Module 4

Gram-Schmidt Orthogonalization Procedure

- The Gram–Schmidt orthogonalization procedure permits the representation of any set of M energy signals, as linear combinations of N orthonormal basis Auctions, where $N < M$.

- That is to say, we may represent the given set of real-valued energy signals $s_1(t), s_2(1), \ldots, s_m(t)$, each of duration T seconds, in the form

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t) \qquad \begin{array}{l} 0 \leqslant t \leqslant T \\ i = 1, 2, \ldots, M \end{array} \tag{1}$$

- where, the coefficients of the expansion are defined by,

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt \qquad \begin{matrix} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{matrix} \tag{2}$$

- The real-valued basis functions $\phi_1 t, \phi_2(t), \ldots, \phi_N(t)$ are orthonormal, that is;

$$\int_0^T \phi_i(t)\phi_j(t)dt = \begin{cases} 1 & \text{if i = j} \\ 0 & \text{if i} \neq \text{j} \end{cases} \tag{3}$$

Stage-1

- Establish whether or not the given set of signals $s_1(t), s_2(t), \ldots, s_M(t)$ is linearly independent.
- If they are not linearly independent, then there exists a set of co-efficients $a_1, a_2, \ldots, a_M$, not all equal to zero, such that we may write

$$a_1 s_1(t) + a_2 s_2(t) + \cdots + a_M s_M(t) = 0 \qquad 0 \leq t \leq T \tag{4}$$

- Suppose, in particular, that $a_M(t) \neq 0$. Then we may express the corresponding signal $s_M(t)$ as

$$s_M(t) = -\left[\frac{a_1}{a_M} s_1(t) + \frac{a_2}{a_M} s_2(t) + \cdots + \frac{a_{M-1}}{a_M} s_{M-1}(t)\right] \tag{5}$$

which implies that the signal $s_M(t)$ may be expressed in terms of the remaining $(M-1)$ signals.

- If they are not linearly independent, then there exists a set of coefficients $b_1, b_2, \ldots, b_{M-1}$, not all equal to zero, such that we may write

$$b_1 s_1(t) + b_2 s_2(t) + \cdots + b_{M-1} s_{M-1}(t) = 0 \qquad 0 \leq t \leq T \tag{6}$$

- Suppose, in particular, that $b_{M-1}(t) \neq 0$. Then we may express the corresponding signal $s_{M-1}(t)$ as linear combination of the remaining $M-2$ signals as

$$s_{M-1}(t) = -\left[\frac{b_1}{b_{M-1}} s_1(t) + \frac{b_2}{b_{M-1}} s_2(t) + \cdots + \frac{b_{M-2}}{b_{M-1}} s_{M-2}(t)\right] \tag{7}$$

- Let $s_1(t), s_2(t), \ldots, s_N(t)$ denote this subset of linearly independent signals, where $N \leq M$.

Stage-2

As a starting point, define the first basis function as

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}} \tag{8}$$

where, E_1 is the energy of the signal $s_1(t)$.

Rearraging Eq. (8) we get,

$$\begin{aligned} s_1(t) &= \sqrt{E_1}\phi_1(t) \\ &= s_{11}\phi_1(t) \end{aligned} \tag{9}$$

where, the coefficient $s_{11} = \sqrt{E_1}$ and $\phi_1(t)$ has unit energy.

To define the second basis function, we define a new intermediate function as

$$g_2(t) = s_2(t) - s_{21}\phi_1(t) \tag{10}$$

which is orthogonal to $\phi_1(t)$ over the interval $0 \le t \le T$.
The second basis function is then given by

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)dt}} \tag{11}$$

Substituting Eq. (11) in Eq. (10) and simplifying we get

$$\begin{aligned}
\phi_2(t) =& \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{\int_0^T (s_2(t) - s_{21}\phi_1(t))^2}} \\
=& \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{\int_0^T [s_2(t)]^2 dt + \int_0^T s_{21}^2\phi_1^2(t)dt - \int_0^T 2\ s_2(t)\ s_{21}\phi_1(t)dt}}
\end{aligned}$$

$$
\begin{aligned}
\phi_2(t) &= \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{\int_0^T s_2^2(t)dt + s_{21}^2 \int_0^T \phi_1^2(t)dt - 2\ s_{21} \int_0^T \ s_2(t)\phi_1(t)dt}} \\
&= \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 + s_{21}^2 - 2\ s_{21} \times s_{21}}} \\
&= \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 + s_{21}^2 - 2\ s_{21}^2}}
\end{aligned}
$$

$$
\phi_2(t) = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}} \tag{12}
$$

where, E_2 is the energy of signal $s_2(t)$ given as

$$
E_2 = \int_0^T s_2^2(t)dt
$$

Continuing in this fashion, we may define

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t) \tag{13}$$

where the coefficients $s_{ij},\ j = 1, 2, \ldots, i-1$, are themselves defined by

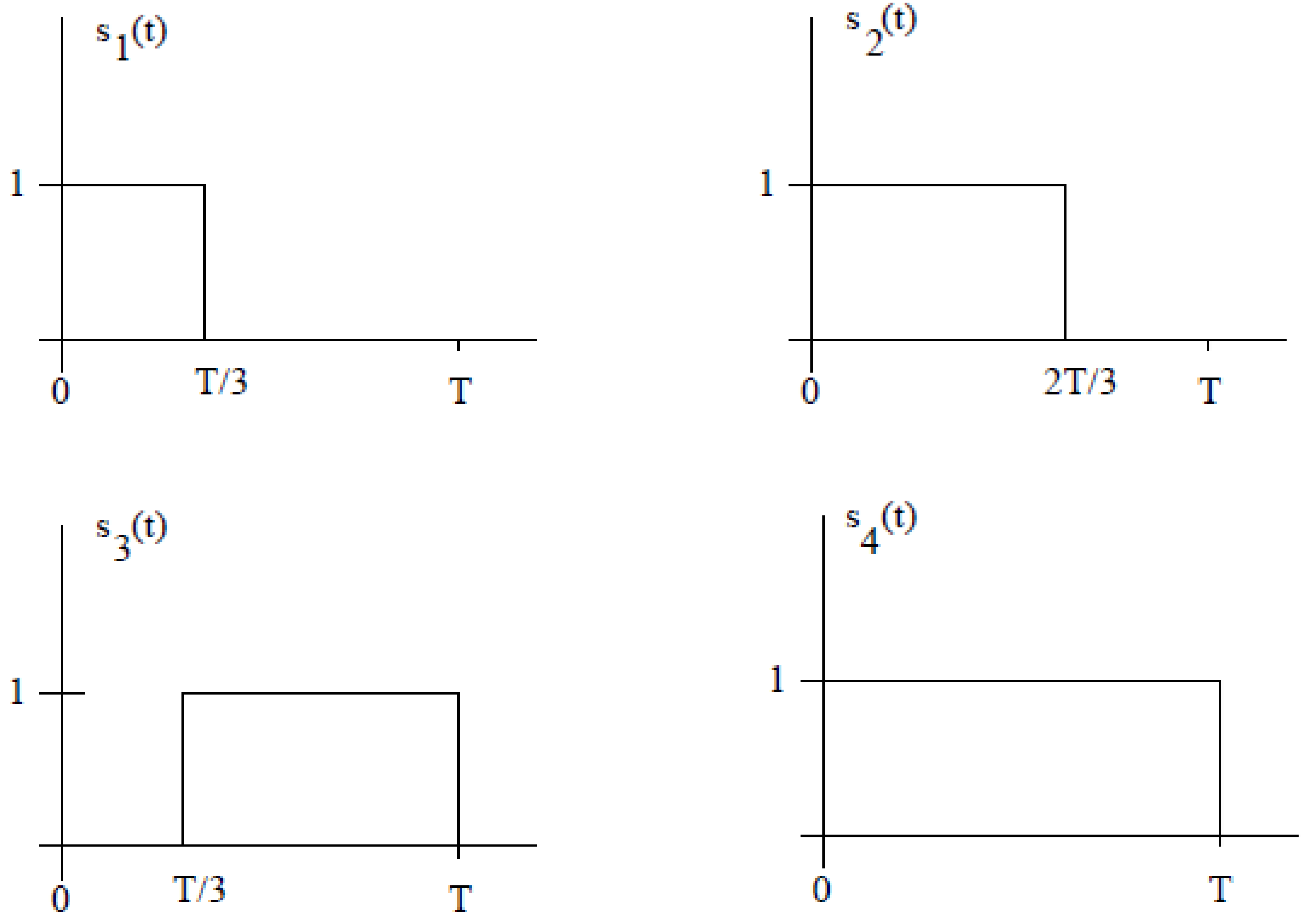
$$s_{ij} = \int_0^T s_i(t)\phi_j(t)dt \tag{14}$$

Then it follows that the set of functions

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)dt}} \tag{15}$$

forms an orthonormal set.

Example. A set of four waveform is illustrated as below. Find an orthonormal set for this set of signals by applying the Gram-Schmidt procedure.



Step 1: This signal set is not linearly independent because

$$s_4(t) = s_1(t) + s_3(t)$$

Therefore, we will use $s_1(t)$, $s_2(t)$, and $s_3(t)$ to obtain the complete set of basis functions.

Step 2:

a)

$$E_1 = \int_0^T s_1^2(t)dt = T/3$$

$$f_1(t) = \frac{s_1(t)}{\sqrt{E_1}} = \begin{cases} \sqrt{3/T} \ , & 0 \leq t \leq T/3 \\ 0 \ , & \text{else} \end{cases}$$

b)

$$
\begin{aligned}
s_{21} &= \int_0^T s_2(t) f_1(t) dt \\
&= \int_0^{T/3} \sqrt{3/T} dt = \sqrt{T/3}
\end{aligned}
$$

$$E_2 = \int_0^T s_2^2(t) dt = 2T/3$$

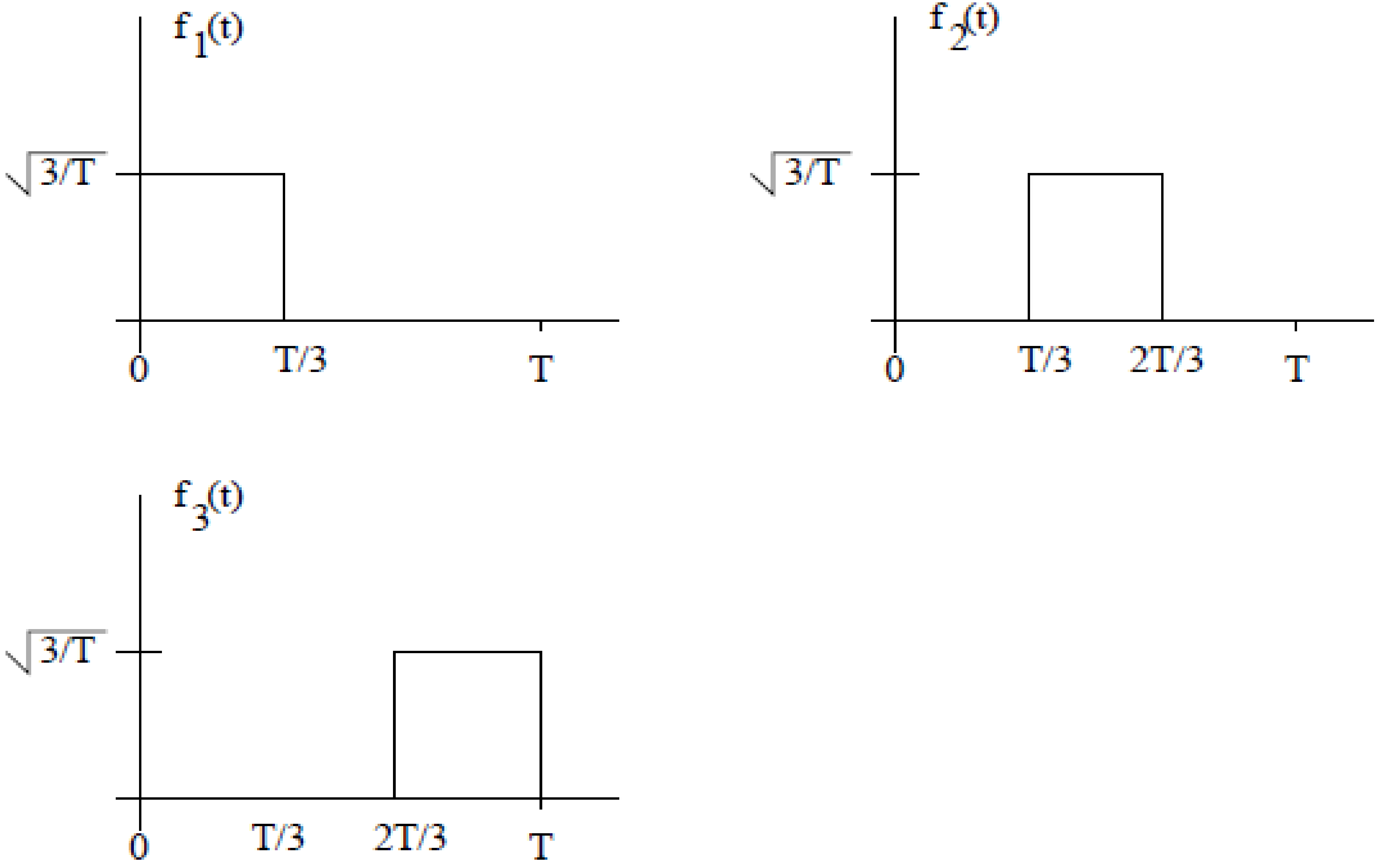
$$
\begin{aligned}
f_2(t) &= \frac{s_2(t) - s_{21} f_1(t)}{\sqrt{E_2 - s_{21}^2}} \\
&= \begin{cases} \sqrt{3/T} \ , & T/3 \le t \le 2T/3 \\ 0 \ , & \text{else} \end{cases}
\end{aligned}
$$

c)

$$
\begin{aligned}
s_{31} &= \int_0^T s_3(t) f_1(t) dt = 0 \\
s_{32} &= \int_0^T s_3(t) f_2(t) dt \\
&= \int_{T/3}^{2T/3} \sqrt{3/T} dt = \sqrt{T/3}
\end{aligned}
$$

$$
\begin{aligned}
g_3(t) &= s_3(t) - s_{31} f_1(t) - s_{32} f_2(t) \\
&= \begin{cases} 1 \ , & 2T/3 \leq t \leq T \\ 0 \ , & \text{else} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f_3(t) &= \frac{g_3(t)}{\sqrt{\int_0^T g_3^2(t) dt}} \\
&= \begin{cases} \sqrt{3/T} \ , & 2T/3 \leq t \leq T \\ 0 \ , & \text{else} \end{cases}
\end{aligned}
$$


f1(t)
√3/T
0
T/3
T
f2(t)
√3/T
0
T/3
2T/3
T
f3(t)
√3/T
0
T/3
2T/3
T

$$
\begin{aligned}
s_1(t) &\leftrightarrow \mathbf{s}_1 = (\sqrt{T/3}, 0, 0) \\
s_2(t) &\leftrightarrow \mathbf{s}_2 = (\sqrt{T/3}, \sqrt{T/3}, 0) \\
s_3(t) &\leftrightarrow \mathbf{s}_3 = (0, \sqrt{T/3}, \sqrt{T/3}) \\
s_4(t) &\leftrightarrow \mathbf{s}_4 = (\sqrt{T/3}, \sqrt{T/3}, \sqrt{T/3})
\end{aligned}
$$