Frequency Response of linear phase FIR filters

Case 1 : Symmetrical Impulse response, N odd

The frequency response of filter with impulse response shown in fig 6.1a is (N = 7).

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n) e^{-j\omega n}$$

This can be split as.

$$H(e^{j\omega}) = \sum_{n=0}^{2} h(n) e^{-j\omega n} + h(3) e^{-j3\omega} + \sum_{n=4}^{6} h(n) e^{-j\omega n}$$

In general N-odd.

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\left(\frac{N-1}{2}\right)\omega} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n) e^{-j\omega n} \quad \text{——— (1)}$$

put $n = N-1-m$, then $\Rightarrow m = N-1-n$.

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\left(\frac{N-1}{2}\right)\omega} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m) e^{-j\omega(N-1-m)} \quad \text{——— (2)}$$

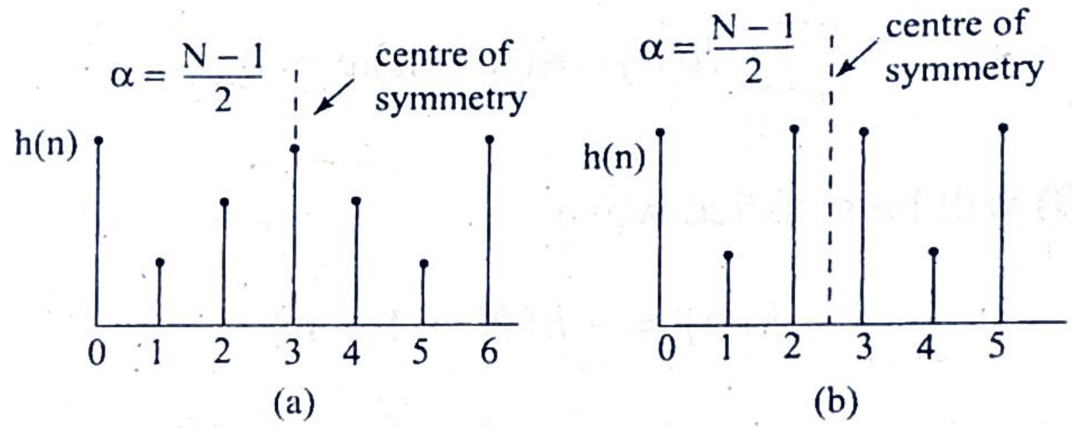

Fig. 6.1 Impulse-response sequence of symmetric sequences for (a) N odd (b) N even.

change the last summation variable as n.

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\left(\frac{N-1}{2}\right)\omega}$$

$$+ \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n) e^{-j\omega(N-1-n)} \quad \text{--- (3)}$$

For symmetrical impulse response.

$$h(n) = h(N-1-n)$$

substituting in (3).

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\left(\frac{N-1}{2}\right)\omega}$$

$$+ \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega(N-1-n)}. \quad \text{--- (4)}$$

$$= e^{-j\left(\frac{N-1}{2}\right)\omega} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n) e^{j\left(\frac{N-1}{2}-n\right)\omega} + h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h(n) e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos\omega\left[\frac{N-1}{2}-n\right] + h\left(\frac{N-1}{2}\right) \right]$$

Let $\frac{N-1}{2} - n = p$ then $\Rightarrow n = \frac{N-1}{2} - p$.

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{p=\frac{N-1}{2}}^{p=1} 2h\left(\frac{N-1}{2}-p\right) \cos\omega p + h\left(\frac{N-1}{2}\right) \right]$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \left[\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \cos\omega n + h\left(\frac{N-1}{2}\right) \right]$$

$$\therefore H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega n \qquad \text{—— (5)}$$

where:

$$a(0) = h\left(\frac{N-1}{2}\right)$$

$$a(n) = 2h\left[\frac{N-1}{2} - n\right].$$

eqn (5) can be rewritten as.

$$H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \bar{H}(e^{j\omega})$$

$$= \bar{H}(e^{j\omega})\, e^{j\theta(\omega)}.$$

where $\bar{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega n.$

$$\theta(\omega) = -\left(\frac{N-1}{2}\right)\omega = -\alpha\omega.$$

$\bar{H}(e^{j\omega}) \rightarrow$ is called zero phase frequency response.

$\rightarrow$ is a real and even function of ω.

To get the magnitude and phase response:

magnitude response $|H(e^{j\omega})|$

$$|H(e^{j\omega})| = |\bar{H}(e^{j\omega})|.$$

and phase

$$\angle H(e^{j\omega}) = \begin{cases} \theta(\omega) = -\alpha\omega & \text{when } \bar{H}(e^{j\omega}) \geq 0 \\ -\alpha\omega + \pi & \text{when } \bar{H}(e^{j\omega}) < 0. \end{cases}$$

$\bar{H}(e^{j\omega}) \longrightarrow$ zerophase frequency response.

$\hookrightarrow$ may take both +ve and −ve value.

$|H(e^{j\omega})| \longrightarrow$ magnitude response.

$\hookrightarrow$ strictly nonnegative.

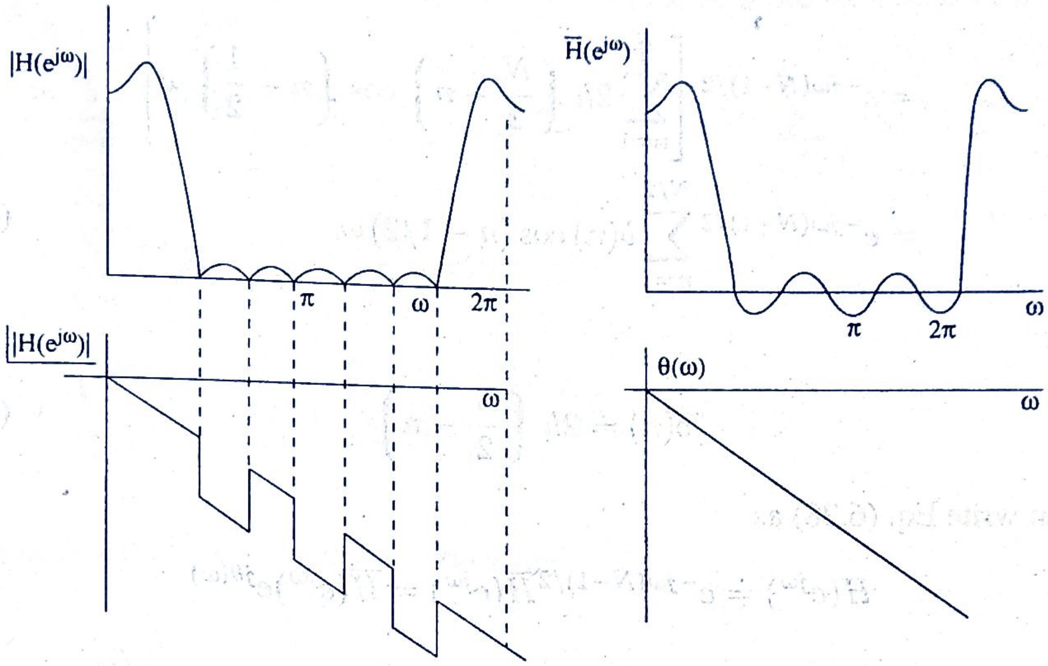

Fig. 6.3 Relation between magnitude response $|H(e^{j\omega})|$ and the zero phase response $\overline{H}(e^{j\omega})$ and between $\angle H(e^{j\omega})$ and $\theta(\omega)$.

Case II: Symmetric impulse response for N even

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \\
&= \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)e^{-j\omega n} \\
&= \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)e^{-j\omega(N-1-n)}
\end{aligned}$$

We know $h(n) = h(N-1-n)$

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega(N-1-n)} \\
&= e^{-j\omega(N-1)/2}\left[\sum_{n=0}^{\frac{N-2}{2}} h(n)e^{j\omega[(N-1)/2-n]} + \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega[(N-1)/2-n]}\right] \\
&= e^{-j\omega(N-1)/2}\left[\sum_{n=0}^{\frac{N-2}{2}} 2h(n)\cos\omega\left(\frac{N-1}{2}-n\right)\right]
\end{aligned}$$

$$= e^{-j\omega(N-1)/2}\left[\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right)\cos\left(n-\frac{1}{2}\right)\omega\right]$$

$$= e^{-j\omega(N-1)/2}\sum_{n=1}^{N/2} b(n)\cos(n-1/2)\,\omega \tag{6.28}$$

where

$$b(n) = 2h\left(\frac{N}{2}-n\right) \tag{6.29}$$

We can write Eq. (6.28) as

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}\overline{H}(e^{j\omega}) = \overline{H}(e^{j\omega})e^{j\theta(\omega)}$$

where

$$\overline{H}(e^{j\omega}) = \sum_{n=1}^{N/2} b(n)\cos\left(n-\frac{1}{2}\right)\omega \tag{6.30}$$

and

$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega \tag{6.31}$$

The frequency response of linear phase filter with symmetric impulse response for N even is shown in Fig. 6.4.

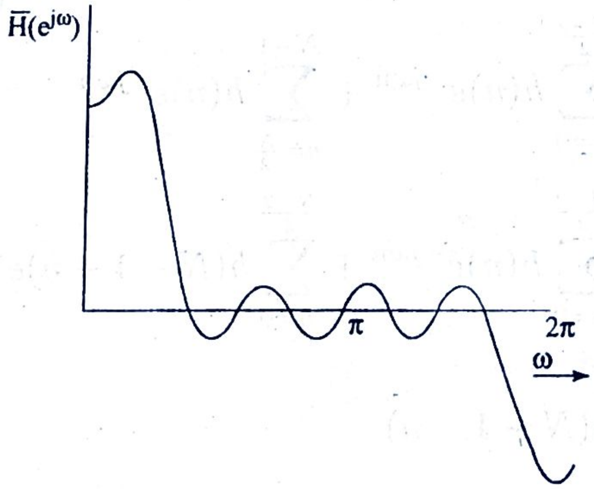

Fig. 6.4 Frequency response for linear phase FIR filter, symmetric impulse response N even.

Case III: Antisymmetric N odd

For this type of sequence

$$h\left(\frac{N-1}{2}\right) = 0$$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega(N-1)/2} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)e^{-j\omega(N-1-n)}$$

We know $h(n) = -h(N-1-n)$, therefore,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega n} - \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega[N-1-n]}$$

$$= e^{-j\omega(N-1)/2}\left[\sum_{n=0}^{\frac{N-3}{2}} h(n)e^{j\omega[(N-1)/2-n]} - \sum_{n=0}^{\frac{N-3}{2}} h(n)e^{-j\omega[(N-1)/2-n]}\right]$$

$$= e^{-j\omega(N-1)/2}j\left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n)\sin\omega\left(\frac{N-1}{2}-n\right)\right]$$

$$= e^{-j\omega(N-1)/2}e^{j\pi/2}\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\sin\omega n$$

$$= e^{-j\omega(N-1)/2}e^{j\pi/2}\sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega n \tag{6.32}$$

where $c(n) = 2h\left(\frac{N-1}{2}-n\right)$ (6.33)

$$H(e^{j\omega}) = \overline{H}(e^{j\omega})e^{-j\omega(N-1)/2}e^{j\pi/2} = \overline{H}(e^{j\omega})e^{j\theta(\omega)}$$

$$\overline{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} c(n)\sin\omega n \tag{6.34}$$

$$\theta(\omega) = \frac{\pi}{2} - \alpha\omega = \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega \tag{6.35}$$

The frequency response of linear phase FIR filter for antisymmetric sequence with N odd is shown in Fig. 6.5.

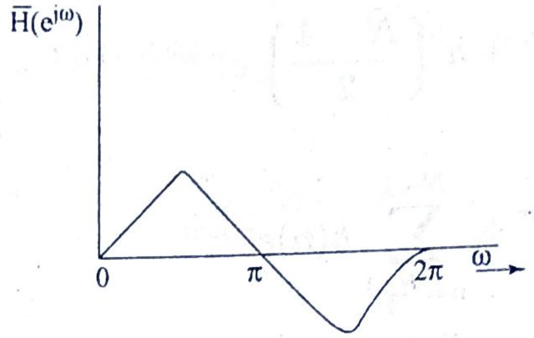

Fig. 6.5 Frequency response of linear phase FIR filter for antisymmetric sequence with N odd

Case IV: *N* even

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)e^{-j\omega(N-1-n)}$$

We have $h(n) = -h(N-1-n)$, therefore

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega n} - \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega(N-1/2)}\left[\sum_{n=0}^{\frac{N-2}{2}} h(n)e^{j\omega[(N-1)/2-n]} - \sum_{n=0}^{\frac{N-2}{2}} h(n)e^{-j\omega[(N-1)/2-n]}\right]$$

$$= e^{-j\omega(N-1)/2}e^{j\pi/2}\left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\sin\omega(n-1/2)\right]$$

$$= e^{-j\omega(N-1)/2}e^{j\pi/2}\left[\sum_{n=1}^{N/2} d(n)\sin\omega\left(n-\frac{1}{2}\right)\right] \tag{6.36}$$

where $d(n) = 2h\left(\frac{N}{2}-n\right)$ (6.37)

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}e^{j\pi/2}\overline{H}(e^{j\omega}) \tag{6.38}$$

$$H(e^{j\omega}) = \overline{H}(e^{j\omega})e^{j\theta(\omega)} = \overline{H}(e^{j\omega})e^{j(\frac{\pi}{2}-\alpha\omega)} \tag{6.39}$$

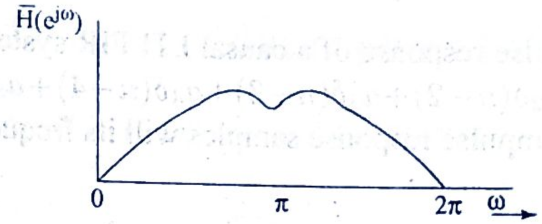

Fig. 6.6 Frequency response of linear phase filter for antisymmetric impulse response with N even.

$$\overline{H}(e^{j\omega}) = \sum_{n=1}^{N/2} d(n) \sin \omega(n - 1/2) \tag{6.40}$$

$$\theta(\omega) = \frac{\pi}{2} - \alpha\omega \tag{6.41}$$

The frequency response of linear phase filter for antisymmetric sequence with N even is shown in Fig. 6.6.

The impulse response of symmetric with odd number of samples can be used to design all types of filters.

The symmetric impulse response having even number of samples cannot be used to design highpass filters.

The frequency response of antisymmetric impulse response is imaginary and this type of filters are most suitable for such filters as Hilbert transformers and differentiators.

Example 6.1 Determine the frequency response of FIR filter defined by $y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$. Calculate the phase delay and group delay.

Solution

Given

$$y(n) = 0.25x(n) + x(n-1) + 0.25x(n-2)$$

Taking Fourier transform on both sides

$$Y(e^{j\omega}) = 0.25X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) + 0.25e^{-2j\omega}X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = 0.25 + e^{-j\omega} + 0.25e^{-2j\omega}$$

$$= e^{-j\omega}(0.25e^{j\omega} + 1 + 0.25e^{-j\omega}) = e^{-j\omega}(1 + 0.5\cos\omega)$$

$$= e^{-j\omega}\overline{H}(e^{j\omega}) \tag{6.41a}$$

Comparing Eq. (6.41a) with Eq. (6.25) we get $\theta(\omega) = -\omega$.

The phase delay $\tau_p = \dfrac{-\theta(\omega)}{\omega} = \dfrac{\omega}{\omega} = 1.$

The group delay $= -\dfrac{d\theta(\omega)}{d\omega} = \dfrac{-d}{d\omega}(-\omega) = 1.$
