Differential and difference equation Representation of (LTI) System

Differential equation → continuous time systems

Difference equation → discrete time systems.

1. Differential equation representation.

General form

$$\boxed{\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)} \quad \text{(1)}$$

$x(t)$ → i/p of the system.

$y(t)$ → o/p of the system.

a_k, b_k → constant coefficients

(N, M) represent order of the system. Often $N \geq M$ & N is the order of the system.

The solution to the differential equation is $y(t)$.

$y(t)$ has two components.
1. Natural response ~~or natural~~
2. forced response.

o/p is expressed as the sum of two components.
(1) homogenous solution.
(2) particular solution.

$$y = y^{(h)} + y^{(P)}$$

$y^{(h)}$ = homogenous solution

$y^{(P)}$ = particular solution

ⓐ homogenous Solution (~~Natural response~~)

The natural response is the system o/p when the i/p is zero. Hence for a continuous time system the natural response $y^{(h)}(t)$ is the solution of the homogenous equation.

Homogenous equation.

put $x(t) = 0$ in eqtn ①;

Homogenous equation is $\displaystyle\sum_{K=0}^{N} a_K \frac{d^K}{dt^K} y^{(h)}(t) = 0$

Homogenous solution is

$$y^{(h)}(t) = \sum_{i=1}^{N} c_i e^{r_i t},$$

where r_i are the N roots of the system's characteristic equation

$$\sum_{K=0}^{N} a_K r^K = 0$$

example: Find the ~~natural response~~ homogenous ~~eq~~ response & for the system discribed by the differential equation.

$$5 \frac{d}{dt} y(t) + 10\, y(t) = 2\, x(t); \quad y(0) = 3.$$

To find natural response, put $x(t) = 0$.

ie Homogenous equation $5 \frac{d}{dt} y(t) + 10\, y(t) = 0.$

replacing $\frac{d^k}{dt^k} y(t)$ by r^k.

$$5r + 10 = 0$$

ie $\quad r = -2$

$\therefore$ Homogenous solution is

$$y^{(n)}(t) = c\, e^{-2t}$$

at $t = 0$, $y(0) = 3$ (given)

$$\Rightarrow \quad 3 = c \cdot e^{-2 \times 0}$$

$$\Rightarrow \quad c = 3$$

$\therefore$ Natural response is $\underline{\underline{y^{n}(t) = 3e^{-2t}}}$

Q2 Find the natural response (homogenous solution) of the system described by the differential equation.

$$\frac{d^2}{dt^2} y(t) + 3\frac{d}{dt} y(t) + 2y(t) = x(t) + \frac{d}{dt} x(t).$$

$$y(0) = 0 \; ; \; \left.\frac{d}{dt} y(t)\right|_{t=0} = 1$$

Ans: Homogenous equation:

$$\frac{d^2}{dt^2} y(t) + 3\frac{d}{dt} y(t) + 2y(t) = 0.$$

$\therefore$ characteristic equation is $r^2 + 3r + 2 = 0$

$\Rightarrow \cancel{r(r+3)+2=0} \quad \Rightarrow r^2 + r + 2r + 2 = 0$

$\Rightarrow (r+1)(r+2) = 0 \quad \Rightarrow r(r+1) + 2(r+1) = 0$

Roots are $r_1 = -1$; $r_2 = -2$

$\therefore$ Natural response $y^n(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$$y^n(t) = c_1 e^{-t} + c_2 e^{-2t}. \quad \text{—— (1)}$$

$y(0) = 0$

$\Rightarrow 0 = c_1 + c_2$ i.e. $c_1 = -c_2$

$\left.\frac{d}{dt} y(t)\right|_{t=0} = 1 \Rightarrow \frac{d}{dt} y^n(t) = -2 c_1 e^{-2t} - 2c_2 e^{-3t}$

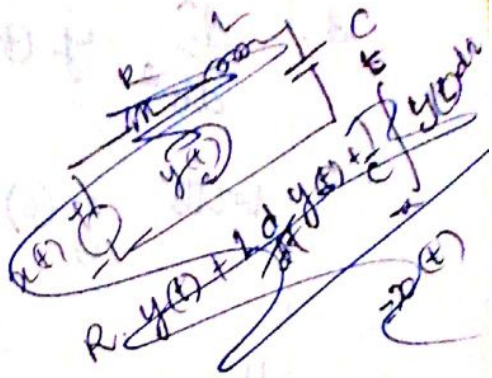
$\Rightarrow 1 = -2c_1 - c_1 - 2c_2$

$1 = + C_2 - 2C_2$

$\Rightarrow 1 = -C_2$

or $C_2 = \underline{\underline{-1}}$ $\therefore C_1 = 1$

$$\boxed{\therefore y^n(t) = e^{-t} - e^{-2t}}$$



Q.2. $\frac{d^2}{dt^2} y(t) + 4y(t) = 3\frac{d}{dt} x(t)$ with $y(0) = -1$ &

$\frac{d}{dt} y(t) \Big|_{t=0} = 1$

Ans: Homogenous equation

$\frac{d^2}{dt^2} y(t) + 4y(t) = 0$

characteristic equation $r^2 + 4 = 0$; $r = \pm\sqrt{-4}$

roots are imaginary $r = \pm 2j$

$r_1 = 2j$ & $r_2 = -2j$

$\therefore$ natural response $y^n(t) = C_1 \cos 2t + C_2 \sin 2t$

$y(0) = -1$; $-1 = C_1$ —— ①

$\frac{d}{dt} y^n(t) = -2C_1 \sin 2t + 2C_2 \cos 2t$.

$\frac{d}{dt} y^n(t) \Big|_{t=0} = 1 = 2C_2 \Rightarrow C_2 = 1/2$.

$$\boxed{\therefore y^n(t) = -\cos 2t + 1/2 \sin 2t}$$

3. $\frac{d^2}{dt^2} y(t) + 2\frac{d}{dt} y(t) + 2y(t) = \frac{d}{dt} x(t)$

with $y(0) = 1$; $\frac{d}{dt} y(t) \Big|_{t=0} = 0$.

Homogenous equation: $\frac{d^2}{dt^2} y(t) + 2\frac{d}{dt} y(t) + 2y(t)$

characteristic equation

$$r^2 + 2r + 2 = 0$$

Solution is ~~com~~ roots are complex.

$$r = -1 \pm j$$

$\therefore$ Natural response

$$y^n(t) = \sum_{i=1}^{2} c_i e^{r_i t}.$$

$$= \sum_{i=1}^{2} c_i e^{(-1 \pm j)t}$$

$$y^n(t) = e^{-t}\left[c_1 \cos t + c_2 \sin t\right]$$

$y(0) = 1 \Rightarrow \quad 1 = c_1$ —— (1)

$$\frac{d}{dt} y^n(t) = e^{-t}\left[-c_1 \sin t + c_2 \cos t\right]$$
$$- e^{-t}\left[c_1 \cos t + c_2 \sin t\right]$$

$\frac{d}{dt} y^n(t) \Big|_{t} = 0 = c_2 - c_1$ —— (2)

from ①

$0 = c_2 - 1$

$\Rightarrow c_2 = 1$ or $c_1 = c_2 = 1$

$$\boxed{\therefore y^n(t) = e^{-t}(\cos t + \sin t)}$$

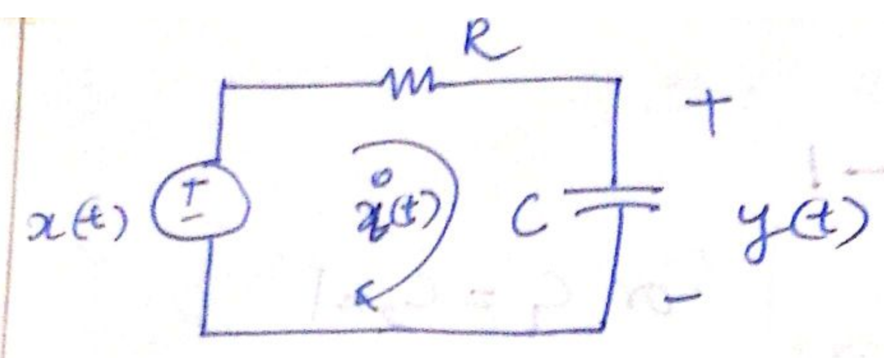
ⓑ Particular Solution

$y^p(t)$, particular solution is obtained by assuming the system o/p has the same general form as the i/p.

	Input $x(t)$	particular solution $y^p(t)$
1.	k	c
2.	t	$c_1 t + c_2$
3.	e^{-at}	$c e^{-at}$
4.	$\cos(\omega t + \phi)$	$c_1 \cos(\omega t) + c_2 \sin(\omega t)$

Q. RC circuit in given figure is ~~depicted~~ described by the differential equation

$$y(t) + RC \frac{d}{dt} y(t) = x(t) \quad \text{——} ①$$



Find the particular solution for this system with an i/p $x(t) = \cos(\omega_0 t)$

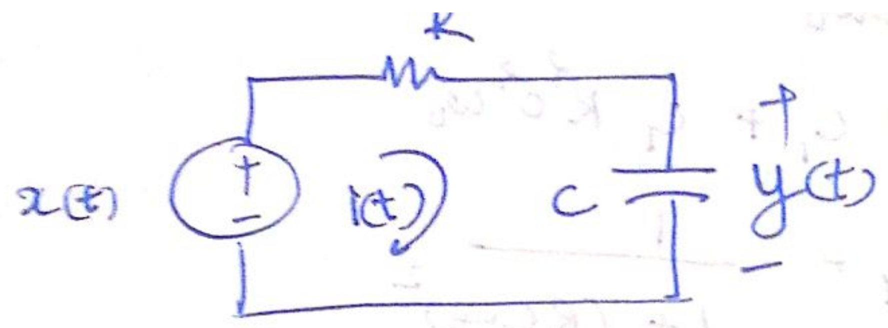
Complete Solution ~~or forced Response~~

$y = y^{p} + y^{h}$

① Find the form of the homogenous solution $y^{(h)}$ from the roots of the characteristic equat.

② Find the particular solution $y^{(p)}$ by assuming that it is of the same form as the input yet is independent of all terms in the homogenous solution.

③ Determine the coefficients of in the homogenous solution so that the complete solution $y = y^{(p)} + y^{(h)}$, satisfies the initial conditions.



Find the complete response of the RC circuit to an input $x(t) = \cos(t)\,u(t)$ Volt.
assuming $R = 1\Omega$, $C = 1F$. and assuming that the initial voltage across the capacitor
$y(0) = 2V$.

Q2. $x(t) = e^{-t} u(t)$

$y(0) = -1/2$, $\left.\frac{d}{dt} y(t)\right|_{t=0} = 1/2$

$$\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6 y(t) = x(t).$$

Find the complete solution.

* Natural response:

is the system o/p for zero i/p. The Natural response assumes zero i/p and thus does not involve a particular solution. Natural response describes the manner in which the system dissipates any stored energy or memory of the past represented by non-zero initial conditions.

eg: Determine the natural response for the system

$$y(0) = 3, \quad \frac{d}{dt} y(t) \Big|_{t=0} = -7$$

$$\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6 y(t) = 2 x(t) + \frac{d}{dt} x(t).$$

homogeneous eqn:

$$\frac{d^2}{dt^2} y(t) + 5 \frac{d}{dt} y(t) + 6 y(t) = 0.$$

characteristic eqtn.

$$r^2+5r+6=0$$

ie $\quad r+2r+3r+6=0$

ie $\quad (r+2)(r+3)=0.$

Roots are $r_1=-2,\ r_2=-3.$

$$y^h(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

$$y(0) = 3 = c_1 + c_2$$ ie

$$\frac{d}{dt} y(t)\Big|_{t=0} = -7 \quad \text{ie} \quad -2c_1 e^{-2t} - 3c_2 e^{-3t} = \frac{d}{dt} y^h(t)$$

ie $\quad -7 = -2c_1 - 3c_2$

$\quad\quad 3 = c_1 + c_2$

$\quad -7 = -2c_1 - 3c_2$

$\quad\quad 6 = 2c_1 + 2c_2$

$\quad -1 = 0 - c_2$

$\Rightarrow c_2 = 1 \quad \therefore \quad c_1 = 3 - c_2 = \underline{\underline{2}}$

$$\boxed{y^n(t) = 3e^{-2t} + 2e^{-3t}}$$ ✓

Forced response

Forced response is the system o/p due to the i/p signal assuming zero initial conditions. Thus the forced response is of the same form as the complete solution of the differential equation. A system with zero initial conditions is said to be 'at rest', since there is no stored energy or memory in the system. The forced response describes the system behavior that is 'forced' by the i/p when the system is at rest.

eg: $x(t) = e^{-t} u(t)$, $\frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = x(t)$.

Homogenous solution: $\frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = 0$

characteristic equation.

$$r^2 + 5r + 6 = 0$$

roots are $r_1 = -2$, $r_2 = -3$.

$$y^h(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$

$x(t) = e^{-t} u(t)$.

particular solution $\quad y^p(t) = C_3 e^{-t} u(t)$.

$y + [C_3 e^{-t} - 5 e^{-t} C_3 + 6 e^{-t} C_3] u(t) = e^{-t} u(t)$

ie $\quad 2C_3 = 1 \quad$ or $\quad C_3 = 1/2$.

$y^p(t) = 1/2\, e^{-t} u(t)$.

$y(t) = 1/2\, e^{-t} u(t) + C_1 e^{-2t} + C_2 e^{-3t}$

at $t=0$, $y(t) = 0$.

$0 = 1/2 + C_1 + C_2$

$\frac{d}{dt} y(t)\Big|_{t=0} = 0. \quad$ ie $\quad 0 = -1/2 - 2C_1 - 3C_2$

ie $\quad -1/2 = C_1 + C_2$

$1/2 = -2C_1 - 3C_2$

~~$-1/2 = C_1 + C_2$~~

$-1 = 2C_1 + 2C_2$

$1/2 = -2C_1 + 3C_2$

$-1/2 = -C_2$

$\Rightarrow C_2 = 1/2, \quad C_1 = -1/2 - C_2 = \underline{\underline{-1}}$

$$\boxed{\therefore\; y^F(t) = \left[1/2\, e^{-t} - 1\, e^{-2t} + 1/2\, e^{-3t}\right] u(t)}$$

Difference equation Representation

$$y(n) = y^{h}(n) + y^{p}(n).$$

complete solution = homogeneous solution + particular solution.

(a) Homogeneous Solution

general form of difference equation

$$\boxed{\sum_{K=0}^{N} a_K\, y[n-K] = \sum_{K=0}^{M} b_K\, x[n-K]}$$

Homogeneous equation is

$$\sum_{K=0}^{N} a_K\, y[n-K] = 0.$$

~~Characteristic equation~~ solution is

$$y^{(h)}[n] = \sum_{i=1}^{N} C_i\, r_i^{n}$$

where r_i are the N roots of the discrete time system's characteristic equation.

$$\sum_{K=0}^{N} a_K\, r^{N-K} = 0.$$

$y[n] - Py[n-1] = x[n]$. $\quad y(0) = 1$

homogeneous equation. $y[n] - Py[n-1] = 0$.

Characteristic equation:

$$r - P = 0$$

Root of the equation: $r = P$.

$$y^h[n] = \cancel{c_1 e^{Pn}} \quad c_1 p^n$$

$$y^h[n] = c_1 P = 1 \Rightarrow c_1 = 1/p.$$

$$y^h[n] = \frac{1}{P} p^n = p^{n-1}$$

Particular Solution

I/p	$x(n)$	particular solution $y^P(n)$
1)	1	c
2)	n	$c_1 n + c_2$
3)	α^n	$c\alpha^n$
4.	$\cos(\Omega n + \phi)$	$c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$

Q. Find the forced response . $x(n) = \frac{1}{2}^n u(n)$

$y[n] - \frac{1}{4} y[n-1] = x(n)$

$y^f(n) = y^h(n) + y^p(n)$

homogeneous equation :

$y[n] - \frac{1}{4} y[n-1] = 0$

characteristic equation $\sum_{K=0}^{N} a_K r^{N-K} = 0$

$N = 1$

$r - \frac{1}{4} = 0$

Roots of the characteristic equation $r = \frac{1}{4}$.

$\therefore y^h(n) = c_1 (\frac{1}{4})^n$.

<u>particular solution.</u>

$x[n] = (\frac{1}{2})^n u(n)$

$y^p[n] = c_2 (\frac{1}{2})^n u(n)$

ie $c_2 (\frac{1}{2})^n - \frac{1}{4} c_2 (\frac{1}{2})^{n-1} = (\frac{1}{2})^n u(0)$

ie $c_2 (\frac{1}{2})^n - \frac{1}{2} c_2 (\frac{1}{2})^n = (\frac{1}{2})^n$

ie $\frac{1}{2} c_2 (\frac{1}{2})^n = (\frac{1}{2})^n$

Equating the coefficients is $\frac{1}{2} c_2 = 1$

$\Rightarrow c_2 = 2$

$y^f[n] = 2(1/2)^n u[n] + C_1(1/4)^n$.

$y[0] = 0$ ie $0 = 2 + C_1$

ie $C_1 = -2$

$\Rightarrow \boxed{y^f[n] = 2(1/2)^n - 2(1/4)^n}$

8. Complete solution

$y[n] - 1/4\, y[n-1] = x[n]$; $x[n] = (1/2)^n u(n)$;

$y[-1] = 8$.

homogenous solution:

$y[n] - 1/4\, y[n-1] = 0$.

characteristic equation.

$r - 1/4 = 0$, $r = 1/4$.

solution: $y^h[n] = C_1(1/4)^n$.

particular solution $y^p(n) = C_2(1/2)^n u(n)$

ie $C_2(1/2)^n u(n) - 1/4 \cdot 2 \cdot C_2(1/2)^n \underset{u(n)}{} = (1/2)^n u(n)$

ie $C_2(1 - 1/2)(1/2)^n = (1/2)^n$

ie equating coefficents,

$\underline{\underline{C_2 = +2}}$

$y[n] = +2(1/2)^n u(n) + C_1(1/4)^n$

$y(-1) = 8$

$\Rightarrow \quad 8 = +2\times(1/2)^{-1} + c_1(1/4)^{-1}$

$\Rightarrow \quad 8 = +2\times 2 + c_1 \times 4$

$\Rightarrow \quad 8 = +4 + c_1 \times 4$

$\Rightarrow \quad \cancel{8 = c_1} \quad c_1 = 1$

$$\boxed{\therefore y[n] = (1/4)^n + 2.(1/2)^n}$$