Derivation of DIF-FFT Algorithm (Radix-2)

In DIF Algorithm the output sequence $X(k)$ is divided into smaller and smaller subsequences.

- To derive the algorithm, we begin by splitting the DFT formula into two summations, of which one involves the sum over the first $N/2$ data points and the second the sum over the last $N/2$ data points.

Thus we obtain.

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ; \; k = 0, 1, \dots N-1$$

$$= \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}.$$

$$\left(n \rightarrow n + N/2 . \quad \therefore \text{limit} \rightarrow \sum_{n=0}^{N/2-1} \right)$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(n+N/2) W_N^{k(n+N/2)}$$

$$\therefore X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + W_N^{Nk/2} \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn} \quad \text{——} ①$$

$$\left[\text{Since } W_N^{kN/2} = \left(W_N^{N/2}\right)^k = (-1)^k\right]$$

$\therefore$ above equation ① can be re written as.

$$X(k) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^k x\left(n+\frac{N}{2}\right)\right] W_N^{kn}.$$

Now let us split $X(k)$ into the even and odd numbered samples. Thus we obtain.

$$X(2k) = \sum_{n=0}^{N/2-1} \left[x(n) + x(n+N/2)\right] W_N^{2kn}$$

$$(\because (-1)^{2k} = 1)$$

and
$$X(2k+1) = \sum_{n=0}^{N/2-1} \left[x(n) - x(n+N/2)\right] W_N^{(2k+1)n}.$$

$$(\because (-1)^{2k+1} = -1)$$

The above two equations can be re written as.

$$X(2k) = \sum_{n=0}^{N/2-1} \left[x(n) + x(n+N/2)\right] W_{N/2}^{kn}. \quad \text{——} ②$$

$$; k = 0, 1 \ldots \frac{N}{2}-1$$

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left\{\left[x(n) - x(n+N/2)\right] W_N^{n}\right\} W_{N/2}^{kn}. \quad \text{——} ③$$

where we have used the fact that

$$W_N^{2kn} = W_{N/2}^{kn}.$$

If we define $\frac{N}{2}$ point sequences $g_1(n)$ and $g_2(n)$ as

$$\left.\begin{aligned} g_1(n) &= x(n) + x(n+N/2) \\ g_2(n) &= [x(n) - x(n+N/2)]\, W_N^{n} \end{aligned}\right\} \longrightarrow (4)$$

$$: n = 0, 1, \ldots \frac{N}{2} - 1$$

then

$$\left.\begin{aligned} X(2k) &= \sum_{n=0}^{N/2-1} g_1(n)\, W_{N/2}^{kn} \\ X(2k+1) &= \sum_{n=0}^{N/2-1} g_2(n)\, W_{N/2}^{kn}. \end{aligned}\right\} \longrightarrow (5).$$

Let $N=8$.

Computation of sequences $g_1(n)$ and $g_2(n)$ according to equation (4) the subsequent use of these sequences to compute the $N/2$ point DFT's are as depicted below.

(4) $\rightarrow$

$n=0 \rightarrow g_1(0) = x(0) + x(4)$

$n=1 \rightarrow g_1(1) = x(1) + x(5)$

$n=2 \rightarrow g_1(2) = x(2) + x(6)$

$n=3 \rightarrow g_1(3) = x(3) + x(7).$

$g_2(0) = [x(0) - x(4)]\, W_8^0.$

$g_2(1) = [x(1) - x(5)]\, W_8^1$

$g_2(2) = [x(2) - x(6)]\, W_8^2$

$g_2(3) = [x(3) - x(7)]\, W_8^3$

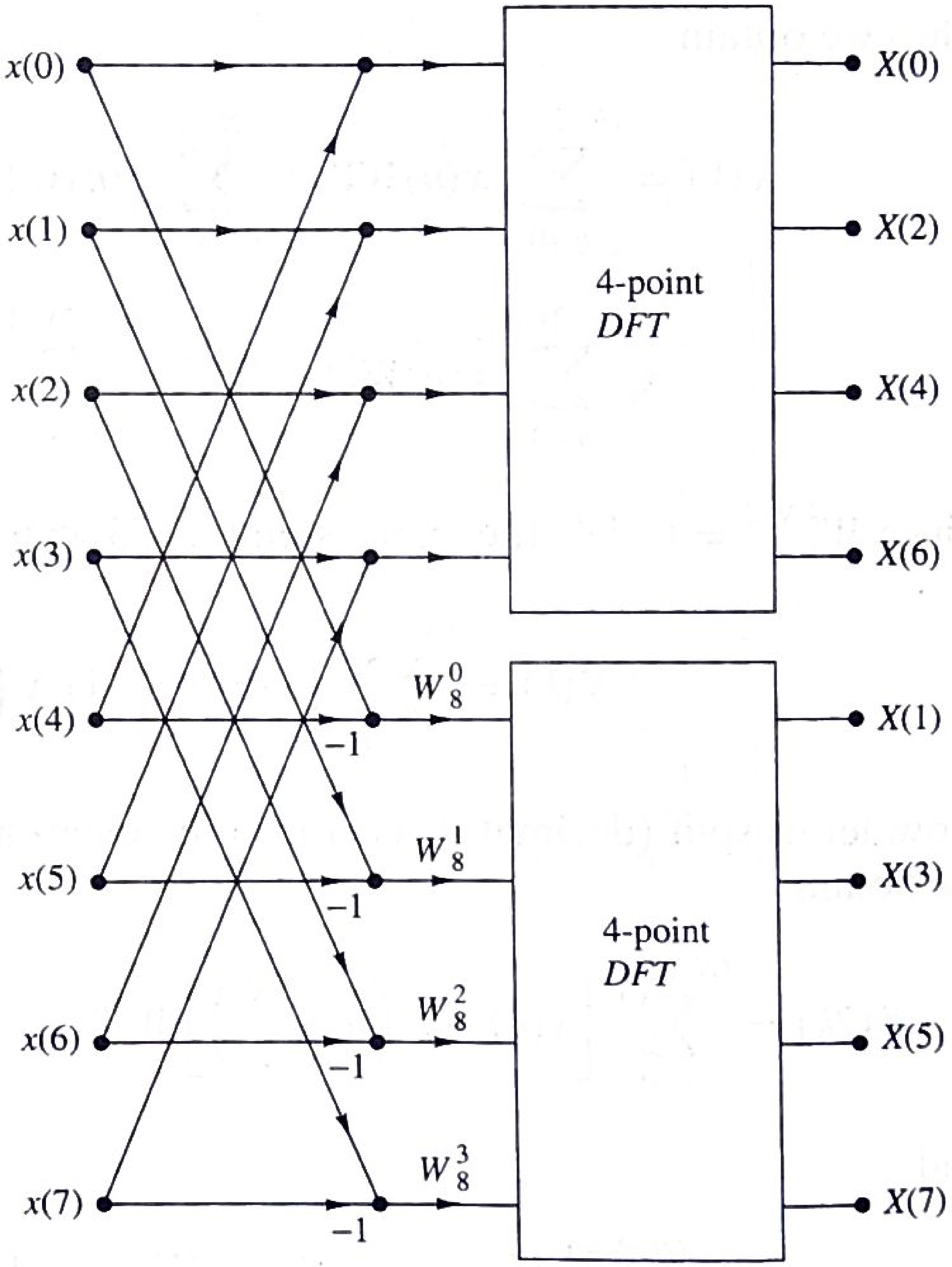
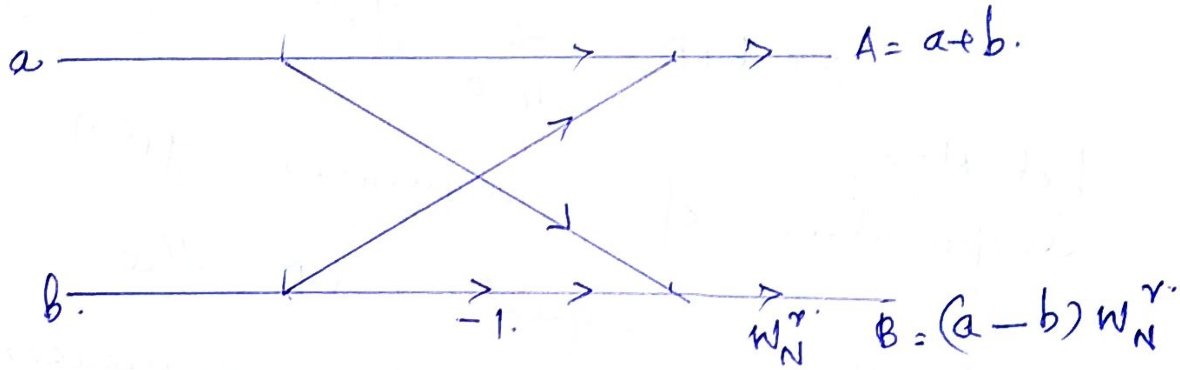


Figure 8.1.9
First stage of the decimation-in-frequency FFT algorithm.

- This computational procedure can be repeated through decimation of $\frac{N}{2}$ point DFTs $X(2k)$ and $X(2k+1)$
- The entire process involves $v = \log_2 N$ stages of decimation, where each stage involves $\frac{N}{2}$ butterflies
- Basic butterfly computation in decimation in frequency FFT algorithm.



The 8-point decimation in frequency Algorithm is shown below.

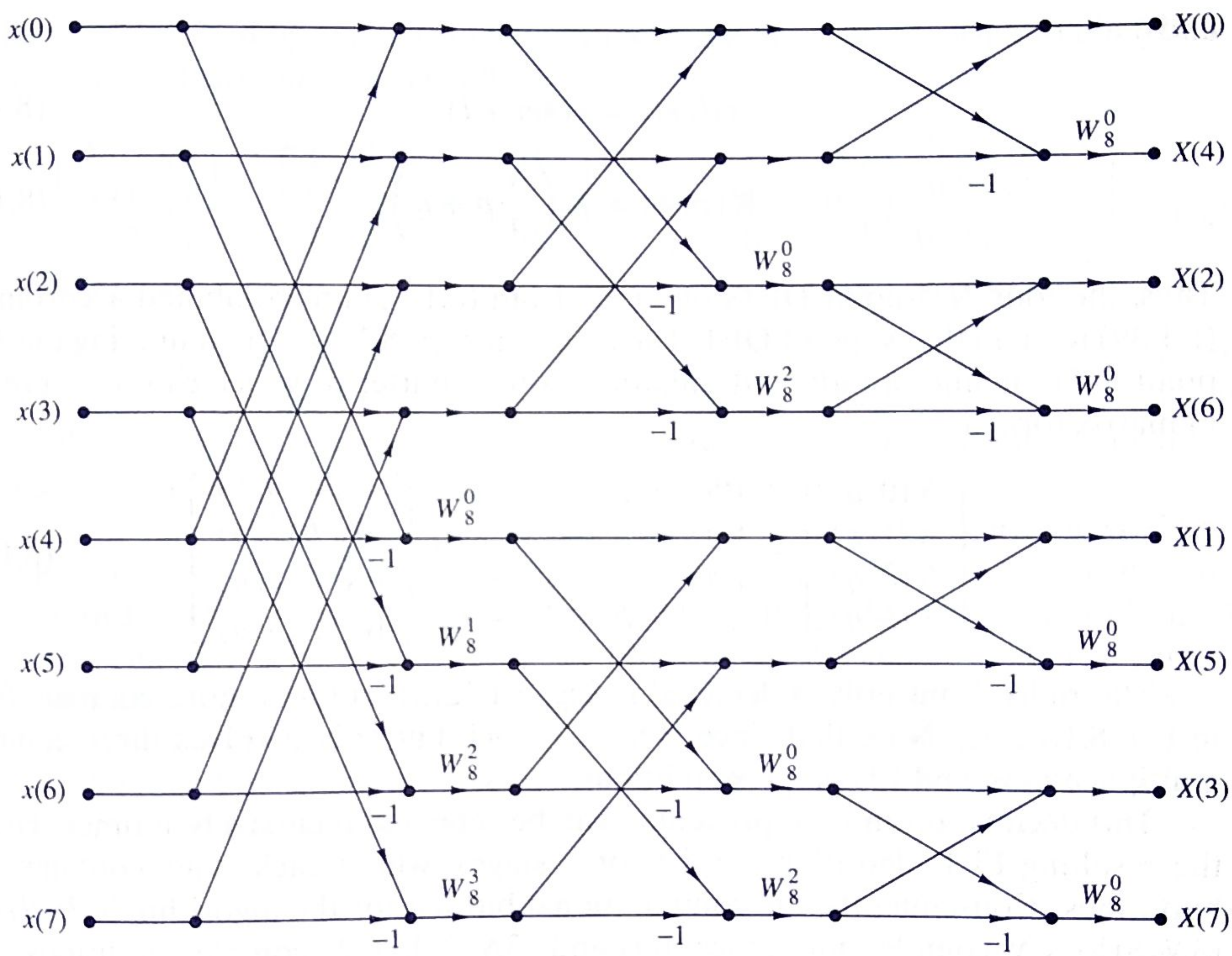


Figure 8.1.11 $N = 8$-point decimation-in-frequency FFT algorithm.