INFORMATION THEORY & CODING

LECTURE 8

CONTENTS

- Quick recap
- Binary Erasure Channel
- Capacity of BEC channel

Binary Erasure Channel

→ BEC is widely used in digital communication.

⇒ has 2 inputs (0,1) and 3 outputs (0, y, 1).

→ whenever an error occurs, the symbol will be received as y, and no decision will be made about the information, but an immediate request will be made through reverse channel, for retransmission of the transmitted symbol till a correct symbol is received at the output.

Why called so?

- Symbol y indicates due to noise, no deterministic decision can be made as to whether the received symbol is 0 or 1. Symbol y indicates that output symbol is erased. Hence the name Binary Erasure Channel.

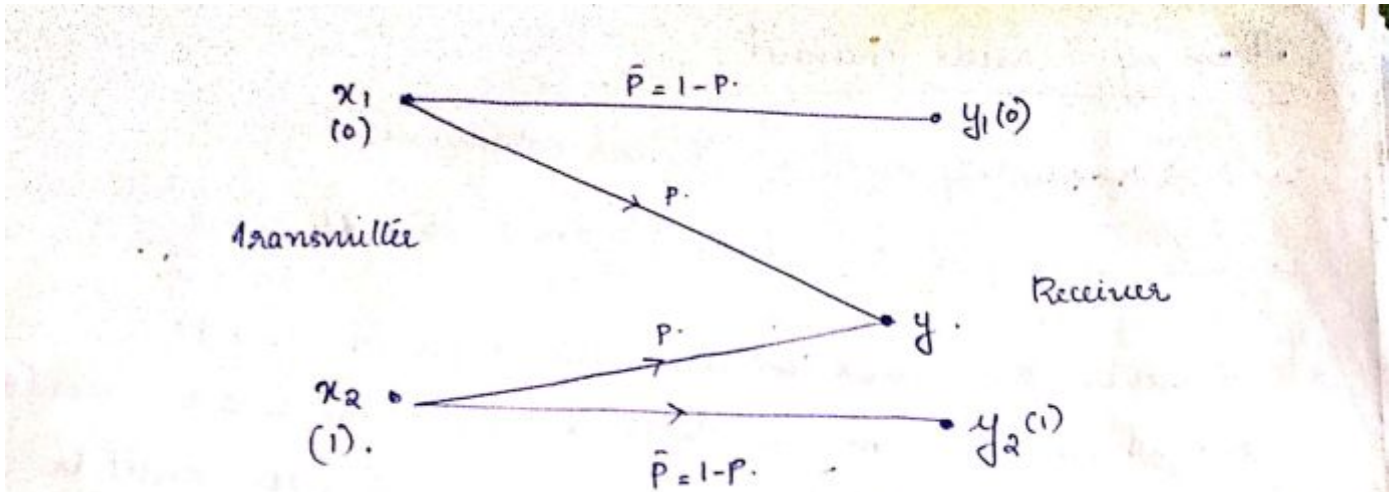
This can be considered as equivalent to error detection and requesting for retransmission (ARQ) method of correction

ARQ = Automatic Repeat Request.

This ensures 100% data recovery.

Disadv of BEC ⇒ Requires a dedicated Reverse Channel to carry out data retransmission

Channel diagram and channel matrix



BEC is also a symmetric channel whose channel matrix is constructed from the channel diagram as.

$$P(y/x) = P(y_j/x_i) = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_2 \end{array} \\ \begin{bmatrix} \bar{P} & P & 0 \\ 0 & P & \bar{P} \end{bmatrix} \end{array}$$

Properties that of symmetric channel

Let $P(x_1) = \omega$ and $P(x_2) = 1 - \omega = \bar{\omega}$

$\omega + \bar{\omega} = 1$

$p + \bar{p} = 1$

$$H(y/x) = h = \sum_{j=1}^{s} p_j \log_2 \frac{1}{p_j}$$

$$\boxed{H(y/x) = p \log \frac{1}{p} + \bar{p} \log \frac{1}{\bar{p}} = h}$$

Find Channel capacity?

$$H(X) = \sum_{i=1}^{2} P(x_i) \log_2 \frac{1}{P(x_i)}$$

$$= w \log_2 \frac{1}{w} + \bar{w} \log_2 \frac{1}{\bar{w}}$$

C= MAX(I(X,Y))

I(X,Y) = H(X)-H(X/Y)

We know that

$P(x_i, y_j) = P(x_i)\, P(y_i|x_i)$.

On multiplying 1st row of matrix by $P(x_1) = w$ and 2nd row by $P(x_2) = \bar{w}$ we get joint probability matrix of BEC

$$P(x, y) = P(x_i, y_j) = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y & y_2 \end{array} \\ \begin{bmatrix} \bar{p}w & pw & 0 \\ 0 & p\bar{w} & \bar{p}\bar{w} \end{bmatrix} \end{array}$$

- From properties of JPM we get

$$P(y_1) = \bar{P}w$$

$$P(y_2) = \bar{P}\bar{w}$$

$$P(y) = Pw + P\bar{w}$$

$$= P(w + \bar{w})$$

$$P(y) = P \cdot 1$$

$$\underline{P(y) = P}$$

Find P(X/Y)

$$P(x|y) = \frac{P(x_i, y_j)}{P(y_j)}$$

P(y1) = pw
P(y) = p
P(y2)= pw

$$P(x|y) = \begin{matrix} \\ x_1 \\ x_2 \end{matrix} \begin{matrix} y_1 & y & y_2 \end{matrix} \begin{bmatrix} \bar{p}w/\bar{p}w & pw/p & 0 \\ 0 & p\bar{w}/p & \bar{p}\bar{w}/\bar{p}\bar{w} \end{bmatrix} = \begin{bmatrix} 1 & w & 0 \\ 0 & \bar{w} & 1 \end{bmatrix}$$

Find H(X/Y)

Equivocation $H(X/Y) = \sum_{i=1}^{n} \sum_{j=1}^{s} P(x_i, y_j) \log \frac{1}{P(x_i/y_j)}$

$$= \sum_{i=1}^{2} \sum_{j=1}^{3} P(x_i, y_j) \log \frac{1}{P(x_i/y_j)}.$$

$$\begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} y_1 & y & y_2 \end{array} \\ \begin{bmatrix} \bar{p}w & pw & 0 \\ 0 & p\bar{w} & \bar{p}\bar{w} \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & w & 0 \\ 0 & \bar{w} & 1 \end{bmatrix}$$

$$= \bar{p}w \cdot \log \frac{1}{1} + pw \cdot \log \frac{1}{w} + 0 + 0 +$$

$$p\bar{w} \cdot \log \frac{1}{\bar{w}} + \bar{p}\bar{w} \log \frac{1}{1}$$

$$H(X/Y) = pw \log \frac{1}{w} + p\bar{w} \log \frac{1}{\bar{w}}$$

$$= p\left[w \log \frac{1}{w} + \bar{w} \log \frac{1}{\bar{w}}\right].$$

Find I(x,y)

Mutual Information of BEC is given by.

$$I(x,y) = H(x) - H(x/y).$$

$$= \left[\omega \log \frac{1}{\omega} + \bar{\omega} \log \frac{1}{\bar{\omega}}\right] - P\left[\omega \log \frac{1}{\omega} + \bar{\omega} \log \frac{1}{\bar{\omega}}\right]$$

$$= H(X) - P\,H(X)$$

$$\boxed{I(x,y) = \bar{P}\,H(x)} \quad = H(X)\left[1 - P\right]$$

$$= H(X) \cdot \bar{P}.$$

Channel capacity of BEC.

$C = \text{Max}(I(x,y))$.

$= \bar{p}\ \text{Max}[H(x)]$

$= \bar{p}\ \text{Max}[\log 2]$.

$= \bar{p} \log 2$.

$C = \bar{p}$.

Even though BEC is a symmetric channel, the mutual information $I(x,y)$ cannot be computed using the form $[H(y) - H(y/x)]$ because $H(y)$ contains the symbol y which is used for error detection & which will be rejected at the receiver.

CONCLUSION

- Binary erasure channel
- Capacity of BEC

THANK YOU