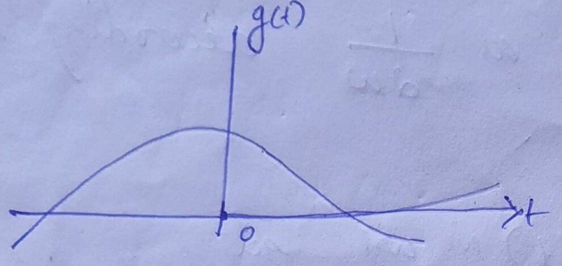
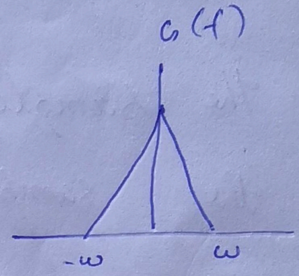
Signal reconstruction & Interpolation formula.

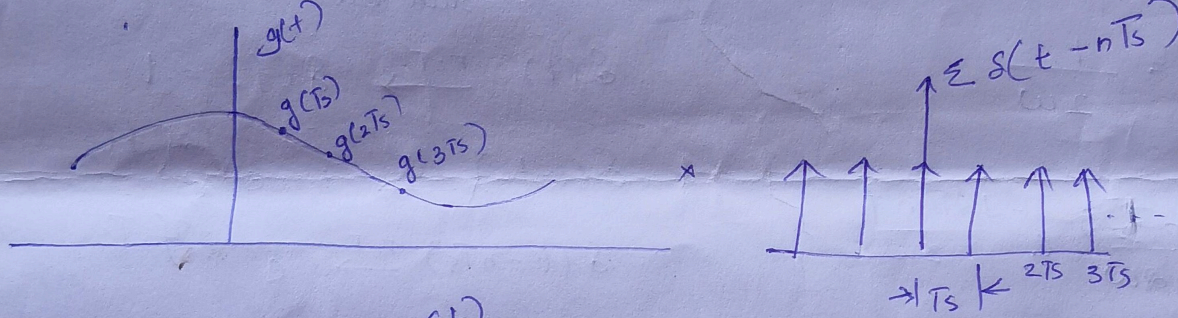
Consider an analog signal $g(t)$ that is continuous in both time and amplitude. This $g(t)$ is bandlimitted to max frequency W Hz

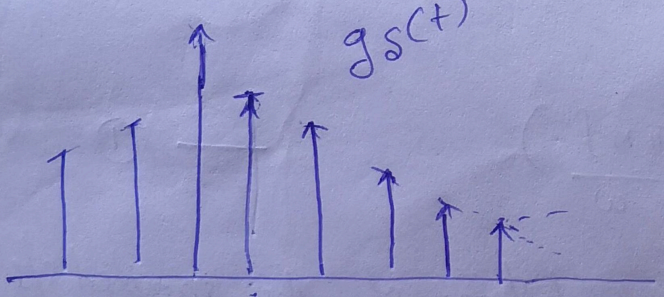


and it's spectrum



Now we are sampling at an interval of 'T_s' seconds by multiplying it with a train of impulses $\Sigma\,\delta(t-nT_s)$





This is the sampled s/l.

In this sampled s/l the amplitude of train of impulses is getting varied according to the amplitude of $g(t)$ at the sampling instants. The sampled s/l is expressed as

$$\boxed{g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t-nT_s)}$$

In this equation we can say that $g(nT_s)$ is the coefficient of $\delta(t-nT_s)$ or coefficient of expansion which is sequence of amplitude or sequence of numbers.

The sampling interval T_s is taken as $\frac{1}{2W}$ according to the sampling theorem

By taking the fourier transform of $g_\delta(t)$ we will get

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g(nT_s)\, e^{-j2\pi n T_s} \longrightarrow$$

w.k.that

$x(t) \xrightarrow{F.T} X(f)$

$x(t-t_0) \xrightarrow{F.T} X(f)\, e^{-j2\pi f t_0}$

such that

Substitute $T_s = \frac{1}{2W}$

$\delta(t) \xrightarrow{F.T} \delta(f) = 1$

$\delta(t-t_0) \xrightarrow{F.T} 1 \cdot e^{-j2\pi f t_0}$

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) e^{-\frac{j2\pi n f}{2W}}$$

$\therefore \delta(t-nT_s) \xrightarrow{F.T} e^{-j2\pi f n T_s}$

$$G_\delta(f) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \exp\left(-\frac{j\pi n f}{W}\right) \quad \text{— ①}$$

This is the frequency domain representation of the sampled s/l.

From the sampling theorem we have got another equation

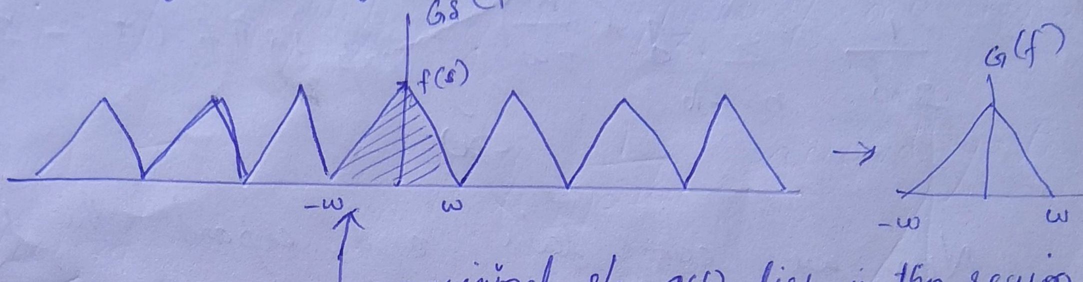
as
$$G_\delta(f) = f_s \sum_{n=-\infty}^{\infty} G(f - nf_s) \quad \text{— ②}$$

from the above summation we are taking only zeroth coefficient to the outside of summation

$$G_\delta(f) = \underset{n=0}{f_s\, G(f)} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} G(f-nf_s) \quad \text{— ③}$$

we have to reconstruct the original signal $g(t)$ from the sampled signal $g_s(t)$; where $g(t)$ is a bandlimited s/l.

Spectrum of $g_s(t)$ is given by



in order to retain this we have to apply 2 conditions in eqn no: 3

(i) $G(f) = 0$ outside the interval $|f| \geq w$

(ii) $f_s = 2w$

$$G_s(f) = f_s\, G(f)$$

$$G_s(f) = 2w\, G(f)$$

$$G(f) = \frac{1}{2w} G_s(f) \quad \text{———} \quad (4)$$

substitute equation no: 1 in eqn no: 4

$$G(f) = \frac{1}{2w} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2w}\right) \exp\left(\frac{-j\pi n f}{w}\right)$$

$$g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi f t)\, df.$$

$$= \int_{-w}^{w} \frac{1}{2w} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2w}\right) \exp\left(\frac{-j\pi n f}{w}\right) \exp(j2\pi f t)\, df$$

$$= \frac{1}{2w} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2w}\right) \int_{-w}^{w} \exp\left(\frac{-j\pi n f}{w}\right) \exp(j2\pi f t)\, df.$$

$g(nT_s)$ → sample values at different time instant

$$= \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \int_{-W}^{W} \exp\left(-\frac{j\pi n f}{W} + j2\pi f t\right) df$$

$$= \frac{1}{2W} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \int_{-W}^{W} \exp j2\pi f\left(t - \frac{n}{2W}\right) df \quad \text{——(5)}$$

Now we can take and expand the itegration part only

$$\int_{-W}^{W} \exp j2\pi f\left(t - \frac{n}{2W}\right) df$$

$\rightarrow$ in the forms of $e^{j\theta}$

$e^{j\theta} = \cos\theta + j\sin\theta$

$$\int_{-W}^{W} \cos 2\pi f\left(t - \frac{n}{2W}\right) + j \sin 2\pi f\left(t - \frac{n}{2W}\right) df$$

$$= 2\int_{0}^{W} \cos 2\pi f\left(t - \frac{n}{2W}\right) df + 0$$

w.k.that

$\int_{-\infty}^{\infty} f(\theta)\, d\theta = 0 \rightarrow$ if the given function is odd.

$\int_{-\infty}^{\infty} f(\theta)\, d\theta = 2\int_{0}^{\infty} f(\theta)\, d\theta \rightarrow$ if the the given function is even.

$\sin\theta$ = odd function

$\cos\theta$ - even function

$$= \cancel{2}\left[\frac{\sin 2\pi f\left(t - \frac{n}{2W}\right)}{\cancel{2}\pi\left(t - \frac{n}{2W}\right)}\right]_{0}^{W}$$

$\sin 0 = 0$; so we can apply upperlimit only.

$$= \frac{\sin 2\pi W\left(t - \frac{n}{2W}\right)}{\pi\left(t - \frac{n}{2W}\right)} \Rightarrow \frac{2W \sin(2\pi W t - n\pi)}{2\pi W t - n\pi}$$

Substitute this integration result in eqn no: 5 ; then

$$g(t) = \frac{1}{\cancel{2W}} \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) \cdot \cancel{2W} \frac{\sin 2\pi W t - n\pi}{2W\pi t - n\pi}$$

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2w}\right) \cdot \frac{\sin 2\pi wt - n\pi}{2\pi wt - n\pi}$$

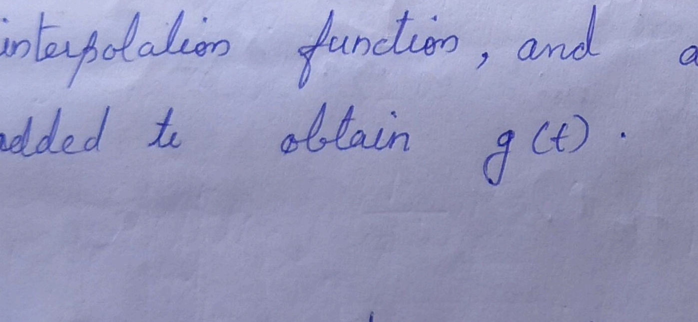
$$\longrightarrow \frac{\sin \pi x}{\pi x} = \text{sinc}\, x$$

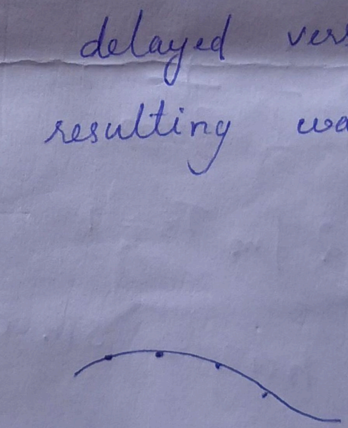
$$\therefore g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2w}\right) \text{sinc}\,(2wt - n)$$

This is known as interpolation formula for reconstructing the original s/l $g(t)$ from the sequence of sample values $g(n/2w)$; with the sinc function playing the role of an interpolation function.

$$\text{Sinc}\, x = \begin{cases} 1 & x = 0 \\ 0 & x = \pm 1, \pm 2 \end{cases} \Rightarrow \text{interpolatory property}$$

Each sample is multiplied by a delayed version of the interpolation function, and all the resulting waveforms are added to obtain $g(t)$.





The process of reconstructing a continuous time s/l $g(t)$ from its samples is called as interpolation. So a signal $g(t)$ band limited to f_m Hz or wHz can be reconstructed completely from its samples. This is achieved by passing the sampled s/l through an ideal low-pass filter of cut-off frequ. f_m Hz or wHz.

gain of the low pass filter will be T_s

where $T_s = \frac{1}{2W} = \frac{1}{2f_m}$ → [Assuming sampling is done at Nyquist rate

Reconstruction and interpolation transfer function may be expressed as

$$H(\omega) = T_s \times rect\left[\frac{\omega}{4\pi f_m}\right]$$

The impulse response $h(t)$ of this filter is the inverse fourier transform of $H(\omega)$

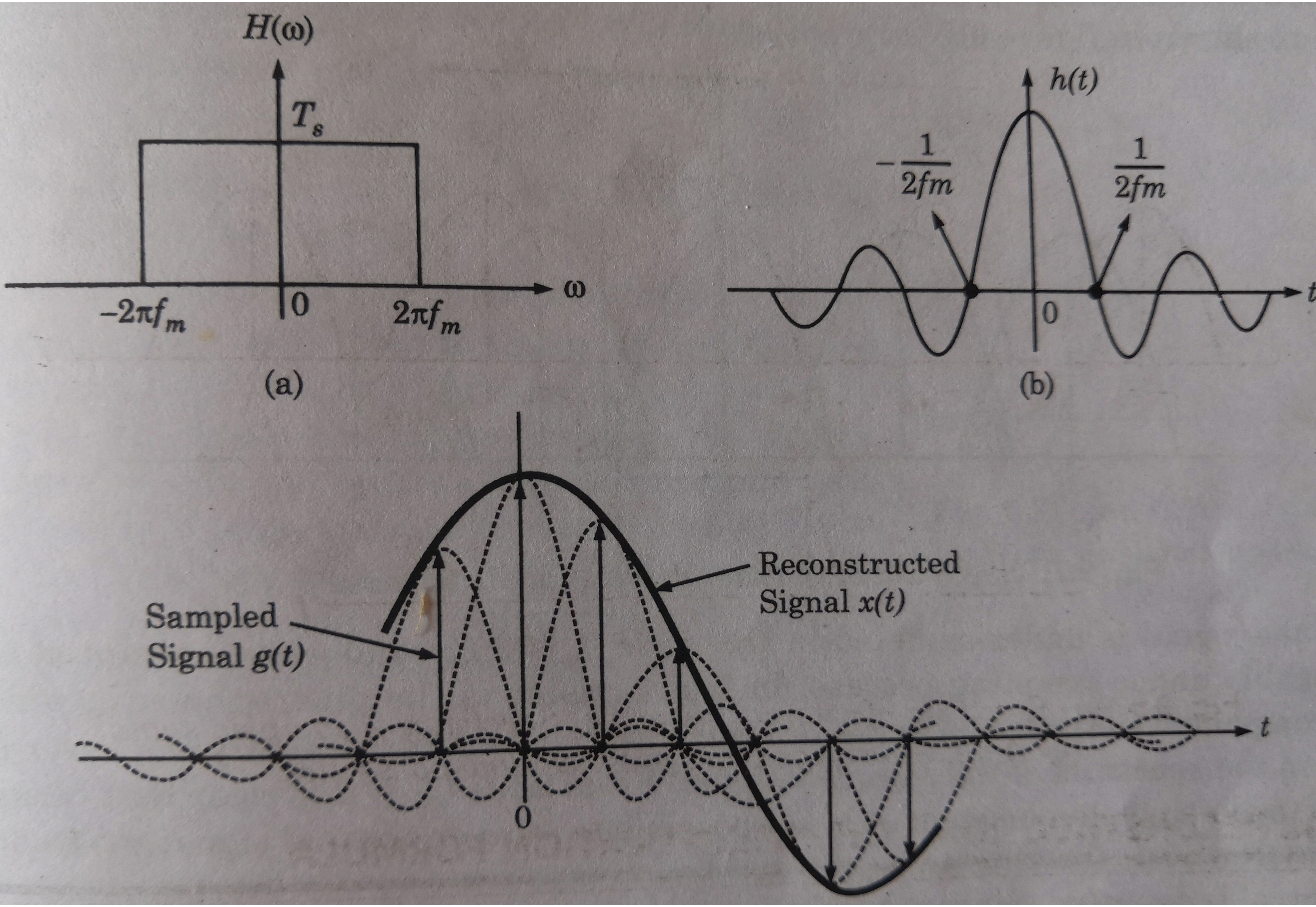
$$h(t) = F^{-1}[H(\omega)] = F^{-1}\left[T_s\, rect\left(\frac{\omega}{4\pi f_m}\right)\right]$$

$$h(t) = 2f_m T_s \,Sinc(2\pi f_m t) \quad \text{—— (A)}$$

$$T_s = \frac{1}{2f_m}$$

$2f_m T_s = 1$ → putting this in eqn A

$$h(t) = sinc(2\pi f_m t)$$


H(ω)
T_s
−2πf_m
0
2πf_m
ω
(a)
h(t)
−1/2fm
1/2fm
0
t
(b)
Reconstructed
Signal x(t)
Sampled
Signal g(t)
0
t