8. Multirate Signal Processing

8.1 Introduction

So far we discussed single rate systems in which single sampling rate is used. These systems that use single sampling rate from A/D converter to D/A converter are known as single rate systems. Now we extend our discussion on systems where the sampling rate of signals are unequal at various parts of the system. The discrete-time systems that process data at more than one sampling rate are known as multirate systems. There are many cases where multirate signal processing is used. They are

1. In high quality data acquisition and storage systems.
2. In audio signal processing. For example a CD is sampled at 44.1 kHz but DAT (digital audio tape) is sampled at 48 kHz. Conversion between DAT (since studio recordings are mostly made on DAT) and CD use multirate signal processing techniques.
3. In video, PAL and NTSC run at different sampling rates. Therefore to watch an American program in Europe, one needs a sampling rate converter.
4. In speech processing to reduce the storage space or the transmitting rate of speech data.
5. In transmultiplexers.
6. Narrowband filtering for fetal ECG and EEG.

The two basic operations in multirate signal processing are decimation and interpolation. Decimation reduces the sampling rate, whereas interpolation increases the sampling rate.

Down sampling (Decimation)
(Sub sampling, sampling rate compression)

- Reduces the sampling rate of a discrete time signal.
- The sampling rate of a discrete time signal can be reduced by a factor M by taking every M^{th} value of the signal and leaving remaining $M-1$ values.
- The output signal $y(n)$ is a down sampled signal $x(n)$ can be represented by.

$$y(n) = x(Mn)$$

- The block diagram representation of the down sampler is as below.

$$x(n) \longrightarrow \boxed{\downarrow M} \longrightarrow y(n) = x(Mn)$$

Let us assume the signal $x(n)$ as shown in Fig. 8.2. The downsampled signal $x(n)$ can be obtained by simply keeping every M-th sample and removing $(M-1)$ in between samples. This process is equal to reducing the sampling rate by a factor M. In Fig. 8.2 the signal $x(n)$ is downsampled by a factor 3 to get a signal $y(n) = x(3n)$.

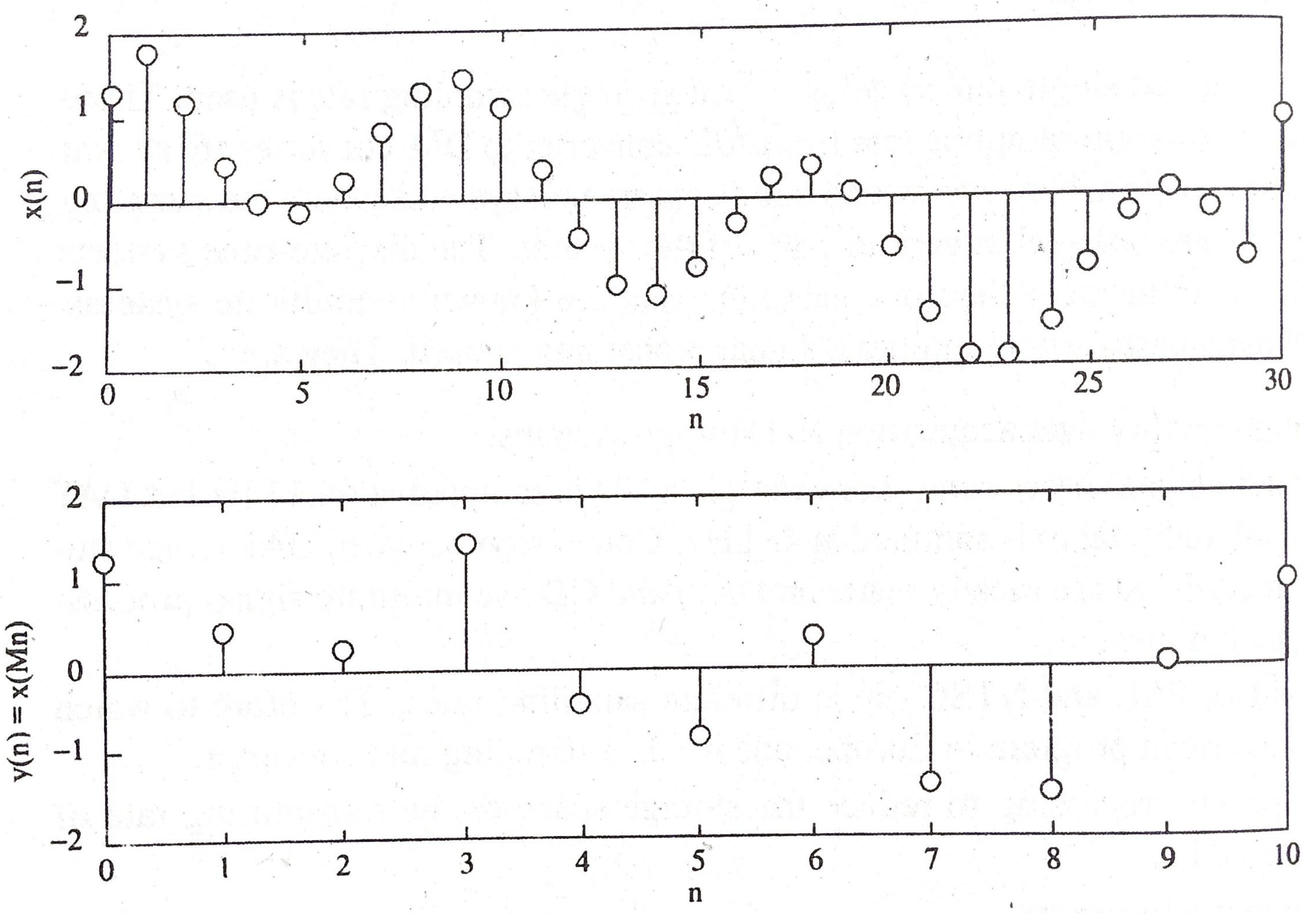


Fig. 8.2 Downsampling

Example: If $x(n) = \{1, -1, 2, 4, 0, 3, 2, 1, 5, \ldots\}$
then $y(n) = x(nM)$ for $M = 2$ is

$$y(n) = \{1, 2, 0, 2, 5, \ldots\}.$$

That is, we left one sample (in general $M - 1$ samples) in between samples of $x(n)$ to generate $y(n)$.

Spectrum of down sampled signal

- If $X(e^{j\omega})$ represents the spectrum of input signal $x(n)$, then the spectrum of down sampled signal $y(n)$, i.e. $Y(e^{j\omega})$ is given by.

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi k}{M}\right)}\right)$$

$$= \frac{1}{M}\left[X\left(e^{j\frac{\omega}{M}}\right) + X\left(e^{j\left(\frac{\omega-2\pi}{M}\right)}\right) + \dots X\left(e^{j\left(\frac{\omega-2\pi(M-1)}{M}\right)}\right)\right]$$

- From above relation we find that if fourier transform of input is $X(e^{j\omega})$ then the fourier transform of output $Y(e^{j\omega})$ is the sum of M uniformly shifted and stretched version of $X(e^{j\omega})$ and magnitude scaled by a factor of $1/M$.

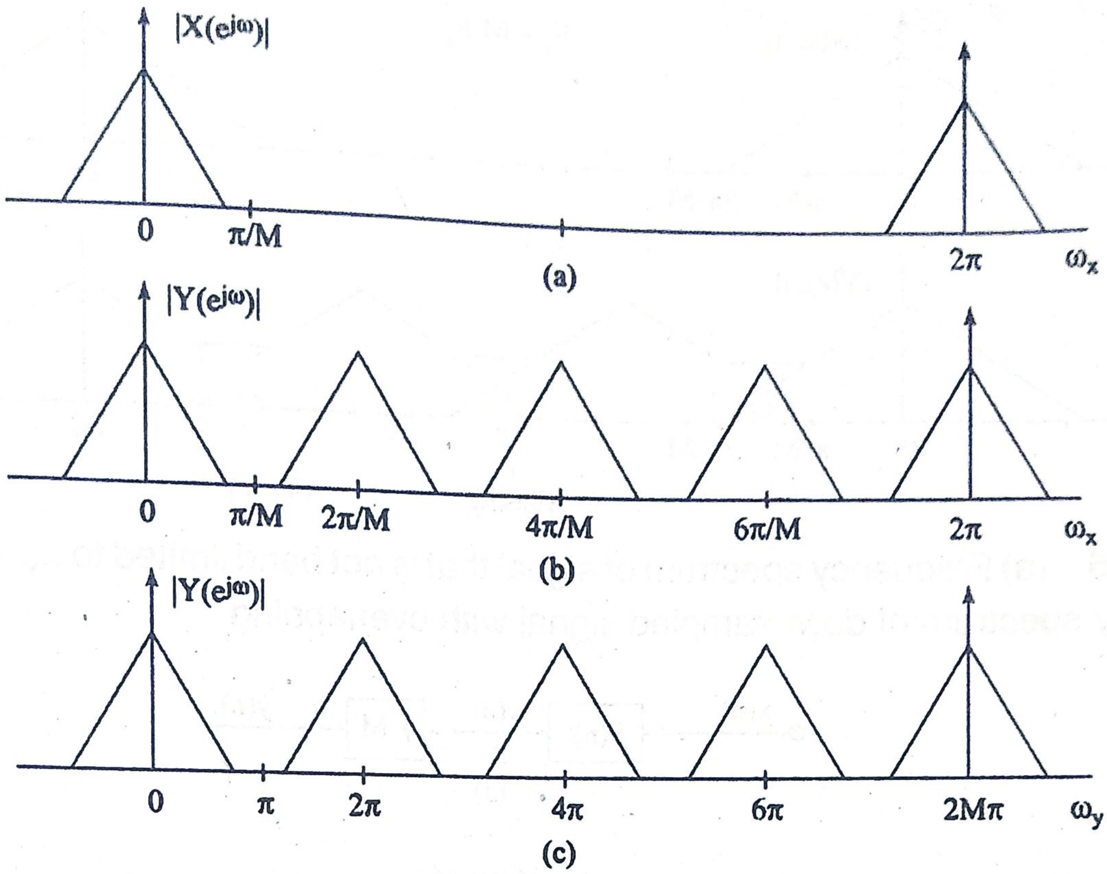
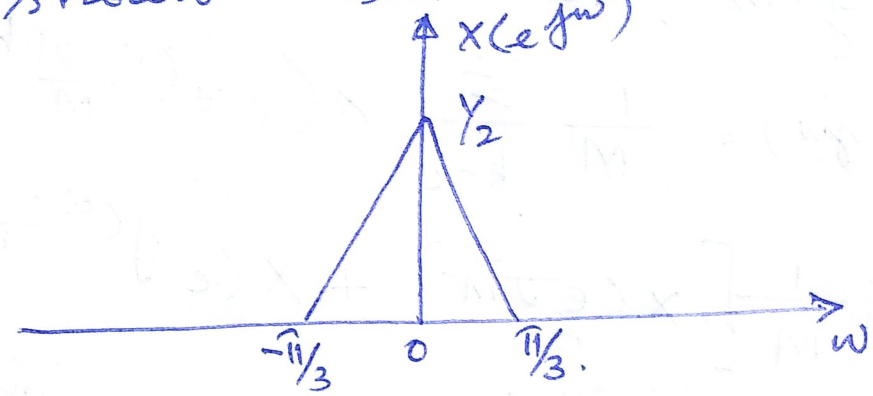


Fig. 8.4 (a) Frequency spectrum of a signal band limited to $\omega = \frac{\pi}{M}$ (b) Frequency spectrum of downsampled signal with respect to ω_x (c) Frequency spectrum of downsampled signal with respect to ω_y

If the spectrum of the original signal $X(e^{j\omega})$ is band limited to $\omega = \frac{\pi}{M}$, the spectrum being periodic with period 2π, the spectrum of the downsampled signal $Y(e^{j\omega})$ according to Eq. (8.12) is the sum of all the uniformly shifted and stretched versions of $X(e^{j\omega})$ scaled by a factor $\frac{1}{M}$. In every interval of 2π, in addition to the original spectrum we find $M-1$ equally spaced replica. Due to the added duplicate spectrum, the period is now $\frac{2\pi}{M}$. In Fig. 8.4b the frequency variable ω_x is related to the original sampling rate. On the otherhand, in Fig. 8.4c the frequency variable ω_y is normalized with respect to reduced sampling rate. The relationship between the original and reduced sampling rates is

$$F_y = MF_x \tag{8.13}$$

Q) Sketch the ~~following~~ frequency response of the output of down sampler (M=3). Frequency response of the input signal is shown below.



Ans: Frequency response of output is given by.

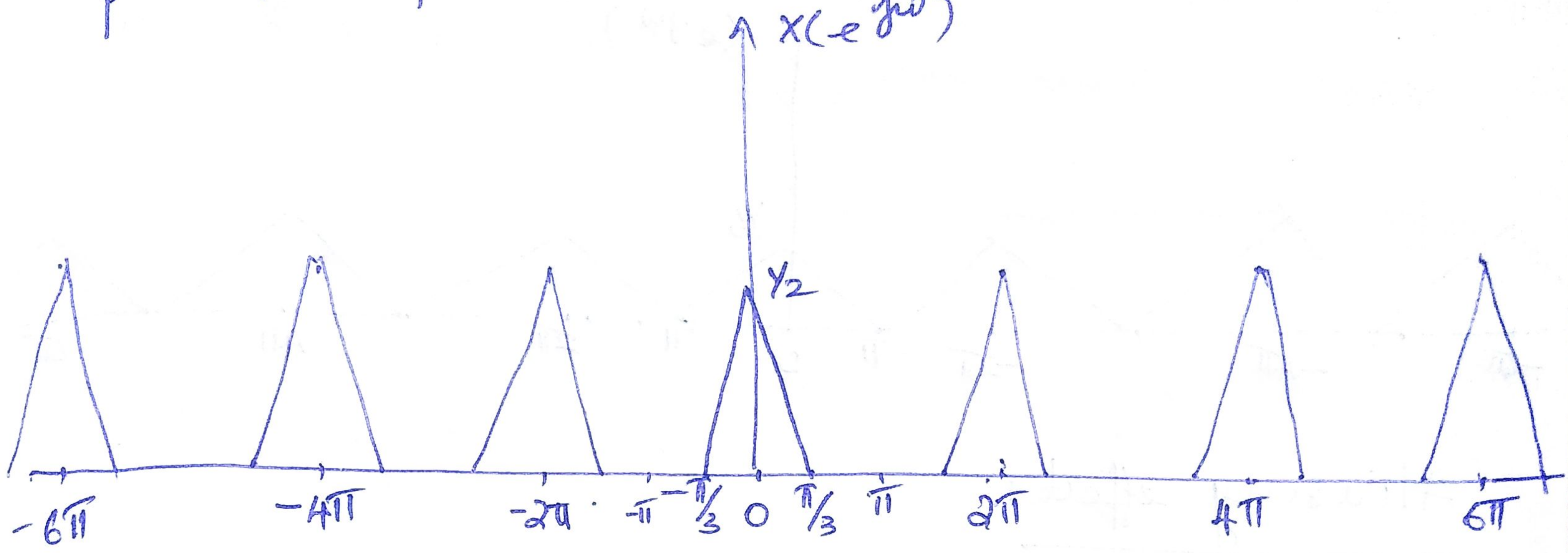
$$Y(e^{j\omega}) = \frac{1}{M} \sum_{k=0}^{M-1} X\left(e^{j\left(\frac{\omega - 2\pi k}{M}\right)}\right)$$

Here $M=3$.

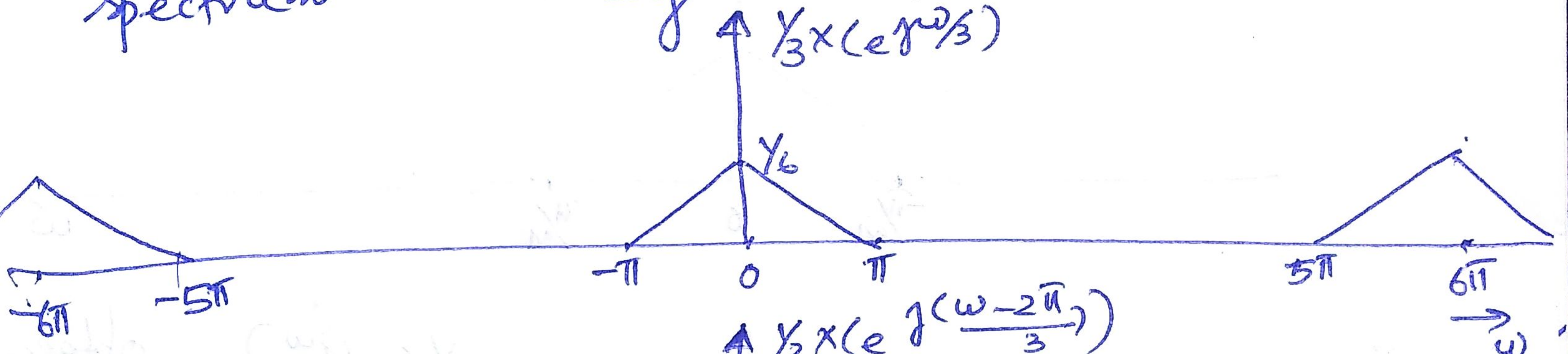
$$\therefore \quad Y(e^{j\omega}) = \frac{1}{3} \sum_{k=0}^{2} X\left(e^{j\left(\frac{\omega - 2\pi k}{M}\right)}\right)$$

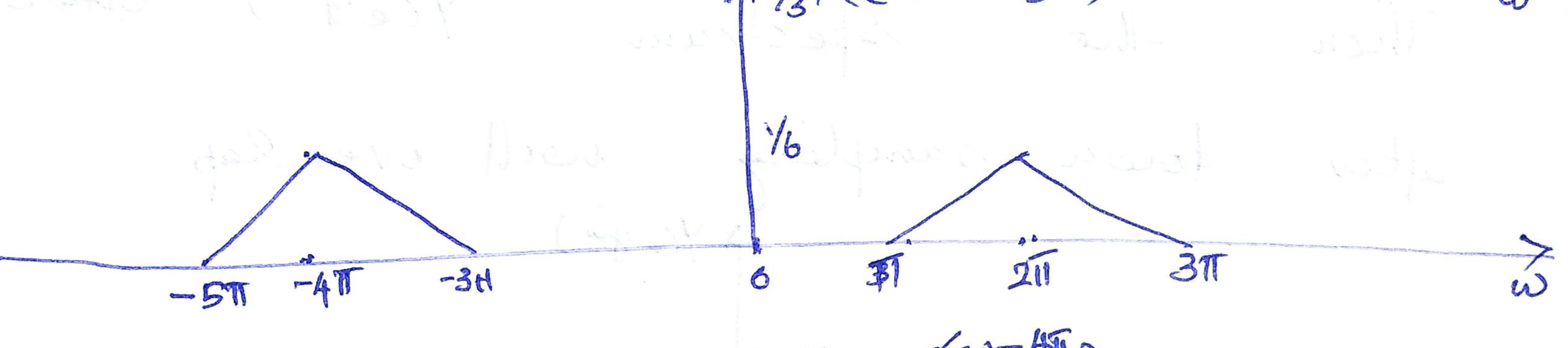
$$= \frac{1}{3}\left[X\left(e^{j\frac{(\omega)}{3}}\right) + X\left(e^{j\left(\frac{\omega-2\pi}{3}\right)}\right) + X\left(e^{j\left(\frac{\omega-4\pi}{3}\right)}\right)\right.$$

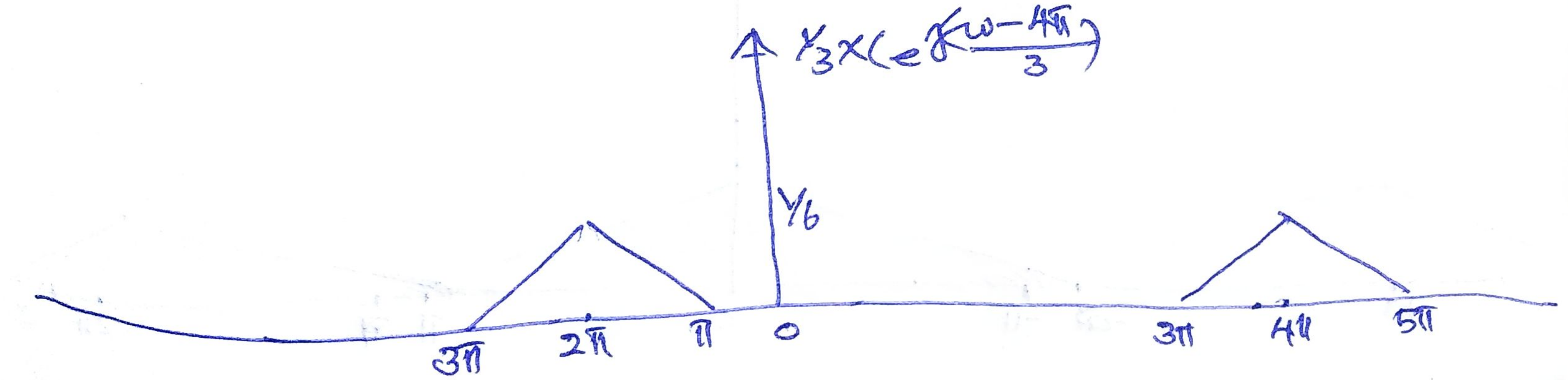
- The input spectrum is periodic with a period of 2π



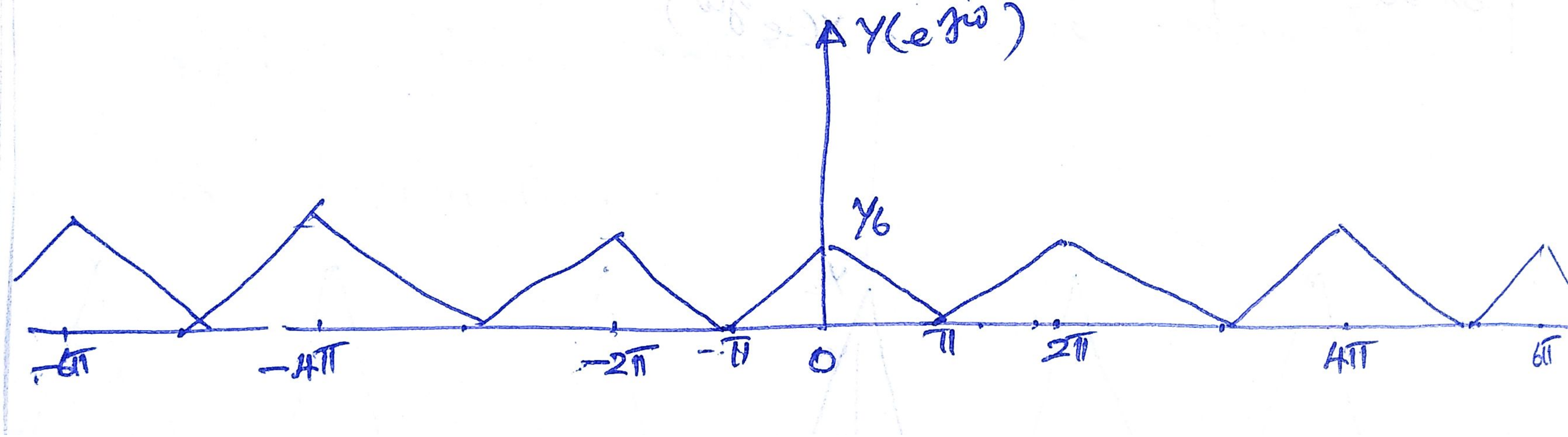
- The output spectrum $Y(e^{j\omega})$ is sum of three shifted and scaled input spectrum they are:







Adding above three spectrum the



Aliasing effect

From Fig. 8.5 we can find that the spectrum obtained after downsampling will overlap if the original spectrum is not band limited to $\omega = \frac{\pi}{M}$. This overlap causes aliasing. Therefore aliasing due to downsampling a signal by a factor of M is absent if and only if the signal $x(n)$ is band limited to $\pm\frac{\pi}{M}$. If the signal $x(n)$ is not band limited to $\pm\frac{\pi}{M}$, then a lowpass filter with cut-off frequency $\frac{\pi}{M}$ is used prior to downsampling as shown in Fig. 8.6a. This filter is known as anti-aliasing filter. Its ideal magnitude response is shown in Fig. 8.6b. The complete process, i.e., filtering and then downsampling is referred to as decimation.

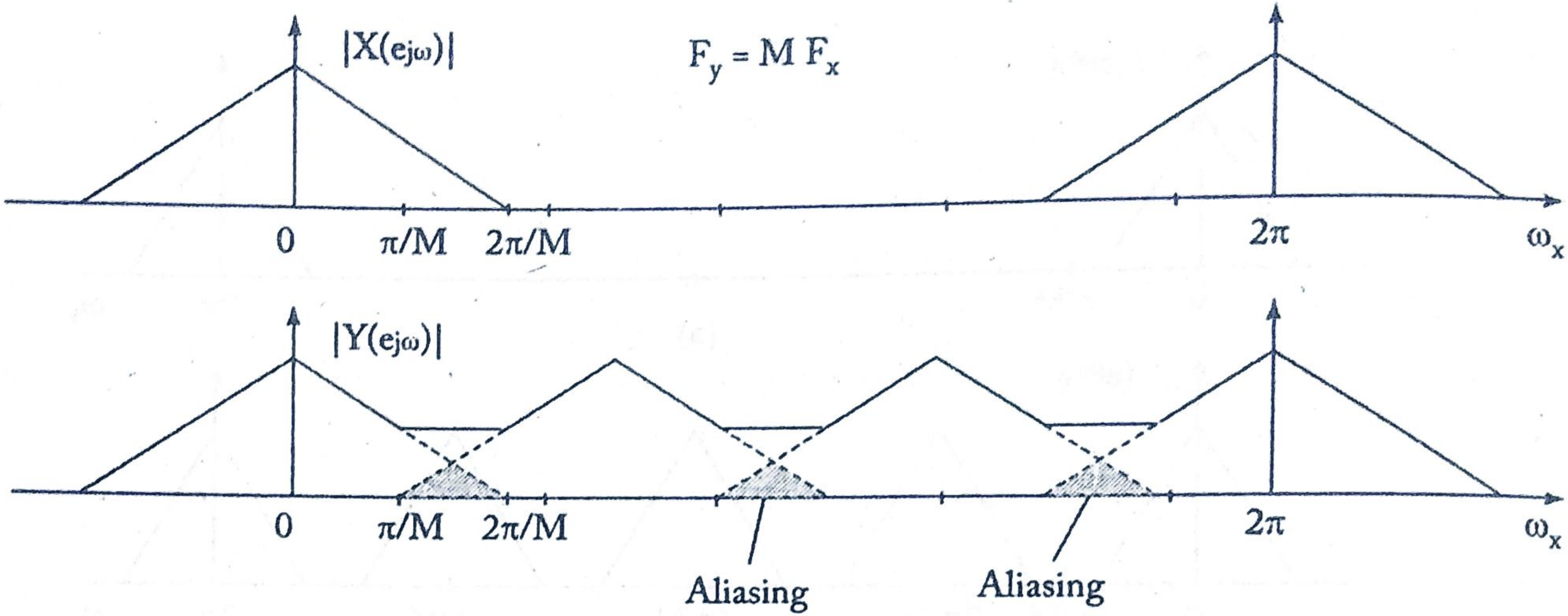
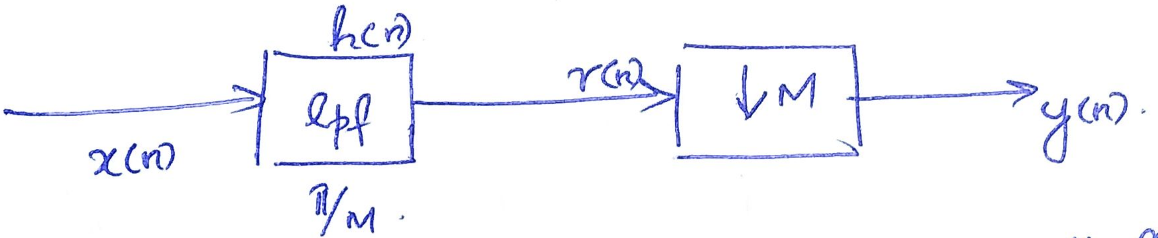


Fig. 8.5 (a) Frequency spectrum of signal that is not band limited to $\omega_c = \frac{\pi}{M}$ (b) Frequency spectrum of downsampled signal with overlapping

This overlapping of high frequency components is known as aliasing effect

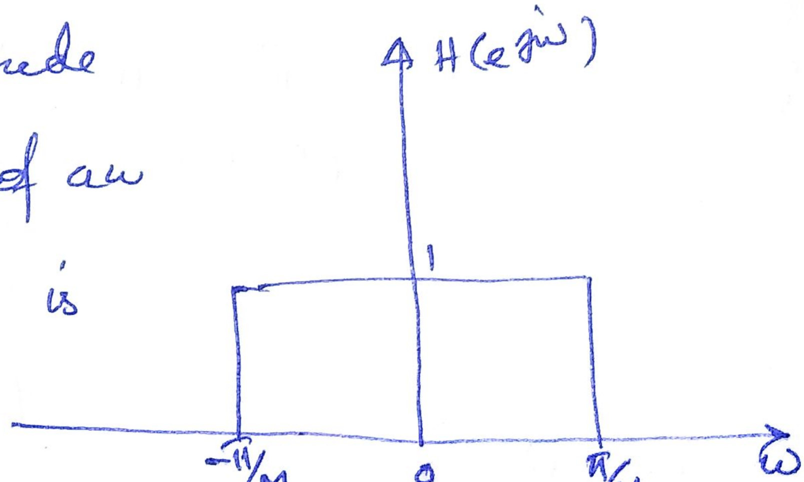
<u>Anti aliasing filter:</u>

If $x(n)$ is not band limited to $\pm \pi/M$, then a ~~lpf~~ lowpass filter (lpf) with cutoff frequency π/M is used prior to the down sampler to remove the high frequency components.



_ This lowpass filter with cutoff frequency π/M is known as anti aliasing filter.

The ideal magnitude response $H(e^{j\omega})$ of an anti aliasing filter is as shown in fig



The complete process of filtering and down sampling is referred to as decimation.

8.4 Upsampling

The sampling rate of a discrete-time signal can be increased by a factor L by placing $L-1$ equally spaced zeros between each pair of samples. Mathematically upsampling is represented by

$$y(n) = \begin{cases} x\left(\frac{n}{L}\right) & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{8.15}$$

and the symbol for upsampler is shown in Fig. 8.12.

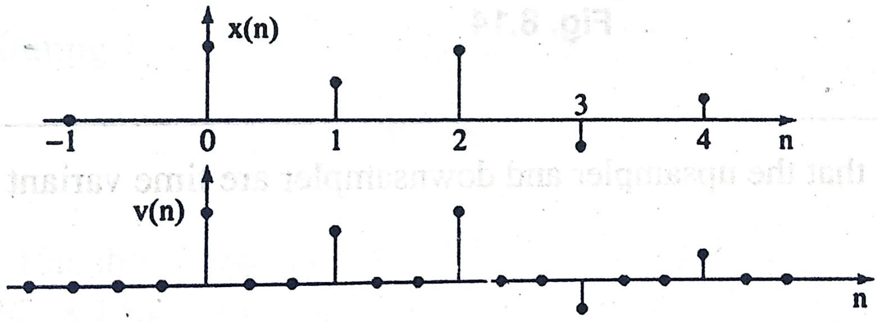


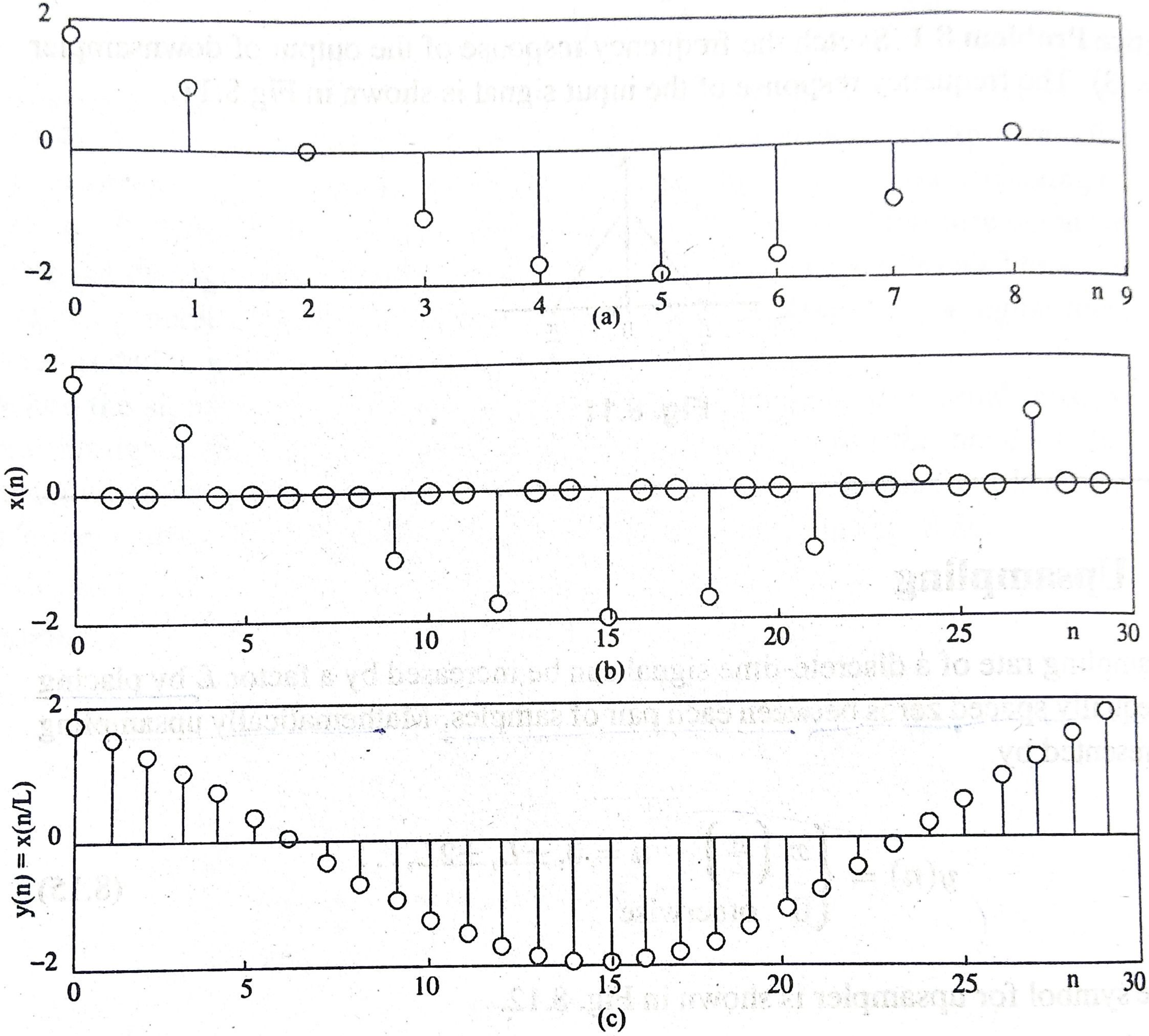
Fig. 8.12

Fig. 8.13 Upsampler

If $x(n) = \{1, 2, 4, -2, 3, 2, 1, \ldots\}$ then the two fold upsampling results in a sequence

$$y(n) = x\left(\frac{n}{2}\right) = \{1, 0, 2, 0, 4, 0, -2, 0, 3, 0, 2, 0, 1, 0, \ldots\}$$

Fig. 8.14 shows the signal $x(n)$ and its three fold upsampled signal $y(n)$. In practice, the zero valued samples inserted by upsampler are replaced with approximate nonzero values using some type of filtering process. This process is called interpolation.



2
0
−2
0
1
2
3
4
5
6
7
8
n
9
(a)
2
x(n)
0
−2
0
5
10
15
20
25
n
30
(b)
2
y(n) = x(n/L)
0
−2
0
5
10
15
20
25
n
30
(c)

Fig. 8.14

8.5 Spectrum of the Upsampled Signal

The z-transform of the signal $y(n)$ is given by

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{L}\right) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} x(n) z^{-Ln} = X(z^L)
\end{aligned}
\tag{8.16}
$$

Substituting $z = e^{j\omega}$ in Eq. (8.16) we get

$$
Y(e^{j\omega}) = X(e^{j\omega L}) \tag{8.17}
$$

Fig. 8.15a shows the spectrum $X(e^{j\omega})$. The spectrum $X(e^{j\omega L})$ is sketched for $L = 3$ in Fig.8.15b. Note that the frequency spectrum $X(e^{j3\omega})$ is three fold repetition

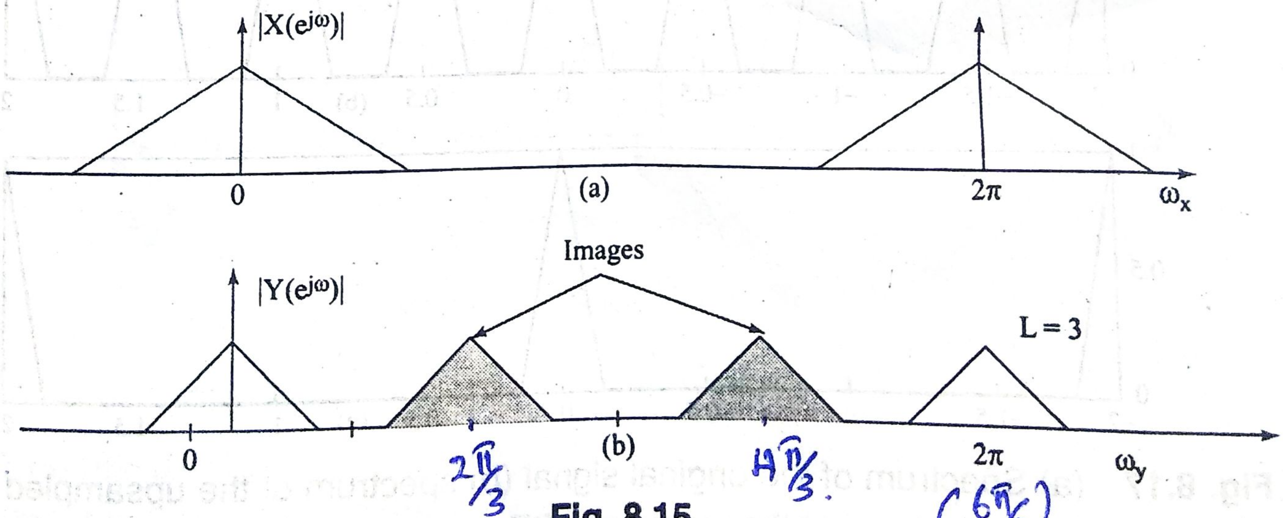


Fig. 8.15

of $X(e^{j\omega})$. That is inserting $L-1$ zeros between successive value of $x(n)$, results in a signal whose spectrum $X(e^{j\omega L})$ is an L-fold periodic repetition of the input signal spectrum $X(e^{j\omega})$.

From Fig. 8.15b we can find that in the interval $-\frac{\pi}{3} < \omega < \frac{\pi}{3}$ the signal $y(n)$ has the same spectrum as $x(n)$ apart from scaling factor. Above this range the spectrum of $y(n)$ has two spectrum of the same form after each one. These are called image spectrum and the phenomena is known as imaging.

8.6 Anti-imaging Filter

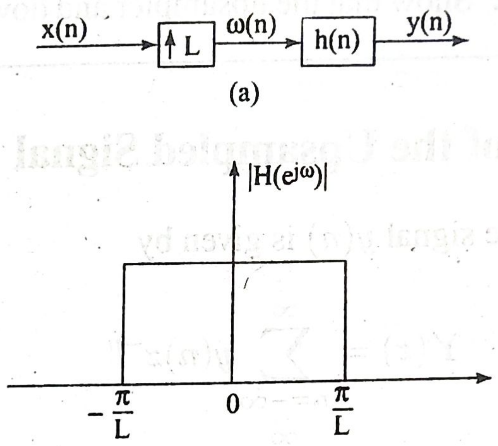


Fig. 8.16 (a)Interpolator with an upsampler and anti-imaging filter (b) Magnitude response of anti-imaging filter.

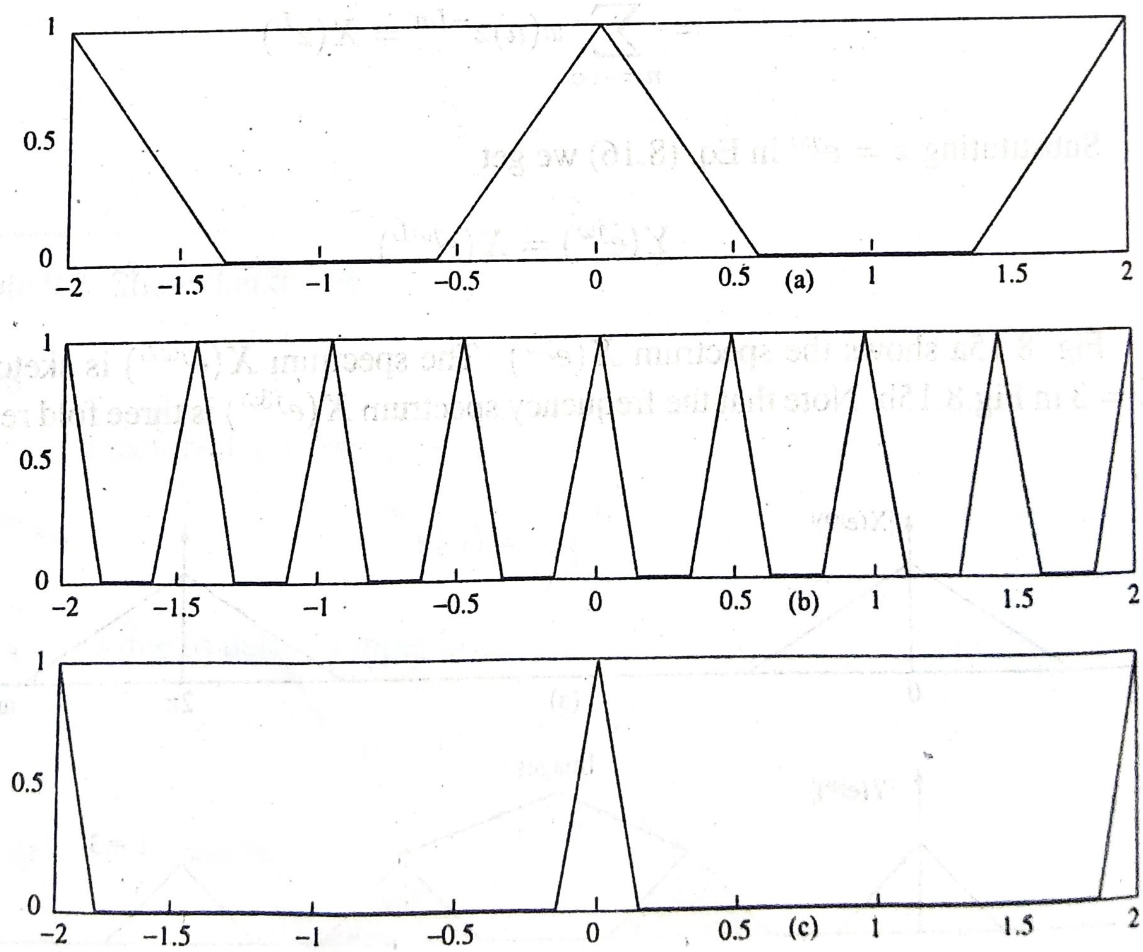


Fig. 8.17 (a) Spectrum of the original signal (b) spectrum of the upsampled signal (c) Spectrum of the signal at the output of LPF.

In previous section we studied that the frequency spectrum of upsampled signal $y(n)$ with a factor L, contains $(L-1)$ additional images of the input spectrum. These $(L-1)$ images are due to the addition of $L-1$ zero samples between successive samples of $x(n)$. Since we are not interested in image spectrum, a lowpass filter with a cutoff frequency $\omega_c = \frac{\pi}{L}$ can be used after upsampler as shown in Fig. 8.16. This filter which is used to remove the image spectrum is known as anti-imaging filter. The complete process of upsampling and filtering is known as interpolation.

Fig. 8.17 shows the spectrum of the original and the spectrum of the signal upsampled by a factor 4. Note that the spectrum of the upsampled signal contains three (in general $L-1$) additional images. When the signal is passed through a lowpass filter with cutoff frequency $\frac{\pi}{L}$ (where $L = 4$ in this case) the resultant spectrum is as shown in Fig. 8.17c.