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Roll. No: 46

1) Express the point $P(3, 45°, 210°)$ in cartesian coordinate system

Given the spherical coordinates $(r, \theta, \phi) = (3, 45°, 210°)$

To convert it to cartesian coordinate

$$x = r\sin\theta\cos\phi = 3\sin(45°)\cos(210°) = -1.837$$

$$y = r\sin\theta\sin\phi = 3\sin(45°)\sin(210°) = -1.061$$

$$z = r\cos\theta = 3\cos(45°) = 2.1213$$

The cartisian coordinate of point P is $(-1.837, -1.061, 2.1213)$

2) Express the point ~~P(~~ $P(1, -4, -3)$ in cylindrical and in Spherical coordinate system.

To express P in cylindrical coordinate system

$$\rho = \sqrt{x^2+y^2} = 4.123$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = -75.96° = 284.036°$$

$$z = z = -3$$

Cylindrical coordinates of P is $(4.123, 284.036°, -3)$

To express in spherical coordinate system

$$r = \sqrt{x^2+y^2+z^2} = 5.099$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} = 126.039°$$

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = -75.963° = 284.036°$$

The sperical coordinates of P is $(5.099, 126.039°, 284.036°)$

3) Find the distance between $A(5, \frac{3\pi}{2}, 0)$ & $B(5, \frac{\pi}{2}, 10)$

To convert it to cartesian coordinates

$x_A = \rho_A \cos\phi_A = 0 \qquad x_B = \rho_B \cos\phi_B = 0$

$y_A = \rho_A \sin\phi_A = -5 \qquad y_B = \rho_B \sin\phi_B = 5$

$z_A = z_A = 0 \qquad z_B = 10$

distance between A & B, $AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}$

$AB = \sqrt{100 + 100} = \underline{10\sqrt{2}}$

4) Determine the divergence of the vector field

$\vec{E} = (a^3 \cos\theta / r^2)\hat{a}_r - (a^3 \sin\theta / r^2)\hat{a}_\theta$

at $(a/2, 0, \pi/2)$

$$\nabla \cdot \vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial (E_\phi)}{\partial\phi}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \times a^3\frac{\cos\theta}{r^2}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-a^3\frac{\sin\theta}{r^2} \times \sin\theta\right) + 0$$

$$= 0 + \frac{1}{r\sin\theta} \times \frac{-a^3}{r^2} \times 2\sin\theta\cos\theta$$

$$= \frac{-2a^3}{r^3}\cos\theta$$

$$\nabla \cdot \vec{E}\Big|_{(a/2, 0, \pi/2)} = \frac{2a^3}{(a/2)^3}\cos 0 = \underline{-16}$$

5. For a vector field $\vec{A} = \rho^2 \hat{a}_\rho + 2z\hat{a}_z$, Verify divergence theorem for the circular cylindrical region enclosed by $\rho = 5$, $z = 0$, $z = 4$.

Divergence theorem $\quad \psi = \oint_S \vec{A}\cdot d\vec{s} = \int_V (\nabla\cdot\vec{A})\, dv$.

RHS $\quad \int_V (\nabla\cdot\vec{A}) dv$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(A_\phi) + \frac{\partial}{\partial z}(A_z)$$

$$= \frac{1}{\rho}\times\frac{\partial}{\partial\rho}(\rho^3) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(0) + \frac{\partial}{\partial z}(2z)$$

$$= 3\rho + 2.$$

$$\int_V (\nabla\cdot\vec{A})\, dv = \int_0^5\int_0^{2\pi}\int_0^4 (3\rho+2)\,\rho\, d\rho\, d\phi\, dz$$

$$= \int_0^4\int_0^{2\pi} \left[3\frac{\rho^3}{3} + \frac{2\rho^2}{2}\right]_0^5 d\phi\, dz$$

$$= \int_0^4\int_0^{2\pi} \left[5^3 + 5^2\right] d\phi\, dz.$$

$$= 150 \times 4 \times 2\pi = \cancel{640\pi}$$

$$= 1200\pi$$

LHS $\quad \oint_S \vec{A}\cdot d\vec{s}$

$$d\vec{s} = \rho\, d\rho\, d\phi\, \hat{a}_z = 5\, d\rho\, d\phi\, \hat{a}_z$$

For top surface $\quad \oint_S \vec{A}\cdot d\vec{s} = \int_0^5\int_0^{2\pi} 2\times4\times\rho\, d\rho\, d\phi = \cancel{\int_0^{2\pi} 8\left[\frac{\rho^2}{2}\right]_0^5 d\phi}$

$$\psi_t = \underline{200\pi}$$

flux through bottom, $\Psi_b = \int_0^5 \int_0^{2\pi} 2 \times 0 \times \rho \, d\rho \, d\phi = 0$

flux through curved surphase.

$$\Psi_s = \int_0^5 \int_0^{2\pi} \cancel{2z} \rho^2 \times \rho \, d\rho \, d\phi = \int_0^{2\pi} \int_0^4 \rho^3 \, dz \, d\phi$$

$$= 125 \times 4 \times 2\pi = 1000\pi$$

$$\Psi_{total} = \int_s \vec{A} \cdot \vec{ds} = 1000\pi + 200\pi = \underline{1200\pi}.$$

Hence divergenie theorem is verified.

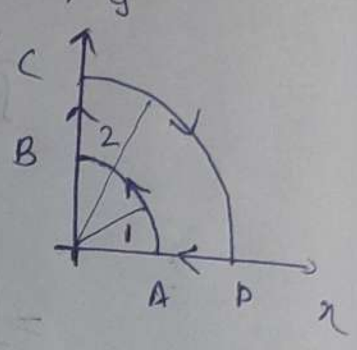
6. Determine the gradient of the field $T = 5\rho e^{-2z} \sin\phi$ at $(2, \frac{\pi}{3}, 0)$

Gradient of T = grad(T) = $\nabla T = \frac{\partial T}{\partial \rho} \hat{a}_\rho + \frac{1}{\rho} \frac{\partial T}{\partial \phi} \hat{a}_\phi + \frac{\partial T}{\partial z} \hat{a}_z$

$$\nabla T = 5 e^{-2z} \sin\phi \, \hat{a}_\rho + \frac{1}{\rho} 5\rho e^{-2z} \cos\phi \, \hat{a}_\phi$$
$$+ -2 \times 5\rho e^{-2z} \sin\phi \, \hat{a}_z.$$

$$\nabla T \Big|_{(2, \frac{\pi}{3}, 0)} = 5 \sin(\tfrac{\pi}{3}) \hat{a}_\rho + 5 e^{-2 \times 0} \cos(\tfrac{\pi}{3}) \hat{a}_\phi$$
$$- 2 \times 5 \times 2 e^{-2 \times 0} \sin(\tfrac{\pi}{3}) \hat{a}_z$$
$$= 4.33 \hat{a}_\rho + 2.5 \hat{a}_\phi - 17.32 \hat{a}_z.$$

7. Assume a vector function $\vec{F} = 5\rho \sin\phi\, \hat{a}_\rho + \rho^2 \cos\phi\, \hat{a}_\phi$. For the contour shown in figure below, verify stokes theorem.



stokes Theorem ⇒ $\oint_L \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot d\vec{s}$.

LHS

$$\oint_L A \cdot dl = \left[\int_A^B + \int_B^C + \int_C^D + \int_D^A \right] \vec{A} \cdot d\vec{l}$$

Along AB, $\rho = 1$, $d\vec{l} = \rho\, d\phi\, \hat{a}_\phi$

$$\int_A^B \vec{A} \cdot d\vec{l} = \int_0^{\pi/2} \rho^2 \cos\phi\, \rho\, d\phi = [\sin\phi]_0^{\pi/2} = 1 \quad -(1)$$

Along BC, $\phi = 90^\circ$ & $d\vec{l} = d\rho\, \hat{a}_\rho$

$$\int_B^C \vec{A} \cdot d\vec{l} = \int_1^2 5\rho \sin\phi = 7.5 \quad -(2)$$

Along CD, $\rho = 2$, $d\vec{l} = \rho\, d\phi\, \hat{a}_\phi$

$$\int_C^D \vec{A} \cdot d\vec{l} = \int_{90}^{0} \rho^2 \cos\phi\, \rho\, d\phi = 8 \times -1 = -8 \quad -(3)$$

Along DA, $\phi = 0$, $d\vec{l} = d\rho\, \hat{a}_\rho$

$$\int_D^A \vec{A} \cdot d\vec{l} = \int_2^1 5\rho \sin\phi\, d\rho = 0 \quad -(4)$$

$$\oint_L \vec{A} \cdot d\vec{l} = \frac{1}{2}.$$

RHS $\int_s (\nabla \times A) \cdot ds \Rightarrow$

$$\nabla \times \vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 5\rho\sin\phi & \rho^3\cos\phi & 0 \end{vmatrix}$$

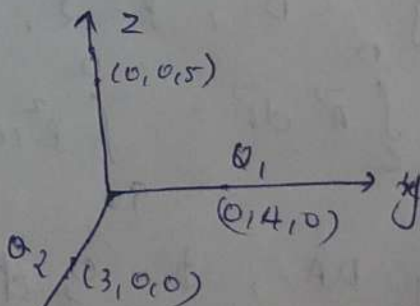
$$= \frac{1}{\rho}\left[3\rho^2\cos\phi - 5\rho\cos\phi\right]\hat{a}_z$$

$$= \cos\phi\,(3\rho - 5)\,\hat{a}_z$$

$$\int_s (\nabla \times \vec{A}) \cdot d\vec{s} = \int_0^{90^\circ}\int_1^2 \cos\phi\,(3\rho - 5)\,\rho\, d\rho\, d\phi$$

$$= \int_0^{90} \frac{-\cos\phi}{2}\, d\phi = \frac{+1}{2}$$

8. Find $\vec{E}$ at $(0,0,5)$ m due to $Q_1 = 0.35\,\mu C$ at $(0,4,0)$ m and $Q_2 = -0.55\,\mu C$ at $(3,0,0)$ m



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{Q_k(\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^3}$$

$n = 2.$

Q_1 is at $4\hat{a}_y$

Q_2 is at $3\hat{a}_x$

$$\vec{E} = 9\times10^9\left[\frac{0.35\times10^{-6}(5\hat{a}_z - 4\hat{a}_y)}{\sqrt{41}^3} + \frac{-0.55\times10^{-6}(5\hat{a}_z - 3\hat{a}_x)}{\sqrt{34}^3}\right]$$

$$= 9\times10^9\left(1.33\times10^{-8}(5\hat{a}_z - 4\hat{a}_y) - 2.774\times10^{-9}(5\hat{a}_z - 3\hat{a}_x)\right)$$

$$= 74.898\,\hat{a}_x - 47.98\,\hat{a}_y - 64.845\,\hat{a}_z$$