Differential Entropy

Differential entropy is used when the binary data is effected by noise and the probability density function of noise is continuous.

$$h(x) = -\int_{-\infty}^{\infty} f_X(x) \log_2 f_X(x)\, dx$$

Differential Entropy of A Gaussian RV

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \longrightarrow (1)$$

$$h(x) = -\int_{-\infty}^{\infty} f_X(x) \log_2 f_X(x) \cdot dx \longrightarrow (2)$$

$$\log_e x = \frac{\log_2 x}{\log_2 e}$$

$$\log_2 x = \log_e x \cdot \log_2 e$$

$\rightarrow \log_2 f_X(x) =$

$$= \log_2 e \cdot \log_e \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$= \log_2 e \left[\log_e 1/\sqrt{2\pi\sigma^2} + \log_e e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right]$$

$$= \log_2 e \left[-\log_e \sqrt{2\pi\sigma^2} - \frac{(x-\mu)^2}{2\sigma^2} \right] \longrightarrow (3)$$

(3) in (2)

$$h(x) = \int_{-\infty}^{\infty} f_X(x) \cdot \log_2 e \left[\log_e \sqrt{2\pi\sigma^2} + \frac{(x-\mu)^2}{2\sigma^2} \right] . dx$$

$$= \log_2 e \log_e \sqrt{2\pi\sigma^2} \underbrace{\int_{-\infty}^{\infty} f_X(x)\, dx}_{\text{area under pdf} = 1} + \log_2 e \int_{-\infty}^{\infty} \frac{(x-\mu)^2}{2\sigma^2} f_X(x) . dx$$

$$= \underbrace{\log_2 e \log_e (2\pi\sigma^2)^{1/2}}_{\log_2 e \cdot \log_e x = \log_2 x .} + \frac{\log_2 e}{2\sigma^2} \underbrace{\int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) . dx}_{\rightarrow (4)} \quad \sigma^2$$

We know

$$Var[X] = E[(X-\mu)^2]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x) dx \longrightarrow (5)$$

$$= \sigma^2 \quad \text{(Variance of G.RV)}$$

(5) in (4)

$$h(x) = \log_2 (2\pi\sigma^2)^{1/2} + \frac{\log_2 e}{2\sigma^2} \times \sigma^2$$

$$= \tfrac{1}{2} \log_2 (2\pi\sigma^2) + \tfrac{1}{2} \log_2 e$$

$$= \tfrac{1}{2} \left[\log_2 (2\pi\sigma^2) + \log_2 e \right] = \tfrac{1}{2} \log_2 \left[2\pi e \sigma^2 \right]$$