100 \times 100}}$$

$$= \frac{3 \times 10^{10}}{100}\sqrt{m^2 \times 25 + n^2 \times 100} = 3 \times 10^8 \sqrt{100\ (m^2 \times 0.25 + n^2)}$$

$$f_c = 3 \times 10^9 \sqrt{m^2 \times 0.25 + n^2}$$

Now for TE_{10} mode, $m = 1$, $n = 0$ so that

$$f_c = 3 \times 10^9 \sqrt{0.25 \times 1^2 + 0} = 3 \times 10^9 \times 0.5$$

$$f_c = 1.5 \times 10^9 \text{ Hz}$$

(ii) For TE_{20} mode, $m = 2$, $n = 0$

$$f_c = 3 \times 10^9 \sqrt{2^2 \times 0.25 + 0} = 3 \times 10^9 \times 2 \times 0.5$$

$$f_c = 3 \times 10^9 \text{ Hz}$$

(iii) For TE_{01} mode $m = 0$, $n = 1$ then

$$f_c = 3 \times 10^9 \sqrt{0 + 1} = 3 \times 10^9 \text{ Hz}$$

(iv) For TE_{02} mode, $m = 0$, $n = 2$ so that

$$f_c = 3 \times 10^9 \sqrt{0 + 2^2} = 6 \times 10^9 \text{ Hz}$$

(v) For TE_{11} mode, $m = 1$, $n = 1$ then

$$f_c = 3 \times 10^9 \sqrt{0.25 \times 1^2 + 1^2} = 3 \times 10^9 \times \sqrt{1.25}$$

$$= 3 \times 10^9 \times 1.1180 = 3.3541 \times 10^9 \text{ Hz}$$

(vi) For TE_{12} mode, $m = 1$, $n = 2$ then

$$f_c = 3 \times 10^9 \sqrt{0.25 \times 1^2 + 2^2}$$

$$= 3 \times 10^9 \sqrt{4.25} = 3 \times 10^9 \times 2.0615$$

$$f_c = 6.1846 \times 10^9 \text{ Hz}$$

(vii) For TE_{21} mode, $m = 2$, $n = 1$ then

$$fc = 3 \times 10^9 \sqrt{0.25 \times 2^2 + 1^2}$$

$$= 3 \times 10^9 \sqrt{2} = 3 \times 10^9 \times 1.4142 = 4.2426 \times 10^9 \text{ Hz}$$

Summarising we have

For TE_{10} mode, $f_c = 1500$ MHz
For TE_{20} mode, $f_c = 3000$ MHz
For TE_{01} mode, $f_c = 3000$ MHz
For TE_{02} mode, $f_c = 6000$ MHz
For TE_{11} mode, $f_c = 3354.1$ MHz
For TE_{12} mode, $f_c = 6184.6$ MHz
For TE_{21} mode, $f_c = 4242.6$ MHz

Out of seven modes shown above, it is obvious that TE_{10} is the dominant mode with the lowest cut-off frequency of $f_c = 1500$ MHz $= 1.5$ GHz. Further if $f = 2\,f_c$ all possible modes are propagated. Now by eqn.

$$v_{ps} = \frac{v_p}{\sqrt{\left\{1 - \left(\frac{\omega_c}{\omega}\right)^2\right\}}}$$

or

$$\frac{v_{ps}}{v_p} = \frac{1}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{1}{\sqrt{1 - \left(\frac{f_c}{2f_c}\right)^2}} \qquad \because f = 2f_c$$

$$= \frac{1}{\sqrt{1 - (0.5)^2}} = \frac{1}{\sqrt{1 - 0.25}} = \frac{1}{\sqrt{0.75}} = \frac{1}{0.8660} = 1.1547 \text{ Ans.}$$

Example 10.77. *An air filled hollow rectangular conducting waveguide has cross-section of 8 × 10 cm. How many TE mode will this waveguide transmit at frequencies below 4 GHz. How these modes are designated and what are their cut-off frequencies.*

Solution. Given that the maximum frequency, the waveguide can transmit is 4 GHz. Hence by eqn. 10.247 (e), the minimum wavelength below which the propagation does occur is given by

$$\lambda_C = \frac{v_p}{f_c} = \frac{3 \times 10^8}{4 \times 10^9} = 0.075 \text{ metres} \qquad f_c = 4 \text{ GHz}$$

The cut-off frequencies in terms of waveguide dimensions are given by eqn. 10.192 (b)

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

Given

$$a = 10 \text{ cm} = 0.10 \text{ metres}$$
$$b = 8 \text{ cm} = 0.08 \text{ metres}$$

Hence

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{0.10}\right)^2 + \left(\frac{n}{0.08}\right)^2}} = \frac{2}{\sqrt{\frac{n^2}{0.0064} + \frac{m^2}{0.01}}}$$

$$= \frac{2}{\sqrt{\frac{n^2 \times 10000}{64} + \frac{m^2 \times 100}{1}}} = \frac{2}{\sqrt{\frac{100\,[(n^2 \times 100) + m^2 \times 64]}{64}}}$$

$$\lambda_c = \frac{2}{\frac{10}{8}\sqrt{n^2 \times 100 + 64\, m^2}} = \frac{1.6}{\sqrt{100\, n^2 + 64\, m^2}} = \frac{1.6}{\sqrt{100\,(m^2 + 0.64\, m^2)}}$$

$$\lambda_c = \frac{0.16}{\sqrt{n^2 + 0.64\, m^2}}$$

Now, for TE_{10} mode, $m = 1,\ n = 0$

$$\lambda_c = \frac{0.16}{\sqrt{0 + 0.64 \times 1}} = \frac{0.16}{0.8} = \frac{16}{80} = \mathbf{0.2\ metres}$$

(ii) For TE_{01} mode, $m = 0,\ n = 1$

Then $$\lambda_c = \frac{0.16}{\sqrt{0 + 1^2}} = \mathbf{0.16\ metres}$$

(iii) For TE_{11} mode, $m = 1,\ n = 1$ so that

$$\lambda_c = \frac{0.16}{\sqrt{0.64 \times 1^2 + 1^2}} = \frac{0.16}{\sqrt{1.64}} = \frac{0.16}{1.280} = \mathbf{0.125\ metres}$$

(iv) For TE_{20} mode, $m = 2,\ n = 0$

$$\lambda_c = \frac{0.16}{\sqrt{0.64 \times 2^2 + 0}} = \frac{0.16}{0.8 \times 2} = \frac{0.16}{1.6} = \frac{16}{160} = 0.1$$

$$\lambda_c = \mathbf{0.1\ metres}$$

(v) For TE_{02} mode, $m = 0,\ n = 2$

Then $$\lambda_c = \frac{0.16}{\sqrt{0.64 \times 0 + 2^2}} = \frac{0.16}{2}$$

$$\lambda_c = \mathbf{0.08\ metres}$$

(vi) For TE_{21} mode, $m = 2,\ n = 1$

Then $$\lambda_c = \frac{0.16}{\sqrt{0.64 \times 2^2 + 1^2}} = \frac{0.16}{\sqrt{3.56}} = \frac{0.16}{1.8868}$$

$$\lambda_c = \mathbf{0.0847\ metres}$$

(vii) For TE_{12} mode, $m = 2,\ n = 1,$

$$\lambda_c = \frac{0.16}{\sqrt{0.64 \times 1^2 + 2^2}} = \frac{0.16}{\sqrt{4.64}} = \frac{0.16}{2.154}$$

$$\lambda_c = \mathbf{0.07428\ metres}$$

Summarising, we have

(i) For TE_{10} mode, $\lambda_c = 0.2$ metres

(ii) For TE_{01} mode, $\lambda_c = 0.16$ metres

(iii) For TE_{11} mode, $\lambda_c = 0.125$ metres

(iv) For TE_{20} mode, $\lambda_c = 0.1$ metres

(v) For TE_{02} mode, $\lambda_c = 0.080$ metres

(vi) For TE_{21} mode, $\lambda_c = 0.0847$ metres

(vii) For TE_{12} mode, $\lambda_c = 0.07428$ metres

It is seen that only first six modes will be transmitted and the last mode being less than 0.075 metres (λ_c) will not be transmitted.

The six modes transmitted are designated as TE_{10}, TE_{01} etc. and their cut off frequencies are given by

$$\lambda_c = \frac{v_p}{f_c} \ ; f_c = \frac{v_p}{\lambda_c} \text{ Hz}$$

(i) For TE_{10} mode, $f_c = \dfrac{3 \times 10^8}{0.2} = 1500$ MHz

(ii) For TE_{01} mode, $f_c = \dfrac{3 \times 10^8}{0.16} = 1875$ MHz

(iii) For TE_{11} mode, $f_c = \dfrac{3 \times 10^8}{0.125} = 2400$ MHz

(iv) For TE_{20} mode, $f_c = \dfrac{3 \times 10^8}{0.1} = 3000$ MHz

(v) For TE_{02} mode, $f_c = \dfrac{3 \times 10^8}{0.08} = 3750$ MHz

(vi) For TE_{21} mode, $f_c = \dfrac{3 \times 10^8}{0.0847} = 3541.91$ MHz.

Example 10.78. *An air filled rectangular waveguide has cross-sectional dimensions ...*
b = 4 cm. Find the ...

$\sqrt{0.5601} = \frac{1}{0.74839} = 1.3362.$

Example 10.79. *The larger dimension of the cross-section of a rectangular wave guide is 2 cm. Find the cut-off frequency and wavelength for the dominant TE mode.*

(Roorkee Univ. Applied Electromagnetic Theory, 1990-91)

Solution. The dominant TE mode in a rectangular waveguide is TE_{10}. Hence from eqn.

$$f_c = \frac{v_p}{2\,ab}\sqrt{(mb)^2 + (na)^2} \quad \text{... 10.247 (4)}$$

For TE_{10} mode, $m = 1,\ n = 0$

$$a = 2 \text{ cm} = 0.02 \text{ m};\ v_p = 3 \times 10^8 \text{ m/s}$$

Then

$$f_c = \frac{v_p}{2\,ab} \times \sqrt{(mb)^2} = \frac{v_p m}{2a}$$

$$= \frac{3 \times 10^8 \times 1}{2 \times 0.02} = \frac{3}{0.04} \times 10^8 = 7.5 \times 10^8 = 7.5 \times 10^9 \text{ Hz}$$

$$f_c = \mathbf{7.5 \text{ GHz}} \quad \textbf{Ans.}$$

and the corresponding wavelength for the dominant mode is

$$\lambda_c = \frac{2\pi}{\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}} = \frac{2\pi}{\frac{m\pi}{a}} \quad \text{... 10.248 (a)}$$

$$\lambda_c = \frac{2a}{m} = \frac{2 \times 0.02}{1} = \mathbf{0.04 \text{ metres}} \quad \textbf{Ans.}$$

Example 10.80. *A hollow rectangular waveguide has dimensions $a = 4$ cm, $b = 2$ cm. Calculate the amount of attenuation, if the frequency is 3 GHz.*

Solution. The critical frequency of the waveguide is given by

$$f_c = \frac{v_p}{2\,ab}\sqrt{(mb)^2 + (na)^2} \quad \text{... 10.247 (d)}$$

For TE_{10} mode

$$f_c = \frac{v_p}{2\,ab}\sqrt{(mb)^2 + 0} = \frac{v_p}{2\,ab} \times mb = \frac{v_p}{2\,a} \cdot m \qquad \left| \begin{array}{l} m = 1\ ;\ n = 0 \\ a = \frac{4}{100} \text{ m} \end{array} \right.$$

$$= \frac{3 \times 10^8 \times 1 \times 100}{2 \times 4} = 0.375 \times 10^{10} \text{ Hz}$$

$$f_c = \mathbf{3.75 \text{ GHz.}}$$

Since the critical frequency is 3.75 GHz so the given frequency 3 GHz will not be propagated and will get attenuated.

The amount of attenuation is given by eqn. 10.246 *i.e.*

$$P = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu\, \varepsilon} = \alpha + j\,\beta$$

$$\alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu_0\, \varepsilon_0} \qquad \Big|\ \because \beta = 0 \text{ since wave is attenuated}$$

fundamental mode TE_{10}

$$\alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2 \mu_0 \varepsilon_0}$$

$\because \mu_0 = 4\pi \times 10^{-7}$ H/m ; $\varepsilon_0 = \frac{10^{-9}}{36\pi}$ F/m ;

$\omega = 2\pi f$; $f = 3$ GHz

$$= \sqrt{\left(\frac{1 \times 3.14 \times 100}{4}\right)^2 - \frac{(2\pi \times 3 \times 10^9)^2 \times 4\pi \times 10^{-7} \times 10^{-9}}{36\pi}}$$

$$= \sqrt{\left(\frac{314}{4}\right)^2 - \frac{4\pi^2 \times 9 \times 10^{18} \times 10^{-16}}{9}}$$

$$= \sqrt{(78.5)^2 - 4 \times 9.8596 \times 10^2} = \sqrt{6162.25 - 3943.84} = \sqrt{2218.41}$$

$\alpha =$ **47.1 nepers/metres**

1 neper = 8.686 *db* and 1 *db* = 0.115 neper

$\alpha = 47.1 \times 8.686\ db =$ **409.1106 db Ans.**

Example 10.81 *If a rectangular ...*