Test whether the random process $X(t) = A\cos(\omega t + \theta)$ is WSS if θ is uniformly distributed in the interval $[-\pi, \pi]$

Inorder to get whether it is a WSS process; we need to check whether

→ mean of the random process is constant

→ Autocorrelation function depends only on the difference between the observation time (τ) and not on the exact time.

Here it is given that in $\cos(\omega t + \theta)$; θ is uniformely distributed over the interval $[-\pi, \pi]$

The pdf of a uniform distribution is given by

$$f(x) = \frac{1}{b-a} \qquad a \le x \le b$$

so

$$f(\theta) = \frac{1}{\pi + \pi} \qquad -\pi < \theta < \pi$$

$$\boxed{f(\theta) = \frac{1}{2\pi} \qquad -\pi < \theta < \pi}$$

(i) To prove $E[X(t)] = \text{constant}$

$$E[X(t)] = \int_{-\infty}^{\infty} X(t)\, f(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} A\cos(\omega t + \theta) \frac{1}{2\pi}\, d\theta$$

$$= \frac{A}{2\pi} \int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta$$

$$= \frac{A}{2\pi} \Big[\sin(\omega t + \theta) \Big]_{-\pi}^{\pi}$$

$$= \frac{A}{2\pi} \left[\sin(\omega t + \pi) - \sin(\omega t - \pi) \right]$$

$$= \underline{\underline{0}}$$

So $E[x(t)]$ or mean of the random process is constant.

(ii) To check whether auto correlation function depends only on τ

$$R_x(\tau) = E[x(t) \cdot x(t+\tau)]$$

$$E[x(t) \cdot x(t+\tau)] = A\cos(\omega t + \theta)\, A\cos(\omega(t+\tau) + \theta)$$

$\hookrightarrow 2\cos A \cos B = \cos(A-B) + \cos(A+B)$

$A = \omega t + \theta \quad B = \omega t + \omega\tau + \theta$

$A - B = -\omega\tau$

$A + B = 2\omega t + \omega\tau + 2\theta$

$$= \frac{A^2}{2} \left[\cos(-\omega\tau) + \cos(2\omega t + \omega\tau + 2\theta) \right]$$

even function

$$= \frac{A^2}{2} \left[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\theta) \right]$$

$$E[x(t)\cdot x(t+\tau)] = \frac{A^2}{2} E[\cos\omega\tau + \cos(\omega 2t + \omega\tau + 2\theta)]$$

$$= \underbrace{\frac{A^2}{2} E[\cos\omega\tau]}_{a} + \underbrace{\frac{A^2}{2} E[\cos(2\omega t + \omega\tau + 2\theta)]}_{b}$$

w. k. that $E[\text{constant}] = \text{constant}$

$\therefore \frac{A^2}{2} E[\cos(\omega\tau)] = \frac{A^2}{2}\cos(\omega\tau)$ Because here θ is the variable which uniformely distributed and no θ function here.

By taking the b part of the equation.

$$\frac{A^2}{2} E[\cos 2\omega t + \omega\tau + 2\theta] \Rightarrow \frac{A^2}{2}\cdot\int_{-\pi}^{\pi} x(t) f_\theta \, d\theta$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi} \cos(2\omega t + \omega\tau + 2\theta)\frac{1}{2\pi}\, d\theta$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}\cos(2\omega t + \omega\tau + 2\theta)\, d\theta$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin(2\omega t + \omega\tau + 2\theta)}{2}\right]_{-\pi}^{\pi}$$

$$= \frac{A^2}{8\pi}\left[\sin(2\omega t + \omega\tau + 2\pi) - \sin(2\omega t + \omega\tau - 2\pi)\right]$$

$$= 0$$

Substituting the results of a part and b part in the equation; we get

$$E[x(t) \cdot x(t+\tau)] = \frac{A^2}{2} \cos(\omega\tau)$$

ie; $$R_x(\tau) = \frac{A^2}{2} \cos(\omega\tau)$$

So Autocorrelation function depends only on the time difference τ.

As the mean is constant and autocorrelation is a function of τ only the given process is a WSS process.