
KTUNOTES
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$$\int_0^T e^{jk\omega_0 t - jm\omega_0 t}\, dt$$

$$= \int_0^T e^{j\omega_0 t (k-m)}\, dt$$

$$= \int_0^T \cos[\omega_0 t (k-m)] + j \sin[\omega_0 t (k-m)]\, dt$$

$$= \frac{1}{j\omega_0 (k-m)} \left[e^{j\omega_0 T (k-m)} - 1 \right]$$

$$= \frac{1}{j\omega_0 (k-m)} \left[e^{j2\pi (k-m)} - 1 \right]$$

$$= \frac{1}{j\omega_0 (k-m)} \left[\cos 2\pi (k-m) + j \sin 2\pi (k-m) - 1 \right]$$

$$= 1/j\omega_0 (k-m) \left[1 + 0 - 1 \right]$$

$$= 1/j\omega_0 (k-m) \times 0 = 0.$$

Hence, proved the given signal is orthogonal.

MODULE - 3

CONTINOUS TIME FOURIER SERIES

→ Fourier transformation is used to convert time domain to frequency domain.

→ If the signal is periodic signal, we use fourier series and if it is aperiodic signal, we use fourier transform.

→ CTFS, CTFT, DTFS, DTFT

- CTFS ↓ Continous with periodic
- CTFT ↓ Continous with aperiodic
- DTFS ↘ discrete with periodic
- DTFT ↘ Discrete with aperiodic

→ In continous case;

$x(t+T) = x(t)$

$T = 2\pi/\omega_0$ → fundamental period

$\cos \omega_0 t$ and $e^{j\omega_0 t}$ are periodic signals.

The eqn of continous time fourier series, is,

CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

→ eqn is called synthesis eqn.

when $K=0$, $x(t) =$ constant

$K = \pm 1$, $x(t) = a_1 e^{j\omega_0 t}$, $a_{-1} e^{-j\omega_0 t}$ } 1st harmonic

$K = \pm 2$, $x(t) = a_2 e^{j\omega_0 2t}$, $a_{-2} e^{-j\omega_0 2t}$ } 2nd harmonic

Multiply both side of synthesis eqn by $e^{-jn\omega_0 t}$, we get,

$$e^{-jn\omega_0 t} x(t) = \sum_{k=-\infty}^{\infty} a_k e^{-jn\omega_0 t} e^{jk\omega_0 t}$$

Integrating,

$$\int_0^T e^{-jn\omega_0 t} x(t)\,dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\,dt$$

$$\int_0^T e^{+j(k-n)\omega_0 t}\,dt = \cos(k-n)\omega_0 t + j\sin(k-n)\omega_0 t$$

$$= 0 + 0 - 1$$

$\int = 0$

$$= \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}$$

$$\therefore \int_0^T e^{-jn\omega_0 t} x(t)\,dt = a_n T$$

Imp
$$a_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\omega_0 t}\,dt$$
→ ~~eqn~~ Cutted

~~analysis eqn~~

CTFS

$$\therefore a_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\,dt$$ → analysis eqn

Calculate the frequency components in the following signals.

(i) $x(t) = \sin\omega_0 t$

Ans. Since, $\sin\omega_0 t$ is continuous periodic signal, we use CTFS eqn.

By eulers theorem,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

when this compare with synthesis eqn, we

$K = 1$, $a_1 = 1/2j$

$K = -1$, $a_{-1} = -1/2j$

K else, $a_k = 0$.

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$k=1$, $x(t) = \sum a_1 e^{\dots}$

$k=-1$, $x(t) = \sum a_{-1} e^{\dots}$

→ Amplitude spectrum is $|a_k|$ Vs K

→ Phase spectrum is $\angle a_k$ Vs K.

We've $a_1 = \frac{1}{2j}$, $a_{-1} = \frac{-1}{2j}$

$|a_1| = \sqrt{(\frac{1}{2})^2} = \frac{1}{2}$

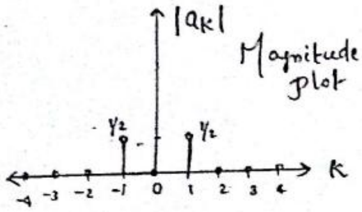
$|a_{-1}| = +\frac{1}{2}$

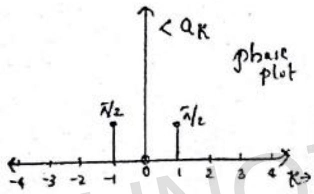
Now, $a_1 = 0 - \frac{j}{2}$

$a_2 = 0 + \frac{j}{2}$

$\angle a_1 = \tan^{-1}\left(\frac{-1/2}{0}\right) = 90^\circ = \pi/2$

$\angle a_2 = \tan^{-1}\left(\frac{1/2}{0}\right) = 90^\circ = \pi/2$





o/p is always discrete.

Q? plot the magnitude & phase spectrum of $x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4})$

Ans. $x(t) = 1 + \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} + \cancel{2}\,\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{\cancel{2}} + \frac{e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}}{2}$

$= 1 + \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} + e^{j\omega_0 t} + e^{-j\omega_0 t} + \frac{1}{2} e^{j(2\omega_0 t + \pi/4)} + \frac{1}{2} e^{-j(2\omega_0 t + \pi/4)}$

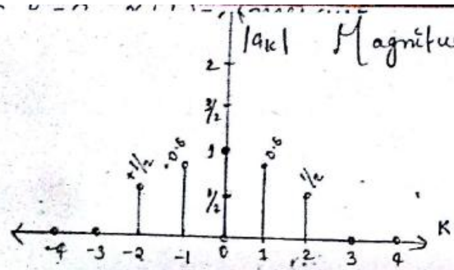
$= 1 + \left(\frac{1}{2j} + 1\right) e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\omega_0 t} + \frac{1}{2} e^{j(2\omega_0 t + \pi/4)} + \frac{1}{2} e^{-j(2\omega_0 t + \pi/4)}$

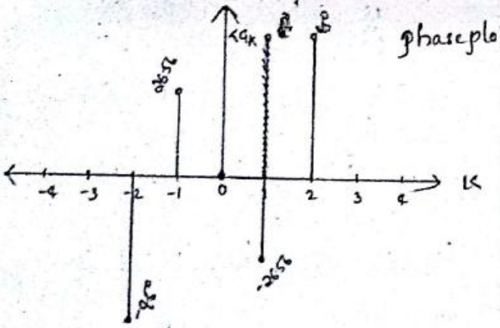
$= 1 + \left(\frac{1}{2j} + 1\right) e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\omega_0 t} + \frac{1}{2} e^{j\pi/4} \cdot e^{j2\omega_0 t} + \frac{1}{2} e^{-j\pi/4} \cdot e^{-j2\omega_0 t}$

Now, compare with the synthesis eqn.,

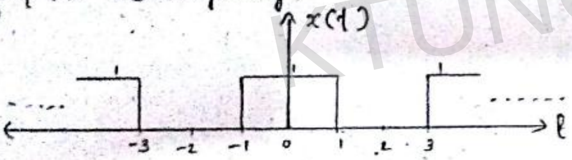
$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, then we get

$a_0 = 1$	$\angle a_0 = 0$	$\lvert a_0\rvert = 1$
$a_1 = 1 + \frac{1}{2j}$	$\angle a_1 = \tan^{-1}(-\frac{1}{2})$	$\lvert a_1\rvert = \sqrt{5/4}$
$a_{-1} = 1 - \frac{1}{2j}$	$\angle a_{-1} = \tan^{-1}(\frac{1}{2})$	$\lvert a_{-1}\rvert = \sqrt{5/4}$
$a_2 = \frac{1}{2} e^{j\pi/4}$	$\angle a_2 = \pi/4 = 45^\circ$	$\lvert a_2\rvert = \frac{1}{2}$
$a_{-2} = \frac{1}{2} e^{-j\pi/4}$	$\angle a_{-2} = \pi/4 = 45^\circ$	$\lvert a_{-2}\rvert = \frac{1}{2}$





Q. Obtain the fourier series representation for the waveform



Ans: Since, it is repeated ; it is periodic.

Here, period, $T = 4$.

$$a_k = \frac{1}{T}\int_{T} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{4}\int_{-2}^{2} x(t)\, e^{-jk\omega_0 t}\, dt$$

(-2 to 2 input)

($\omega_0 = 2\pi/T = 2\pi/4 = \pi/2$)

$$= \frac{1}{4}\int_{-1}^{1} 1\cdot e^{-jk\pi/2\, t}\, dt$$

(-2 to -1 = 0, -1 to 1 = 1, 1 to 2 = 0. So neglect the limit -2 to -1 and 1 to 2, limit changes to -1 to 1)

$$= \frac{1}{4}\left[\frac{e^{-jk\pi/2\, t}}{-jk\pi/2}\right]_{-1}^{1}$$

$$= \frac{1}{4}\left[\frac{e^{-jk\pi/2} - e^{jk\pi/2}}{-jk\pi/2}\right]$$

$$= \frac{1}{2}\left[\frac{e^{-jk\pi/2} - e^{jk\pi/2}}{2\times -jk\pi/2}\right]$$

$$= \frac{1}{2}\times\left[\frac{e^{jk\pi/2} - e^{-jk\pi/2}}{2j(k\pi/2)}\right]$$

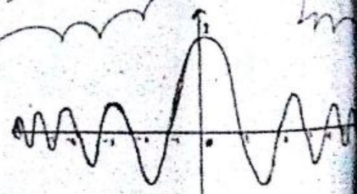
$$= \frac{1}{2}\,\frac{\sin(k\pi/2)}{k\pi/2}$$

($\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$)

$$\therefore x(k) = a_k = \frac{1}{2}\,\frac{\sin(k\pi/2)}{k\pi/2}$$

$$x(k) = \frac{1}{2}\,\text{sinc}\left(\frac{k}{2}\right)$$

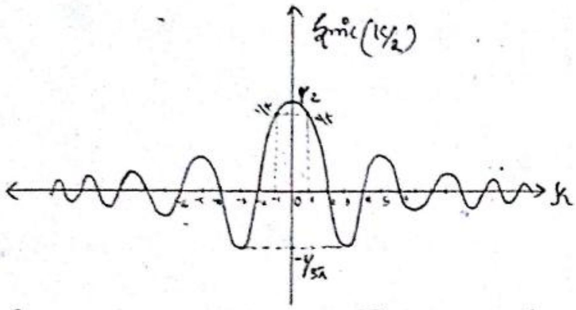
Now, we have to plot its magnitude & phase



($\text{sinc}\,\theta = \frac{\sin\pi\theta}{\pi\theta}$)

(Sinc θ = 1, θ = 0; 0, θ = ±nπ n = integer)

K	0	1	-1	± 2	± 3	± 4
a_K	$1/2$	$1/\pi$	$1/\pi$	0	$-1/3\pi$	0



V.Imp

✡ CONVERGENCE OF FOURIER SERIES:

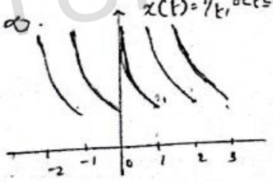
• Dirichlets Condition :

• For the existing of fourier series there are 3 sufficient Conditions

(i) $x(t)$ absolutely Integrable.

ie, $\int |x(t)| dt < \infty$.

$x(t) = 1/t, \; 0 < t \leq T$



eg:

$$\int x(t) dt < \infty$$

$$\int_0^1 1/t \, dt = [\log t]_0^1$$

$$= \log 1 - \log 0 = 0 - (-\infty) = \infty.$$

Hence it is not absolutely integrable

ie, It is not exist.

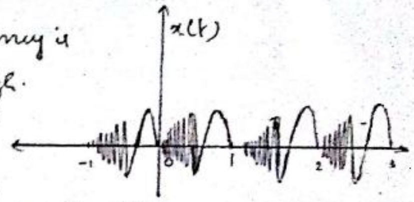
(ii) $x(t)$ must have finite numbers of maxima and minima over a period.

eg:

$$x(t) = \sin 2\pi/t, \quad 0 \leq t \leq 1$$

when $t = 0.001$, $\sin 2000\pi$

Hence, frequency is very high. We cannot count. Hence, no. of maxima & minima is infinite.

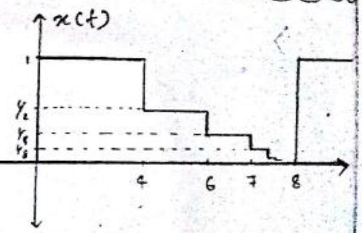


Hence, the condition is not satisfied. It is not exist.

(iii) $x(t)$ must have finite no. of finite discontinuities over a period.

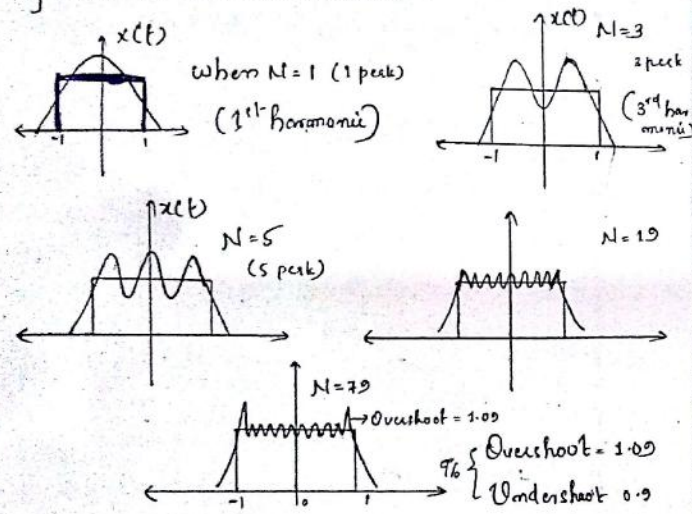
Example:

Here, there are infinite no. of discontinuities. Hence, the fourier series doesn't exist.



{Here, one period = 8, ∴ there are infinite no. of discontinuities over a period}.

Gibbs Phenomenon :



when harmonics increases, there is only a ripple at the discontinuity points.

FOURIER SERIES (TYPES):

(i) Exponential FS:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}$$

(ii) Trigonometric FS:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\omega_0 nt + \sum_{n=1}^{\infty} b_n \sin\omega_0 nt$$

{ actually { $x(t) = \sum_{k=0}^{\infty} a_k \cos k\omega_0 t + \sum_{k=0}^{\infty} b_k \sin k\omega_0 t$ }

when $k = 0$, we get a_0 & $b_0 = 0$ }

where, $a_0 = \frac{1}{T}\int_T x(t)\,dt$

$$a_n = \frac{2}{T}\int_T x(t)\cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T}\int_T x(t)\sin n\omega_0 t\, dt$$

(iii) Polar fourier Series :

$$x(t) = d_0 + \sum_{n=1}^{\infty} d_n \cos(n\omega_0 t + \theta_n)$$

Compare trigonometric FS with Polar FS,

we get, $d_0 = a_0 = \frac{1}{T}\int_T x(t)\,dt$

magnitude $d_n = \sqrt{a_n^2 + b_n^2}$

Phase $\theta_n = \tan^{-1}(b_n/a_n)$

Imp

→ Properties of CTFS :

1) LINEARITY :

If $x(t) \xleftrightarrow{FS} X(K)$

$y(t) \leftrightarrow Y(K)$,

then $A\,x(t) + B\,y(t) \xleftrightarrow{FS} A\,X(K) + B\,Y(K)$

Proof :

We've $X(k) = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$ (a_k)

$Y(k) = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$

Let $z(t)$ be a new signal,

$z(t) = A\,x(t) + B\,y(t)$

then $Z(k) = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt$

$= \frac{1}{T}\int_T [A\,x(t) + B\,y(t)]\, e^{-jk\omega_0 t}\, dt$

$= \frac{1}{T}\int_T A\,x(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}\int_T B\,y(t)\, e^{-jk\omega_0 t}\, dt$

$= A\,\frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt + B\,\frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$

$z(t) = A\,X(k) + B\,Y(k)$

ie, $A\,x(t) + B\,y(t) = A\,X(k) + B\,Y(k)$

Hence, proved.

2. TIME SHIFTING :

If $x(t) \longleftrightarrow X(k)$

then $x(t - t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0}\, X(k)$

Proof :

$X(k) = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

Let $y(t) = x(t - t_0)$

Then $Y(k) = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$

$Y(k) = \frac{1}{T}\int_T x(t - t_0)\, e^{-jk\omega_0 t}\, dt$

Let $\tau = t - t_0 \Rightarrow t = \tau + t_0$

$d\tau = dt$

then $Y(k) = \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau + t_0)}\, d\tau$

$= \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 \tau} \cdot e^{-jk\omega_0 t_0}\, d\tau$

$= e^{-jk\omega_0 t_0}\left[\frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 \tau}\, d\tau\right]$

$Y(k) = e^{-jk\omega_0 t_0}\, X(k)$

Hence proved.

$y(t) = x(t - t_0)$

$\therefore y(t) \xleftrightarrow{FS} Y(k)$

$x(t - t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t_0} X(k)$

$Y(k) = e^{-jk\omega_0 t_0} X(k)$

8. TIME REVERSAL :

If $x(t) \overset{FS}{\longleftrightarrow} X(k)$

then $x(-t) \longleftrightarrow X(-k)$

Proof :

Let $y(t) = x(-t)$

then $Y(k) = \frac{1}{T}\int_0^T y(t)\, e^{-j\omega_0 k t}\, dt$

$= \frac{1}{T}\int_0^T x(-t)\, e^{-j\omega_0 k t}\, dt$

Now, let $\tau = -t$

$d\tau = -dt$

when $t \to 0$, $\tau \to 0$

$t \to T$, $\tau = -T$

Then, $Y(k) = \frac{1}{T}\int_0^{-T} x(\tau)\, e^{j\omega_0 k \tau} (-d\tau)$

$= \frac{1}{T}\int_{-T}^{0} x(\tau)\, e^{j\omega_0 k \tau}\, d\tau$

$\left\{ \because \int_0^{-T} (-d\tau) = \int_{-T}^{0} d\tau \right\}$

$= \frac{1}{T}\int_{-T}^{0} x(\tau)\, e^{-j\omega_0 (-k)\tau}\, d\tau$

$Y(k) = X(-k) \Rightarrow x(-t) \longleftrightarrow X(-k)$

Hence, proved.

$\left\{ \begin{array}{l} \text{eg: } x(t) = \sin\omega_0 t \\ X(k) = \{-3, 2, 4\} \\ x(-t) = ? \quad X(-k) = \{4, 2, -3\} \end{array} \right.$

-1 0 +1

9. TIME DIFFERENTIATION :

If $x(t) \overset{FS}{\longleftrightarrow} X(k)$

then, $\frac{d}{dt} x(t) \longleftrightarrow j\omega_0 k\, X(k)$

Proof :

We have,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{j\omega_0 k t} \quad \ldots\ldots (1)$$

then $\frac{d}{dt} x(t) = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} X(k)\, e^{j\omega_0 k t}\right]$

$= \sum_{k=-\infty}^{\infty} X(k)\, \frac{d}{dt} e^{j\omega_0 k t}$

$\frac{d}{dt} x(t) = \sum_{k=-\infty}^{\infty} X(k)\, j\omega_0 k\, e^{j\omega_0 k t} \quad \ldots\ldots (2)$

Comparing (1) & (2),

$\frac{d}{dt} x(t) \longleftrightarrow j\omega_0 k\, X(k)$

Hence, proved.

$$\frac{d^n}{dt^n} x(t) \longleftrightarrow (j\omega_0 k)^n X(k)$$

gen...

5. TIME SCALING:

If $x(t) \xleftrightarrow{FS} X(k)$

then $x(\alpha t) \xleftrightarrow{FS} X(k)\Big|_{T \to T/\alpha}$

Proof:

Let $y(t) = x(\alpha t)$

$$Y(k) = \frac{1}{T}\int_0^T y(t)\, e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{T/\alpha}\int_0^{T/\alpha} x(\alpha t)\, e^{-j\omega_0 k \alpha t}\, \alpha\, dt$$

Put $m = \alpha t$

$dm = \alpha\, dt$

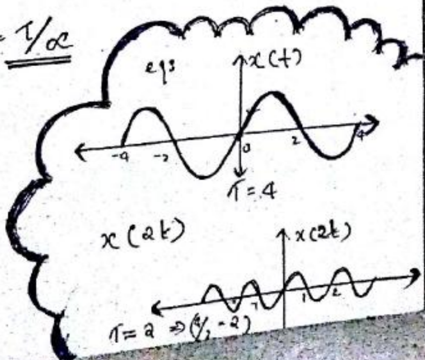
$dt = dm/\alpha$

$$= \frac{1}{(T/\alpha)}\int_0^{T/\alpha} x(m)\, e^{-j\omega_0 k m}\, dm$$

$$= X(K)\Big|_{T = T/\alpha}$$

Hence, proved.

Example:



6. FREQUENCY SHIFTING:

If $x(t) \xleftrightarrow{F.S} X(k)$

then $e^{j\omega_0 K_0 t}\, x(t) \xleftrightarrow{F.S} X(K - K_0)$

Proof:

Let $y(t) = e^{j\omega_0 K_0 t}\, x(t)$.

$$Y(K) = \frac{1}{T}\int_0^T y(t)\, e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{T}\int_0^T x(t)\, e^{j\omega_0 K_0 t} \cdot e^{-j\omega_0 k t}\, dt$$

$$= \frac{1}{T}\int_0^T x(t)\, e^{-j\omega_0 (K - K_0) t}\, dt$$

$$Y(K) = X(K - K_0)$$

Hence, proved.

7. MULTIPLICATION PROPERTY:

If $x(t) \xleftrightarrow{FS} X(K)$

$y(t) \xleftrightarrow{F.S} Y(K)$

then

$$x(t)\,y(t) \xleftrightarrow{FS} X(K) \circledast Y(K)$$

↳ Circular convolution

$*$ → linear convolution

Proof :

Let $z(t) = x(t)\,y(t)$

$$Z(K) = \frac{1}{T}\int_T z(t)\, e^{-j\omega_0 K t}\, dt$$

$$= \frac{1}{T}\int_T x(t)\,y(t)\, e^{-j\omega_0 K t}\, dt$$

$$= \frac{1}{T}\int_T \sum_{l=-\infty}^{\infty} X(l)\, e^{j\omega_0 l t} \cdot y(t) \cdot e^{-j\omega_0 K t}\, dt$$

$$= \sum_{l=-\infty}^{\infty} X(l) \left[\frac{1}{T}\int_T y(t)\, e^{-j\omega_0 (K-l) t}\, dt \right]$$

$$= \sum_{l=-\infty}^{\infty} X(l)\, Y(K-l)$$

$$Z(K) = X(K) \circledast Y(K)$$

$$\sum_{K=-\infty}^{\infty} x[K]\, y[n-K] = x[n] * y[n]$$

Hence, proved.

8. CONVOLUTION PROPERTY:

If $x(t) \xleftrightarrow{FS} X(K)$

$y(t) \xleftrightarrow{FS} Y(K)$

then,

$$x(t) \circledast y(t) \longleftrightarrow T\, X(K)\, Y(K)$$

Proof:

Let $z(t) = x(t) \circledast y(t)$

$$= \int_{l=T} x(l)\, y(t-l)\, dl$$

$$Z(K) = \frac{1}{T}\int_{t=T} z(t)\, e^{-j\omega_0 K t}\, dt$$

$$= \frac{1}{T}\int \left(x(t) \circledast y(t)\right) e^{-j\omega_0 K t}\, dt$$

$$= \frac{1}{T}\int_{t=T}\int_{l=T} x(l)\, y(t-l)\, dl\; e^{-j\omega_0 K t}\, dt$$

$$= \frac{1}{T}\int_{l=T} x(l) \int_{t=T} y(t-l)\, e^{-j\omega_0 K t}\, dt\, dl \quad \ldots (1)$$

Now, $\int_{t=T} y(t-l)\, e^{-j\omega_0 K t}\, dt$

put $m = t-l$, $t = m+l$

$$= \int_{m=T} y(m)\, e^{-j\omega_0 K (m+l)}\, dm$$

$$= e^{-j\omega_0 K l} \times \frac{T}{T} \int_{m=T} y(m)\, e^{-j\omega_0 K m}\, dm$$

$$= e^{-j\omega_0 K l} \times T \times Y(K) \quad \ldots (2)$$

(2) in (1) $\Rightarrow$

$$Z(K) = \frac{1}{T}\int_{l=T} x(l)\, T\, Y(K)\, e^{-j\omega_0 K l}\, dl$$

$$= T\, Y(K) \left[\frac{1}{T}\int_T x(l)\, e^{-j\omega_0 K l}\, dl \right]$$

Hence, proved.

9. POWER OF A SIGNAL:

Q) State parseval's theorem for Continuous periodic Signal.

Proof:

$$\frac{1}{T}\int |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

we've $x(t)\,x^*(t) = |x(t)|^2$

$(a+jb)(a-jb) = |a+jb|^2 = a^2+b^2$

$$P = \frac{1}{T}\int x(t)\,x^*(t)\,dt \quad \ldots\ldots (1)$$

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\,e^{j\omega_0 k t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} X^*(k)\,e^{-j\omega_0 k t} \quad \ldots\ldots (2)$$

Substitute (2) in (1),

$$P = \frac{1}{T}\int x(t)\sum_{k=-\infty}^{\infty} X^*(k)\,e^{-j\omega_0 k t}\,dt$$

$$= \sum_{k=-\infty}^{\infty} X^*(k)\times\frac{1}{T}\int x(t)\,e^{-j\omega_0 k t}\,dt$$

$$= \sum_{k=-\infty}^{\infty} X^*(k)\,X(k)$$

$$= \sum_{k=-\infty}^{\infty} |X(k)|^2$$

- Parseval's theorem states that the total average power in a periodic signal equals the sum of the average power in all of its harmonic components.

{Power spectral density = $|X(k)|^2$ v/s K }

☆ Symmetry Condition:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\omega_0 t\,dt$$

$$\left.\begin{aligned} a_n &= \frac{2}{T}\int_0^{T/2}[x(t)+x(-t)]\cos n\omega_0 t\,dt \\ b_n &= \frac{2}{T}\int_0^{T/2}[x(t)-x(-t)]\sin n\omega_0 t\,dt \end{aligned}\right\}\text{Symmetry.}$$

Even:

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$$

$b_n = 0$.

Odd:

$a_n = 0$

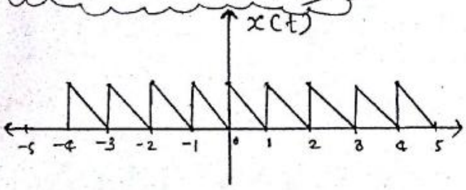
$b_n = 4/T \int_0^{T/2} x(t) \sin n\omega_0 t \, dt$

★ HALFWAVE SYMMETRY:

$x(t) = x(t \pm T/2)$

Here, $T=2$

$x(t) = x(t \pm 1)$



☆ CTFT

- DTFS - Discrete time periodic stable
- DTFT - " aperiodic " — Discrete case
- ZT - " unstable
- CTFS - Continuous time periodic stable
- CTFT - " aperiodic stable — Continuous
- LT - " unstable

◉ Synthesis Equation: (CTFT)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega$$

◉ Analysis eqn: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

◉ CONVERGENCE OF FOURIER TRANSFORM (existence)

(i) $x(t)$ should be absolutely Integrable.

ie, $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$

(ii) $x(t)$ should have finite number of maxima and minima in any finite interval.

(iii) $x(t)$ should have finite no. of finite discontinuities in any finite interval.

? Calculate the fourier coefficients of a continuous time signal given by,

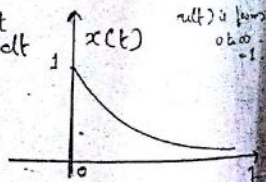
$x(t) = e^{-at} u(t)$, where $a>0$

plot its magnitude and phase spectrum.

Ans. We've to calculate $X(j\omega)$. {fourier coefficients}

We've, $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

$u(t)$ is from 0 to ∞ -1



$X(j\omega) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j\omega t} dt$

$= \int_0^{\infty} e^{-at} \times 1 \times e^{-j\omega t} dt$ (Since $u(t)$ is from 0 to ∞)

$= \int_0^{\infty} e^{-(a+j\omega)t} dt = \left[\frac{e^{-(a+j\omega)t}}{a+j\omega} \right]_0^{\infty}$

$$= -\frac{1}{a+j\omega}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{a+j\omega}(0-1)$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

Magnitude is given by,

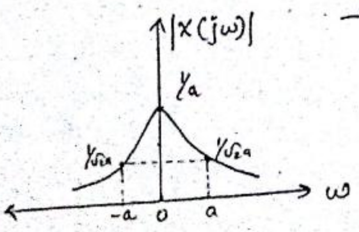
$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

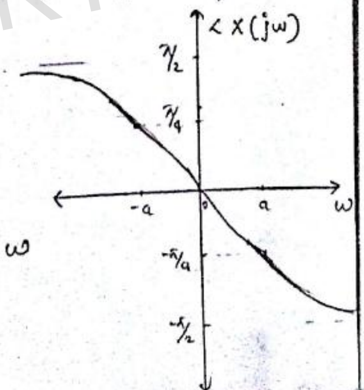
Now, phase, $\angle X(j\omega) = 0 - \tan^{-1}(\omega/a) = -\tan^{-1}(\omega/a)$

$\omega = 0$, $|X(j\omega)| = 1/a$, $\angle X(j\omega) = 0$

$\omega = a$, $|X(j\omega)| = 1/\sqrt{2}a$, $\angle X(j\omega) = -\pi/4$

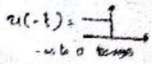
$\omega = -a$, $|X(j\omega)| = 1/\sqrt{2}a$, $\angle X(j\omega) = \pi/4$





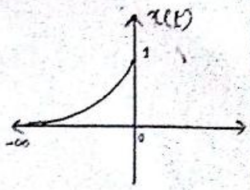
? $x(t) = e^{at}\, u(-t)$, $a > 0$

Ans. We've



$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{at}\, u(-t)\, e^{-j\omega t}\, dt$$

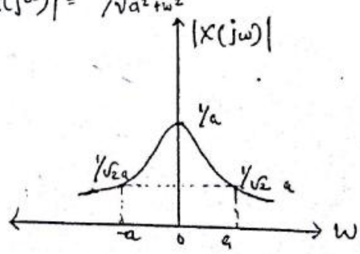


$$= \int_{-\infty}^{0} e^{at} \cdot e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{(-j\omega+a)t}\, dt = \left[\frac{e^{(a-j\omega)t}}{a-j\omega}\right]_{-\infty}^{0}$$

$$X(j\omega) = \frac{1}{a-j\omega}\left(e^{0} - e^{-\infty}\right) = \frac{1}{a-j\omega}$$

$$|X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

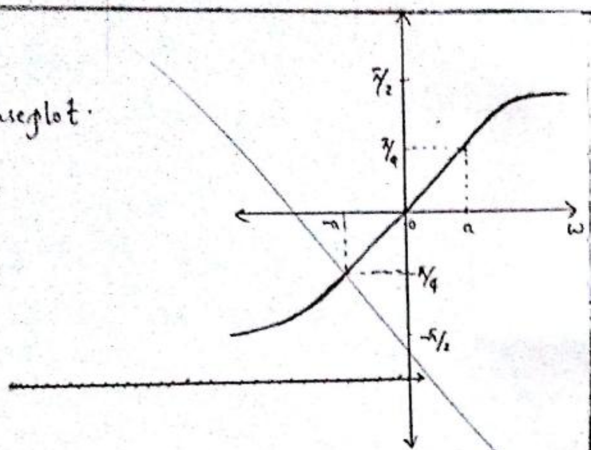


$\omega = 0$, $|X(j\omega)| = 1/a$

$\omega = a$, $|X(j\omega)| = 1/\sqrt{2}a$

$\omega = -a$, $1/\sqrt{2}a$

Phaseplot.



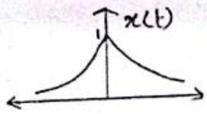
$$\angle X(j\omega) = 0 - \tan^{-1}(-\omega/a) = \tan^{-1}(\omega/a)$$

$\omega = 0$, $\angle X(j\omega) = \tan^{-1}(0/a) = 0$

$\omega = a$, $\angle X(j\omega) = \tan^{-1}(a/a) = \pi/4$

$\omega = -a$, $\angle X(j\omega) = \tan^{-1}(-a/a) = \tan^{-1}(-1) = -\tan^{-1}(1) = -\pi/4$

? $x(t) = e^{-a|t|}$, $a<1$



Ans: $X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$X(j\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt$$

when $t<0$, $x(t) = e^{-a|t|} = e^{at}$

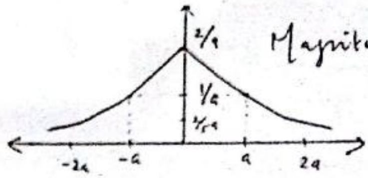
when $t>0$, $x(t) = e^{-a|t|} = e^{-at}$

$$= \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$$

$$X(j\omega) = 2a/(a^2+\omega^2)$$

$$\therefore |X(j\omega)| = 2a/(a^2+\omega^2)$$

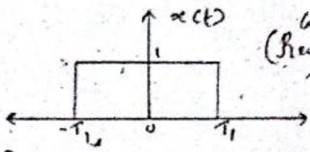
$$\angle X(j\omega) = 0$$



Magnitudeplot

v.imp

? Find the fourier transform of signal given by $x(t)$.



Continuous (Rectangular Signal)

Ans: we've to find $X(j\omega)$.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T_1}^{T_1} 1 \cdot e^{-j\omega t}\, dt$$

$$= \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-T_1}^{T_1} = \left[\frac{e^{-j\omega T_1} - e^{j\omega T_1}}{-j\omega}\right]$$

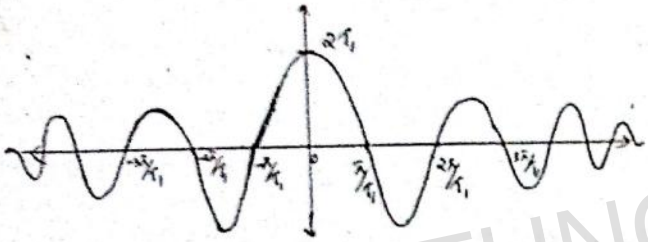
$$= \left[\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}\right]$$

$$= \frac{2}{\omega} \sin \omega T_1$$

$$= \frac{2 \times \sin\left(\frac{\pi \omega T_1}{\pi}\right)}{\pi \left(\frac{\omega T_1}{T_1 \pi}\right)}$$

We've $\operatorname{sinc}\theta = \frac{\sin \pi\theta}{\pi\theta}$, $\theta = 1, 2, 3 \ldots$ for sinc zero $\left\{ \pi\theta = \pm n\pi, \; \theta = \pm\frac{n\pi}{\pi} = 1, 2, 3 \ldots \right\}$ $\pi\theta = \pm n\pi$, $\theta = \pm n$

$$\therefore X(j\omega) = 2 T_1 \operatorname{sinc}\left(\frac{\omega T_1}{\pi}\right).$$



$\frac{\omega T_1}{\pi} = 1, 2, 3 \ldots \Rightarrow \omega = \frac{\pi}{T_1}, \frac{2\pi}{T_1}, \frac{3\pi}{T_1} \ldots$

→ Fourier transform of a continuous rectangular function is a sinc function

for square

$$\left\{ \begin{array}{l} x(t) - e^{-j\theta} \\ e^{-j\theta/2}\left(e^{j\theta/2} - e^{-j\theta/2}\right) \\ 2j \frac{e^{-j\theta/2}}{2j}\left(e^{j\theta/2} - e^{-j\theta/2}\right) = 2j\, e^{-j\theta/2} \sin\theta/2 \end{array} \right\}$$

2\. Find the fourier transform of impulse fn.

Ans. $x(t) = \delta(t)$.

Properties of $\delta(t)$ are;

$$\int_{-\infty}^{\infty} x(t) \cdot \delta(t)\, dt = x(0) \ldots (1)$$

And $$\int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$$

$x(t)\cdot\delta(t) = x(0)$

$x(t)\cdot\delta(t-t_0) = x(t_0)$

Now, fourier transform,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt \quad \ldots\ldots (2)$$

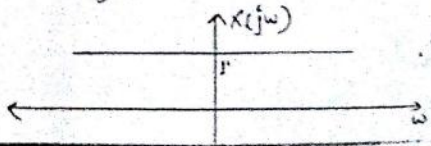
Compare (1) & (2), then $x(t) = e^{-j\omega t}$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \delta(t)\, x(t)\, dt = x(0)$$

$$= e^{-j\omega \times 0} = e^0 = 1.$$

$\therefore X(j\omega) = 1$.



Continuous time fourier transform = $X(j\omega)$

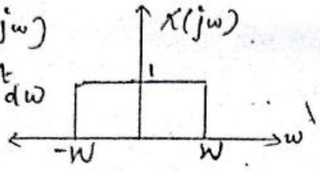
fourier series = $x(t)$

Q Calculate the signal $x(t)$ whose fourier transform is given as follows.

$$X(j\omega) = \begin{cases} 1 & , |\omega| < W \\ 0 & , |\omega| > W \end{cases}$$

Ans $x(t) = $ IFT of $X(j\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$



$$= \frac{1}{2\pi}\int_{-W}^{W} 1 \cdot e^{j\omega t}\, d\omega$$

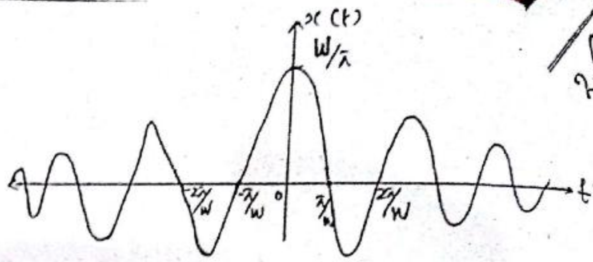
$$= \frac{1}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-W}^{W}$$

$$= \frac{1}{2\pi}\left[\frac{e^{jWt} - e^{-jWt}}{jt}\right]$$

$$= \frac{1}{\pi t}\sin Wt \qquad \text{sinc}\,\theta = \frac{\sin \pi\theta}{\pi\theta}$$

$$= \frac{1}{\pi}\,\frac{\sin\left(\frac{\pi W t}{\pi}\right)}{\frac{\pi W t}{W\pi}}$$

$$x(t) = \frac{W}{\pi}\,\text{sinc}\left(\frac{Wt}{\pi}\right)$$



✯ PROPERTIES OF CTFT:

1. LINEARITY:

If $x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$

$y(t) \overset{FT}{\longleftrightarrow} Y(j\omega)$

then $a\,x(t) + b\,y(t) \overset{FT}{\longleftrightarrow} a\,X(j\omega) + b\,Y(j\omega)$.

• Proof:

We have $a\,X(j\omega) = \int_{-\infty}^{\infty} a\,x(t)\, e^{-j\omega t}\, dt$

$b\,Y(j\omega) = \int_{-\infty}^{\infty} b\,y(t)\, e^{-j\omega t}\, dt$

Now $a\,X(j\omega) + b\,Y(j\omega) = \int_{-\infty}^{\infty} a\,x(t)\, e^{-j\omega t}\, dt + \int_{-\infty}^{\infty} b\,y(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} (a\,x(t) + b\,y(t))\, e^{-j\omega t}\, dt$$

$$= a\,X(j\omega) + b\,Y(j\omega)$$

2. TIME SHIFTING:

If $x(t) \Leftrightarrow X(j\omega)$

then, $x(t-t_0) \Leftrightarrow e^{-j\omega t_0} X(j\omega)$

Proof:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Let $y(t) = x(t-t_0)$

$$\text{then } Y(j\omega) = \int_{-\infty}^{\infty} x(t-t_0) e^{-j\omega t} dt$$

Let $t - t_0 = \tau$

$t = t_0 + \tau$, $dt = d\tau$

$t = -\infty, \tau = -\infty$, $t = \infty, \tau = \infty$

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_0)} d\tau$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau) e^{-j\omega \tau} d\tau$$

$$= e^{-j\omega t_0} X(j\omega)$$

$$Y(j\omega) = e^{-j\omega t_0} X(j\omega)$$

Hence, proved.

3. FREQUENCY SHIFTING:

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then $e^{j\omega_0 t} x(t) \xleftrightarrow{FT} X(j(\omega - \omega_0))$

Proof:

Let $y(t) = e^{j\omega_0 t} x(t)$

$$\text{then, } Y(j\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-jt(\omega - \omega_0)} dt$$

$$Y(j\omega) = X(j(\omega - \omega_0))$$

Hence, proved.

4. TIME REVERSAL:

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then $x(-t) \xleftrightarrow{FT} X(-j\omega)$

Proof:

Let $y(t) = x(-t)$

$$\text{then } Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega_0 t}\, dt$$

Let $\tau = -t$, $\Rightarrow d\tau = -dt$.

when $t \to -\infty$, $\tau \to \infty$

$t \to \infty$, $\tau \to -\infty$

Then $Y(j\omega) = \int_{\infty}^{-\infty} x(\tau)\, e^{j\omega\tau} (-d\tau)$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\tau(-\omega)}\, d\tau$$

$$Y(j\omega) = X(-j\omega)$$

Hence, proved.

5. TIME SCALING:

If $x(t) \overset{FT}{\longleftrightarrow} X(j\omega)$

then $x(\alpha t) \overset{FT}{\longleftrightarrow} \frac{1}{|\alpha|} X\left(\frac{j\omega}{\alpha}\right)$

Proof:

Case-1: $\alpha > 0$

$y(t) = x(\alpha t)$

then $Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} x(\alpha t)\, e^{-j\omega t}\, dt$$

let $m = \alpha t$

$t = m/\alpha \Rightarrow dt = dm/\alpha$

when $t = -\infty$, $m = -\infty$,

$t = \infty$, $m = \infty$

$Y(j\omega) = \int_{-\infty}^{\infty} x(m)\, e^{-j\omega(m/\alpha)}\, dm/\alpha$

$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} x(m)\, e^{-j\omega m/\alpha}\, dm$$

$$Y(j\omega) = \frac{1}{\alpha} X(j\omega/\alpha)$$

- Case-2: $\alpha < 0$

$y(t) = x(-\alpha t)$

let $m = -\alpha t$, $t = -m/\alpha$

$dm = -\alpha\, dt$

$t = -\infty$, $m = \infty$, & $t = \infty$, $m = -\infty$

$Y(j\omega) = \int_{\infty}^{-\infty} x(m)\, e^{-j\omega - m/\alpha} \left(-dm/\alpha\right)$

$$= \frac{1}{\alpha} \int_{-\infty}^{\infty} x(m)\, e^{-j\omega m(-w/\alpha)}\, dm$$

$$Y(j\omega) = \frac{1}{\alpha}\, X(-j\omega/\alpha)$$

From these two, generally

$$x(\alpha t) \longleftrightarrow \frac{1}{|\alpha|}\, X(j\omega/\alpha)$$

Hence, proved.

6. TIME DIFFERENTIATION:

If $x(t) \xleftrightarrow{FT} X(j\omega)$.

then $\frac{d}{dt}x(t) \Leftrightarrow j\omega\, X(j\omega)$.

Proof:

Let $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

$$\frac{d}{dt}x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, \frac{d}{dt} e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\, X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{d}{dt}x(t) = j\omega\, X(j\omega)$$

Hence proved.

7. DIFFERENTIATION IN FREQUENCY:

If $x(t) \Leftrightarrow X(j\omega)$

$-jt\, x(t) \Leftrightarrow \frac{d}{d\omega}X(j\omega)$

Proof:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\frac{d}{d\omega}X(j\omega) = \int_{-\infty}^{\infty} x(t)\, \left(\frac{d}{d\omega} e^{-j\omega t}\right) dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot -jt\; e^{-j\omega t}\, dt$$

$$= -jt \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\frac{d}{d\omega}X(j\omega) = -jt\, x(t)$$

Hence, proved.

8. CONVOLUTION IN TIME:

If $x_1(t) \Leftrightarrow X_1(j\omega)$

$x_2(t) \Leftrightarrow X_2(j\omega)$

then $x_1(t) * x_2(t) \Leftrightarrow X_1(j\omega) \cdot X_2(j\omega)$

Proof:

We've $y(t) = x_1(t) * x_2(t)$

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau$$

Now, $Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \underbrace{\left[\int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j\omega t}\, dt \right]}_{\text{By time shifting property}} d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau} \cdot X_2(j\omega)\, d\tau$$

$$= X_2(j\omega) \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau$$

$Y(j\omega) = X_2(j\omega) \cdot X_1(j\omega)$

Hence, proved.

9: MULTIPLICATION IN TIME:

If $x_1(t) \longleftrightarrow X_1(j\omega)$

$x_2(t) \longleftrightarrow X_2(j\omega)$

then, $x_1(t) \cdot x_2(t) = \frac{1}{2\pi}\left(X_1(j\omega) * X_2(j\omega)\right)$

Proof:

Let $y(t) = x_1(t) \cdot x_2(t)$.

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\lambda)\, e^{j\lambda t}\, d\lambda\, x_2(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\lambda) \left(\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, e^{j\lambda t}\, dt \right) d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\lambda) \underbrace{\left(\int_{-\infty}^{\infty} x_2(t)\, e^{-jt(\omega-\lambda)}\, dt \right]}_{\text{freq. shifting}} d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(j\lambda)\, X_2(j(\omega-\lambda))\, d\lambda$$

We've $\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau = x_1(t) * x_2(t)$

$\therefore Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(X_1(j\omega) * X_2(j\omega) \right)$

Hence, proved.

DUALITY

If $x(t)$ ...

then

10. DUALITY PROPERTY:

If $x(t) \xleftrightarrow{FT} X(\omega)$

then

$$X(t) \longleftrightarrow 2\pi\, x(-\omega)$$

Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$2\pi\, x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Now, put $t = -t$

then $2\pi\, x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$

Now, interchanging t & ω,

$$2\pi\, x(-\omega) = \underbrace{\int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt}_{FT \text{ of } X(t)}$$

ie, $X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$.

Hence, proved.

11. PARSEVAL'S THEOREM:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

Proof:

CTFS DTFS
CTFT DTFT
Energy

$$|x(t)|^2 = x(t)\; x^*(t)$$

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\; x^*(t)\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, dt$$

$$x^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t}\, dt$$

$$\therefore \Rightarrow \int_{-\infty}^{\infty} \left[x(t) \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega)\, e^{-j\omega t}\, d\omega\right] dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(j\omega) \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \Big]\, d\omega$$

$$= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} X^*(j\omega)\, X(j\omega)\, d\omega\right.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega$$

Hence proved.

12. INTEGRATION PROPERTY:

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then $\int_{-\infty}^{t} x(\tau)\, d\tau \longleftrightarrow \frac{X(j\omega)}{j\omega} + \pi X(0)\, \delta(\omega)$

Proof :

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

we've $u(t-\tau) = 1, \ (t-\tau) > 0$

$\qquad\qquad\quad\ \ 0, \ (t-\tau) < 0$

$\Rightarrow \int_{-\infty}^{t} x(\tau) \times 1\, d\tau$

Now, we have $x(t) * u(t) \leftrightarrow X(j\omega) \cdot U(j\omega)$

$X(j\omega) \cdot U(j\omega)$;

But $U(j\omega) = 1/j\omega + \pi\,\delta(\omega)$

$= X(j\omega)\left[1/j\omega + \pi\,\delta(\omega)\right]$

$= X(j\omega)/j\omega + \pi\, X(j\omega)\,\delta(\omega)$

But from product property,

$$\left\{ \begin{array}{l} x(t)\,\delta(t) = x(0)\,\delta(t) \\ x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0). \end{array} \right\}$$

$= X(j\omega)/j\omega + \pi\, X(0)\,\delta(\omega)$ //

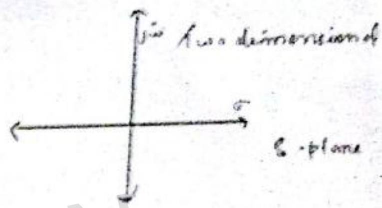
Hence, proved.

✡ LAPLACE TRANSFORM

$x(t) \xleftrightarrow{LS} X(s)$

where $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

Now, ILT, $x(t) = 1/2\pi \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds$.



$(s = \sigma + j\omega)$

diverging signals are unstable

Converging signals are stable

we've $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-(\sigma + j\omega)t}\, dt$

$= \int_{-\infty}^{\infty} \left(x(t)\, e^{-\sigma t}\right) e^{-j\omega t}\, dt$

$= F\left\{ x(t) \cdot e^{-\sigma t} \right\}$.

$\left(\because F\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \right)$

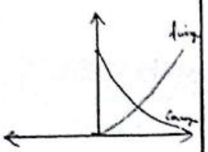
$\therefore L\{x(t)\} = F\{x(t)\, e^{-\sigma t}\}$.

This is the relation b/w fourier & lapl[ace]

→ We know that, Laplace transform is only for unstable signals. Therefore, when $x(t)$ is multiplied with $e^{-\sigma t}$, then it also become unstable (since $\mathcal{L}\{x(t)\} = F\{x(t)e^{-\sigma t}\}$)

Therefore, $e^{-\sigma t}$ is called convergence factor.

$$\begin{cases} t\to\infty,\ x(t)\to 0 \Rightarrow \text{convergence} \\ t\to\infty,\ x(t)\to\infty \Rightarrow \text{divergence} \end{cases}$$

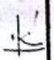


* Region of Convergence (ROC):

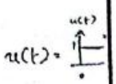
→ Fourier transform existing points of $x(t)$ is called region of convergence.

→ or, The points at which the $x(t)$ is stable & convergence, is called region of convergence.

? Calculate the Laplace transform of a signal $x(t) = e^{-at}\,u(t)$.



Ans. $X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt$

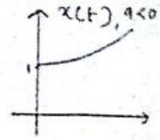


$$= \int_{-\infty}^{\infty} e^{-at}\,u(t)\,e^{-st}\,dt$$

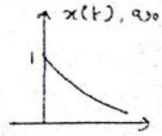
∴ limit is from 0 to ∞.

$$= \int_0^{\infty} e^{-at}\,e^{-st}\,dt = \int_0^{\infty} e^{-(a+s)t}\,dt$$

$$= \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^{\infty}$$



$$X(s) = \frac{1}{s+a}$$



when $a<0$, $x(t) = e^{at}u(t)$ → positive slope

when $a<0$, $x(t) = e^{-at}u(t)$ → negative / decaying slope

For convergence, we want $a>0$.

$$\therefore X(s) = \int_0^{\infty} e^{-at}\,e^{-st}\,dt$$

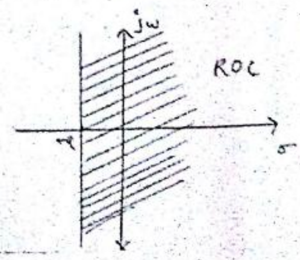
$$= \int_0^{\infty} e^{-at}\,e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_0^{\infty} e^{-(a+\sigma)t}\,e^{-j\omega t}\,dt$$

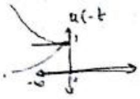
Here ∴ for convergence, $(a+\sigma)>0$

$\sigma > -a$

∴ ROC → $Re\{s\} > -a$



? $x(t) = -e^{-at}\, u(-t)$.



Ans. $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

$= \int_{-\infty}^{0} -e^{-at}\, u(-t)\, e^{-st}\, dt$

$= \int_{-\infty}^{0} -e^{-at}\, e^{-st}\, dt$

$= \left[\dfrac{-e^{-(a+s)t}}{-(a+s)} \right]_{-\infty}^{0}$

$= \left[\dfrac{e^{0}}{-(a+s)} - \cancel{\dfrac{e^{\infty}}{-(a+s)}} - \dfrac{e^{-(-a-s)\infty}}{-(a+s)} \right]$

$X(s) = \dfrac{1}{-(a+s)} = \dfrac{1}{s+a}$

$\therefore \cancel{X(s) = -1/(s+a)}$

$x(t) = e^{-at}$
- Signal rightside $= a > 0$ for convergence
- Signal leftside $= a < 0$ for convergence

Now, $a < 0$, $x(t) = e^{at}\, u(-t)$

~~$a > 0$, $x(t) = e^{-at}\, u(-t)$~~

$\cancel{X(s) = \int_{-\infty}^{0} e^{-at}\, e^{-st}\, dt}$

$\cancel{= \int_{-\infty}^{0} e^{-at}\, e^{-(\sigma+j\omega)t}\, dt}$

$= \int_{-\infty}^{0} e^{-(a+\sigma)t}\, e^{-j\omega t}\, dt$

$X(s) = -\int_{-\infty}^{0} e^{-at}\, e^{-\sigma t}\, e^{-j\omega t}\, dt$

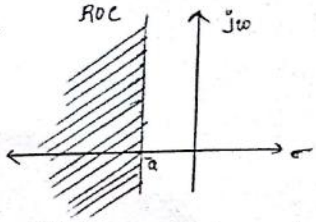
$= -\int_{-\infty}^{0} e^{-(a+\sigma)t}\, e^{-j\omega t}\, dt$

for convergence

$\Rightarrow a + \sigma < 0$

$\sigma < -a$

$\therefore Re\{s\} < -a$



x(t)	X(s)	ROC
$x(t) = e^{-at}\, u(t)$	$1/(s+a)$	$\sigma > -a$
$x(t) = -e^{-at}\, u(-t)$	$1/(s+a)$	$\sigma < -a$

? $x(t) = e^{at}\, u(t)$

Ans. $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$

$= \int_{0}^{\infty} e^{at}\, e^{-st}\, dt$

$= \int_{0}^{\infty} e^{(-s+a)t}\, dt$

$= \left[\dfrac{e^{(-s+a)t}}{-s+a} \right]_{0}^{\infty} = \left[\dfrac{e^{-[s-a]t}}{-s+a} \right]_{0}^{\infty}$

$= \left[\dfrac{e^{-\infty}}{-s+a} - \dfrac{e^{0}}{-s+a} \right] = 0 - \dfrac{1}{-[s-a]} = \dfrac{1}{s-a}$

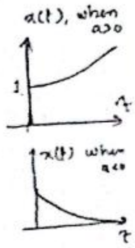
time domain = $x(t)\ \forall t$

freq domain = $X(j\omega)\ \forall \omega$

Now, for

$$X(s) = \frac{1}{s-a}$$

Now, for convergence, $a<0$ $\cancel{a>0}$



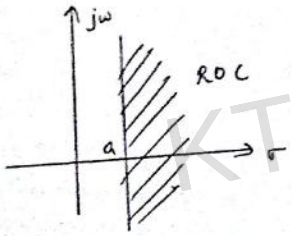
$$\therefore X(s) = \int_0^\infty e^{at}\, e^{-st}\, dt$$

$$= \int_0^\infty e^{at}\, e^{-(\sigma+j\omega)t}\, dt$$

$$= \int_0^\infty e^{(a-\sigma)t}\, e^{-j\omega t}\, dt$$

$\therefore$ for convergence, $(a-\sigma)<0$

$a < \sigma \Rightarrow \sigma > a.$



? $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$. Find Laplace transform.

Ans. $x_1(t) = 3e^{-2t}u(t)$, $x_2(t) = 2e^{-t}u(t)$

$$X_1(s) = \int_0^\infty 3e^{-2t}\, e^{-st}\, dt$$

$$= 3\int_0^\infty e^{-(2+s)t}\, dt$$

$$= 3\left[\frac{e^{-(2+s)t}}{-(2+s)}\right]_0^\infty$$

$$= 3\left[\frac{e^{-\infty}}{-(2+s)} - \frac{e^{0}}{-(2+s)}\right]$$

$$= 3\left[0 + \frac{1}{(2+s)}\right] = \frac{3}{2+s}$$

$$\therefore X_1(s) = \frac{3}{2+s}$$

Now, $X_2(s) = \int_0^\infty 2\, e^{-t}\, e^{-st}\, dt$

$$= 2\int_0^\infty e^{-(s+1)t}\, dt$$

$$= 2\left[\frac{e^{-(s+1)t}}{-(s+1)}\right]_0^\infty$$

$$= 2\left[\frac{e^{-\infty}}{-(s+1)} - \frac{e^{0}}{-(s+1)}\right]$$

$$= 2\left[0 + \frac{1}{s+1}\right]$$

$$X_2(s) = \frac{2}{s+1}$$

$$\therefore X(s) = X_1(s) - X_2(s)$$

$$= \frac{3}{2+s} - \frac{2}{s+1} = \frac{3s+3-4-2s}{2s+2+s^2+s}$$

$$= \frac{s-1}{s^2+3s+2} = \frac{s-1}{(s+1)(s+2)}$$

ROC of $X_1(s)$ is,

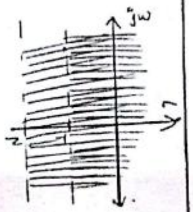
$x(t)$ is in the form $x(t) = e^{-at}u(t)$

∴ ROC $= \sigma > -a$

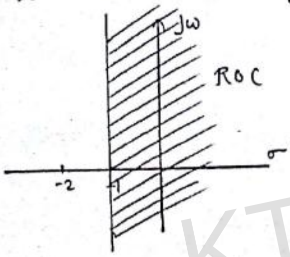
ROC of $X_1(s) = \sigma > -2$

ROC of $X_2(s)$ is, $\sigma > -a$

∴ $\sigma > -1$



∴ Final ROC is given by,



? $x(t) = e^{-2t}u(t) + e^{-t}\cos 3t\, u(t)$

Ans. $x(t) = e^{-2t}u(t) + e^{-t}\left(\frac{e^{3jt}+e^{-3jt}}{2}\right)u(t)$

$x(t) = e^{-2t}u(t) + \frac{e^{-t}e^{j3t}}{2}u(t) + \frac{e^{-t}e^{-3jt}}{2}u(t)$

$X_1(s) = \int_0^\infty e^{-2t}e^{-st}dt$

$= \left[\frac{e^{-(2+s)t}}{-(2+s)}\right]_0^\infty = \left[0 - \frac{1}{-(2+s)}\right]$

$X_1(s) = 1/(2+s)$

Now, $X_2(s) = \int_0^\infty \frac{e^{-t}e^{j3t}}{2}e^{-st}dt$

$= \frac{1}{2}\int_0^\infty e^{-(1-3j+s)t}dt$

$= \frac{1}{2}\left[\frac{e^{-(1-3j+s)t}}{-(1-3j+s)}\right]_0^\infty$

$= \frac{1}{2}\left[0 - \frac{1}{-(1-3j+s)}\right]$

$X_2(s) = \frac{1}{2(1-3j+s)}$

$X_3(s) = \int_0^\infty \frac{e^{-t}e^{-3jt}}{2}e^{-st}dt$

$= \frac{1}{2}\left[\frac{e^{-(1+3j+s)t}}{-(1+3j+s)}\right]_0^\infty$

$X_3(s) = \frac{1}{2(1+3j+s)}$

$X(s) = \frac{1}{2+s} + \frac{1}{2(1-3j+s)} + \frac{1}{2(1+3j+s)}$

$= \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}$

ROC of $X_1(s) = \sigma > -2$

ROC of 2nd term is, $\sigma > -1$

$\therefore$ Final ROC is, $\sigma > -1$

Zeros & Poles :

→ The points at which $X(s) = 0$, for a rational LT, are called zero. (Root of numerator = zeros) $X(s) = 0 \Rightarrow P(s) = 0$. $\left(X(s) = \frac{P(s)}{Q(s)} \right)$

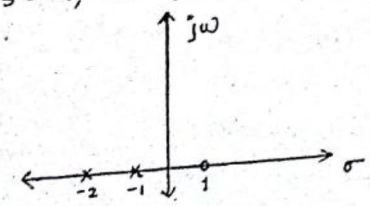
→ The points at which $X(s) = \infty$, is called poles. (roots of denominator).

→ zeros are represented as 'o' and poles as 'x'.

Example; $X(s) = s-1 / s^2+3s+2$

zeros : $s = 1$ $(X(s) = 0)$

poles : $s = -1, -2$ $(X(s) = \infty)$



Here, poles = 2, zeros = 1

ie, zeros < poles, then we can say that (2-1) zeros at $s = \infty$.

. when $P < Z$, then $(Z-P)$ poles at $s = \infty$.

Q $X(s) = 2s^2+5s+12 / (s^2+2s+10)(s+2)$

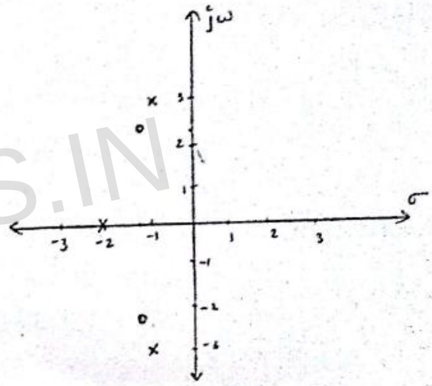
Ans. Zeros, $\Rightarrow 2s^2+5s+12 = 0$

$s_1 = -1.25 \pm j\,2.1$

Poles, $\Rightarrow (s^2+2s+10) = 0$

$s = -1+3j, \; -1-3j, \; -2$

$\therefore$ poles $= -1+3j, \; -1-3j, \; -2$



PROPERTIES ROC :

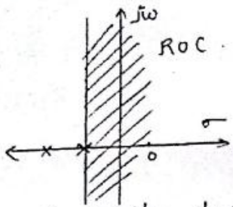
→ (i) ROC of $X(s)$ consists of strips parallel to the $j\omega$ axis in the s-plane.

→ (ii) For rational laplace transform, ROC doesn't contains any poles.

Example, $X(s) = s-1/(s+2)(s+1)$

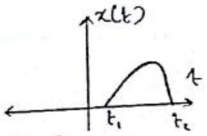
ROC is $\sigma > -1$

i.e, there is no poles in ROC.



(iii) If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

Example:



- Here, $x(t)$ have finite duration ~~x(t) have~~ $\int_{-\infty}^{\infty} |x(t)|dt < \infty$ absolutely integrable and (area) or it is absolutely integrable, then ROC is the entire s-plane.

(iv) If $x(t)$ is right sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all values of s for which $Re\{s\} > \sigma_0$ will also be in the ROC.

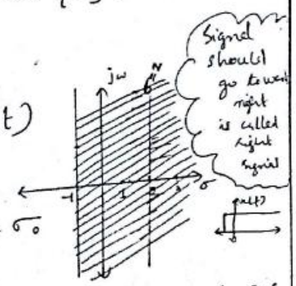
- Example; $x(t) = e^{-t} u(t)$

Here, ROC $= \sigma > -1$

Now, $Re\{s\} = \sigma_0 \Rightarrow \sigma = \sigma_0$

i.e, let $\sigma = 2$ in the ROC

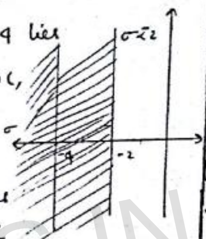
then $Re\{s\} > \sigma_0 \Rightarrow \sigma > \sigma_0 \Rightarrow \sigma > 2$ also in the ROC.



(V) If $x(t)$ is left sided signal and if the line $Re\{s\} = \sigma_0$ is in the ROC, then all the values of s for which $Re\{s\} < \sigma_0$ will also be in the ROC.

- Example; $x(t) = -e^{-2t} u(-t)$

Let $\sigma = -4$ lies in the ROC, then $\sigma < -4$ is also lies in the ROC



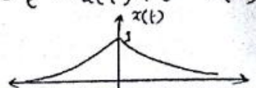
(vi) If $x(t)$ is a two sided signal and if the line $Re\{s\} = \sigma_0$ is in the ROC, then, the ROC will consist of a strip in the s-plane that includes the line $Re\{s\} = \sigma_0$.

Example; $x(t) = e^{-a|t|}$

when $a > 0$, $a = 2$

$x(t) = e^{-2|t|}$

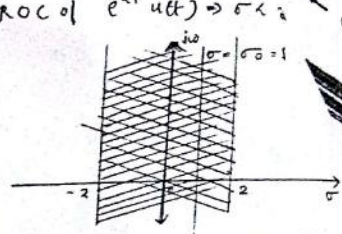
$x(t) = e^{-2t} u(t) + e^{2t} u(-t)$



$\therefore X(s) = 1/s+2 + 1/s$ ... i.e region the right pole.

ROC of $e^{-\alpha t} u(t) \Rightarrow \sigma >$

ROC of $e^{\alpha t} u(-t) \Rightarrow \sigma <$



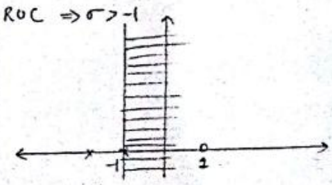
$\sigma = \sigma_0 = 1$ is in the ROC, then $\sigma_0 = 1$ is also in b/w -2 to 2 (in ROC).

(vii) If laplace transform of $x(t)$ is rational, ~~and its~~ ~~ROC~~ then its ROC is bounded by poles or $x \to \infty$.

Example: $x(t) = e^{-2t} u(t) + e^{-t} u(t)$

$X(s) = s-1/(s+1)(s+2)$

ROC $\Rightarrow \sigma > -1$



(viii) If the laplace transform of $x(t)$ is rational, and ~~then~~ if $x(t)$ is right sided, then ROC is

Example

Scanned by CamScanner

the region in the s-plane to the right of the rightmost pole.

Example : $X(s) = \frac{s-1}{(s+1)(s+2)}$

(Some example).

- If $x(t)$ is left sided, then the ROC is the region in the s-plane to the left of the leftmost pole.

☆ INVERSE LAPLACE TRANSFRM

- $x(t) = e^{-at}u(t)$

  LT $= 1/(s+a)$, ROC $= \sigma > -a$
- $x(t) = -e^{-at}u(-t)$

  LT $= 1/(s+a)$, ROC $= \sigma < -a$
- $x(t) = e^{at}u(t)$

  LT $= 1/(s-a)$, ROC $= \sigma > a$
- $x(t) = -e^{at}u(-t)$,

  LT $= 1/(s-a)$, ROC $= \sigma < a$
- $x(t) = u(t)$ $\{ e^{-at}u(t) = e^{0t}u(t) = u(t)$ $(\because a=0)$ $\therefore$ LT $= 1/(s+0) = 1/s$

  LT $= 1/s$, ROC $= \sigma > 0$

? $X(s) = \frac{1}{(s+1)(s+2)}$, $\sigma > -1$ $\quad Re\{s\} > -1$

Find ILT (ie, $x(t)$).

Ans. $X(s) = A/(s+1) + B/(s+2)$

$= A(s+2) + B(s+1)$

$= s(A+B) + 2A + B$

$s = 0 \Rightarrow A = +1,\ B = -1$

$X(s) = 1/(s+1) - 1/(s+2)$

$x(t) = e^{-t}u(t) - e^{-2t}u(t)$.

? $X(s) = \frac{1}{(s+1)(s+2)}$, $\sigma < -2$

Ans. $X(s) = A/(s+1) + B/(s+2)$

$A = 1,\ B = -1$

$X(s) = 1/(s+1) - 1/(s+2)$

$\therefore x(t) = -e^{-t}u(-t) - -e^{-2t}u(-t)$

$x(t) = -e^{-t}u(-t) + e^{-2t}u(-t)$

? $X(s) = \frac{1}{(s+1)(s+2)}$, $-2 < \sigma < -1$

Ans. $-1 < \sigma < -2 \Rightarrow \sigma > -2$ & $\sigma < -1$

$X(s) = A/(s+1) + B/(s+2)$

$A = 1,\ B = -1$

$\therefore X(s) = 1/(s+1) + -1/(s+2)$

$\therefore x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$

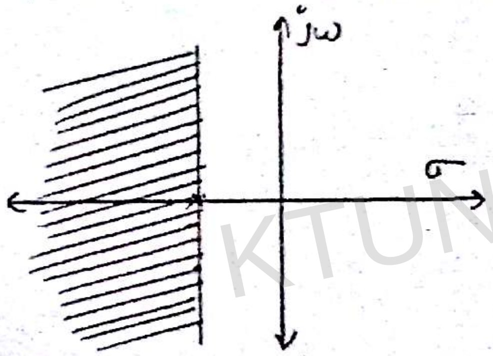
$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$

★ STABILITY:

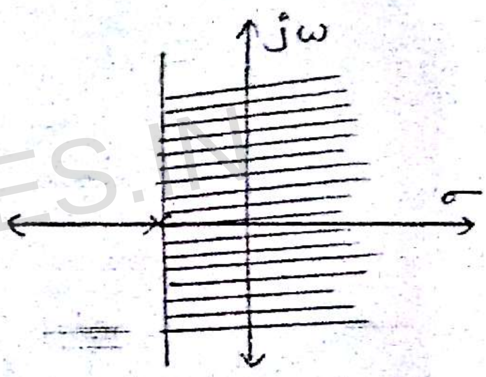
→ For an LTI system, if impulse response $h(t)$ is absolutely integrable, then the system is stable.

i.e, $\int_{-\infty}^{\infty} h(t)\,dt < \infty \Rightarrow$ stable.

→ In the case of laplace transform, $H(s)$ should include the $j\omega$ axis, for stability.
That is, if a system includes $j\omega$ in the ROC, then the system is a stable system and otherwise, it is unstable.



doesn't includes jω.
It is not stable



Includes jω.
Hence, it is stable

Here, $H(s)$ = transfer function
$= L\{h(t)\}$

$x(t) \rightarrow h(t) \rightarrow y(t)$

$y(t) = x(t) * h(t)$

$Y(s) = X(s) \cdot H(s) \Rightarrow H(s) = Y(s)/X(s)$

→ Transfer function is the ratio of laplace transform of output to the laplace transform of input.

$$H(s) = \frac{Y(s)}{X(s)} = \frac{L\{o/r\}}{L\{i/r\}}$$

Q) A transfer function $H(s) = \frac{s-1}{(s+1)(s-2)}$ is given. Calculate the impulse response $h(t)$.

Ans. We have to find ILT.

$$\frac{s-1}{(s+1)(s-2)} = A/(s+1) + B/(s-2)$$

$$(s-1) = A(s-2) + B(s+1)$$

when $s = 2 \Rightarrow 1 = B(3) \Rightarrow B = 1/3$

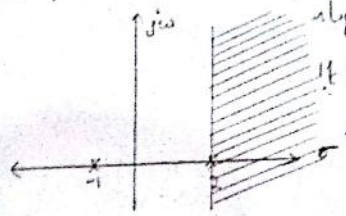
when $s = -1 \Rightarrow -2 \Rightarrow A(-3) = A = 2/3$

$$H(s) = \frac{2}{3(s+1)} + \frac{1}{3(s-2)}$$

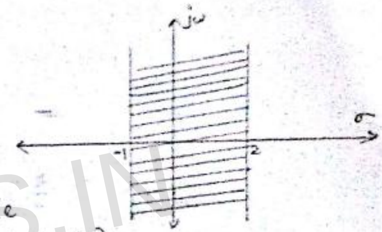
(1) $h(t) = 2/3\, e^{-t} u(t) + 1/3\, e^{2t} u(t)$

$\Rightarrow$ Unstable

Here, poles are, -1 & 2

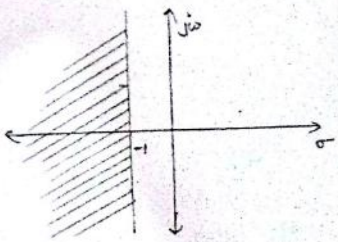


(2) $h(t) = 2/3\, e^{-t} u(t) - 1/3\, e^{2t} u(-t)$ $\begin{cases} \sigma > -1 \\ \sigma < 2 \end{cases}$



$\Rightarrow$ Stable (includes jω)

(3) $h(t) = -2/3\, e^{-t} u(-t) - 1/3\, e^{2t} u(-t)$ $\begin{cases} \sigma < 2 \\ \sigma < -1 \end{cases}$



$\Rightarrow$ Unstable. (doesn't include jω)

ional trans
nly if all
lf half of the
have negative

→ Transfer function :

ansform LTI system, it is causal when input , for $t<0$.

$=0$, for $t<0$ ⇒ right sided signal.

ie, ROC should be right side of the right most poles, in the case of causal signals.

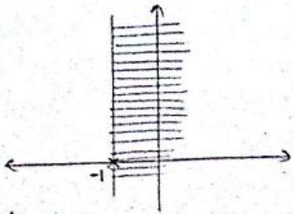
Note → ROC for a causal system is the right half plane.

→ For a system with rational transfer function (s/m fn), causality of system is equivalent to the ROC being the right half plane to the right of the right most pole.

Example: $h(t) = e^{-t}\,u(t)$. Check whether the system is causal.

Ans. $H(s) = \frac{1}{s+1}$

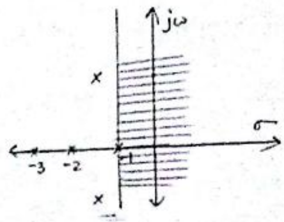
ROC, $\sigma > -a \Rightarrow \sigma > -1$



ie, ROC being the right of the rightmost pole. Hence the system is both causal & stable bcz, ROC contains $j\omega$ axis.

Imp: A causal system with rational transfer function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left half of the s-plane. That is, all of the poles have negative real parts.

for both causal & stable, all the poles should be in the left half



$x(t)$	$X(s)$	ROC
• $\cos\omega_0 t\, u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\sigma > 0$
• $\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\sigma > 0$
• $e^{-\alpha t}\cos\omega_0 t\, u(t)$	$\frac{(s+\alpha)}{(s+\alpha)^2+\omega_0^2}$	$\sigma > -\alpha$
• $e^{-\alpha t}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$	$\sigma > -\alpha$
• $t^n\, u(t)$	$\frac{n!}{s^{n+1}}$	not defined

? $X(s) = \dfrac{s^2+12}{s(s+2)(s+3)}$

Calculate $x(t)$.

Ans $\dfrac{s^2+12}{s(s+2)(s+3)} = \dfrac{A}{s} + \dfrac{B}{s+2} + \dfrac{C}{s+3}$

$s^2+12 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$

when $s=-2$

$16 = B(-2)(-2+3) \Rightarrow 16 = B(-2)$

$B = 16/-2 = -8$

when $s=-3$,

$21 = 0 + 0 + -3C(-1)$

$21 = 3C \Rightarrow C = 21/3 = 7$

when $s=0$, $12 = A(2)(3)$

$A = 12/6 = 2$

$\therefore A = 2,$

$B = -8,$

$C = 7.$

$\therefore \dfrac{s^2+12}{s(s+2)(s+3)} = \dfrac{2}{s} + \dfrac{-8}{s+2} + \dfrac{7}{s+3}$

$x(t) = 2u(t) - 8e^{-2t}u(t) + 7e^{-3t}u(t)$

$x(t) = [2 - 8e^{-2t} + 7e^{-3t}]u(t)$

ROC, $\sigma > 0$ $\{\sigma>0, \sigma>-2, \sigma>-3\} \Rightarrow \sigma>0$ (com

? Calculate $x(t)$ of a signal whose $X(s)$ is given by, $X(s) = \dfrac{20}{(s+3)(s^2+8s+25)}$

Ans $\dfrac{20}{(s+3)(s^2+8s+25)} = \dfrac{A}{s+3} + \dfrac{Bs+C}{s^2+8s+25}$

when $s=-3$, $A = \dfrac{20}{9-24+25} = \dfrac{20}{10} = 2$

put $s=0$, $\dfrac{20}{3(25)} = \dfrac{A}{3} + \dfrac{C}{25}$

$\dfrac{20}{3\cdot 25} = \dfrac{2}{3} + \dfrac{C}{25}$

$\dfrac{20}{75} - \dfrac{2}{3} = \dfrac{C}{25} \Rightarrow C = -10$

when $s=1$,

$\dfrac{20}{4\cdot 34} = \dfrac{A}{4} + \dfrac{B+C}{34}$

$\dfrac{5}{34} = \dfrac{2}{4} + \dfrac{B-10}{34}$

$\dfrac{5}{34} - \dfrac{2}{4} + \dfrac{10}{34} = \dfrac{B}{34}$

$\left(\dfrac{15}{34} - \dfrac{2}{4}\right) \times 34 = B \Rightarrow B = -2$

$\therefore X(s) = \dfrac{2}{s+3} + \dfrac{-2s-10}{s^2+8s+25}$

$$= \frac{2}{s+3} + \frac{-2s-10}{(s+4)^2+9}$$

$$= \frac{2}{s+3} + \frac{-2s-10}{[s^2+8s+16]+9}$$

$$= \frac{2}{s+3} + \frac{-2s-10}{(s+4)^2+9}$$

$$= \frac{2}{s+3} + \frac{-2(s+4)-2}{(s+4)^2+9}$$

$$= \frac{2}{s+3} - \frac{2(s+4)}{(s+4)^2+9} - \frac{2}{(s+4)^2+9}$$

$$= \frac{2}{s+3} - \frac{2(s+4)}{(s+4)^2+9} - \frac{2\times 3}{3\,[(s+4)^2+9]}$$

$$x(t) = 2e^{-3t}u(t) - 2e^{-4t}\cos 3t\, u(t) - \frac{2}{3} e^{-4t}\sin 3t\, u(t)$$

$$\frac{1}{s} \longleftrightarrow u(t)$$

$$\frac{1}{s^2} \longleftrightarrow t\,u(t)$$

$$\frac{1}{s+1} \longleftrightarrow e^{-t}u(t)$$

$$\frac{1}{(s+1)^2} \longleftrightarrow e^{-t}u(t) \times t = te^{-t}u(t)$$

$$\frac{1}{s^3} \longleftrightarrow \frac{t^2}{2}\, u(t)$$

$$\frac{1}{(s+1)^3} \longleftrightarrow e^{-t}u(t) \times \frac{t^2}{2} = \frac{t^2}{2}e^{-t}u(t)$$

$$\frac{1}{(s-1)^3} \longleftrightarrow \frac{t^2}{2}\, e^{t}u(t)$$

☆

Unilateral LT	Bilateral LT
• $X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$	• $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$
• No need to specify ROC	• ROC is must
• Only causal	• Causal & non causal
• Evaluation of inverse is same	• Evaluation of inverse is same
• Applications: Used to solve linear constant coefficient differential equations with non zero initial condition.	• Application: System stability, causality.

Bilateral is $X(s) = \int$

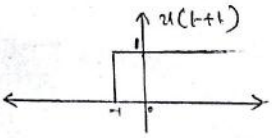
? Find unilateral LT (ULT) and bilateral LT (BLT) of $x(t)$

$$x(t) = e^{-\alpha(t+1)}u(t+1)$$

Ans. BLT,

$$X(s) = \int_{-\infty}^{\infty} e^{-\alpha(t+1)}\, u(t+1)\, e^{-st}\, dt$$

$$= \int_{-1}^{\infty} e^{-\alpha(t+1)} \times 1\; e^{-st}\, dt$$



$$= \int_{-1}^{\infty} e^{-\alpha t} \cdot e^{-\alpha} \cdot e^{-st}\, dt$$

$$= e^{-\alpha} \int_{-1}^{\infty} e^{-t(\alpha+s)}\, dt = e^{-\alpha} \left[\frac{e^{-t(\alpha+s)}}{-(\alpha+s)} \right]_{-1}^{\infty}$$

$$= \frac{e^{-\alpha}}{-(\alpha+s)} \left[e^{-\infty} - e^{(\alpha+s)} \right]$$

$$= \frac{e^{-\alpha} \times \cancel{-}\, e^{(\alpha+s)}}{\cancel{-}(\alpha+s)} = \frac{e^{-\alpha}\, e^{(\alpha+s)}}{\alpha+s}$$

$$= \frac{\cancel{e^{-\alpha}} \cdot \cancel{e^{\alpha}} \cdot e^{s}}{\alpha+s} = \underline{\underline{e^{s}/_{\alpha+s}}}$$

Now, ULT,

$$X(S) = \int_{0}^{\infty} e\, x(t)\, e^{-st}\, dt$$

This is right side ; 0 to ∞

$$= \int_{0}^{\infty} e^{-\alpha(t+1)}\, u(t+1)\, e^{-st}\, dt$$

$$= e^{-\alpha} \int_{0}^{\infty} e^{-t(\alpha+s)}\, dt$$

$$= e^{-\alpha} \left[\frac{e^{-t(\alpha+s)}}{-(\alpha+s)} \right]_{0}^{\infty} = \frac{e^{-\alpha}}{-(\alpha+s)} [0-1]$$

$$X(s) = \frac{e^{-\alpha}}{s+\alpha}$$

✱ PROPERTIES OF LAPLACE TRANSFORM:

(1) LINEARITY.

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ with ROC $= R_1$

and $x_2(t) \xleftrightarrow{LT} X_2(s)$ with ROC $= R_2$.

then $\alpha x_1(t) + \beta x_2(t) \longleftrightarrow \alpha X_1(s) + \beta X_2(s)$ with ROC $= R_1 \cap R_2$.

PROOF:

$$\alpha X_1(s) = \int_{-\infty}^{\infty} \alpha\, x_1(t) \cdot e^{-st}\, dt \quad \ldots\ldots (1) \quad ROC = R_1$$

$$\beta X_2(s) = \int_{-\infty}^{\infty} \beta\, x_2(t)\, e^{-st}\, dt \quad \ldots\ldots (2) \quad ROC = R_2.$$

$$(1)+(2) \Rightarrow \alpha X_1(s) + \beta X_2(s) = \int_{-\infty}^{\infty} \alpha x_1(t)\, e^{-st} dt + \int_{-\infty}^{\infty} \beta x_2(t)\, e^{-st}$$

$$= \int_{-\infty}^{\infty} [\alpha\, x_1(t) + \beta\, x_2(t)]\, e^{-st}\, dt$$

ie, $\alpha x_1(t) + \beta x_2(t) \longleftrightarrow \alpha X(s) + \beta X_2(s)$

2. TIME SHIFTING.

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

then $x(t-t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$ with ROC = R

PROOF :

$$LT\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0) e^{-st} dt$$

put $t - t_0 = \tau$ thus $dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau+t_0)} d\tau = e^{-st_0} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau$$

$$= e^{-st_0} X(s)$$

Similarly, $x(at+b) \xleftrightarrow{LT} \frac{1}{|a|} e^{sb/a} X(s/a)$

3. SHIFT IN S-DOMAIN:

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

then $e^{s_0 t} \cdot x(t) \xleftrightarrow{LT} X(s-s_0)$ with ROC = R + Re{s_0}

Proof:

$$LT\{x(t) e^{s_0 t}\} = \int_{-\infty}^{\infty} x(t) \cdot e^{s_0 t} e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-(s-s_0)t} dt$$

$$= X(s-s_0)$$

4. TIME SCALING:

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

then $x(at) \xleftrightarrow{LT} \frac{1}{|a|} X(s/a)$ with ROC = |a| R

PROOF:

By definition,

$$LT[x(at)] = \int_{-\infty}^{\infty} x(at) e^{-st} dt$$

put $at = \tau$, thus $dt = d\tau/a$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s/a \, \tau} \, d\tau/a = 1/a \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s/a \, \tau} d\tau$$

$$= 1/a \, X(s/a)$$

As 'a' is real constant may be positive or negative

So, generalising it,

$$L[x(at)] = \frac{1}{|a|} \cdot X(s/a).$$

5. TIME REVERSAL:

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

then $x(-t) \xleftrightarrow{LT} X(-s)$ with ROC = − R.

PROOF:

Let $y(t) = x(-t)$.

$$Y(s) = \int_{-\infty}^{\infty} y(t) \cdot e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(-t) \cdot e^{-st}\, dt$$

put $-t = m$, $-dt = dm$

when $t \to -\infty$, $m \to \infty$

$t \to \infty$, $m \to -\infty$

$$\therefore Y(s) = \int_{\infty}^{-\infty} x(m) \cdot e^{sm} (-dm)$$

$$= \int_{-\infty}^{\infty} x(m)\, e^{sm}\, dm = \int_{-\infty}^{\infty} x(m) \cdot e^{-(-m)s}\, dm$$

$$= X(-s)$$

ie, $x(-t) \xleftrightarrow{LT} X(-s)$

6. DIFFERENTIATION IN TIME:

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

then $dx(t)/dt \longleftrightarrow sX(s)$ with ROC containing R.

PROOF:

By definition of inverse laplace transform,

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) \cdot e^{st}\, ds.$$

differentiating bothsides, we obtain

$$dx(t)/dt = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} [s\, X(s)]\, e^{st}\, ds$$

$$= LT^{-1}[s\, X(s)]$$

$$\therefore dx(t)/dt \longleftrightarrow s\, X(s).$$

7. INTEGRATION IN TIME DOMAIN:

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

then $\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{LT} X(s)/s$ with ROC = $R \cap Re\{s\}$

PROOF:

Consider the convolution $x(t) * u(t)$

$$= \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$$

Since $u(t-\tau) = 1$, for $t > 0$ (or $\tau < t$)

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$L\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = L[x(t) * u(t)] = X(s) \cdot U(s) = \frac{X(s)}{s}$$

Thus $\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{LT} X(s)/s$

(viii) CONVOLUTION IN TIME DOMAIN:

Convolution in time domain corresponds to multiplication in S-domain

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ with ROC $= R_1$

and $x_2(t) \xleftrightarrow{LT} X_2(s)$ with ROC $= R_2$

then $x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s) \cdot X_2(s)$ with ROC $= R_1 \cap R_2$.

PROOF:

$$L\left[x_1(t) * x_2(t)\right] = \int_{-\infty}^{\infty} \left[x_1(t) * x_2(t)\right] e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, e^{-st}\, d\tau \cdot dt \quad \frac{e^{s\tau}}{e^{s\tau}}$$

On rearranging, we get,

$$= \int_{-\infty}^{\infty} x_1(\tau)\, e^{-s\tau} d\tau \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-s(t-\tau)} dt$$

$$= X_1(s) \cdot X_2(s).$$

9. MULTIPLICATION IN TIME DOMAIN:

Multiplication in time domain corresponds to Convolution in S-domain.

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ with ROC $= R_1$

and $x_2(t) \xleftrightarrow{LT} X_2(s)$ with ROC $= R_2$.

then $x_1(t) \cdot x_2(t) \xleftrightarrow{LT} \frac{1}{2\pi j}\left[X_1(s) * X_2(s)\right]$

with ROC containing $R_1 \cap R_2$

PROOF:

$$LT\left[x_1(t) \cdot x_2(t)\right] = \int_{-\infty}^{\infty} x_1(t) \cdot x_2(t) \cdot e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x_1(t) \left[\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X_2(s_0)\, e^{s_0 t} ds_0\right] e^{-st} dt$$

On rearranging we get,

$$\frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X_2(s_0) \cdot \int_{-\infty}^{\infty} \left\{x_1(t)\, e^{s_0 t}\right\} e^{-st} dt \cdot ds_0.$$

From the use of frequency shifting property, we get,

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X_2(s_0) \cdot X_1(s-s_0)\, ds_0$$

$$= \frac{1}{2\pi j}\left\{X_2(s) * X_1(s)\right\} = \frac{1}{2\pi j}\left\{X_1(s) * X_2(s)\right\}$$

Thus, $x_1(t) \cdot x_2(t) \xleftrightarrow{LT} \frac{1}{2\pi j}\left\{X_1(s) * X_2(s)\right\}$

10. DIFFERENTIATION IN - S - DOMAIN:

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

$-t \cdot x(t) \xleftrightarrow{LT} \frac{d}{ds} X(s)$ with ROC = R

PROOF:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

differentiating with respect to 's',

$$\frac{d}{ds} X(s) = \int_{-\infty}^{\infty} x(t) \cdot \{-t\} \cdot e^{-st} dt$$

$$\frac{d}{ds} X(s) = \int_{-\infty}^{\infty} \{-t \cdot x(t)\} \cdot e^{-st} dt$$

$$= LT \{-t\, x(t)\}.$$

$\therefore -t\, x(t) \xleftrightarrow{LT} \frac{d}{ds} X(s)$.

Similarly, $(-t)^n x(t) \xleftrightarrow{LT} d^n X(s)/ds^n$.

Hence, proved.

11. INTEGRATION IN S-DOMAIN:

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

then $x(t)/t \xleftrightarrow{LT} \int_{-\infty}^{\infty} X(s) \cdot ds$

PROOF:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Integrating both the sides w.r.t. s, from s to

$$\int_{s}^{\infty} X(s) ds = \int_{s}^{\infty} \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt \cdot ds = \int_{-\infty}^{\infty} x(t) \left[\int_{s}^{\infty} e^{-st} ds\right] dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \left[e^{-st}/-t\right]_{s}^{\infty} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \left[e^{-st}/t\right] dt$$

$$\int_{s}^{\infty} X(s) ds = \int_{s}^{\infty} (x(t)/t)\, e^{-st} dt = LT\left[x(t)/t\right]$$

$\therefore x(t)/t \xleftrightarrow{LT} \int_{s}^{\infty} X(s) \cdot ds$.

11. INTEGRATION IN S-DOMAIN:

If $x(t) \xleftrightarrow{LT} X(s)$ with ROC = R

then $\frac{x(t)}{t} \xleftrightarrow{LT} \int_{s}^{\infty} X(s) ds$.

PROOF:

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

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Integrating bothsides w.r.t. s, from $s \to \infty$,

$$\int_s^{\infty} X(s)\,ds = \int_s^{\infty}\int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt\cdot ds$$

$$= \int_{-\infty}^{\infty} x(t)\left[\int_s^{\infty} e^{-st}\,ds\right]dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot\left[e^{-st}/-t\right]_s^{\infty}dt$$

$$= \int_{-\infty}^{\infty} x(t)\left[e^{-st}/t\right]dt$$

$$\int_s^{\infty} X(s)\,ds = \int_s^{\infty}\left[\frac{x(t)}{t}\right]e^{st}\,dt = LT\left[\frac{x(t)}{t}\right]$$

$$\therefore \frac{x(t)}{s} \xleftrightarrow{LT} \int_s^{\infty} X(s)\,ds.$$

MODULE - IV

ANALYSIS OF LTI SYSTEM

(Continuous time)

* DIFFERENTIAL EQUATION :

$$\sum_{K=0}^{N} a_K \frac{d^K y(t)}{dt^K} = \sum_{K=0}^{M} b_K \frac{d^K x(t)}{dt^K} \quad \dots (1)$$

CT → differential eqn. $dy(t)/dt$

DT → difference eqn. $y[n]$, $y[n-1]$

When N=M=1

$$\sum_{K=0}^{1} a_K \frac{d^K y(t)}{dt^K} = \sum_{K=0}^{1} b_K \frac{d^K x(t)}{dt^K}$$

$x(t) \to \boxed{h(t)} \to y(t)$

$x[n] \to \boxed{h[n]} \to y[n]$

$$a_0 y(t) + a_1 \frac{dy(t)}{dt} = b_0 x(t) + b_1 \frac{dx(t)}{dt}$$

• FREQUENCY RESPONSE :

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{FT\{o/p\}}{FT\{i/p\}}$$

Transfer fun^n:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{L\{o/p\}}{L\{i/p\}}$$

with nil initial condition

The frequency response of (1) is given by,

$$\sum_{K=0}^{N} a_K (j\omega)^K\, Y(j\omega) = \sum_{K=0}^{M} b_K (j\omega)^K X(j\omega)$$

(by fourier transforming on bothsides).

Now, $Y(j\omega)\left[\sum_{K=0}^{N} a_K (j\omega)^K\right] = X(j\omega)\left[\sum_{K=0}^{M} b_K (j\omega)^K\right.$