LIKELIHOOD FUNCTION

We learnt received vector/observation vector

X =

Cov of X_j's = 0 .

Eg: Let cov (X1,X2) = 0 means they are uncorrelated or statistically independent

If 2 fns are independent, joint prob density fn = pdt of p.d.f of its individual elements.

Joint prob density fn of vector X = $f_X()$

Conditional pdf of vector X = $f_X(s_{ij})$ or $f_X(_i)$

Source emits m_i symbols where i varies from 1,…,

M. (m1,m2,…,m_M)

This m_i is encoded to obtain the w/f $s_i(t)$.

m1 ----$s_1(t)$

m2 ----$s_2(t)$, etc

Since the elements of X are indep.,the conditional pdf of X (given that s/g $s_i(t)$ or symbol m_i was transmitted) is the conditional pdf of its individual elements.

$f_X(_i) = f_{X1}(_1)\ f_{X2}(_2)........\ f_{XN}(_N)$

$f_X(_i) =$ $\quad$ i=1,2,...,M -----(4)

X_j is a Gaussian RV with mean S_{ij} and variance = $N_0/2$.

If Y is a Gaussian RV, then pdf

X_j is a Gaussian R.V. as it contains AWGN.

If Y is a Gaussian RV, then pdf is as fo

$$f_Y(y) = \frac{1}{\sigma_y\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma_y^2}}$$

$$f_{x_j}(x) = f_{x_j}(x_j|m_i) = \frac{1}{\sigma_{x_j}\sqrt{2\pi}}\, e^{-\frac{(x_j-\mu_{x_j})^2}{2\sigma_{x_j}^2}}$$

$$= \frac{1}{\sqrt{2\pi \times \frac{N_0}{2}}}\, e^{-\frac{(x_j-s_{ij})^2}{2\times N_0/2}}$$

$$f_{x_j}(x_j|m_i) = \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{(x_j-s_{ij})^2}{N_0}}$$

$$4 \Rightarrow f_X(x|m_i) = \prod_{j=1}^{N} \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{(x_j-s_{ij})^2}{N_0}}$$ from eqn 4.

$$= (\pi N_0)^{-N/2}\, e^{\sum_{j=1}^{N} -\frac{(x_j-s_{ij})^2}{N_0}}$$

$$= (\pi N_0)^{-N/2} \, e^{\sum_{j=1}^{N} -\frac{(x_j - s_{ij})^2}{N_0}}$$

$$f_X(x|m_i) = (\pi N_0)^{-N/2} \, e^{\left[-\frac{1}{N_0} \sum_{j=1}^{N} (x_j - s_{ij})^2\right]}$$

★ Conditional pdf of vector x

$$f_X(x|m_i) = (\pi N_0)^{-N/2} \, e^{\left[-\frac{1}{N_0} \sum_{j=1}^{N} (x_j - s_{ij})^2\right]}$$

$f_X(x|m_i)$ depends on s_{ij} or m_i.

Likelihood function $L(m_i)$

- Used to decode s/gs properly

- Rather than finding likelihood function, it is convenient to use log likelihood fn.
- Log likelihood fn.

$$l\left(m_i\right) = \log L\left(m_i\right)$$

- = log

  = log { }

  = log +]

log   is indep of msg s/g /symbol. So we neglect this term.

i from 0 to M

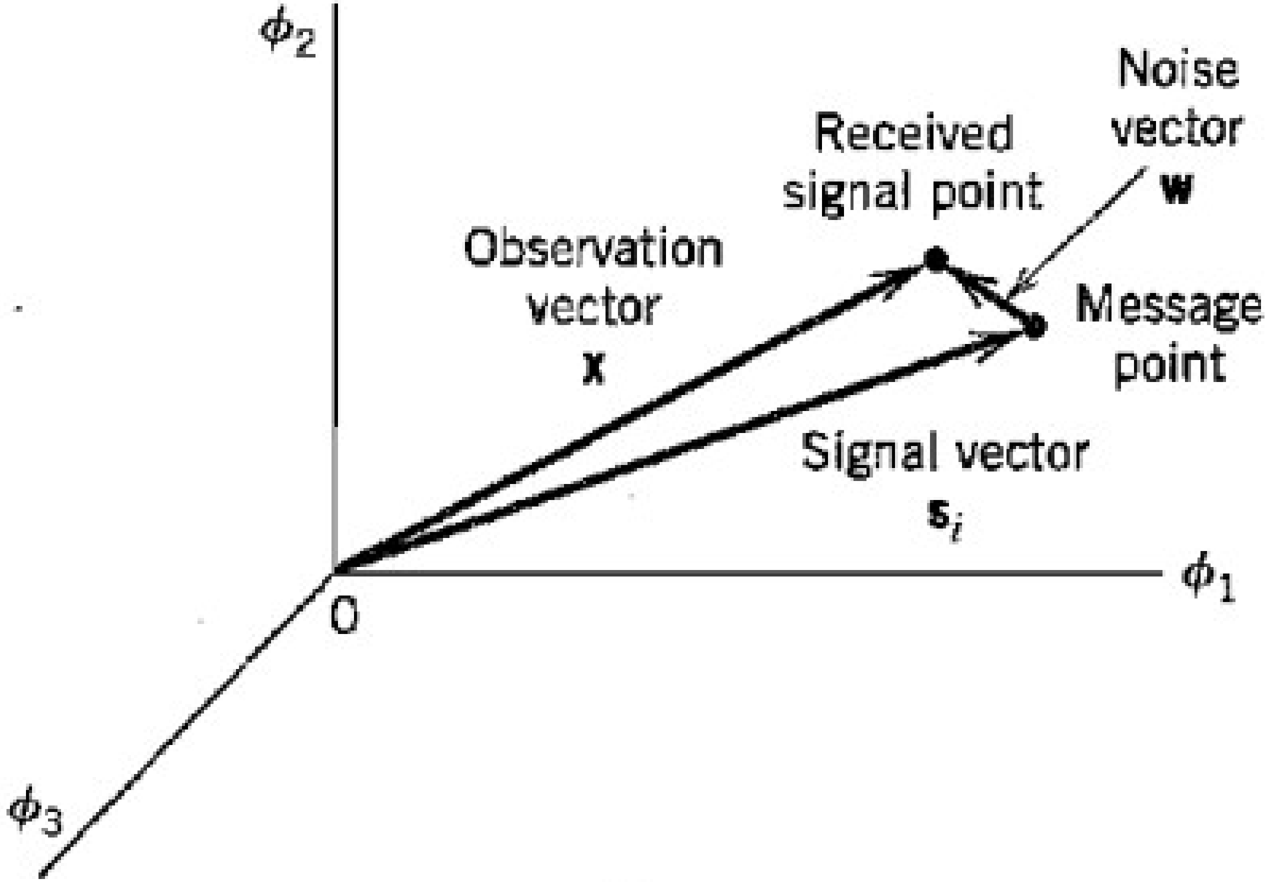
To get error free decoding or to minimise error, log likelihood fn shud be maximized.

MAXIMUM LIKELIHOOD DECODING

- Used for coherent detection of s/gs in presence of noise.
- We have a source which emits m_i symbols. i =1, …, M
- Every symbol is emitted with equal probability.
- $P(m_i)$ =
- m_i is encoded to $s_i(t)$ and transmitted thru AWGN channel
- Rxd s/g x(t).

We can represent $S_i(t)$ as a vector in Euclidean space.

- Signal vector s_i =
- Coordinates -
- Observation/rxd vector =


ϕ_2
Noise vector
$\mathbf{w}$
Received signal point
Observation vector
$\mathbf{x}$
Message point
Signal vector
$\mathbf{s}_i$
ϕ_1
O
ϕ_3

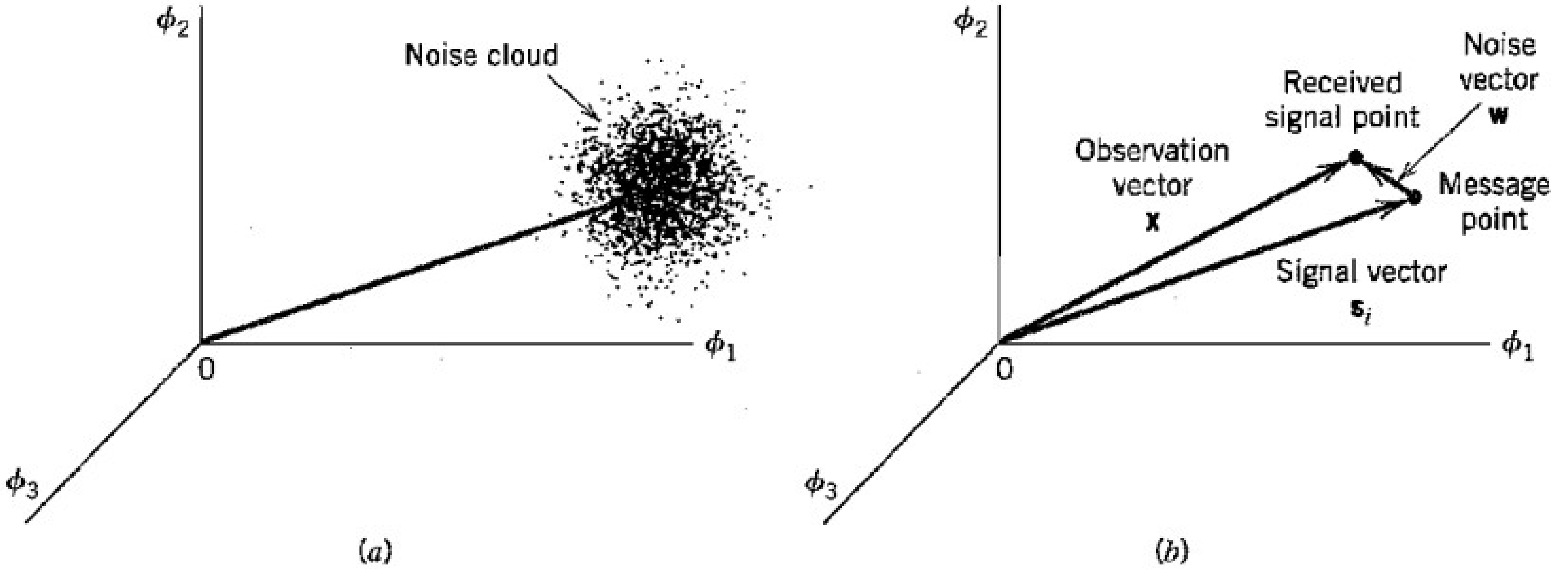


FIGURE 5.7 Illustrating the effect of noise perturbation, depicted in (*a*), on the location of the received signal point, depicted in (*b*).

- Noise cloud / Gaussian distributed cloud.
- Aim is to set a decoding method to detect the msg symbol from the rxd s/g.
- Detection should be error free or with least error.
- Now we will state the detection problem.
- Given the observation vector perform a mapping from to an estimate of the txd s/g, so that error prob shud be min.
- Suppose given the observation vector we make the decision $_i$. The prob of error in this

$$P_e(m_i|\,\mathbf{x}) = P(m_i \text{ not sent}\,|\,\mathbf{x})$$
$$= 1 - P(m_i \text{ sent}\,|\,\mathbf{x})$$

- Decision making criterion is to minimize the prob of error in mapping each given obs vector x into a decision.
- Inorder to minimize $P_e(m_i|x)$, we have to maximize $P(m_i \text{ sent}|x)$.

So we may state the **optimum decision rule** :

Set $_i$ if $P(m_i$ sent$|)$ $P(m_k$ sent$|)$ for all k

That is for eg: if 4 symbols m_1, m_2, m_3 and m_4 are txd.

Then we will find each of the probs

ie, $P(m_1$ sent$|)$

$P(m_2$ sent$|)$

$P(m_3$ sent$|)$

$P(m_4$ sent$|)$

Select the highest prob case. Say if $P(m$ sent$|)$

Set $_i$ if $P(m_i \text{ sent} | \ ) \ P(m_k \text{ sent} | \ )$

for all k

----------(1)

- This decision rule is known as **Maximum a posteriori probability rule (MAP rule).**
- According to Bayes theorem ,

$$P(A/B) =$$

Apply Bayers rule to $P(m_i \text{ sent} | x)$

$$P(m_i | \ ) = \quad \text{---------(2)}$$

Set $_i$

if $P(m_i \text{ sent} | \ )$

- Applying Bayers rule and restating eqn(1) interms of prob density fn. , we get

Set $_i$

------(3)

Where p_k is the a priori prob of transmitting symbol $m_{k,}$

is the conditional pdf of random obs vector X given the txn of symbol m_k and) is the unconditional pdf of X.

In eqn (3)

-) is indep of txd symbol
- $p_k = p_i$ when all source symbols are txd with equal prob. It is a constant.

So we can write,

Set $_i$

for k=i. -----(4)

- We know that Likelihood fn
- So the rule can be restated as

Set i

for k=i. ------(5)

It is convenient to use log likelihood fn. So

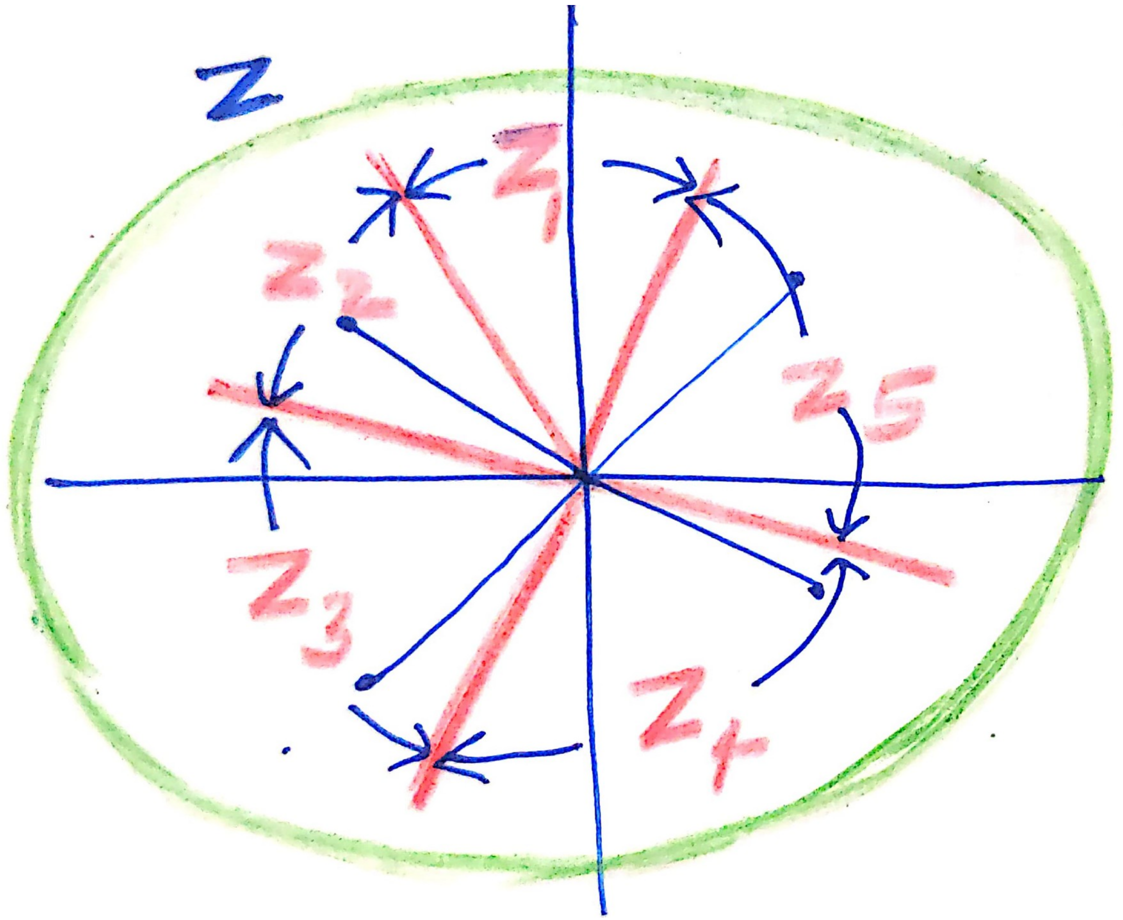
Set i

for k=i. -----(6)

- This 6^{th} eqn is called as maximum likelihood rule and the device for implementing this rule is called max likelihood decoder.

Graphical interpretation of max likelihood decision rule

Let Z denote N dimensional space for representing all possible obs vectors x and this space is called as observation space.



m_1 --- x lies in z_1
m_2 --- x lies in z_2
m_3 --- x lies in z_3
m_4 --- x lies in z_4

- Detection method:
- If obs vector lies in Z_1, then we can say that txd symbol is $m_{1.}$
- If obs vector lies in Z_2, then we can say that txd symbol is $m_{2.}$ and so on.
- Let total obs space Z is divided into M decision region $Z_1, Z_2, \ldots, Z_M$, the eqn (6) rule can be restated as

Observation vector x lies in region Z_i

for k=i. --------(7)

- We know that

We need to maximize

term should be –ve term so that becomes +ve. That means we need to minimize the summation term.

7^{th} rule is restated as

Observation vector x lies in region Z_i

for k=i. --------(8)

- We know that

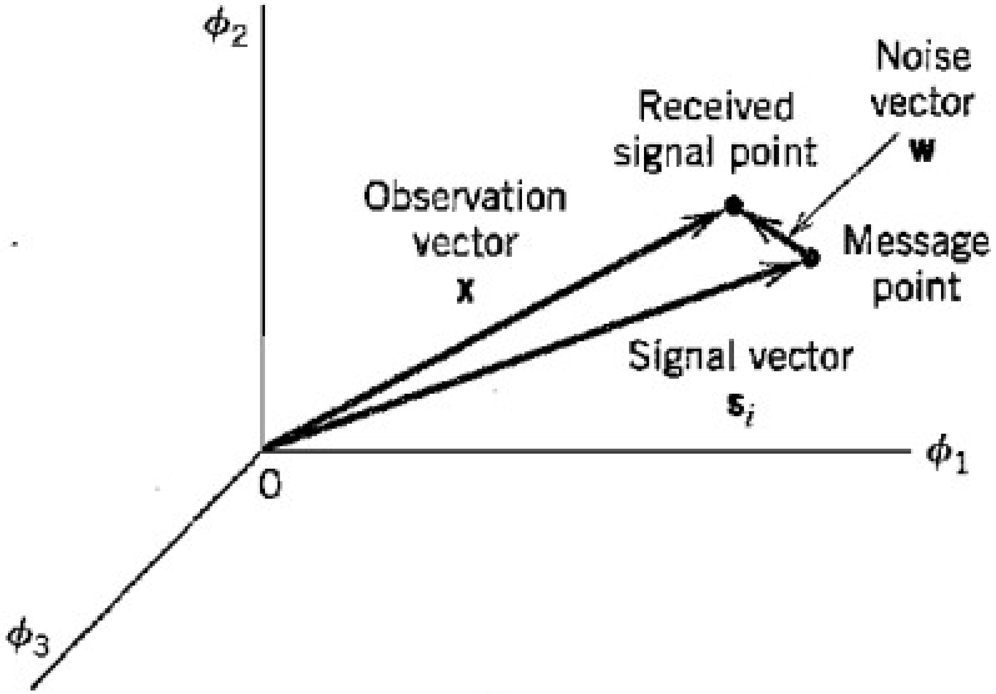
$|^2$

Where $|$ is the Euclidean distance btw rxd s/g pt and msg pt.

8^{th} rule can be restated as

Observation vector x lies in region Z_i

for k=i. -------(9)



- Eqn (9) states that ML decision rule is simply to choose the msg pt closest to the received s/g point.
- We need to minimize

- E_k is the energy of the transmitted s/g.
- The first summation of this expansion is indep of index k and therefore may be ignored. This term contains only rxd vector.
- We need to minimise
- We need to make it (–ve) term.

- It is eqvt to maximizing

8^{th} rule can be restated as

Observation vector x lies in region Z_i

for k=i. -------(10)

is the inner product of obs vector x and s/g vector .

This rule is used to implement ML decoder in correlation receiver.

Fig. shows eg: of decision regions for M=4 s/gs and N=2 dimensions, assuming that s/gs are txd with equal energy E and equal probability.

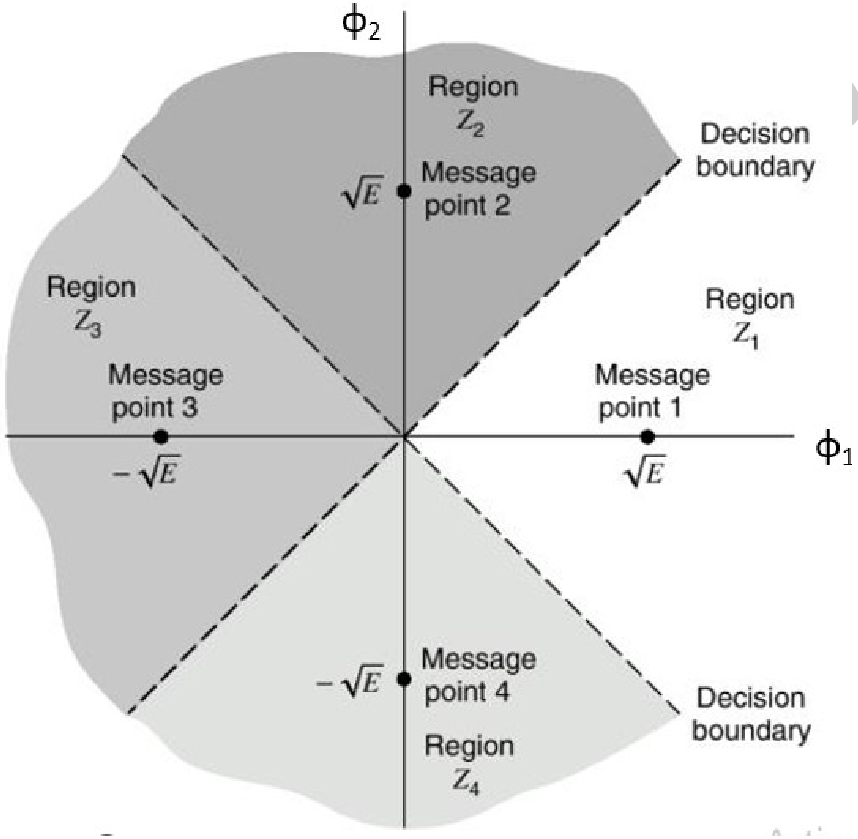


FIGURE 5.8 Illustrating the partitioning of the observation space into decision regions for the case when $N = 2$ and $M = 4$; it is assumed that the M transmitted symbols are equally likely.