Example 5.1 Given the specification $\alpha_p = 1\,\text{dB}$; $\alpha_s = 30\,\text{dB}$; $\Omega_p = 200\,\text{rad/sec}$; $\Omega_s = 600\,\text{rad/sec}$. Determine the order of the filter.

Solution

From Eq. (5.25)

$$A = \frac{\lambda}{\varepsilon} = \left(\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right)^{0.5}$$

$$= \left(\frac{10^{3} - 1}{10^{0.1} - 1}\right)^{0.5} = 62.115$$

From Eq. (5.26)

$$k = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = \frac{1}{3}$$

From Eq. (5.27)

$$N \geq \frac{\log A}{\log 1/k}$$

$$\geq \frac{\log 62.115}{\log 3} = 3.758$$

Rounding off N to the next higher integer we get $N = 4$.

Example 5.2 Determine the order and the poles of lowpass Butterworth filter that has a 3 dB attenuation at 500 Hz and an attenuation of 40 dB at 1000 Hz.

Solution

Given data $\alpha_p = 3\,\text{dB}$; $\alpha_s = 40\,\text{dB}$; $\Omega_p = 2 \times \pi \times 500 = 1000\pi\,\text{rad/sec}$.
$\Omega_s = 2 \times \pi \times 1000 = 2000\pi\,\text{rad/sec}$.

The order of the filter

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\dfrac{10^4 - 1}{10^{0.3} - 1}}}{\log \dfrac{2000\pi}{1000\pi}} = 6.6$$

Rounding 'N' to nearest higher value we get $N = 7$.

The poles of Butterworth filter are given by

$$s_k = \Omega_c e^{j\phi_k} = 1000\pi e^{j\phi_k} \quad k = 1, 2, \ldots 7$$

where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2, \ldots 7.$

Example 5.3 Prove that $\Omega_c = \dfrac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \dfrac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$

Solution

The magnitude square function of Butterworth analog lowpass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}} \tag{5.28}$$

From Eq. (5.15b) we know

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}}$$

Comparing Eq. (5.15b) and Eq. (5.28) we get

$$1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N} = 1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N} = \left(\frac{\Omega}{\Omega_c}\right)^{2N} \tag{5.29}$$

Simplifying above Eq. (5.29) by substituting Eq. (5.17) we obtain

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{0.1\alpha_p} - 1 \tag{5.30}$$

Further simplifying Eq. (5.30) we get

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} \tag{5.31}$$

$$= \frac{\Omega_p}{\varepsilon^{1/N}} \tag{5.32}$$

From Eq. (5.19) we have

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

$$\Omega_s = \Omega_p \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]^{1/2N}$$

$$= \Omega_c(10^{0.1\alpha_p} - 1)^{1/2N} \cdot \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}\right]^{1/2N}$$

$$\Rightarrow \quad \Omega_c = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}} \tag{5.32a}$$

Therefore

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$$

Example 5.10 For the given specifications $\alpha_p = 3\,\text{dB}$; $\alpha_s = 15\,\text{dB}$; $\Omega_p = 1000\,\text{rad/sec}$ and $\Omega_s = 500\,\text{rad/sec}$ design a highpass filter.

Solution

First we design a normalized lowpass filter and then use suitable transformation to get the transfer function of a highpass filter.

For lowpass filter	For highpass filter
$\Omega_c = \Omega_p = 500\,\text{rad/sec}$	$\Omega_c = \Omega_p = 1000\,\text{rad/sec}$
$\Omega_s = 1000\,\text{rad/sec}$	$\Omega_s = 500\,\text{rad/sec}$

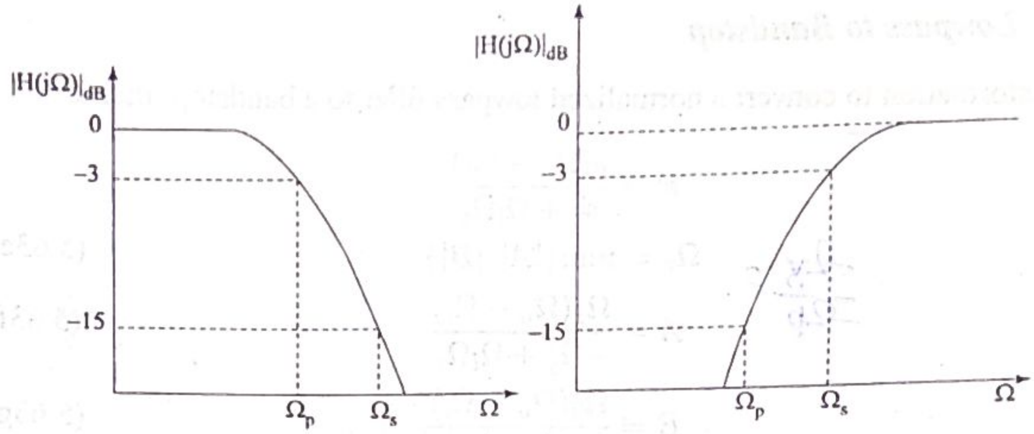

Fig. 5.17 Lowpass to highpass transformation

Lowpass filter specifications

$$\Omega_c = \Omega_p = 500\,\text{rad/sec}; \quad \alpha_p = 3\,\text{dB}$$
$$\Omega_s = 1000\,\text{rad/sec}; \quad \alpha_s = 15\,\text{dB}$$

We have

$$N = \frac{\log \dfrac{\lambda}{\varepsilon}}{\log 1/k}$$
$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = 5.533$$
$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = 1$$
$$k = \frac{\Omega_p}{\Omega_s} = 0.5$$

Therefore $N = \frac{\log 5.533}{\log 2} = 2.468$. Approximating to next higher integer we have $N = 3$.

$H(s)$ for $\Omega_c = 1$ rad/sec and $N = 3$ is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

To get highpass filter having cutoff frequency

$$\Omega_c = \Omega_p = 1000\,\text{rad/sec}$$

Substitute $s \to \dfrac{1000}{s}$

$$H_a(s) = H(s)\Big|_{s \to \frac{1000}{s}}$$
$$= \frac{1}{(s+1)(s^2+s+1)}\Bigg|_{s \to \frac{1000}{s}}$$
$$= \frac{s^3}{(s+1000)[s^2+1000s+(1000)^2]}$$

Example 5.11 For the analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$ determine $H(z)$ using impulse invariance method. Assume $T = 1$ sec.

Solution

Given $H(s) = \dfrac{2}{(s+1)(s+2)}$

Using partial fraction we can write

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2}$$
$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$
$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

$$\boxed{\begin{aligned} A &= (s+1)\frac{2}{(s+1)(s+2)}\Big|_{s=-1} \\ &= 2 \\ B &= (s+2)\frac{2}{(s+1)(s+2)}\Big|_{s=-2} \\ &= -2 \end{aligned}}$$

Using impulse invariance technique we have, if

$$H(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k} \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_kT}z^{-1}}$$

i.e., $(s - p_k)$ is transformed to $1 - e^{p_kT}z^{-1}$.

There are two poles $p_1 = -1$ and $p_2 = -2$. So

$$H(z) = \frac{2}{1-e^{-T}z^{-1}} - \frac{2}{1-e^{-2T}z^{-1}}$$

For $T = 1$ sec

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}}$$
$$= \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.1353z^{-1}}$$
$$H(z) = \frac{0.465z^{-1}}{1-0.503z^{-1}+0.04976z^{-2}}$$

Example 5.12 Using impulse invariance with $T = 1$ sec determine $H(z)$ if $H(s) = \dfrac{1}{s^2+\sqrt{2}s+1}$

Solution

Given $H(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s^2 + \sqrt{2}s + 1}\right]$$

$$= L^{-1}\left[\frac{1}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= L^{-1}\left[\sqrt{2} \cdot \frac{\frac{1}{\sqrt{2}}}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= \sqrt{2}L^{-1}\left[\frac{\frac{1}{\sqrt{2}}}{\left(s + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\right] = \sqrt{2}e^{-t/\sqrt{2}}\sin(t/\sqrt{2})$$

Let $t = nT$

$$h(nT) = \sqrt{2}\, e^{-nT/\sqrt{2}} \sin\frac{nT}{\sqrt{2}}$$

If $T = 1$ sec

$$h(n) = \sqrt{2}\, e^{-n/\sqrt{2}} \sin\frac{n}{\sqrt{2}}$$

$$H(z) = Z[h(n)] = \sqrt{2}\left[\frac{e^{-1/\sqrt{2}}z^{-1}\sin 1/\sqrt{2}}{1 - 2e^{-1/\sqrt{2}}z^{-1}\cos\frac{1}{\sqrt{2}} + e^{-\sqrt{2}}z^{-2}}\right]$$

$$= \frac{0.453z^{-1}}{1 - 0.7497z^{-1} + 0.2432z^{-2}}$$

Example 5.13 Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period $T = 1$ sec.

Solution

From the table 5.1, for $N = 3$, the transfer function of a normalised Butterworth filter is given by

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{A}{s+1} + \frac{B}{s + 0.5 + j0.866} + \frac{C}{s + 0.5 - j0.866}$$

$$A = (s+1)\frac{1}{(s+1)(s^2+s+1)}\bigg|_{s=-1} = \frac{1}{(-1)^2-1+1} = 1$$

$$B = (s+0.5+j0.866)\frac{1}{(s+1)(s+0.5+j0.866)(s+0.5-j0.866)}\bigg|_{s=-0.5-j0.866}$$

$$= \frac{1}{(-0.5-j0.866+1)(-j0.866-j0.866)}$$

$$= \frac{1}{-j1.732(0.5-j0.866)} = \frac{1}{-j0.866-1.5}$$

$$= \frac{-1.5+j0.866}{3} = -0.5+j0.288$$

$$C = B^* = -0.5-j0.288$$

Hence

$$H(s) = \frac{1}{s+1} + \frac{-0.5+0.288j}{s+0.5+j0.866} + \frac{-0.5-0.288j}{s+0.5-j0.866}$$

$$= \frac{1}{s-(-1)} + \frac{-0.5+0.288j}{s-(-0.5-j0.866)} + \frac{-0.5-0.288j}{s-(-0.5+j0.866)}$$

In impulse invariant technique

$$\text{if} \quad H(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}, \quad \text{then} \quad H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T}z^{-1}}$$

Therefore,

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j0.288}{1-e^{-0.5}e^{-j0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5}e^{j0.866}z^{-1}}$$

$$= \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$

Example 5.14 Apply impulse invariant method and find $H(z)$ for $H(s) = \dfrac{s+a}{(s+a)^2+b^2}$.

Solution The inverse Laplace transform of given function is

$$h(t) = \begin{cases} e^{-at}\cos(bt) & \text{for} \quad t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Sampling the function produces

$$h(nT) = \begin{cases} e^{-anT}\cos(bnT) & \text{for} \quad n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
H(z) &= \sum_{n=0}^{\infty} e^{-anT}\cos(bnT)z^{-n} \\
&= \sum_{n=0}^{\infty}\left[e^{-anT}z^{-n}\left(\frac{e^{jbnT}+e^{-jbnT}}{2}\right)\right] \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\left[(e^{-(a-jb)T}z^{-1})^n + (e^{-(a+jb)T}z^{-1})^n\right] \\
&= \frac{1}{2}\left[\frac{1}{1-e^{-(a-jb)T}z^{-1}} + \frac{1}{1-e^{-(a+jb)T}z^{-1}}\right] \\
&= \frac{1-e^{-aT}\cos(bT)z^{-1}}{1-2e^{-aT}\cos(bT)z^{-1}+e^{-2aT}z^{-2}}
\end{aligned}$$

Example 5.15 An analog filter has a transfer function $H(s) = \dfrac{10}{s^2+7s+10}$. Design a digital filter equivalent to this using impulse invariant method for $T = 0.2$ sec.

Solution

Given

$$\begin{aligned}
H(s) &= \frac{10}{s^2+7s+10} \\
&= \frac{-3.33}{s+5} + \frac{3.33}{s+2} = \frac{-3.33}{s-(-5)} + \frac{3.33}{s-(-2)}
\end{aligned}$$

Using Eq. (5.81b) we have

$$\begin{aligned}
H(z) = T\left[\frac{-3.33}{1-e^{-5T}z^{-1}} + \frac{3.33}{1-e^{-2T}z^{-1}}\right] &= 0.2\left[\frac{-3.33}{1-e^{-1}z^{-1}} + \frac{3.33}{e^{-0.4}z^{-1}}\right] \\
&= \left[\frac{-0.666}{1-0.3678z^{-1}} + \frac{0.666}{1-0.67z^{-1}}\right] \\
&= \frac{0.2012z^{-1}}{1-1.0378z^{-1}+0.247z^{-2}}
\end{aligned}$$

Practice Problem 5.7 An analog filter has a transfer function

$$H(s) = \frac{5}{s^3+6s^2+11s+6}$$

Design a digital filter equivalent to this using impulse invariant method for $T = 1$ sec.

Example 5.16 Apply bilinear transformation to $H(s) = \dfrac{2}{(s+1)(s+2)}$ with $T = 1$ sec and find $H(z)$.

Solution

Given $H(s) = \dfrac{2}{(s+1)(s+2)}$

Substitute $s = \dfrac{2}{T}\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$ to get $H(z)$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{2}{(s+1)(s+2)}\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Given $T = 1$ sec

$$H(z) = \frac{2}{\left\{2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$= \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

Example 5.17 Using the bilinear transform, design a highpass filter, monotonic in passband with cutoff frequency of 1000 Hz and down 10 dB at 350 Hz. The sampling frequency is 5000 Hz.

Solution

Given $\alpha_p = 3\,\text{dB}; \omega_c = \omega_p = 2 \times \pi \times 1000 = 2000\pi$ rad/sec

$$\alpha_s = 10\,\text{dB}; \quad \omega_s = 2 \times \pi \times 350 = 700\pi \text{ rad/sec}$$

$$T = \frac{1}{f} = \frac{1}{5000} = 2 \times 10^{-4} sec$$

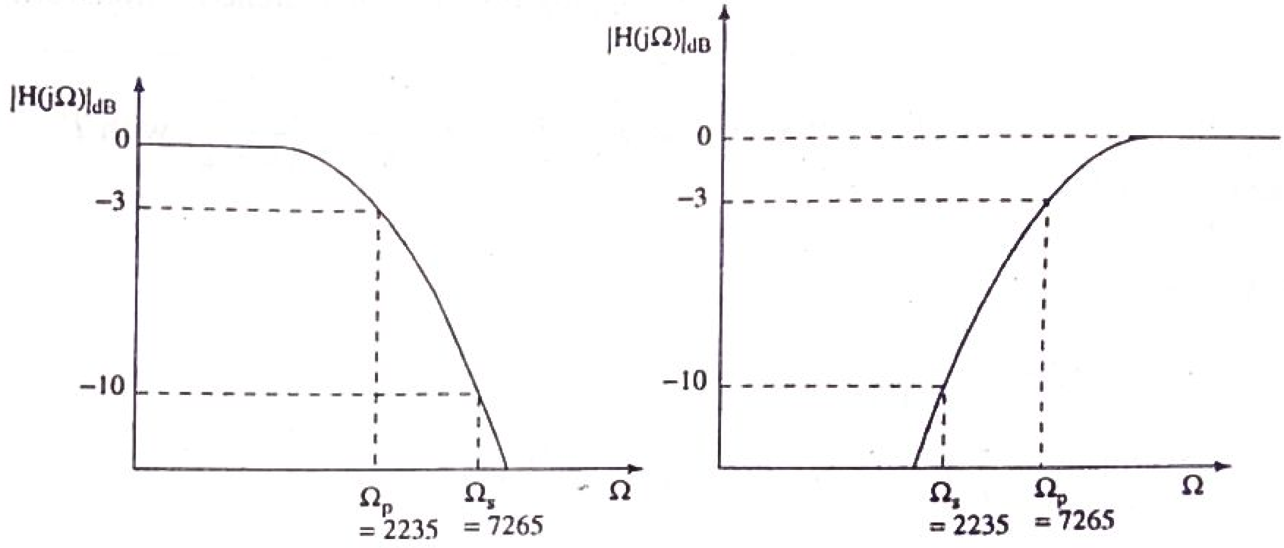

Fig. 5.27

The characteristics are monotonic in both passband and stopband. Therefore, the filter is Butterworth filter.

Prewarping the digital frequencies we have

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p T}{2} = \frac{2}{2 \times 10^{-4}}\tan\frac{(2000\pi \times 2 \times 10^{-4})}{2}$$
$$= 10^4 \tan(0.2\pi) = 7265 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s T}{2} = \frac{2}{2 \times 10^{-4}}\tan\frac{(700\pi \times 2 \times 10^{-4})}{2}$$
$$= 10^4 \tan(0.07\pi) = 2235 \text{rad/sec}$$

First we design a lowpass filter for the given specifications and use suitable transformation to obtain transfer function of highpass filter.

The order of the filter

$$N = \frac{\log\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\dfrac{\Omega_s}{\Omega_p}} = \frac{\log\sqrt{\dfrac{10^{0.1(10)}-1}{10^{0.1(3)}-1}}}{\log\dfrac{7265}{2235}} = \frac{\log 3}{\log 3.25} = \frac{0.4771}{0.5118} = 0.932$$

Therefore, we take $N = 1$.

The first-order Butterworth filter for $\Omega_c = 1$ rad/sec is $H(s) = \dfrac{1}{1+s}$

The highpass filter for $\Omega_c = \Omega_p = 7265$ rad/sec can be obtained by using the transformation

$$s \rightarrow \frac{\Omega_c}{s}$$

$$i.e., \; s \rightarrow \frac{(7265)}{s}$$

The transfer function of highpass filter

$$H(s) = \left.\frac{1}{s+1}\right|_{s=\frac{7265}{s}}$$

$$= \frac{s}{s+7265}$$

Using bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \left.\frac{s}{s+7265}\right|_{s=\frac{2}{2\times 10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{10000\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 7265}$$

$$= \frac{0.5792(1-z^{-1})}{1-0.1584z^{-1}}$$

Example 5.18 Determine $H(z)$ that results when the bilinear transformation is applied to $H_a(s) = \dfrac{s^2 + 4.525}{s^2 + 0.692s + 0.504}$

Solution

In bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Assume $T = 1$ sec.

Then

$$H(z) = \frac{\left[2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 4.525}{4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]^2 + 0.692 \times 2 \times \left[\frac{1-z^{-1}}{1+z^{-1}}\right] + 0.504}$$

$$= \frac{1.4479 + 0.1783z^{-1} + 1.4479z^{-2}}{1 - 1.18752z^{-1} + 0.5299z^{-2}}$$