ECT 305
ANALOG AND DIGITAL COMMUNICATION

Module-3

Source Coding

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Syllabus

- Source coding theorems
- Waveform coding.
- Sampling and Quantization.
- Pulse code modulation,
  - Transmitter and receiver.
- Companding.
  - Practical 15 level A and mu-law companders.
- DPCM transmitter and receiver.
  - Design of linear predictor.
  - Wiener-Hopf equation.
- Delta modulation.
  - Slope overload.

Source Coding Theorem(Noiseless coding theorem)

- Need for efficiently representing data generated by a discrete source.
- Source Encoding
- Source Encoder: An efficient source encoder has the knowledge of the statistics of the source
    - If some symbols occur more frequently than others we need to assign a shorter code.
    - The rare source symbols needed to assigned with longer code symbols
- This is called variable length code.
- Eg : Morse Code
- E is represented by '.'
- Q is represented by '--.-'

Source Coding Theorem(Noiseless coding theorem)

- Source encoder needs to satisfy two functional requirements
  - The code words must be produced by the encoder are in binary form
  - The source code is uniquely decodable so that the original source sequence can be reconstructed perfectly from the encoded binary sequence.
- Consider then the scheme shown in Figure 5.3 that depicts a discrete memoryless source whose output sk is converted by the source encoder into a sequence of 0s and 1s, denoted by bk.
- We assume that the source has an alphabet with K different symbols and that the kth symbol sk occurs with probability pk, k = 0, 1,…., K – 1.

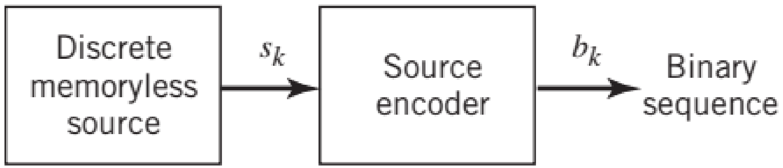


Figure 5.3 Source encoding.

Source Coding Theorem(Noiseless coding theorem)

- Let the binary codeword assigned to symbol s_k by the encoder have length l_k, measured in bits.
- We define the average codeword length of the source encoder as

$$\bar{L} = \sum_{k=0}^{K-1} p_k l_k$$

- In physical terms, the parameter $\bar{L}$ represents the average number of bits per source symbol used in the source encoding process.
- Let L_{min} denote the minimum possible value of L. We then define the coding efficiency of the source encoder as

$$\eta = \frac{L_{\min}}{\bar{L}}$$

- Since $\bar{L} \geq L_{\min}$ the efficiency is less than 1
- The source encoder is said to be efficient when efficiency approaches 1

Source Coding Theorem(Noiseless coding theorem)

- What is the minimum value of L_{min}
- Shannon's first theorem: the source-coding theorem,
- Given a discrete memoryless source whose output is denoted by the random variable S, the entropy H(S) imposes the following bound on the average codeword length for any source encoding scheme:

$$\bar{L} \geq H(S)$$

- According to this theorem, the entropy H(S) represents a fundamental limit on the average number of bits per source symbol necessary to represent a discrete memoryless source, in that it can be made as small as but no smaller than the entropy H(S).
- Thus, setting L_{min} = H(S),
- defining the efficiency of a source encoder in terms of

$$\eta = \frac{H(S)}{\bar{L}}$$

Source Coding theorem: Theorem on channel capacity

- The entropy of the input symbols is given by

$$H(X) = -\sum_{i=1}^{m} P(x_i)\, log_2 P(x_i)$$

- When we consider a discrete memoryless channel accepting symbols at the rate of r_s symbols/seconds, The average rate at which information is going into the channel is given by

$$R_{in} = H(X) r_s \; bits/sec$$

- At the receiver it is not possible to reconstruct the input symbol sequence with certainty
- This is due to errors introduced when the signals pass through the channel
- These errors are in fact introduced due to the noise present in the channel
- Thus some amount of information is lost in the channel due to noise
- Hence the net amount of information is given by the mutual information I(A,B)

$$I(X;Y) = H(X) - H(X|Y) \; b/symbol$$

Source Coding theorem: Theorem on channel capacity

- Then the average rate of information transmission Rt is then given by

$$R_t = I(X,Y) r_s \; bit/sec$$

$$R_t = [H(X) - H(X|Y)] r_s \; bit/sec$$

- Capacity of a discrete memoryless noisy channel is defined as the maximum possible rate of information transmission over the channel
- The maximum rate of transmission occurs when the source is matched to the channel
- There fore the channel capacity is defined as

$$C = Max[R_t]$$

$$C = Max[[H(X) - H(X|Y)] r_s]$$

Source Coding theorem: Theorem on channel capacity

- **Shannon's Theorem on channel capacity : Shannon's Second Theorem**
- It states that When the rate of information transmission $R_t \leq C$ then there exits a coding technique which enables transmission over a channel with as small a probability of error as possible even in the presence of noise in the channel
- **Information Capacity Theorem : Shannon's Third Theorem**
- The information capacity of a continuous channel of bandwidth B hertz perturbed by additive white Gaussian noise of power spectral density $N_0/2$ and limited in bandwidth to B is given by

$$C = B\, log_2\left(1 + \frac{P}{N_0\, B}\right)\ bits\ per\ second$$

- Where P is the average transmitted power

Sampling

- A message signal may originate from a digital source or analog source.
- If the message signal happens to be analog in nature as in speech signal, then it has to be converted into digital form before it can be transmitted by digital means.
- The sampling process is the first process performed in analog to digital conversion.
- Followed by quantizing and encoding.
- In sampling process a continuous time signal is converted into a discrete time signal by measuring the signal at periodic instants of time
- For the sampling process to be of practical utility, it is necessary that we choose the sampling rate properly so that the discrete time signal resulting from the process uniquely defines the original continuous time

Low pass Sampling Theorem

- Let $x(t)$ be a bandlimited lowpass signal bandlimited to W Hz, i.e. $X(f) = 0\ for\ |f| \geq W$.
- Then it is possible to recover $x(t)$ completely without any distortion whatsoever from its samples, if sampling interval T_s is such that $T_s \leq \frac{1}{2W}$ or $f_s \geq 2W$.
- $x(t)$ can be expressed in terms of its samples $x(kT_s)$ as

$$x(t) = 2BT_s \sum_{k=-\infty}^{\infty} x(k\,T_s) sinc\ 2B(t - kT_s)$$

- Where B is any frequency such that $W \leq B \leq (f_s - W)$
- x(t) is the weighted sum of an infinite number of interpolating functions $sinc2\,B\,(t - kT_s)$with $x(kT_s)$ as the weightage given to the sinc function delayed by an amount of time kT_s
- $sinc2\,B\,(t - kT_s)$is the impulse response of an ideal low pass filter whose cut off frequency is B Hz and pass band gain is 1.

Impulse or Ideal Sampling

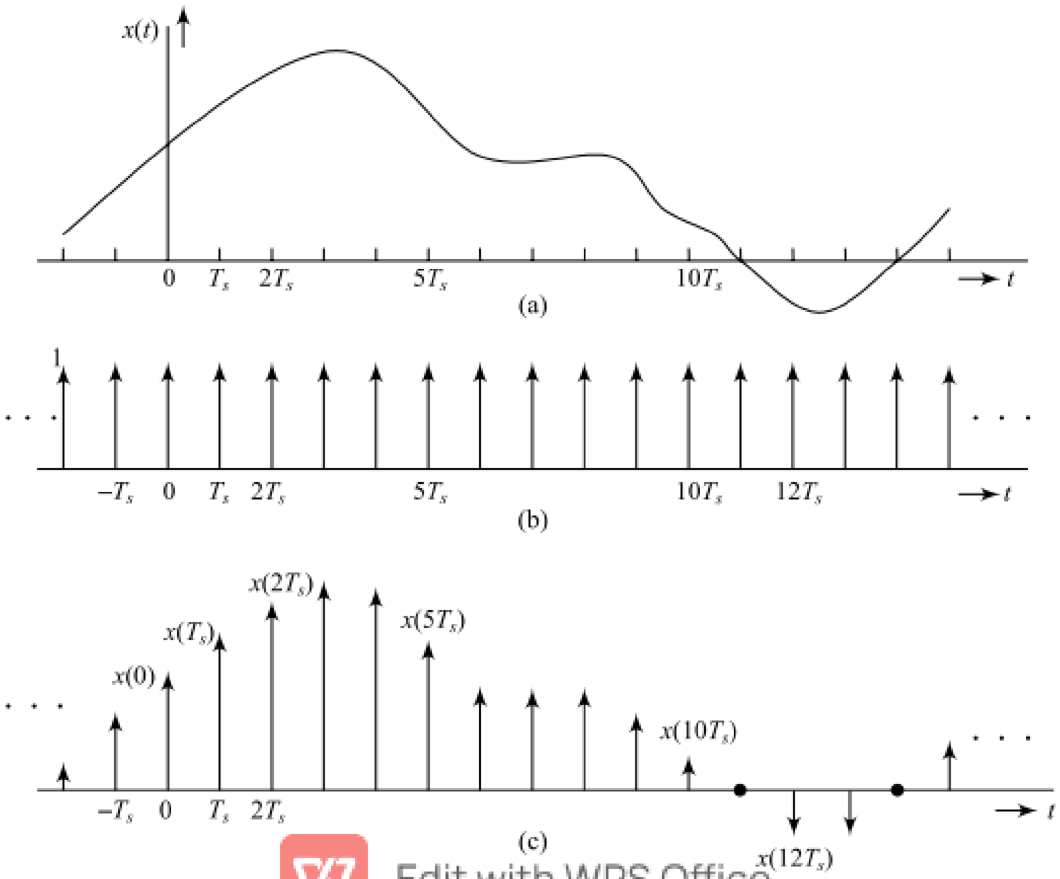


Fig. 8.9 (a) $x(t)$, (b) $s(t)$, (c) $x_\delta(t)$

Impulse or Ideal Sampling

- We will consider a sampling function that is a sequence of unit impulse.

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

- Where $\delta(t - nT_s)$ is a Dirac delta function located at time nT_s
- $s(t)$ is called Dirac comb or ideal sampling function.
- If we model the sampling process as multiplication of $x(t)$ by the sampling function $s(t)$ we have the sampled version is given by

$$x_\delta(t) = x(t).s(t)$$
$$X_\delta(f) = X(f) * S(t)$$

- To find $S(f)$, we know that $s(t)$ is a periodic function with period of T_s
- Hence we may write a Fourier series expansion as

$$s(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_s t} \quad where \ f_s = \frac{1}{T_s} \quad -\infty < t < \infty$$

Impulse or Ideal Sampling

- Where

$$c_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-j2\pi n f_s t}\, dt = f_s$$

$$s(t) = f_s \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t}$$

Taking Fourier transform on both sides

$$S(f) = f_s \mathcal{F}\left[\sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t}\right] = f_s \sum_{n=-\infty}^{\infty} \mathcal{F}[e^{j2\pi n f_s t}]$$

$$= f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s)$$

$$S(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s)$$

$$S(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - n f_s)$$

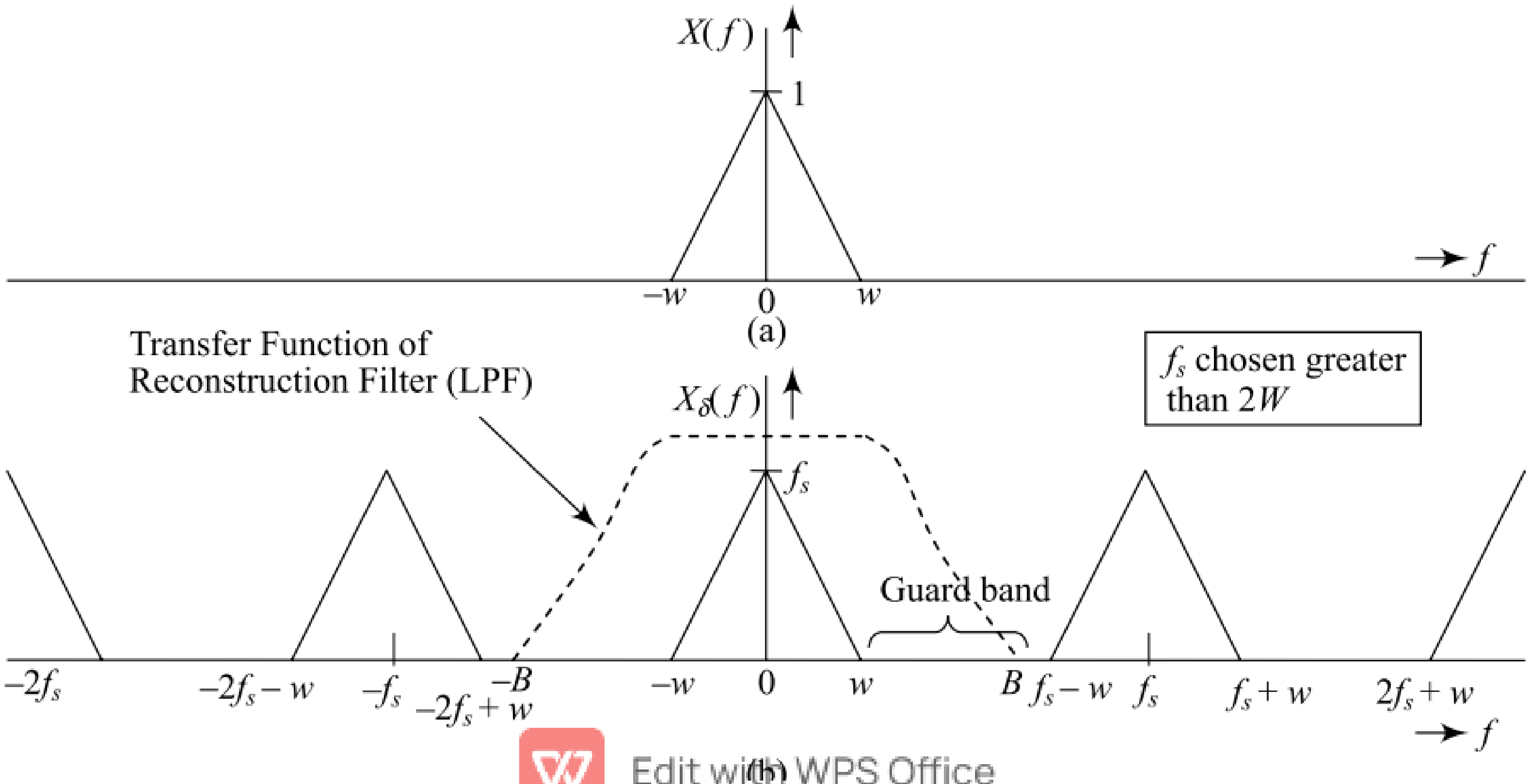
And we know $X(f) * \delta(f - n f_s) = X(f - n f_s)$

Thus

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - n f_s)$$

Impulse or Ideal Sampling

- $X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s)$



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Impulse or Ideal Sampling

- The above equation tells us that the spectrum of ideally sampled version of x(t)is nothing but a periodic repetition of $X(f)$ and is scaled by the factor fs.
- From the spectrum representation it is clear that it is possible to recover X(f) from $X_\delta(f)$ if the sampling rate much larger that twice the maximum frequency of the signal.
- So we can use a lowpass filter whose pass band gain is constant over the range of frequencies 0Hz to WHz
- If the sampling is done less than the nyquist rate $f_s < 2W$ then $f_s - w < W$
- This cause the spectra to overlap and its is not possible to recover x(T) even if we use an ideal LPF.
- Because of overlapping the high frequency components of x(t) reappear as low frequency components.
- This phenomenon is called aliasing, it is also called frequency folding effect.

Impulse or Ideal Sampling

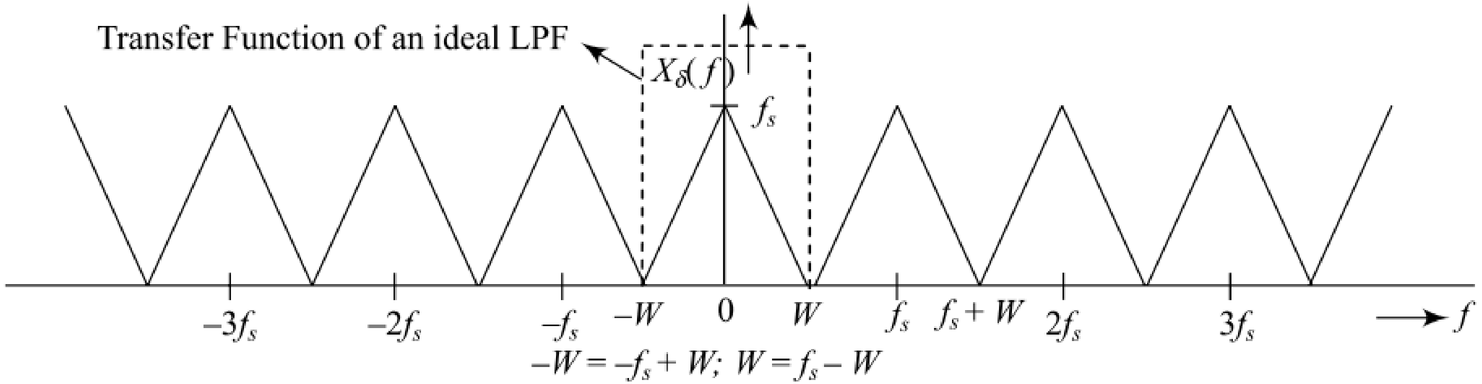


Fig. 8.11 *Spectrum of $x_\delta(t)$ when $f_s = 2W$*

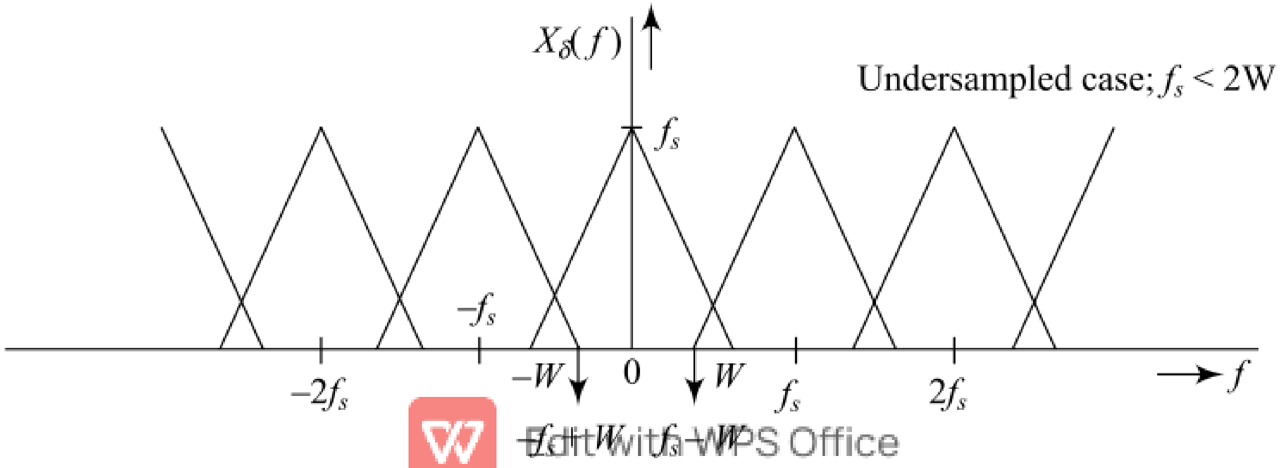


Fig. 8.12 *Spectrum of $x_\delta(t)$ when $f_s < 2W$*

Impulse or Ideal Sampling

1. $X_\delta(f)$ is a repetitive version of $X(f)$, with $X(f)$ repeating itself at regular intervals of f_s, the sampling frequency.
2. If $f_s > 2W$, then there is a guard band and it is easy to separate out $X(f)$ from $X_\delta(f)$, i.e., easy to recover $x(t)$ from $x_\delta(t)$ using a practical LP filter.
3. If $f_s = 2W$, i.e., Nyquist rate, no guard band exists and an ideal LPF is needed to recover $x(t)$ from $x_\delta(t)$.
4. If $f_s < 2W$, aliasing takes place and it is not possible to recover $x(t)$ from $x_\delta(t)$ without distortion.
5. To avoid aliasing, it should be ensured that
   (a) $x(t)$ is strictly band limited
   (b) f_s is greater than $2W$

Reconstruction

let us assume $f_s = 2W$ and that an ideal LPF is used to recover $X(f)$ form $X_\delta(f)$. Let the ideal LPF have a gain of T_s in the pass band and let it introduce τ sec time delay. Then, we can write down its transfer function $H(f)$ as

$$H(f) = T_s \Pi(f/f_s) e^{-j\omega\tau}$$

Hence, the spectrum of the output of the filter is

$$Y(f) = X_\delta(f) \cdot H(f) = T_s f_s X(f) e^{-j\omega\tau}$$

$$\therefore \qquad Y(f) = X(f) e^{-j\omega\tau}$$

or taking the inverse Fourier transform on both sides,

$$y(t) = x(t - \tau)$$

Reconstruction

We now consider the reconstruction operation in the time domain.
$x_\delta(t)$ is a sequence of weighted impulses given by

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

This weighted sequence, when given as the input to the ideal LPF with impulse response $h(t)$, gives an output signal $y(t)$ given by

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h(t - nT_s)$$

where $h(t)$, the impulse response of the ideal LPF is given by

$$h(t) = \mathcal{F}^{-1}[H(f)] = \mathcal{F}^{-1}[T_s \Pi(f/2W)e^{-j\omega\tau}]$$
$$= 2BT_s \operatorname{sinc} 2B(t - \tau)$$

In our case, the cut-off frequency B of the LPF = $W = f_s/2$.

$\therefore$
$$h(t) = 2\frac{f_s}{2}T_s \operatorname{sinc} 2B(t - \tau)$$
$$= \operatorname{sinc} 2B(t - \tau)$$

Reconstruction

Taking the time delay τ introduced by the LPF equal to zero, and substituting for $h(t)$ in Eq. (8.35) using Eq. (8.37), we have

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\,2B(t - nT_s) \tag{8.38}$$

But from Eq. (8.22), we realize that RHS of Eq. (8.38) is nothing but $x(t)$, since $B = \dfrac{f_s}{2} = \dfrac{1}{2T_s}$ in this case.

$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\,2B(t - nT_s) \tag{8.39}$$

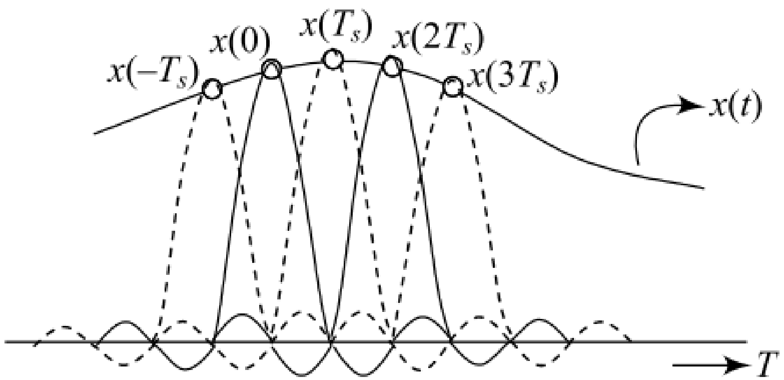


Fig. 8.19 *Reconstruction of x(t) from its samples*

Pulse Code Modulation

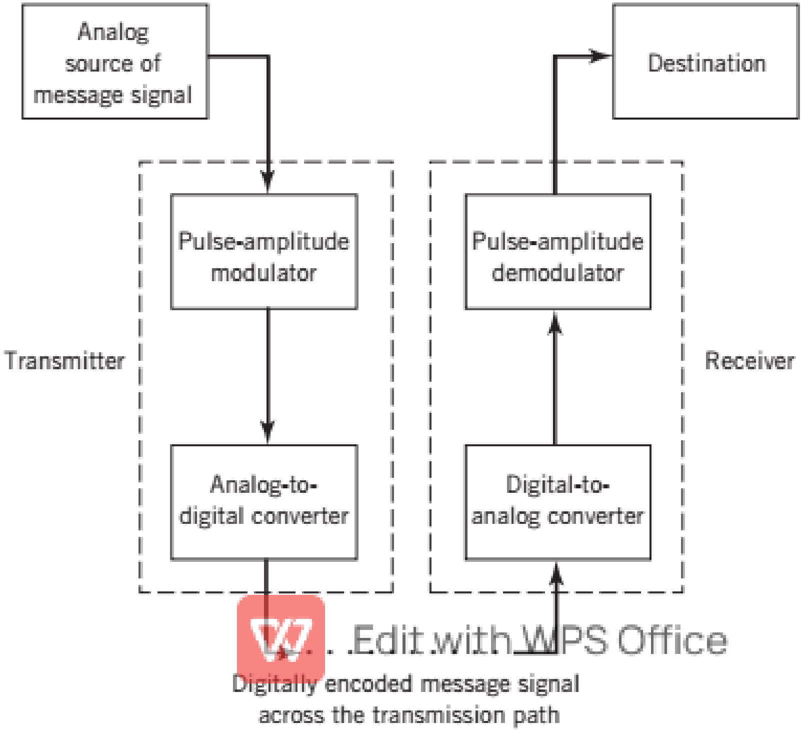
- After sampling the next step in its digital transmission is the generation of a coded version of the signal
- Pulse code modulation is one such method .
- In this method of signal coding the message signal is sampled and the amplitude of each sample is rounded off to the nearest one of a finite set of discrete levels so that both time and amplitude are represented in discrete form.
- This allows the message to be transmitted by means of a digital waveform
- This is the first method to be developed for the digital coding of waveforms.
- **Advantages of digital representations of analog signals are**
  - Ruggedness to transmission noise and interference
  - Efficient regeneration of the coded signal along the transmission path
  - Potential for communication privacy and security through encryption.

Pulse Code Modulation

- Disadvantages
  - Increased cost of transmission bandwidth requirement
  - Increased system complexity
- Waveform Coders
- PCM belongs to a class of signal coders known as waveform coders in which an analog signal is approximated by mimicking the amplitude versus time waveform.
- **Pulse Code Modulation**
- PCM are complex in that the message signal is subjected to a large number of operations.
- The essential operations are
  - Sampling
  - Quantizing
  - Encoding
- This process in done in a circuit called analog to digital converter.

Pulse Code Modulation

- Regeneration of impaired signals occurs at intermediate points along the transmission path.
- At the receiver the essential operation consists of one last stage of regeneration followed by decoding and demodulation of the train of quantized samples.
- These operations are performed by digital to analog converter.



PCM TRANSMITTER AND RECEIVER

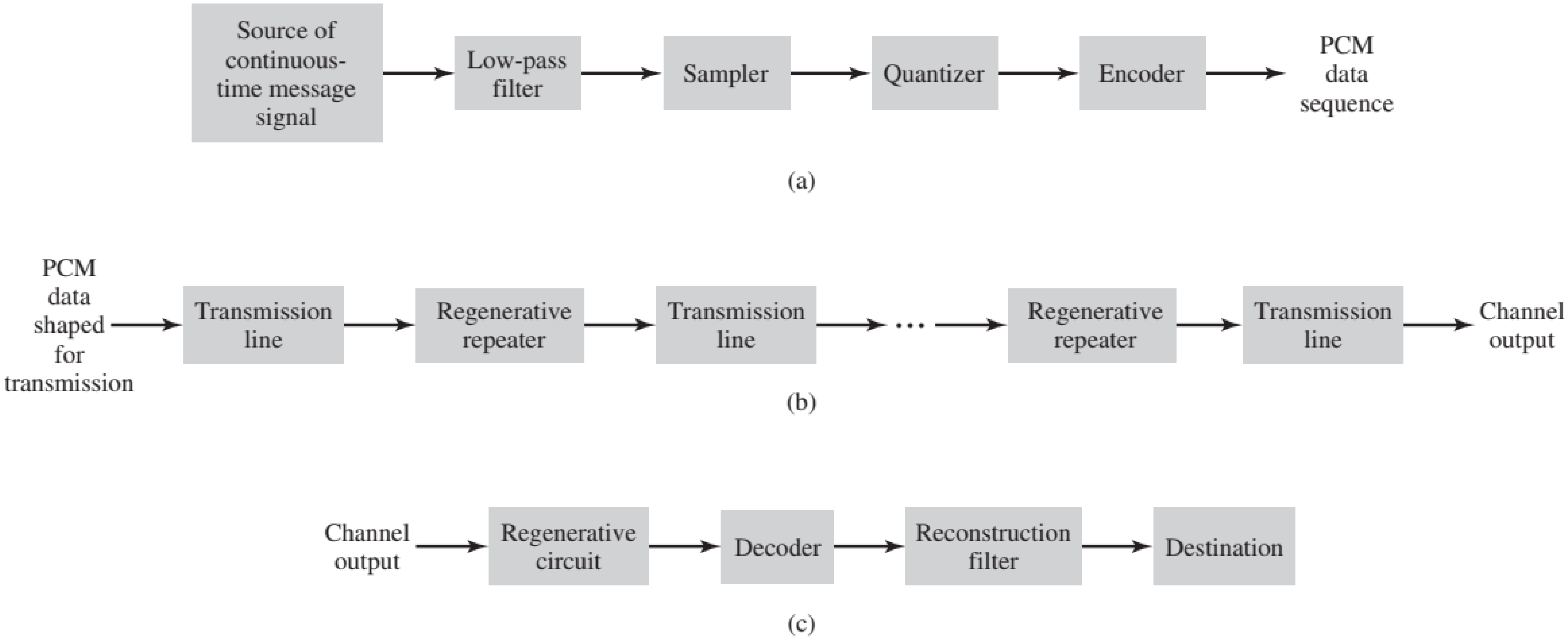


FIGURE 5.11 The basic elements of a PCM system: (*a*) Transmitter, (*b*) transmission path, connecting the transmitter to the receiver, and (*c*) receiver.

Pulse Code Modulation

- **Sampling**
- The incoming message wave is sampled with a train of narrow rectangular pulses so as to closely approximate the instantaneous sampling process.
- For perfect reconstruction the sampling rate is twice highest frequency component W of the message wave.
- In practice a low pass pre-alias filter is used at the front end of the sampler in order to exclude frequencies greater than W before sampling.
- Thus the application of sampling permits the reduction of the continuously varying message wave to a limited number of discrete values per second.

Pulse Code Modulation

- **Quantization**
- Typically, an analog message signal (e.g., voice) has a continuous range of amplitudes and, therefore, its samples have a continuous amplitude range.
- Within the finite amplitude range of the signal, we find an infinite number of amplitude levels
- In actual fact, however, it is not necessary to transmit the exact amplitudes of the samples.
- This means that the message signal may be approximated by a signal constructed of discrete amplitudes selected on a minimum error basis from an available set.
- if we assign the discrete amplitude levels with sufficiently close spacing, then we may make the approximated signal practically indistinguishable from the original message signal.
- <u>Quantization is the process of transforming the sample amplitude m(nTs) of a message signal m(t) at time t = nTs into a discrete amplitude v(nTs) taken from a finite set of possible amplitudes.</u>
- The quantizing process means the straight relationship between input and output is replaced transfer characteristic that is staircase like in appearance

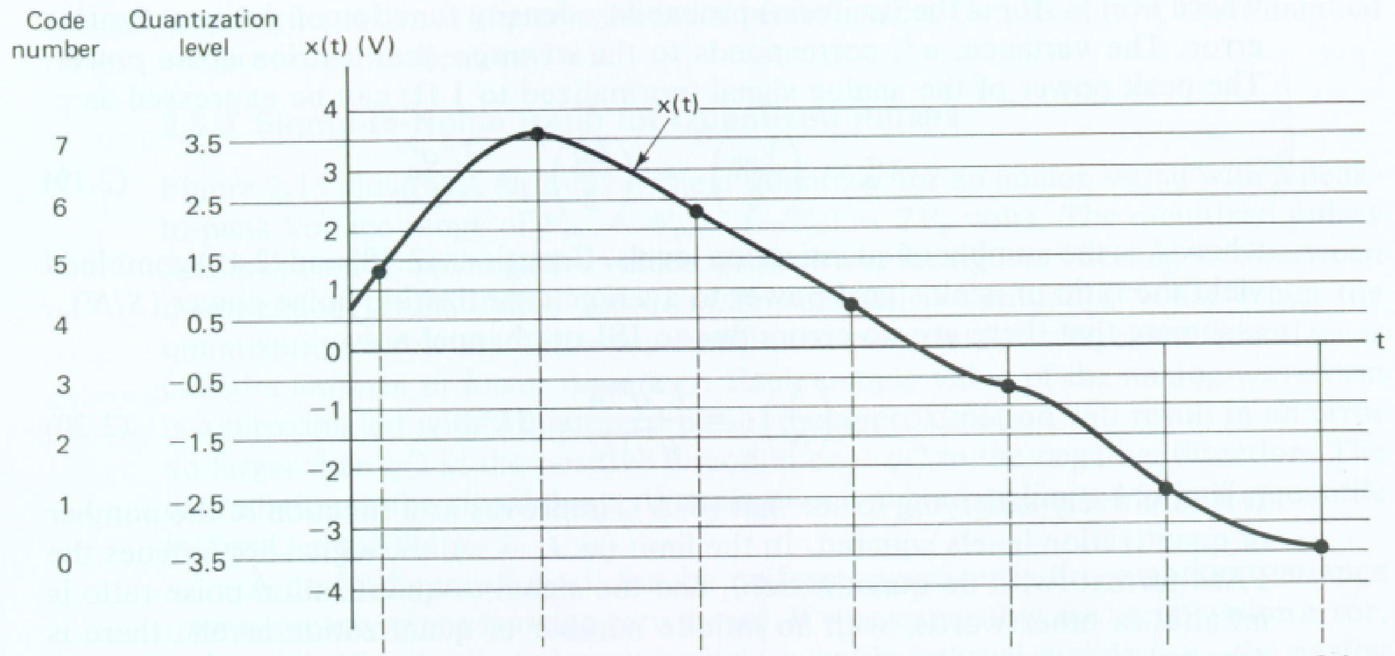
Pulse Code Modulation

- **Quantization**
- Steps

1. The peak to peak range of input samples is subdivide into a finite set of decision levels or decision thresholds that are aligned along the risers of the staircase
2. The output is assigned a discrete value selected from a finite set of representation levels that are aligned along the treads of the staircase.

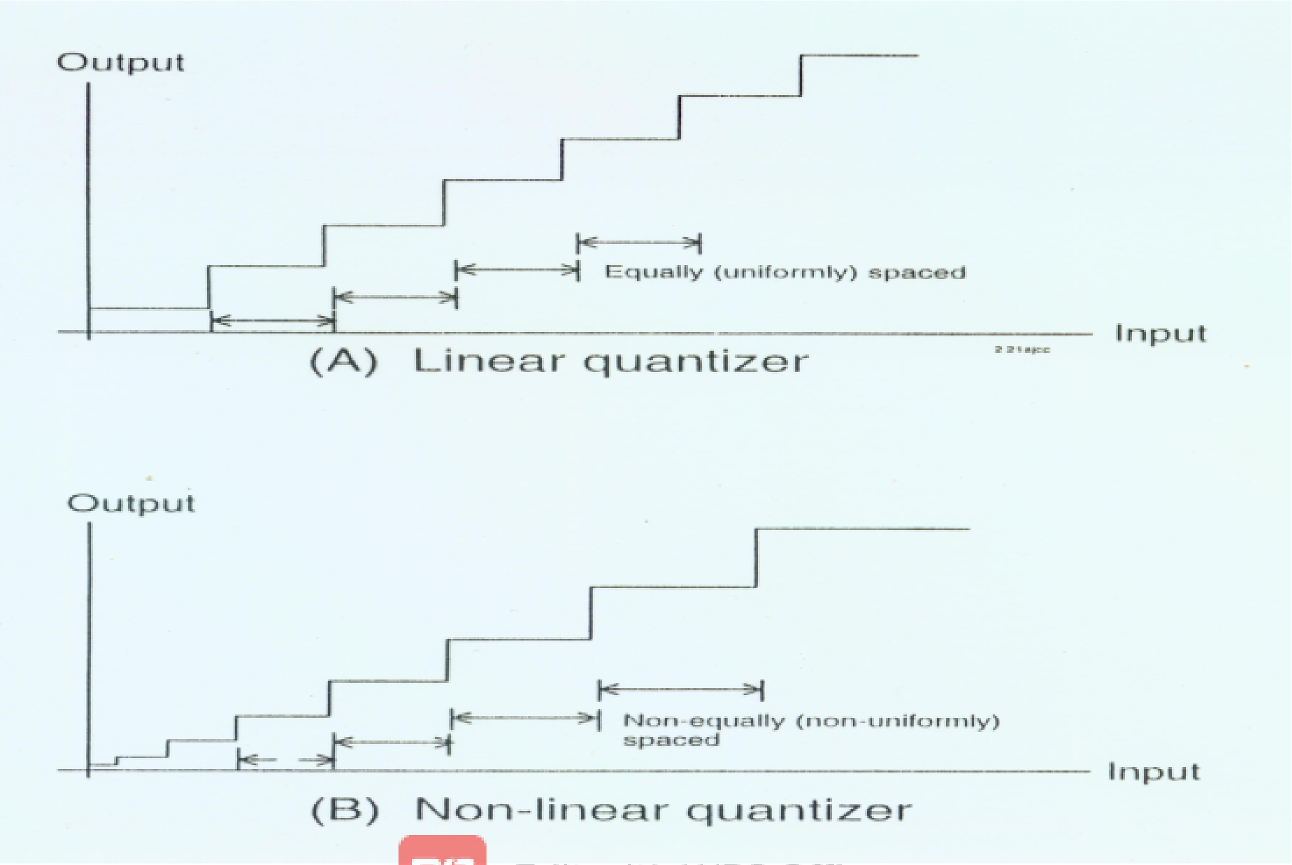
- In case of a uniform quantizer the separation between the decision thresholds and the separation between the representation levels of the quantizier have a common value called step size.

Quantization

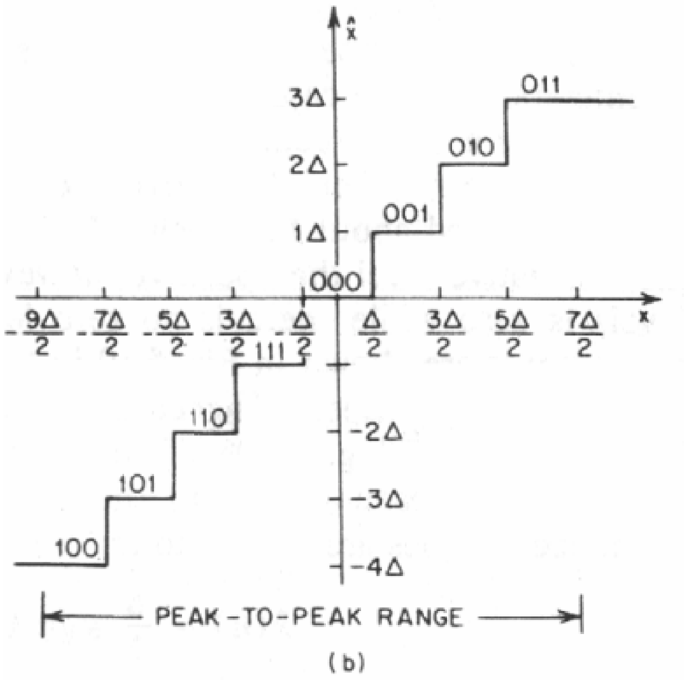


Natural sample value	1.3	3.6	2.3	0.7	−0.7	−2.4	−3.4
Quantized sample value	1.5	3.5	2.5	0.5	−0.5	−2.5	−3.5
Code number	5	7	6	4	3	1	0
PCM sequence	101	111	110	100	011	001	000

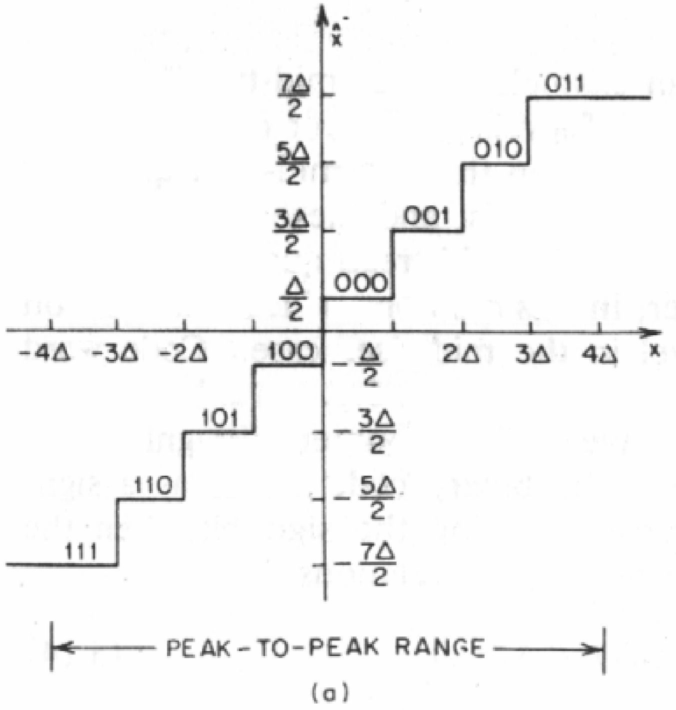
Quantization



Quantization



L = odd, Mid - Tread



L = even, Mid - Riser

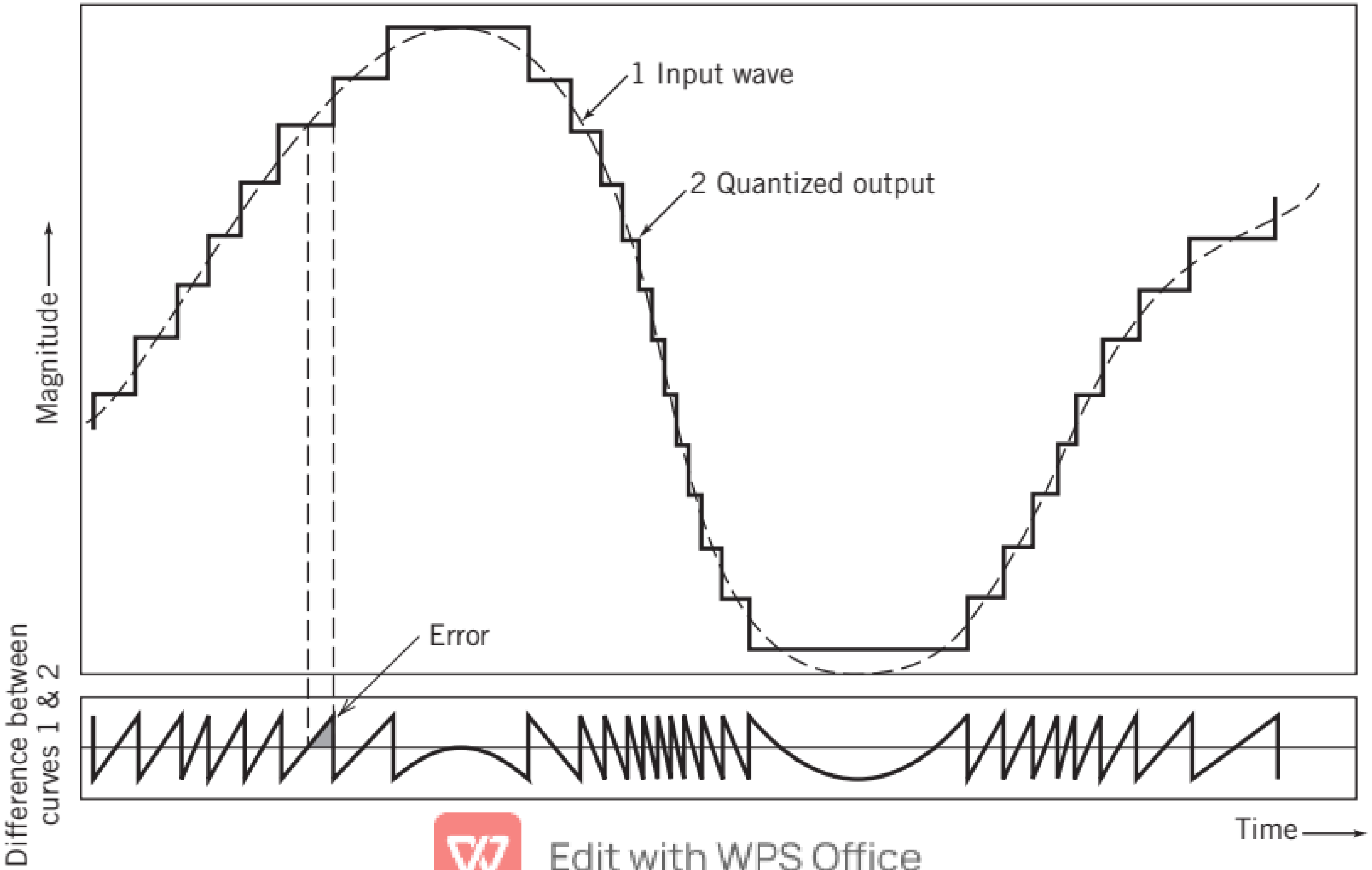
Quantization

- **Symmetric quantizer of the midtread type**
  - According to the staircase like transfer characteristic the decision thresholds of the quantizer are located at $\pm\Delta/2$, $\pm 3\Delta/2$, $\pm 5\Delta/2$ and the representation levels are located at 0, $\pm\Delta$, $\pm 2\Delta$ ….where Δ is the step size.
  - Origin lies in the middle of a tread of the staircase
- **Symmetric quantizer of the midriser type**
  - According to the staircase like transfer characteristic the decision thresholds of the quantizer are located at 0, $\pm\Delta$, $\pm 2\Delta$ …. and the representation levels are located at $\pm\Delta/2$, $\pm 3\Delta/2$, $\pm 5\Delta/2$where Δ is the step size.
  - Origin lies in the middle of a riser of the staircase

Quantization error

- **Quantization error**
- Inevitably, the use of quantization introduces an error defined as the difference between the continuous input sample m and the quantized output sample v.
- The error is called quantization noise
- The maximum instantaneous value of error is half of one step size and the total range of variation is from minus half a step to plus half a step

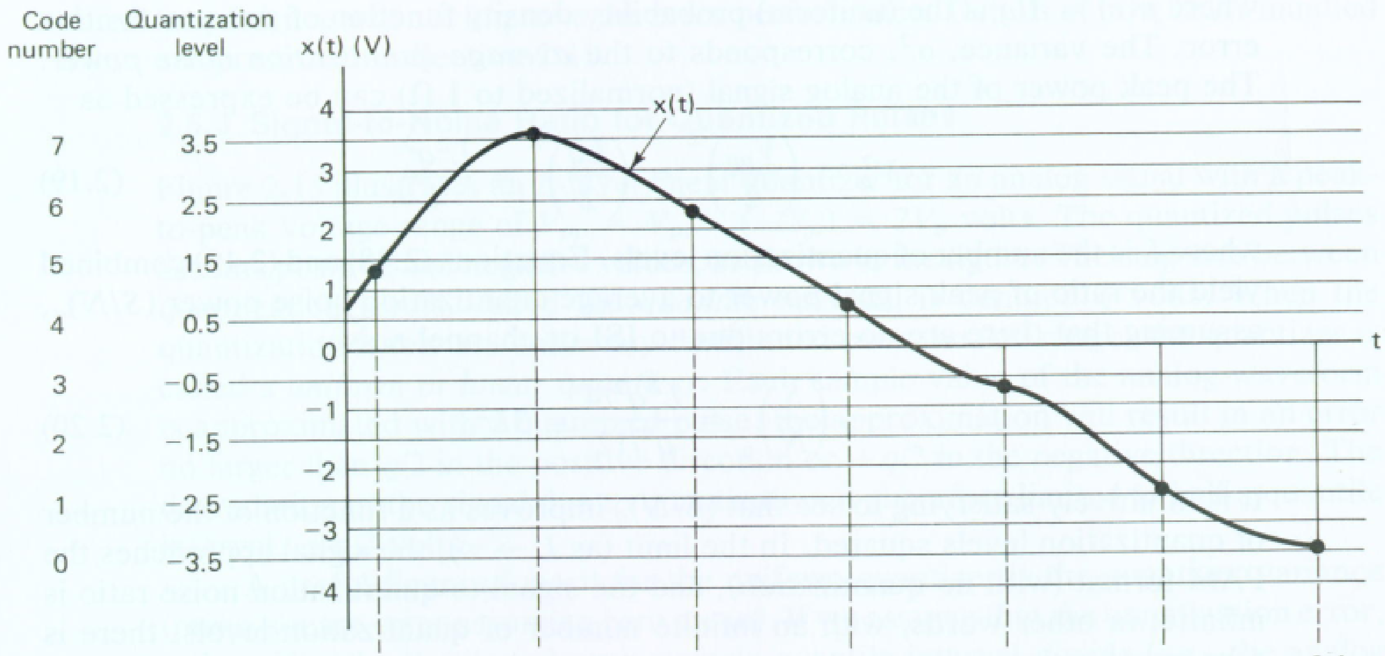
Quantization error



Encoding

- Through the combined use of sampling and quantization, the specification of an analog message signal becomes limited to a discrete set of values, but not in the form best suited to transmission over a telephone line or radio link.
- To exploit the advantages of sampling and quantizing for the purpose of making the transmitted signal more robust to noise, interference, and other channel impairments, we require the use of an encoding process to translate the discrete set of sample values to a more appropriate form of signal.
- Any plan for representing each of this discrete set of values as a particular arrangement of discrete events constitutes a code.
- Table 6.2 describes the one-to-one correspondence between representation levels and codewords for a binary number system for $R = 4$ bits per sample.
- In practice, the binary code is the preferred choice for encoding for the following reason:
- **The maximum advantage over the effects of noise encountered in a communication system is obtained by using a binary code because a binary symbol withstands a relatively high level of noise and, furthermore, it is easy to regenerate.**

Quantization



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Code number	5	7	6	4	3	1	0
PCM sequence	101	111	110	100	011	001	000

Encoding

Table 6.2 Binary number system for $T = 4$ bits/sample

Ordinal number of representation level	Level number expressed as sum of powers of 2	Binary number
0		0000
1	2^0	0001
2	2^1	0010
3	$2^1 + 2^0$	0011
4	2^2	0100
5	$2^2 + 2^0$	0101
6	$2^2 + 2^1$	0110
7	$2^2 + 2^1 + 2^0$	0111
8	2^3	1000
9	$2^3 + 2^0$	1001
10	$2^3 + 2^1$	1010
11	$2^3 + 2^1 + 2^0$	1011
12	$2^3 + 2^2$	1100
13	$2^3 + 2^2 + 2^0$	1101
14	$2^3 + 2^2 + 2^1$	1110
15	$2^3 + 2^2 + 2^1 + 2^0$	1111

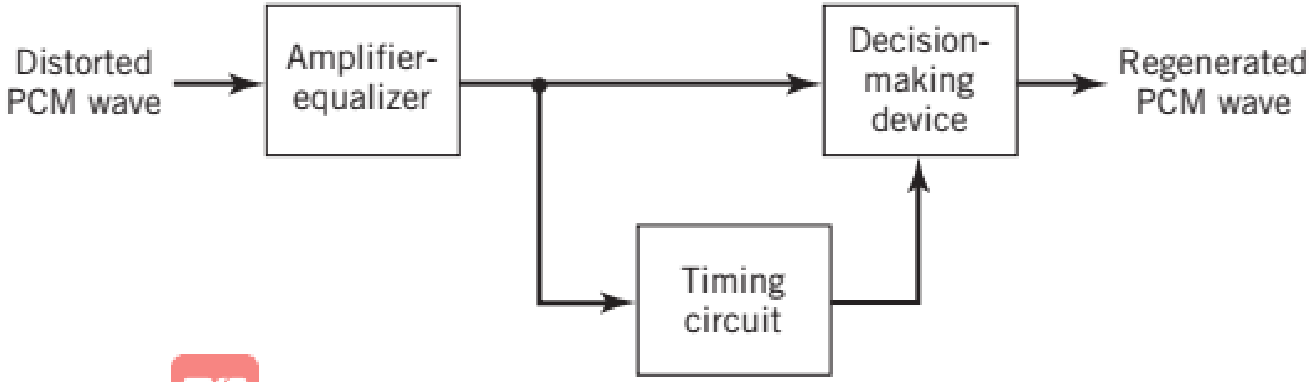
Encoding

- The last signal-processing operation in the transmitter is that of line coding, the purpose of which is to represent each binary codeword by a sequence of pulses;
  - for example, symbol 1 is represented by the presence of a pulse and symbol 0 is represented by absence of the pulse.
- Suppose that, in a binary code, each codeword consists of R bits.
- Then, using such a code, we may represent a total of 2^R distinct numbers.
  - For example, a sample quantized into one of 256 levels may be represented by an 8-bit codeword.

Regeneration

- The most important feature of a PCM systems is its ability to control the effects of distortion and noise produced by transmitting a PCM signal through the channel, connecting the receiver to the transmitter.
- This capability is accomplished by reconstructing the PCM signal through a chain of regenerative repeaters, located at sufficiently close spacing along the transmission path.
- As illustrated in Figure 6.15, three basic functions are performed in a regenerative repeater: equalization, timing, and decision making.

Figure 6.15
Block diagram of regenerative repeater.



Regeneration

- The equalizer shapes the received pulses so as to compensate for the effects of amplitude and phase distortions produced by the non-ideal transmission characteristics of the channel.
- The timing circuitry provides a periodic pulse train, derived from the received pulses, for sampling the equalized pulses at the instants of time where the SNR ratio is a maximum.
- Each sample so extracted is compared with a predetermined threshold in the decision-making device.
  - In each bit interval, decision is then made on whether the received symbol is 1 or 0 by observing whether the threshold is exceeded or not.
  - If the threshold is exceeded, a clean new pulse representing symbol 1 is transmitted to the next repeater; otherwise, another clean new pulse representing symbol 0 is transmitted.
- In this way, it is possible for the accumulation of distortion and noise in a repeater span to be almost completely removed, provided that the disturbance is not too large to cause an error in the decision-making process.
- Ideally, except for delay, the regenerated signal is exactly the same as the signal originally transmitted.

Regeneration

- In practice, however, the regenerated signal departs from the original signal for two main reasons:
  - The unavoidable presence of channel noise and interference causes the repeater to make wrong decisions occasionally, thereby introducing bit errors into the regenerated signal.
  - If the spacing between received pulses deviates from its assigned value, a jitter is introduced into the regenerated pulse position, thereby causing distortion.

Decoding

- The first operation in the receiver is to regenerate (i.e., reshape and clean up) the received pulses one last time.
- These clean pulses are then regrouped into code words and decoded (i.e., mapped back) into a quantized PAM signal.
- The decoding process involves generating a pulse whose amplitude is the linear sum of all the pulses in the code word; each pulse is weighted by its place value ($2^0, 2^1, 2^2, 2^3 \ldots$) in the code, where R is the number of bits per sample
- The sequence of decoded samples represents an estimate of the sequence of compressed samples produced by the quantizer in the transmitter.
  - We use the term “estimate” here to emphasize the fact that there is no way for the receiver to compensate for the approximation introduced into the transmitted signal by the quantizer.
  - Moreover, other sources of noise include bit errors and jitter produced along the transmission path.

Reconstruction

- The final operation in the receiver is to recover the message signal.
- This operation is achieved by passing the output through a low-pass reconstruction filter whose cutoff frequency is equal to the message bandwidth.
- Recovery of the message signal is intended to signify estimation rather than exact reconstruction.
- Pulse-code modulation is a source-encoding strategy, by means of which an analog signal emitted by a source is converted into digital form.

Quantization Noise

- Inevitably, the use of quantization introduces an error defined as the difference between the continuous input sample m and the quantized output sample v.
- The error is called quantization noise
- Let the quantizer input m be the sample value of a zero-mean random variable M. (If the input has a nonzero mean, we can always remove it by subtracting the mean from the input and then adding it back after quantization.)
- A quantizer, denoted by g(.), maps the input random variable M of continuous amplitude into a discrete random variable V; their respective sample values m and v are related by the nonlinear function g(.)
- Let the quantization error be denoted by the random variable Q of sample value q. We may thus write

$$q = m - v$$

$$Q = M - V$$

Quantization Noise

- With the input M having zero mean and the quantizer assumed to be symmetric it follows that the quantizer output V and, therefore, the quantization error Q will also have zero mean.
- Thus, for a partial statistical characterization of the quantizer in terms of output signal-to-(quantization) noise ratio, we need only find the mean-square value of the quantization error Q.
- Consider, then, an input m of continuous amplitude, which, symmetrically, occupies the range $[-m_{max}, m_{max}]$.
- Assuming a uniform quantizer of the midrise type illustrated, we find that the step size of the quantizer is given by

$$\Delta = \frac{2m_{\max}}{L}$$

- where L is the total number of representation levels.

Quantization Noise

- For a uniform quantizer, the quantization error Q will have its sample values bounded by $-\Delta/2 \le q \le \Delta/2$.
- If the step size Δ is sufficiently small (i.e., the number of representation levels L is sufficiently large), it is reasonable to assume that the quantization error Q is a uniformly distributed random variable and the interfering effect of the quantization error on the quantizer input is similar to that of thermal noise, hence the reference to quantization error as quantization noise.
- We may thus express the probability density function of the quantization noise as

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \le \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

- For this to be true, however, we must ensure that the incoming continuous sample does not overload the quantizer.
- Then, with the mean of the quantization noise being zero, its variance σ_Q^2 is the same as the mean-square value; that is,

Quantization Noise

$$\sigma_Q^2 = \mathbb{E}[Q^2]$$

$$= \int_{-\Delta/2}^{\Delta/2} q^2 f_Q(q)\, dq$$

Substituting (6.26) into (6.27), we get

$$\sigma_Q^2 = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} q^2\, dq$$

$$= \frac{\Delta^2}{12}$$

Typically, the L-ary number k, denoting the kth representation level of the quantizer, is transmitted to the receiver in binary form. Let R denote the *number of bits per sample* used in the construction of the binary code. We may then write

$$L = 2^R \tag{6.29}$$

or, equivalently,

$$R = \log_2 L \tag{6.30}$$

Quantization Noise

Hence, substituting (6.29) into (6.25), we get the step size

$$\Delta = \frac{2m_{\max}}{2^R} \tag{6.31}$$

Thus, the use of (6.31) in (6.28) yields

$$\sigma_Q^2 = \frac{1}{3}m_{\max}^2 2^{-2R} \tag{6.32}$$

Let P denote the average power of the original message signal $m(t)$. We may then express the *output signal-to-noise ratio* of a uniform quantizer as

$$\begin{aligned}(\mathrm{SNR})_O &= \frac{P}{\sigma_Q^2} \\ &= \left(\frac{3P}{m_{\max}^2}\right)2^{2R}\end{aligned} \tag{6.33}$$

Equation (6.33) shows that the output signal-to-noise ratio of a uniform quantizer $(\mathrm{SNR})_O$ increases *exponentially* with increasing number of bits per sample R, which is intuitively satisfying.

Quantization Noise

Sinusoidal Modulating Signal

Consider the special case of a full-load sinusoidal modulating signal of amplitude A_m, which utilizes all the representation levels provided. The average signal power is (assuming a load of 1 Ω)

$$P = \frac{A_m^2}{2}$$

The total range of the quantizer input is $2A_m$, because the modulating signal swings between $-A_m$ and A_m. We may, therefore, set $m_{max} = A_m$, in which case the use of (6.32) yields the average power (variance) of the quantization noise as

$$\sigma_Q^2 = \frac{1}{3}A_m^2 2^{-2R}$$

Quantization Noise

Thus, the output signal-to-noise of a uniform quantizer, for a full-load test tone, is

$$(\mathrm{SNR})_O = \frac{A_m^2/2}{A_m^2 2^{-2R}/3} = \frac{3}{2}(2^{2R}) \tag{6.34}$$

Expressing the signal-to-noise (SNR) in decibels, we get

$$10 \log_{10}(\mathrm{SNR})_O = 1.8 + 6R \tag{6.35}$$

The corresponding values of signal-to-noise ratio for various values of L and R, are given in Table 6.1. For sinusoidal modulation, this table provides a basis for making a quick estimate of the number of bits per sample required for a desired output signal-to-noise ratio.

Quantization Noise

Table 6.1 Signal-to-(quantization) noise ratio for varying number of representation levels for sinusoidal modulation

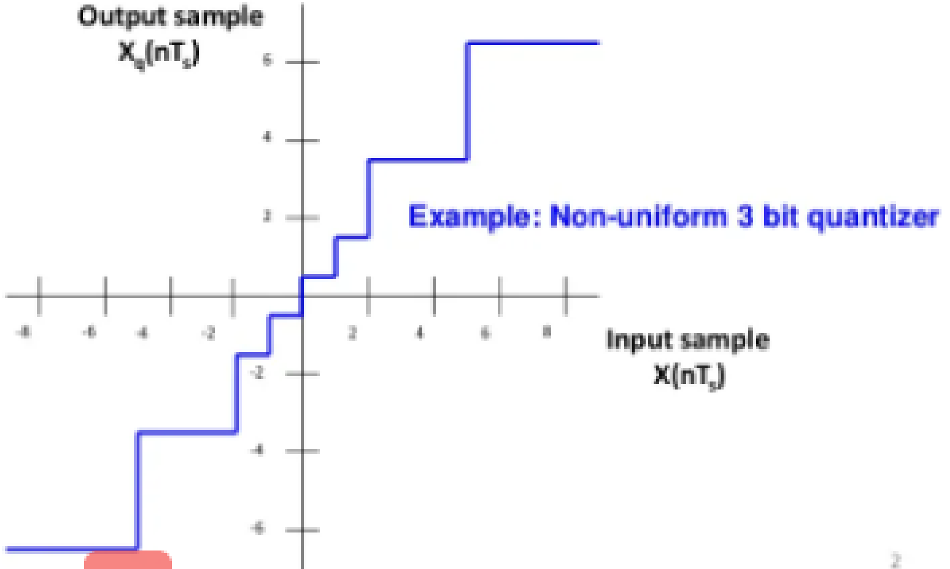
No. of representation levels L	No. of bits per sample R	SNR (dB)
32	5	31.8
64	6	37.8
128	7	43.8
256	8	49.8

Non-Uniform Quantization

- In telephonic communication, however, it is preferable to use a variable separation between the representation levels for efficient utilization of the communication channel.
- Consider, for example, the quantization of voice signals. Typically, we find that the range of voltages covered by voice signals, from the peaks of loud talk to the weak passages of weak talk, is on the order of 1000 to 1.
- By using a nonuniform quantizer with the feature that the step size increases as the separation from the origin of the input–output amplitude characteristic of the quantizer is increased, the large end-steps of the quantizer can take care of possible excursions of the voice signal into the large amplitude ranges that occur relatively infrequently.
- In other words, the weak passages needing more protection are favored at the expense of the loud passages. In this way, a nearly uniform percentage precision is achieved throughout the greater part of the amplitude range of the input signal.
- The end result is that fewer steps are needed than would be the case if a uniform quantizer were used; hence the improvement in channel utilization.

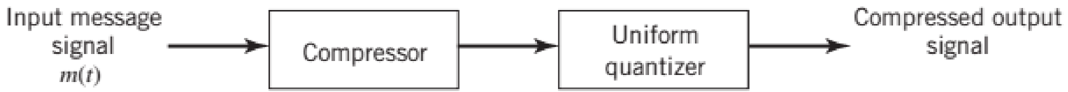
Non-Uniform Quantization

- Many signals such as speech have a non – uniform distribution.
  - The amplitude is more likely to be close to zero than to be at higher levels.
- ***Nonuniform quantizers*** have unequally spaced levels
  - The spacing can be chosen to optimize the SNR for a particular type of signal.



Non-Uniform Quantization- μ-law

- Assuming memoryless quantization, the use of a nonuniform quantizer is equivalent to passing the message signal through a compressor and then applying the compressed signal to a uniform quantizer, as illustrated in Figure

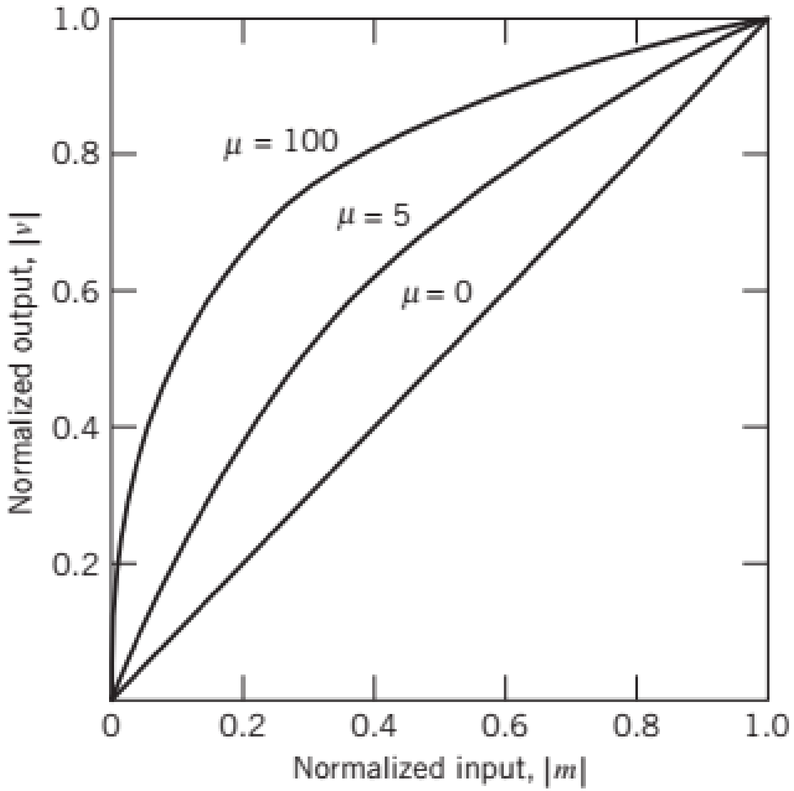


- A particular form of compression law that is used in practice is the so-called μ-law, which is defined by

$$|v| = \frac{\ln(1+\mu|m|)}{\ln(1+\mu)}$$

- where ln, i.e., $\log_e$, denotes the natural logarithm, m and v are the input and output voltages of the compressor, and μ is a positive constant.
- It is assumed that m and, therefore, v are scaled so that they both lie inside the interval [–1, 1].
- The μ-law is plotted for three different values of μ in Figure 6.14a.
- The case of uniform quantization corresponds to μ = 0.

Non-Uniform Quantization- μ-law



Non-Uniform Quantization- μ-law

- For a given value of μ, the reciprocal slope of the compression curve that defines the quantum steps is given by the derivative of the absolute value |m| with respect to the corresponding absolute value |v|; that is,

$$\frac{d|m|}{d|v|} = \frac{\ln(1+\mu)}{\mu}(1+\mu|m|)$$

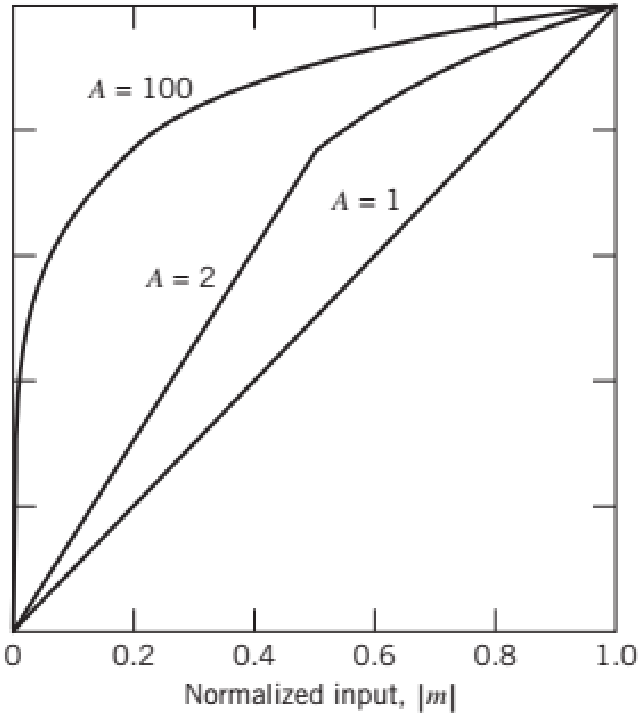
- it is apparent that the μ -law is neither strictly linear nor strictly logarithmic.
- Rather, it is approximately linear at low input levels corresponding to $\mu|m| << 1$ and approximately logarithmic at high input levels corresponding to $\mu|m| >> 1$.

Non-Uniform Quantization- A-law

- Another compression law that is used in practice is the so-called A-law, defined by

$$|v| = \begin{cases} \dfrac{A|m|}{1+\ln A}, & 0 \le |m| \le \dfrac{1}{A} \\ \dfrac{1+\ln(A|m|)}{1+\ln A}, & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

- where A is another positive constant
- The case of uniform quantization corresponds to A = 1.



Non-Uniform Quantization- A-law

- The reciprocal slope of this second compression curve is given by the derivative of |m| with respect to |v|, as shown by

$$\frac{d|m|}{d|v|} = \begin{cases} \dfrac{1+\ln A}{A}, & 0 \le |m| \le \dfrac{1}{A} \\[2ex] (1+\ln A)|m|, & \dfrac{1}{A} \le |m| \le 1 \end{cases}$$

Compander

- To restore the signal samples to their correct relative level, we must, of course, use a device in the receiver with a characteristic complementary to the compressor.
- Such a device is called an expander.
- Ideally, the compression and expansion laws are exactly the inverse of each other.
- With this provision in place, we find that, except for the effect of quantization, the expander output is equal to the compressor input.
- The cascade combination of a compressor and an expander, depicted in Figure 6.13, is called a compander.

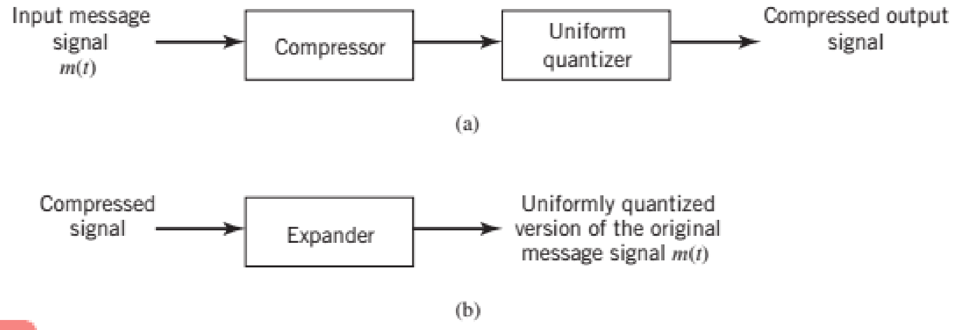


Figure 6.13
(a) Nonuniform quantization of the message signal in the transmitter. (b) Uniform quantization of the original message signal in the receiver.

Compander

- For both the μ-law and A-law, the dynamic range capability of the compander improves with increasing μ and A, respectively.
- The SNR for low-level signals increases at the expense of the SNR for high-level signals.
- To accommodate these two conflicting requirements (i.e., a reasonable SNR for both low- and high-level signals), a compromise is usually made in choosing the value of parameter μ for the μ -law and parameter A for the A-law.
- The typical values used in practice are μ = 255 for the μ -law and A = 87.6 for the A-law.

Differential Pulse-Code Modulation

- we recognize that when a voice or video signal is sampled at a rate slightly higher than the Nyquist rate, the resulting sampled signal is found to exhibit a high degree of correlation between adjacent samples.
- The meaning of this high correlation is that, in an average sense, the signal does not change rapidly from one sample to the next, with the result that the difference between adjacent samples has an average power that is smaller than the average power of the signal itself.
- When these highly correlated samples are encoded as in a standard PCM system, the resulting encoded signal contains redundant information.
- Redundancy means that symbols that are not absolutely essential to the transmission of information are generated as a result of the encoding process.
- By removing this redundancy before encoding, we obtain a more efficient encoded signal, compared to PCM.
- Now, if we know a sufficient part of a redundant signal, we may infer the rest, or at least make the most probable estimate.
- In particular, if we know the past behavior of a signal up to a certain point in time, it is possible to make some inference about its future values;
- such a process is commonly called **prediction**

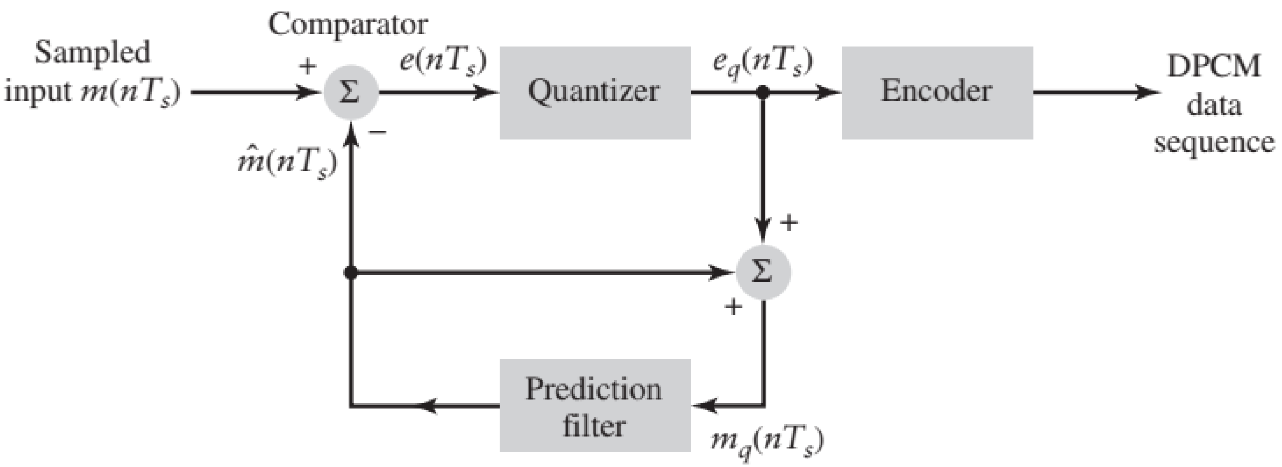
Differential Pulse-Code Modulation

- Suppose then a message signal $m(t)$ is sampled at the rate $f_s = 1/T_s$ to produce a sequence of correlated samples T_s seconds apart; this sequence is denoted by $m(nT_s)$.
- The fact that it is possible to predict future values of the signal provides motivation for the differential quantization scheme shown in Fig. 5.18(a).
- In this scheme, the input signal to the quantizer is defined by

$$e(nT_s) = m(nT_s) - \hat{m}(nT_s)$$

- which is the difference between the input sample $m(nT_s)$and a prediction of it, denoted by $\widehat{m}(nT_s)$
- This predicted value is produced by using a prediction filter whose input, as we see, consists of a quantized version of $m(nT_s)$.
- The difference signal $e(nT_s)$ is called the prediction error, since it is the amount by which the prediction filter fails to predict the incoming message signal exactly

Differential Pulse-Code Modulation- Trasnsmitter



Differential Pulse-Code Modulation

- By encoding the quantizer output in Fig. 5.18(a), we obtain a variation of PCM, which is known as differential pulse-code modulation (DPCM).
- It is this encoded signal that is used for transmission.

The quantizer output may be expressed as

$$e_q(nT_s) = e(nT_s) + q(nT_s) \tag{5.35}$$

where $q(nT_s)$ is the quantization error. According to Fig. 5.18(*a*), the quantizer output $e_q(nT_s)$ is added to the predicted value $\hat{m}(nT_s)$ to produce the prediction-filter input

$$m_q(nT_s) = \hat{m}(nT_s) + e_q(nT_s) \tag{5.36}$$

Substituting Eq. (5.35) into (5.36), we get

$$m_q(nT_s) = \hat{m}(nT_s) + e(nT_s) + q(nT_s) \tag{5.37}$$

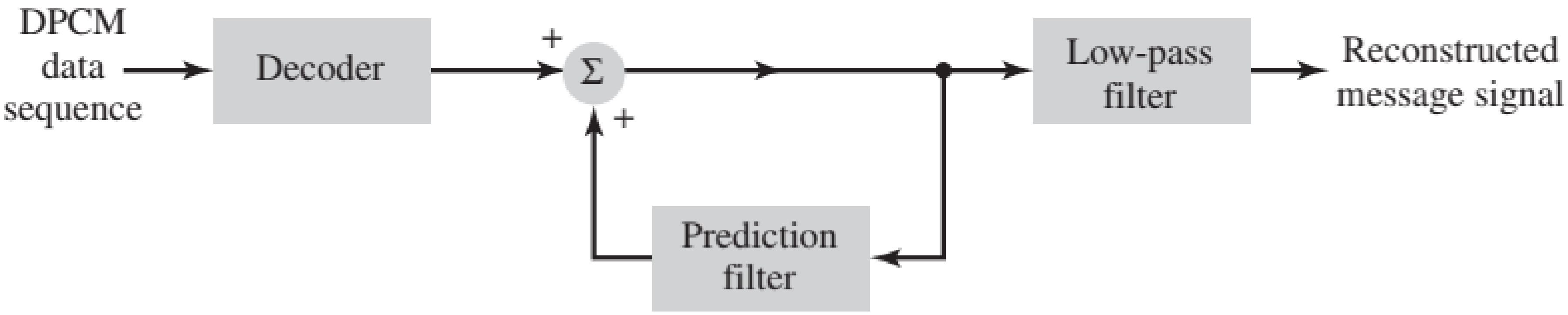
However, from Eq. (5.34) we observe that the sum term $\hat{m}(nT_s) + e(nT_s)$ is equal to the sampled message signal $m(nT_s)$. Therefore, we may rewrite Eq. (5.37) as

$$m_q(nT_s) = m(nT_s) + q(nT_s) \tag{5.38}$$

Differential Pulse-Code Modulation

which represents a quantized version of the message sample $m(nT_s)$. That is, irrespective of the properties of the prediction filter, the quantized signal $m_q(nT_s)$ at the prediction filter input differs from the sampled message signal $m(nT_s)$ by the quantization error $q(nT_s)$. Accordingly, if the prediction is good, the average power of the prediction error $e(nT_s)$ will be smaller than the average power of $m(nT_s)$, so that a quantizer with a given number of levels can be adjusted to produce a quantization error with a smaller average power than would be possible if $m(nT_s)$ were quantized directly using PCM.

DPCM Receiver



DPCM Receiver

- The receiver for reconstructing the quantized version of the message signal is shown in Fig. 5.18(b).
- It consists of a decoder to reconstruct the quantized error signal.
- The quantized version of the original input is reconstructed from the decoder output using the same prediction filter in the transmitter of Fig. 5.18(a).
- In the absence of channel noise, we find that the encoded signal at the receiver input is identical to the encoded signal at the transmitter output.
- Accordingly, the corresponding receiver output is equal to $m_q(nT_s)$ which differs from the original input $m(nT_s)$ only by the quantization error $q\,(nT_s)$incurred as a result of quantizing the prediction error $e(nT_s)$
- Finally, an estimate of the original message signal is obtained by passing the sequence through a low-pass reconstruction filter.

Wiener Filter for Waveform estimation

- Linear estimation involves the estimation of some desired response as a linear filtered version of an input signal.
- Let $X_n, X_{n-1} \ldots X_{n-M}$ denote random samples drawn from a noisy signal X(t) with a sampling T seconds
- Xn is the sample drawn from the process $X(t)$ at time $t = nT$ and Xn-1 is the sample drawn at $t = (n-1)T$
- The sequence of samples $X_n, X_{n-1} \ldots$Xn-M is applied to a tapped delay line filter.
- The filter consists of three sets of elements
  - Delay elements, each of which produces a delay of T seconds
  - Multipliers each of which multiplies its respective tap input by a coefficient
  - Adders for summing multiplier outputs
- Let ho,h1 …hm denote filter coefficeints used to weight the tap inputs Xn,Xn-1….Xn-M repectively

Wiener Filter for Waveform estimation

- Let Yn denote the output of the filter which is defined by the convolution

$$Y_n = \sum_{k=0}^{M} h_k X_{n-K}$$

- We wish to design the filter in such a way that the difference between a desired response D_n and filter output Y_n is minimized in some statistical sense.

$$\varepsilon_n = Dn - Yn$$

- The difference ε_n is called the estimation error, which is a random variable.
- In wiener theory , the filter is optimized by minimizing the mean square value of the estimation error
- Let

$$\xi = E[\varepsilon_n{}^2]$$

- The mean squared error ξ is real and positive scalar quantity
- We need to minimize the value of ξ

Wiener Filter for Waveform estimation

- Substituting and expanding the expression

$$\xi = E[Dn^2] - 2[Dn\, Yn] + E[Yn^2]$$

- Substituting Yn and interchanging the orders of summation and expectation we get

$$\xi = E[Dn^2] - 2\sum_{k=0}^{M} h_k E[DnX_{n-k}] + \sum_{k=0}^{M}\sum_{m=0}^{M} h_k h_m E[X_{n-k}X_{n-m}]$$

- Assuming that the sample X_n and Dn are drawn from zero mean jointly stationary processes we may interpret the three expectations as follows

1. The expectation $E[Dn^2]$ is equal to the variance of the desired response

$$\sigma_D^2 = E[Dn^2]$$

2. The expectation $E[DnX_{n-k}]$ is equal to the cross correlation function between the desired response and the input signal for a lag of k samples

$$R_{DX}(k) = E[DnX_{n-k}] \quad k = 0,1, \dots . M$$

Wiener Filter for Waveform estimation

3. The expectation $E[X_{n-k}X_{n-m}]$ is equal to the autocorrelation function of the input signal for a lag of (m-k) samples

$$R_X(m-k) = E[X_{n-k}X_{n-m}] \; k = 0,1, \dots . M$$

- Hence we can rewrite the equation as

$$\xi = \sigma_D^2 - 2\sum_{k=0}^{M} h_k R_{DX}(k) + \sum_{k=0}^{M}\sum_{m=0}^{M} h_k h_m R_X(m-k)$$

- To obtain the minimum value of ξ we need to differentiate the above equation with respect to h_k and equate it to zero

$$\frac{\partial \xi}{\partial h_k} = -2R_{DX}(k) + 2\sum_{m=0}^{M} h_m R_X(m-k)$$

- Equating the result to zero, we obtain the optimum value of filter coefficients

Wiener Filter for Waveform estimation-Wiener Hopf Equation

- Let the optimum value of the filter coefficients be denoted by $h_{o0}, h_{o1} \ldots\ldots h_{oM}$
- They are given by the solution of the set of simultaneous equation

$$\boxed{\sum_{m=0}^{M} h_m R_X(m-k) = R_{DX}(k) \quad k = 0,1,\ldots M}$$

- This set of equations is called the **discrete time version of the Wiener Hopf equation**
- A filter whose coefficients defined using the above equation is optimum in the mean square sense
- This filter is called a Weiner filter
- Let ξ_{min} denote the minimum mean squared error Substituting the values of $h_{o0}, h_{o1} \ldots\ldots h_{oM}$ for the filter coefficients and then simplifying the expression for the minimum mean squared error we get

$$\xi_{min} = \sigma_D^2 - \sum_{m=0}^{M} h_{ok} R_X(k)$$

Linear Prediction

- Prediction constitutes a special form of estimation
- Specifically the requirement is to use a finite set of present and past samples of a stationary process to predict z sample of the process in the future.
- We say prediction is linear if it is a linear combination of the given samples of the process.
- The filter designed to perform the prediction is called a predictor.
- The difference between the actual sample of the process at the future time of interest and the predictor output is called the prediction error.
- According to the wiener filter theory a predictor is designed to minimize the mean square value of the prediction error.
- Consider the random samples $X_{n-1}, X_{n-2} \ldots \ldots X_{n-M}$ drawn from a stationary process $X(t)$
- Suppose the requirements is to make a prediction of the sample X_n

Linear Prediction

- Let $\widehat{X_n}$ denote the random variable resulting from the prediction
- We thus write

$$\widehat{X_n} = \sum_{k=1}^{M} h_{ok} X_{n-k}$$

- Where $h_{o0}, h_{o1} \ldots\ldots h_{oM}$ are the optimum predictor coefficients
- We refer to M the number of delay elements as its order.
- We may set up the normal equations for the predictor coefficients by minimizing the mean square value of the predictor error.
- It is a special case of Wiener filter.
- The variance of the sample X_n viewed as the desired response equals

$$\sigma_X^2 = E\left[{X_n}^2\right] = R_X(0)$$

- Where it is assumed that X_n has zero mean

Linear Prediction

- The cross correlation function of X_n acting as the desired response and X_{n-k} acting as the kth tap input of the predictor is given by

$$E[\,X_n\,X_{n-k}] = R_X(k) \quad k = 1,2,\dots.M$$

- The autocorrelation function of the predictor's tap input X_{n-k} with another tap input X_{n-m} is given by

$$E[\,X_{n-k}\,X_{n-m}] = R_X(m-k) \quad k,m = 1,2,\dots.M$$

- The above equations are analogues to the equations of wiener filter
- Thus we get

$$\sum_{k=1}^{M} h_{ok} R_X(m-k) = R_X(k) \;\; k = 1,2,\dots.M$$

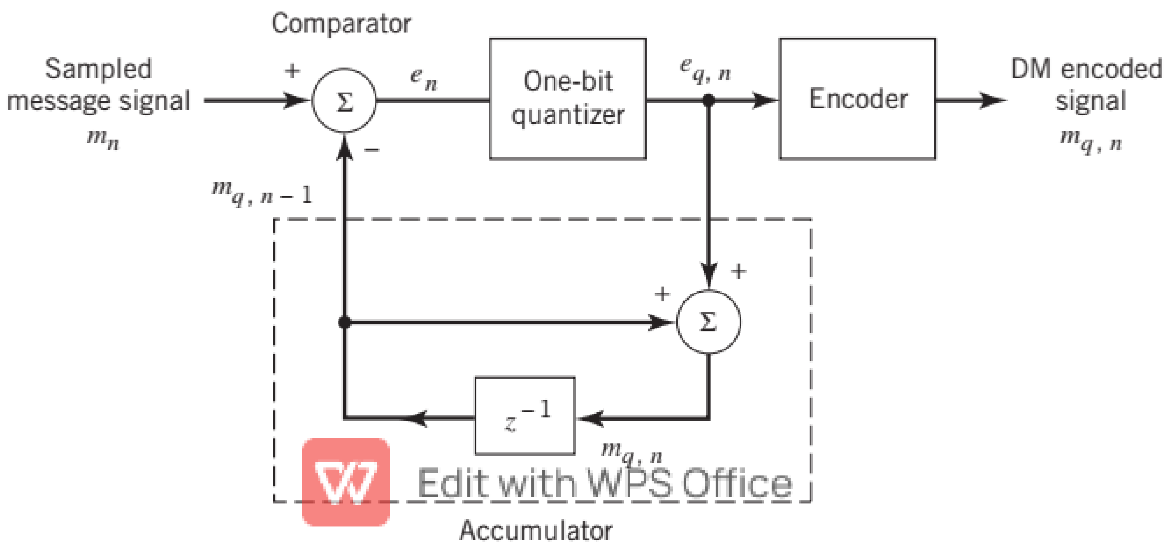
- Thus we need only the autocorrelation function of the signal for different lags in order to solve the normal equations for the predictor coefficients
-

Delta Modulation

- In choosing DPCM for waveform coding, we are, in effect, economizing on transmission bandwidth by increasing system complexity, compared with standard PCM.
- In other words, DPCM exploits the complexity–bandwidth tradeoff. However, in practice, the need may arise for reduced system complexity compared with the standard PCM.
- To achieve this other objective, transmission bandwidth is traded off for reduced system complexity, which is precisely the motivation behind DM.
- Thus, whereas DPCM exploits the complexity–bandwidth tradeoff, DM exploits the bandwidth–complexity tradeoff.
- With the bandwidth–complexity tradeoff being at the heart of DM, the incoming message signal m(t) is oversampled, which requires the use of a sampling rate higher than the Nyquist rate.
- Accordingly, the correlation between adjacent samples of the message signal is purposely increased so as to permit the use of a simple quantizing strategy for constructing the encoded signal.

DM Transmitter

- In the DM transmitter, system complexity is reduced to the minimum possible by using the combination of two strategies:
  - Single-bit quantizer, which is the simplest quantizing strategy; the quantizer acts as a hard limiter with only two decision levels, namely $\pm\Delta$.
  - Single unit-delay element, which is the most primitive form of a predictor is the front-end block labelled z^{-1}, which acts as an accumulator.
- Thus, replacing the multilevel quantizer and the FIR predictor in the DPCM transmitter



DM Transmitter

- From this figure, we may express the equations underlying the operation of the DM transmitter by the following set of equations:

$$
\begin{aligned}
e_n &= m_n - \hat{m}_n \\
&= m_n - m_{q,n-1}
\end{aligned}
$$

$$
\begin{aligned}
e_{q,n} &= \Delta \operatorname{sgn}[e_n] \\
&= \begin{cases} +\Delta & \text{if } e_n > 0 \\ -\Delta & \text{if } e_n < 0 \end{cases}
\end{aligned}
$$

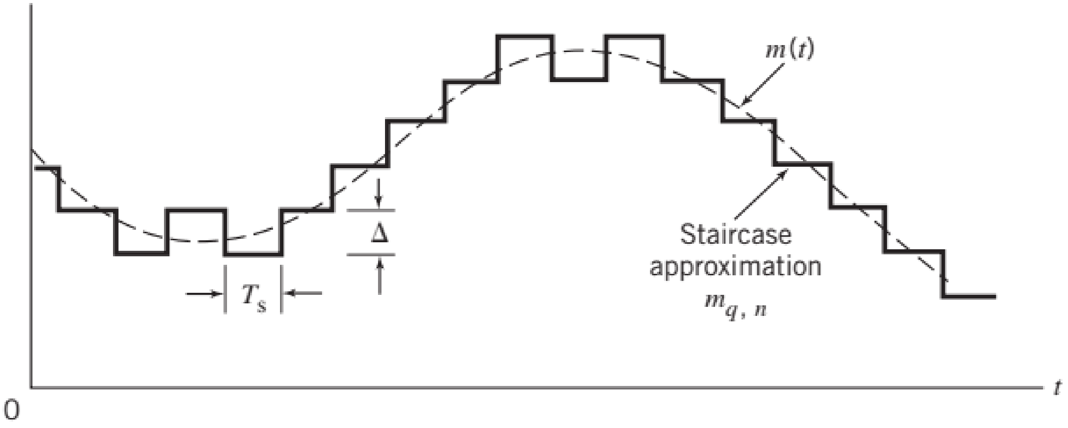
$$
m_{q,n} = m_{q,n-1} + e_{q,n}
$$

According to (6.95) and (6.96), two possibilities may naturally occur:

1. The error signal e_n (i.e., the difference between the message sample m_n and its approximation $\hat{m}_n$) is positive, in which case the approximation $\hat{m}_n = m_{q,n-1}$ is increased by the amount Δ; in this first case, the encoder sends out symbol 1.
2. The error signal e_n is negative, in which case the approximation $\hat{m}_n = m_{q,n-1}$ is reduced by the amount Δ; in this second case, the encoder sends out symbol 0.

DM Transmitter

From this description it is apparent that the delta modulator produces a staircase approximation to the message signal, as illustrated in Figure 6.22a. Moreover, the rate of data transmission in DM is equal to the sampling rate $f_s = 1/T_s$, as illustrated in the binary sequence of Figure 6.22b.

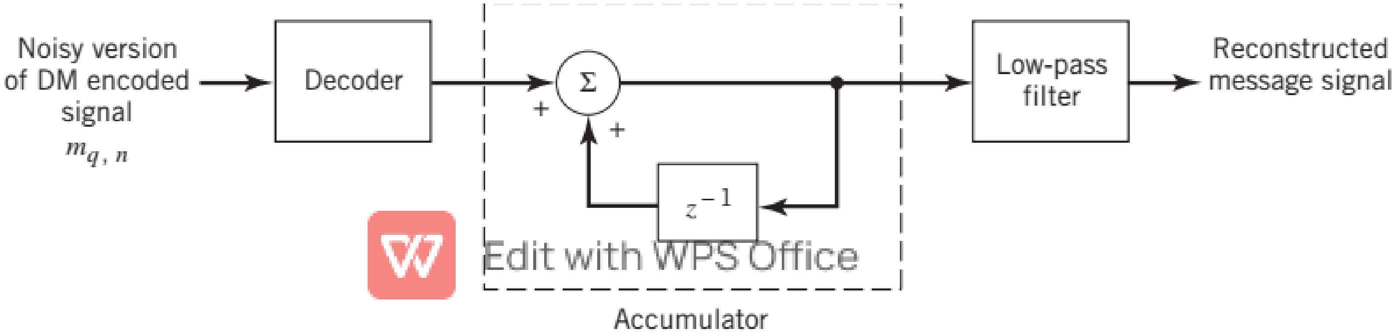


(a)

Binary sequence at modulator output 0 0 1 0 1 1 1 1 1 0 1 0 0 0 0 0 0

DM Receiver

- we may construct the DM receiver of as a special case of the DPCM receiver
- Working through the operation of the DM receiver, we find that reconstruction of the staircase approximation to the original message signal is achieved by passing the sequence of positive and negative pulses (representing symbols 1 and 0, respectively) through the block labelled "accumulator."
- Under the assumption that the channel is distortion less, the accumulated output is the desired mq,n given that the decoded channel output is eq,n.
- The out-of-band quantization noise in the high-frequency staircase waveform in the accumulator output is suppressed by passing it through a low-pass filter with a cutoff frequency equal to the message bandwidth.



Quantization Errors in DM

- DM is subject to two types of quantization error:
  - slope overload distortion and
  - Granular noise
- we observe that the equation is the digital equivalent of integration, in the sense that it represents the accumulation of positive and negative increments of magnitude Δ.

$$m_{q,n} = m_{q,n-1} + e_{q,n}$$

- Moreover, denoting the quantization error applied to the message sample mn by qn, we may express the quantized message sample as

$$m_{q,n} = m_n + q_n$$

- With this expression for mq,n at hand, we find from that the quantizer input is

$$e_n = m_n - (m_{n-1} + q_{n-1})$$

- Thus, except for the delayed quantization error q_{n-1} , the quantizer input is a first backward difference of the original message sample

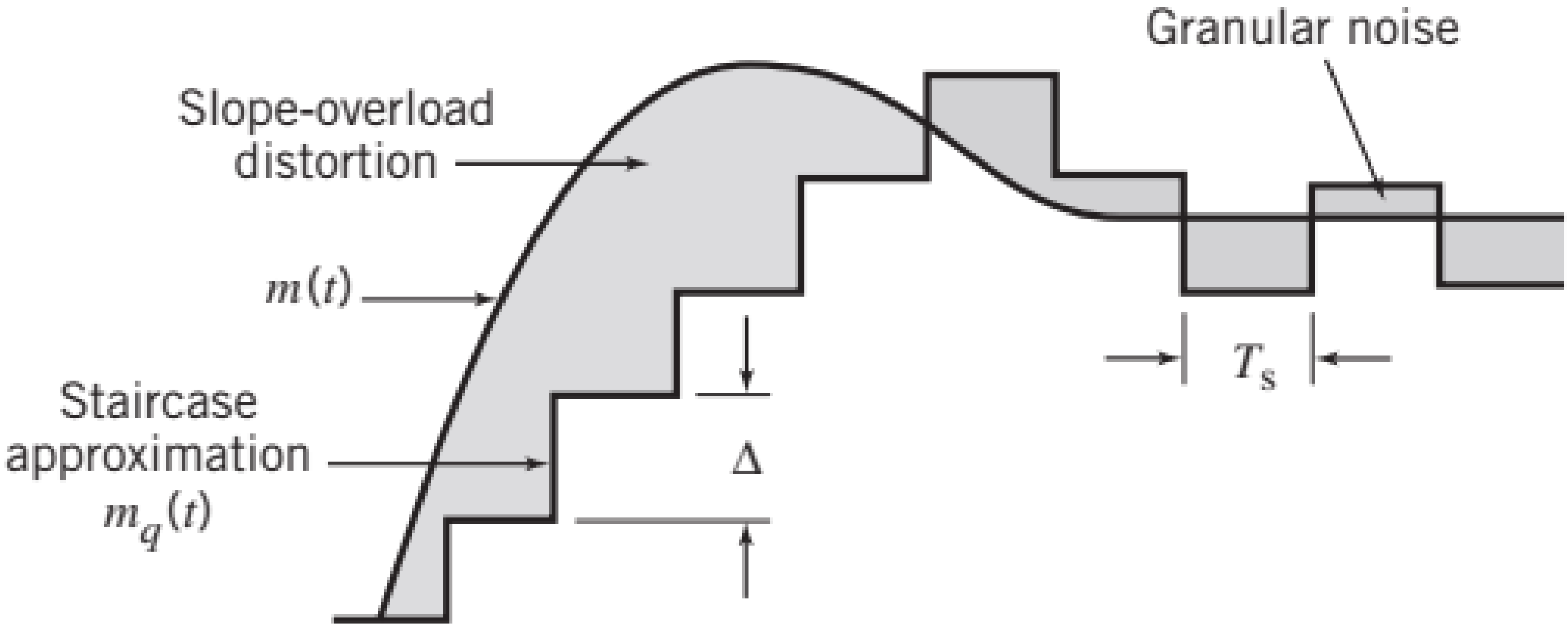
Quantization Errors in DM

- This process is the inverse of the digital integration process carried out in the DM transmitter.
- If, then, we consider the maximum slope of the original message signal m(t), it is clear that in order for the sequence of samples {mq,n} to increase as fast as the sequence of message samples {mn} in a region of maximum slope of m(t), we require that the condition be satisfied

$$\frac{\Delta}{T_s} \geq \max\left|\frac{dm(t)}{dt}\right|$$

- Otherwise, we find that the step-size Δ is too small for the staircase approximation mq(t) to follow a steep segment of the message signal m(t), with the result that mq(t) falls behind m(t),
- This condition is called slope overload, the resulting quantization error is called slope-overload distortion (noise)
- Note that since the maximum slope of the staircase approximation mq(t) is fixed by the step size Δ, increases and decreases in mq(t) tend to occur along straight lines.
- For this reason, a delta modulator using a fixed step size is often referred to as a linear delta modulator.

Quantization Errors in DM



Quantization Errors in DM

- In contrast to slope-overload distortion, granular noise occurs when the step size Δ is too large relative to the local slope characteristics of the message signal m(t),
- thereby causing the staircase approximation mq(t) to hunt around a relatively flat segment of m(t);
- This phenomenon is also illustrated in the tail end of Figure.
- Granular noise is analogous to quantization noise in a PCM system
- We thus see that there is a need to have large step size to accommodate a wide dynamic range
- Whereas a small step size is required for the accurate representation of relatively low level signals
- It is therefore clear that the choice of optimum step size that minimizes the mean square value of the quantizing error in a linear delta modulator will be the result of a compromise between slope overload distortion and granular noise.