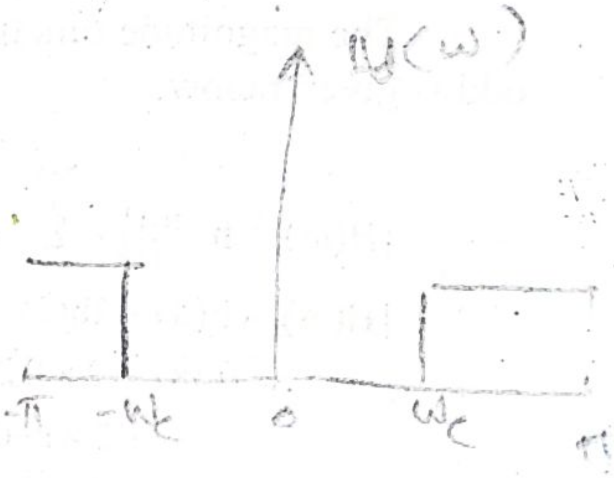
Design a highpass filter using hamming window, with a cut-off frequency of 1.2 radians/sec and N = 9.

SOLUTION



The desired frequency response for highpass filter is

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \ -\pi \le \omega \le -\omega_c \ \ \& \ \ \omega_c \le \omega \le \pi \\ 0 & ; \ \text{otherwise} \end{cases}$$

The $H_d(n)$ is obtained by inverse fourier transform of $H_d(\omega)$

By definition of inverse fourier transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega c} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega c}^{\pi} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega c} e^{j\omega(n-\alpha)} d\omega + \frac{1}{2\pi}\int_{+\omega c}^{\pi} e^{j\omega(n-\alpha)} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega c(n-\alpha)} - e^{-j\pi(n-\alpha)} + e^{j\pi(n-\alpha)} - e^{j\omega c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{2j} - \frac{e^{j\omega c(n-\alpha)} - e^{-j\omega c(n-\alpha)}}{2j}\right]$$

$$= \frac{1}{\pi(n-\alpha)} [\sin (n-\alpha)\,\pi - \sin\, \omega_c(n-\alpha)]$$

$\alpha = \frac{N-1}{2}$

When $n = \alpha$, the terms $\frac{\sin (n-\alpha)\,\pi}{(n-\alpha)}$ and $\frac{\sin \omega_c(n-\alpha)}{(n-\alpha)}$ become 0 / 0 which is indeterminate.

Hence, $h_d(n) = \frac{\sin\,(n-\alpha)\,\pi - \sin\, \omega_c(n-\alpha)}{\pi\,(n-\alpha)}$; for $n \neq \alpha$

L' Hospital rule

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin A\theta}{\theta} = A$$

For $n = \alpha$, $h_d(n)$ can be evaluated using L'Hospital rule.

$$\therefore \text{When } n = \alpha,\ h_d(n) = \underset{n \to \alpha}{\text{Lt}} \frac{\sin\,(n-\alpha)\pi - \sin \omega_c(n-\alpha)}{\pi\,(n-\alpha)}$$

$$= \frac{1}{\pi}\left[\underset{n \to \alpha}{\text{Lt}} \frac{\sin \pi\,(n-\alpha)}{n-\alpha} - \underset{n \to \alpha}{\text{Lt}} \frac{\sin\, \omega_c(n-\alpha)}{n-\alpha}\right] = \frac{1}{\pi}(\pi - \omega_c) = 1 - \frac{\omega_c}{\pi}$$

$$\therefore h_d(n) = 1 - \frac{\omega_c}{\pi}\ ;\quad \text{for } n = \alpha$$

The window sequence for hamming window is given by

$$w_H(n) = 0.54 - 0.46 \cos\left(\tfrac{2\pi n}{N-1}\right)\ ;\quad \text{for } n = 0 \text{ to } (N-1)$$

$$\therefore h(n) = h_d(n)\, w_H(n) = \frac{1}{\pi(n-\alpha)} [\sin \pi\,(n-\alpha) - \sin\,(n-\alpha)\,\omega_c]$$

$$\left[0.54 - 0.46 \cos\left(\tfrac{2\pi n}{N-1}\right)\right]\ ;\ \text{for } n \neq \alpha$$

$$= \left(1 - \frac{\omega_c}{\pi}\right)\left(0.54 - 0.46 \cos\left(\tfrac{2\pi n}{N-1}\right)\right);\ \text{for } n = \alpha$$

Given that $N = 9$; $\omega_c = 1.2$ rad/sec. $\quad \therefore \alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4$

In this example both n and α are integers. Hence $(n-\alpha)$ is also an integer and $(n-\alpha)\pi$ will be an integral multiple of π.

$\therefore \sin(n-\alpha)\,\pi = 0$. Also $(N-1) = 8$

$$\therefore h(n) = \frac{-\sin(n-\alpha)\,\omega_c}{\pi(n-\alpha)}\left[0.54 - 0.46 \cos\frac{n\pi}{4}\right]\ ;\ \text{for } n \neq 4$$

$$\text{and } h(n) = \left(1 - \frac{\omega_c}{\pi}\right)\left[0.54 - 0.46 \cos\frac{n\pi}{4}\right];\ \text{for } n = 4$$

$$\text{When } n = 0\ ;\ h(0) = \frac{-\sin\,((-4)\times 1.2)}{\pi \times (-4)}[0.54 - 0.46 \cos 0] = 0.0063$$

$$\text{When } n = 1\ ;\ h(1) = \frac{-\sin\,((-3)\times 1.2)}{\pi \times (-3)}\left[0.54 - 0.46 \cos\frac{\pi}{4}\right] = 0.0101$$

$$\text{When } n = 2\ ;\ h(2) = \frac{-\sin\,((-2)\times 1.2)}{\pi \times (-2)}\left[0.54 - 0.46 \cos\frac{2\pi}{4}\right] = -0.0581$$

$$\text{When } n = 3\ ;\ h(3) = \frac{-\sin\,((-1)\times 1.2)}{\pi \times (-1)}\left[0.54 - 0.46 \cos\frac{3\pi}{4}\right] = -0.2567$$

$$\text{When } n = 4\ ;\ h(4) = \left(1 - \frac{1.2}{\pi}\right)\left[0.54 - 0.46 \cos\frac{4\pi}{4}\right] = 0.6180$$

When n = 5 ; $h(5) = \frac{-\sin(1\times 1.2)}{\pi\times 1}\left[0.54 - 0.46\cos\frac{5\pi}{4}\right] = -0.2567$

When n = 6 ; $h(6) = \frac{-\sin(2\times 1.2)}{\pi\times 2}\left[0.54 - 0.46\cos\frac{6\pi}{4}\right] = -0.0581$

When n = 7 ; $h(7) = \frac{-\sin(3\times 1.2)}{\pi\times 3}\left[0.54 - 0.46\cos\frac{7\pi}{4}\right] = 0.0101$

When n = 8 ; $h(8) = \frac{-\sin(4\times 1.2)}{\pi\times 4}\left[0.54 - 0.46\cos\frac{8\pi}{4}\right] = 0.0063$

From the above calculations it can be observed that the impulse response is symmetrical with centre of symmetry at n = 4. The magnitude response for linear phase FIR filters, when N is odd and h(n) is symmetrical is given by

$$|H(\omega)| = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2}-n\right)\cos\omega n$$

$$= h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$

$$= 0.618 + 2\times(-0.2567)\cos\omega + 2\times(-0.0581)\cos 2\omega + 2\times(0.0101)\cos 3\omega + 2\times 0.0063\cos 4\omega$$

$$= 0.618 - 0.5134\cos\omega - 0.1162\cos 2\omega + 0.0202\cos 3\omega + 0.0126\cos 4\omega$$

The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} = \sum_{n=0}^{8} h(n)z^{-n} = \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=5}^{8} h(n)z^{-n}$$

$$= \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=0}^{3} h(8-n)z^{-(8-n)}$$

$$= \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=0}^{3} h(n)z^{-(8-n)}$$

$$= \sum_{n=0}^{3} h(n)\left[z^{-n} + z^{-(8-n)}\right] + h(4)z^{-4} \qquad \text{... (3.3.1)}$$