MODULE - 1

Introduction to Electromagnetics Theory - Review of Vector Calculus - Curl, divergence, gradient. Rectangular, Cylindrical and Spherical coordinate Systems. Expression of curl, divergence and Laplacian in cartesian, Cylindrical and Spherical coordinate system. Electric field and magnetic field, Review of coulomb's law, Gauss law and Ampere's Current law. Poissen and Laplace equations, Determination of E and V using Laplace equation.

MODULE - I

Review of Vector Calculus:

Scalar and Vectors

A Scalar is a quantity that has only magnitude.

eg: time, mass, distance, temperature, electric potential etc.

A Vector is a quantity that has both magnitude and direction.

eg: Velocity, force, displacement, electric field intensity.

EM (Electro-magnetic) theory is essentially a study of particular fields.

A field is a function that specifies a particular quantity everywhere in a region. If the quantity is scalar or Vector the field is said to be scalar or Vector field.

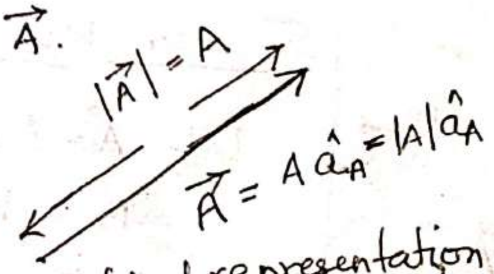
eg: Scalar fields are : temperature distribution in a building
Sound intensity in a theater.

Vector fields are : Gravitational force on a body in space.
Velocity of raindrops in the atmosphere.

Unit Vector: $\hat{a}_A$

It is defined as Vector whose magnitude is Unity and direction is along $\vec{A}$.

$$\boxed{\hat{a}_A = \frac{\vec{A}}{|\vec{A}|} = \frac{A}{A}}$$



Graphical representation of Vector

Thus $\vec{A} = A\hat{a}_A$.

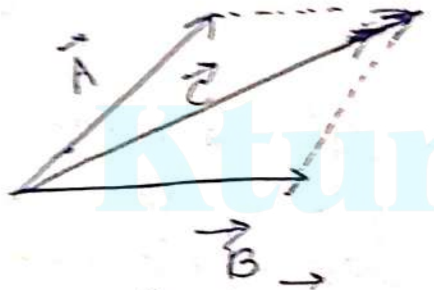
A Vector $\vec{A}$ in Cartesian (or rectangular) Co-ordinate System is

$$\vec{A} = A_x\hat{a_x} + A_y\hat{a_y} + A_z\hat{a_z}.$$

$$A \text{ or } |\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

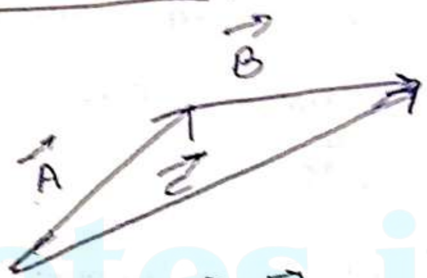
$$\text{Unit Vector } \hat{a_A} = \frac{\vec{A}}{|\vec{A}|} = \frac{A_x\hat{a_x} + A_y\hat{a_y} + A_z\hat{a_z}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Vector Addition and Subtraction:



Vector $\vec{C} = \vec{A} + \vec{B}$

parallelogram rule



$\vec{C} = \vec{A} + \vec{B}$

head-to-tail rule.

Let the two Vectors $\vec{A}$ and $\vec{B}$ represented in 3-D axes.

$$\vec{A} = A_x\hat{a_x} + A_y\hat{a_y} + A_z\hat{a_z}$$

$$\vec{B} = B_x\hat{a_x} + B_y\hat{a_y} + B_z\hat{a_z}.$$

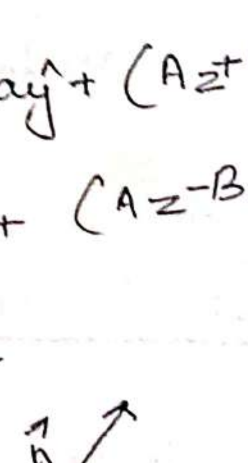
$$\vec{A} + \vec{B} = (A_x + B_x)\hat{a_x} + (A_y + B_y)\hat{a_y} + (A_z + B_z)\hat{a_z}$$

$$\vec{A} - \vec{B} = (A_x - B_x)\hat{a_x} + (A_y - B_y)\hat{a_y} + (A_z - B_z)\hat{a_z}.$$

Scalar Multiplication (Dot Product)

Let $\vec{A} = A_x\hat{a_x} + A_y\hat{a_y} + A_z\hat{a_z}$

$\vec{B} = B_x\hat{a_x} + B_y\hat{a_y} + B_z\hat{a_z}$



$$\boxed{\vec{A}\cdot\vec{B} = AB\cos\theta \text{ or } |AB|\cos\theta}$$

$$\vec{A}\cdot\vec{B} = (A_x \hat{a_x} + A_y \hat{a_y} + A_z \hat{a_z})\cdot(B_x \hat{a_x} + B_y \hat{a_y} + B_z \hat{a_z})$$

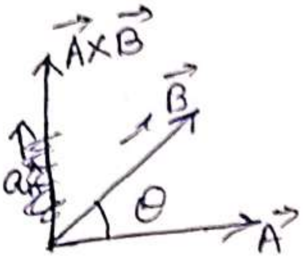
$$\boxed{\vec{A}\cdot\vec{B} = (A_x B_x + A_y B_y + A_z B_z)}$$

$\because \hat{a_x}\cdot\hat{a_x} = \hat{a_y}\cdot\hat{a_y} = \hat{a_z}\cdot\hat{a_z} = 1$

$\hat{a_x}\cdot\hat{a_y} = \hat{a_y}\cdot\hat{a_z} = \hat{a_z}\cdot\hat{a_x} = 0$

If $\vec{A}\cdot\vec{B} = 0$, then $\vec{A}$ and $\vec{B}$ are said to be orthogonal (or perpendicular) with each other.

Vector Multiplication (Cross Product)



Let $\vec{A} = A_x \hat{a_x} + A_y \hat{a_y} + A_z \hat{a_z}$

$\vec{B} = B_x \hat{a_x} + B_y \hat{a_y} + B_z \hat{a_z}$

$$\boxed{A\times B = AB\sin\theta\, \hat{a_n}}$$

where $\hat{a_n}$ is the unit vector normal to the plane containing $\vec{A}$ and $\vec{B}$. The direction of $\hat{a_n}$ is taken as the direction of the right thumb when the fingers of the right hand rotate from $\vec{A}$ to $\vec{B}$.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\hat{a_x} - (A_x B_z - A_z B_x)\hat{a_y} + (A_x B_y - A_y B_x)\hat{a_z}.$$

Two vectors $\vec{A}$ and $\vec{B}$ are parallel, if $A\times B = 0$.

$\hat{a_x}\times\hat{a_x} = \hat{a_y}\times\hat{a_y} = \hat{a_z}\times\hat{a_z} = 0$

$\hat{a_x}\times\hat{a_y} = \hat{a_z}$

$\hat{a_y}\times\hat{a_z} = \hat{a_x}$

$\hat{a_z}\times\hat{a_x} = \hat{a_y}$

Scalar Triple Product.

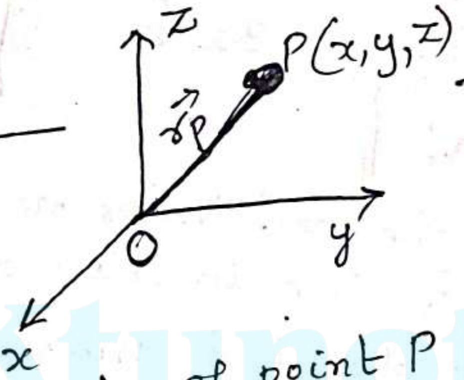
$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

Vector Triple Product

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

Position Vector (r_p)

Position Vector of a point is the directed distance from the origin 'O' to 'P'



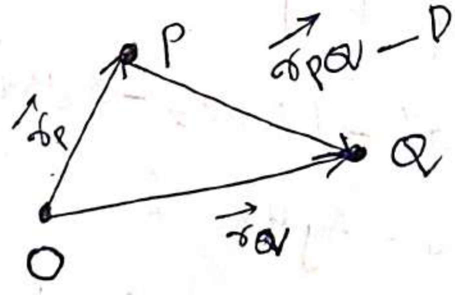
$$\vec{r_p} = OP = x\hat{a_x} + y\hat{a_y} + z\hat{a_z}.$$

Position Vector of point P is useful in defining its position in space.

Distance Vector.

It is the displacement from one point to another.

$\vec{r_{PQ}}$ — Distance Vector



Consider two point P and Q, the distance Vector is the displacement from P to Q,

Hence $r_{PQ} = \vec{r_Q} - \vec{r_P}$

$$= (x_Q - x_P)\hat{a_x} + (y_Q - y_P)\hat{a_y} + (z_Q - z_P)\hat{a_z}.$$

③

Problems:

1) Points P and Q are located at (0,2,4) and (−3,1,5). Calculate
(a) The position Vector P
(b) The distance Vector from P to Q.
(c) The distance between P and Q.

Sol: a) $\vec{r_P} = 0\hat{a_x} + 2\hat{a_y} + 4\hat{a_z} = \underline{\underline{2\hat{a_y} + 4\hat{a_z}}}$

b) $\vec{r_{PQ}} = \vec{r_Q} - \vec{r_P} = (-3\hat{a_x} + \hat{a_y} + 5\hat{a_z}) - (2\hat{a_y} + 4\hat{a_z})$

$= -3\hat{a_x} - \hat{a_y} + 1\hat{a_z} = \underline{\underline{-3\hat{a_x} - \hat{a_y} + \hat{a_z}}}$

c) Distance $d = |\vec{r_{PQ}}| = \sqrt{3^2+1^2+1} = \underline{\underline{3.317}}$.

1.10 Given points P(1,−3,5), Q(2,4,6) and R(0,3,8). find (a) the position Vectors of P and R (b) the distance Vector $\vec{r_{QR}}$ (c) the distance between Q and R.

② $\vec{A} = \hat{a_x} + 3\hat{a_z}$, $\vec{B} = 5\hat{a_x} + 2\hat{a_y} - 6\hat{a_z}$. Find angle between A and B.

$$\vec{A}\cdot\vec{B} = (A_x \cdot B_x) + (A_y \cdot B_y) + (A_z \cdot B_z)$$

$$= 5 + 0 - 18 = \underline{\underline{-13}}$$

$$|A| = \sqrt{1+3^2} = \sqrt{10}$$

$$|B| = \sqrt{25+4+36} = \sqrt{65}$$

angle between A and B $= \dfrac{\vec{A}\cdot\vec{B}}{|A|\cdot|B|} = \dfrac{-13}{\sqrt{10}\sqrt{65}} = \underline{\underline{\dfrac{-13}{\sqrt{650}}}}$

Co-ordinate Systems.

A point or a Vector can be represented in any co-ordinate System, which may be orthogonal or non-orthogonal

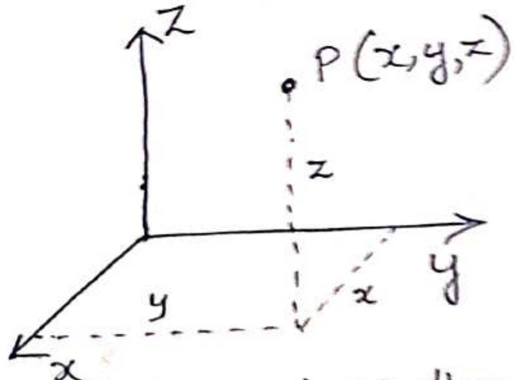
3 Major Co-ordinate Systems

① Cartesian (rectangular)

② Cylindrical

③ Spherical.

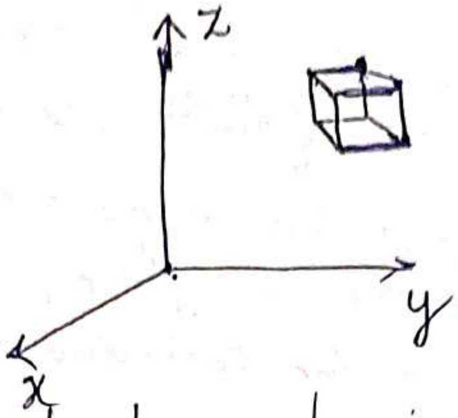
Cartesian Co-ordinate System.



A Vector $\vec{A}$ in this System is Written as,

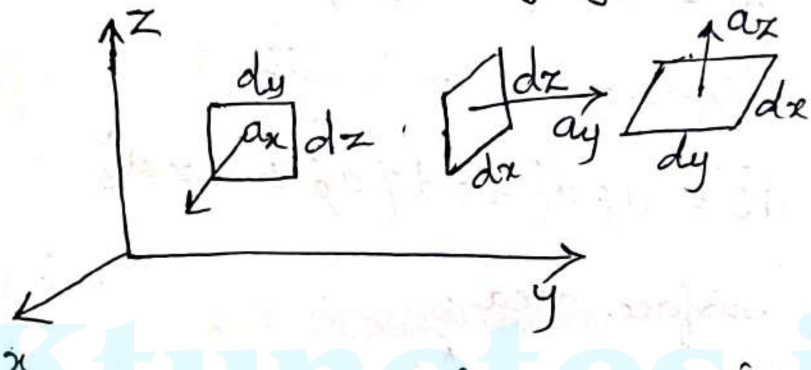
$$\vec{A} = A_x\hat{a_x} + A_y\hat{a_y} + A_z\hat{a_z}.$$

Differential Length, Area and Volume.



differential displacement is given by

$$\vec{dl} = dx\,\hat{a_x} + dy\,\hat{a_y} + dz\,\hat{a_z}$$



Differential normal surface area is given by

$$\vec{dS_x} = dy\,dz\,\hat{a_x}$$

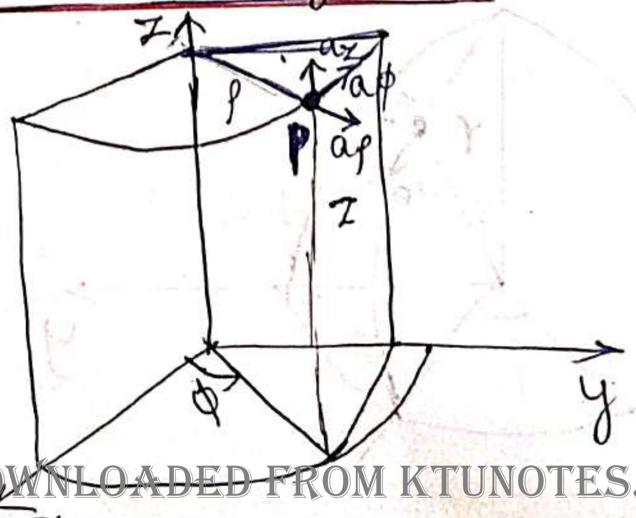
$$\vec{dS_y} = dx\,dz\,\hat{a_y}$$

$$\vec{dS_z} = dx\,dy\,\hat{a_z}.$$

Differential Volume is given by

$$dv = dx\,dy\,dz.$$

Cylindrical Co-ordinate Systems:



A point P is represented as $P(\rho, \phi, z)$

'ρ' is the radius of the Cylinder passing through P or radial distance from the Z-axis

'ϕ' is called azimuthal angle measured from the x axis in the X-Y plane.

'Z' is same as in the Cartesian Co-ordinate system.

Any Vector will be represented as,

$$\vec{A} = A_\rho \hat{a_\rho} + A_\phi \hat{a_\phi} + A_z \hat{a_z}.$$

Differential displacement or length.

$$\vec{dl} = d\rho\, \hat{a_\rho} + \rho\, d\phi\, \hat{a_\phi} + dz\, \hat{a_z}$$

Differential Surface area.

$$\vec{ds} = d\rho\, dz\, \hat{a_\phi}$$

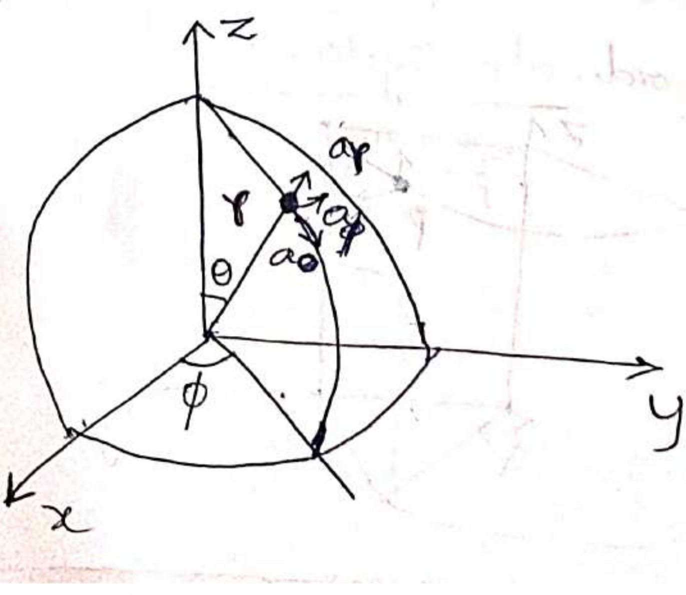
$$= \rho\, d\rho\, d\phi\, \hat{a_z}$$

$$= \rho\, d\phi\, dz\, \hat{a_\rho}$$

Differential Volume

$$dV = \rho\, d\phi\, d\rho\, dz.$$

Spherical Co-ordinate System.



Any point P can be represented as $P(r, \theta, \phi)$ r is the distance from the origin to P. (5)

θ is the angle between z-axis and the position Vector at P.

ϕ measured from x-axis, same azimuthal angle as in cylindrical system.

A Vector in this system is represented by,

$$\vec{A} = A_r \hat{a_r} + A_\theta \hat{a_\theta} + A_\phi \hat{a_\phi}.$$

Differential displacement,

$$dl = d_r \hat{a_r} + r d\theta \hat{a_\theta} + r \sin\theta d\phi \hat{a_\phi}.$$

Differential surface,

$$dS = r^2 \sin\theta \, d\theta d\phi \, \hat{a_r}$$

$$= r \sin\theta \, d\phi dr \, \hat{a_\theta}$$

$$= r dr d\theta \, \hat{a_\phi}$$

Differential Volume

$$dV = r^2 \sin\theta \, dr d\theta d\phi.$$

Coordinate Transformation:

1. Conversion of cartesian to cylindrical.

Given	Transform
x	$\rho = \sqrt{x^2+y^2}$
y	$\phi = \tan^{-1}(y/x)$
z	z

2. Conversion of cylindrical to cartesian.

Given	Transform
ρ	$x = \rho \cos\phi$
ϕ	$y = \rho \sin\phi$
z	z

3. Conversion of Cartesian to Spherical.

Given	Transform
x	$r = \sqrt{x^2+y^2+z^2}$
y	$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$
z	$\phi = \tan^{-1}(y/x)$

4. Conversion of Spherical to Cartesian.

Given	Transform
r	$x = r\sin\theta\cos\phi$
θ	$y = r\sin\theta\sin\phi$
ϕ	$z = r\cos\theta$.

5. Conversion of Cylindrical to Spherical.

Given	Transform.
ρ	$r = \sqrt{\rho^2+z^2}$
ϕ	$\theta = \tan^{-1}(\rho/z)$
z	$\phi = \phi$.

6. Conversion of Spherical to Cylindrical.

Given	Transform.
r	$\rho = r\sin\theta$
θ	$\phi = \phi$
ϕ	$z = r\cos\theta$.

DEL operator (∇ operator)

The differential Vector Operator is Called del(∇) defined as,

$$\nabla = \frac{\partial}{\partial x}\hat{a_x} + \frac{\partial}{\partial y}\hat{a_y} + \frac{\partial}{\partial z}\hat{a_z}$$ (cartersian Co-ordinate System)

$$\nabla = \frac{\partial}{\partial \rho}\hat{a_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}\hat{a_\phi} + \frac{\partial}{\partial z}\hat{a_z}$$ (cylindrical Co-ordinate System)

$$\nabla = \frac{\partial}{\partial r}\hat{a_r} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{a_\phi}$$ (Spherical Co-ordinate System).

There are 4 possible operations with ∇

1) Gradient of a scalar ∇V
2) Divergence of a Vector $\vec{A}$ as $\nabla \cdot \vec{A}$
3) Curl of a Vector $\vec{A}$ as $\nabla \times \vec{A}$
4) Laplacian of a scalar V as $\nabla^2 V$

① Gradient

The gradient of any scalar is the maximum space rate of change of that function

eg: If V represents electric potential then ∇V represents the potential gradient,

$$\nabla V = \frac{\partial V}{\partial x}\hat{a_x} + \frac{\partial V}{\partial y}\hat{a_y} + \frac{\partial V}{\partial z}\hat{a_z}$$ - cartesian co-ordinate.

Gradient of a scalar is a Vector

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a_\phi} + \frac{\partial V}{\partial z}\hat{a_z}$$ - Cylindrical Co-ordinate

$$\nabla V = \frac{\partial V}{\partial r}\hat{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a_\phi}$$ Spherical Co-ordinate.

Find the gradient for the following

1. $U = x^2y + xyz$
2. $V = \rho z \sin\phi + z^2\cos^2\phi + \rho^2$

1. $$\nabla U = \frac{\partial}{\partial x}(x^2y + xyz)\hat{a_x} + \frac{\partial}{\partial y}(x^2y+xyz)\hat{a_y} + \frac{\partial}{\partial z}(x^2y + xyz)\hat{a_z}.$$

$$= (2xy + yz)\hat{a_x} + (x^2 + xz)\hat{a_y} + xy\,\hat{a_z}$$

2. $V = \rho z \sin\phi + z^2\cos^2\phi + \rho^2$

$\cos^2\phi = 1 - \sin^2\phi$

$\frac{\partial}{\partial\phi}(1-\sin^2\phi$

$= -2\sin\phi\cos\phi$

$= -\sin 2\phi$

$$\nabla V = \frac{\partial V}{\partial \rho}\hat{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a_\phi} + \frac{\partial V}{\partial z}\hat{a_z}$$

$$= (z\sin\phi + 2\rho)\hat{a_\rho} + \frac{1}{\rho}(\rho z\cos\phi + z^2(-\sin 2\phi))\hat{a_\phi} + (\rho\sin\phi + 2z\cos^2\phi)a_z.$$

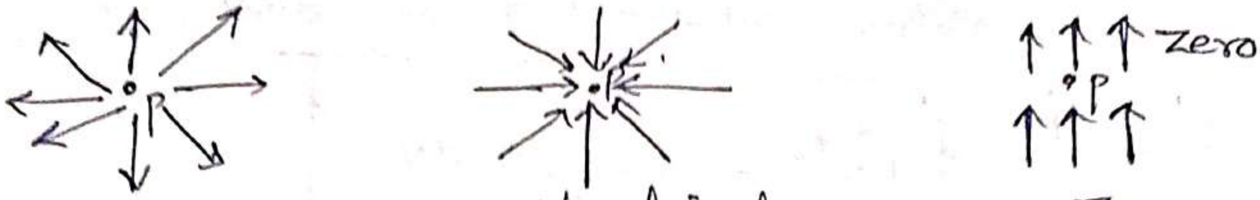
Divergence of a Vector

$$\nabla\cdot\vec{A} = \lim_{\Delta V \to 0} \frac{\oint \vec{A}\cdot ds}{\Delta V}$$

ΔV - small differential Volume.

The divergence of Vector $\vec{A}$ at a given point P is the Outward flux per Unit Volume as the Volume Shrinks about P.

physically, it is a measure of how much the field diverges or emanates from that point.



Positive divergence   Negative divergence   Zero divergence

A Vector is said to be solenoidal if its divergence is zero.

In cartesian Co-ordinate System

$$\nabla \cdot \vec{A} = \left(\frac{\partial}{\partial x}\hat{a_x} + \frac{\partial}{\partial y}\hat{a_y} + \frac{\partial}{\partial z}\hat{a_z}\right) \cdot \left(A_x \hat{a_x} + A_y \hat{a_y} + A_z \hat{a_z}\right)$$

$$\boxed{\nabla \cdot \vec{A} = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z}$$

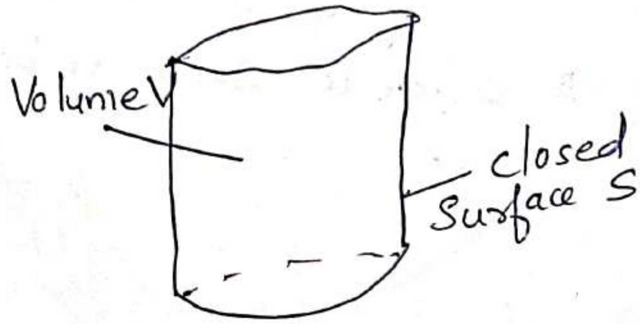
In cylindrical Co-ordinates,

$$\boxed{\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}}$$

In Spherical Co-ordinates.

$$\boxed{\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial (r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta} A_\theta \sin\theta + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}}$$

Divergence Theorem.



It states that the total outward flux of a Vector field $\vec{A}$ through the closed surface S is the same as the Volume integral of the divergence of A

$$\boxed{\oint_S \vec{A} \cdot ds = \int_V \nabla \cdot A \, dv}$$

<u>Curl of a Vector.</u>

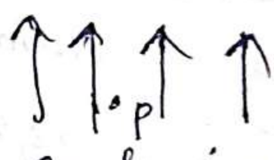
The curl of a Vector is a measure of the tendency of the Vector to rotate or twist.

It is defined as a Vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is normal to the area.

$$\text{Curl } \vec{A} = \nabla \times \vec{A} = \lim_{\Delta S \to 0} \left(\frac{\oint_L \vec{A} \cdot dl}{\Delta S} \right)_{max} a_n .$$

Where ΔS is the area bounded by the Curve L.

Curl is directed Out of the page (right-hand rule).



Curl is zero

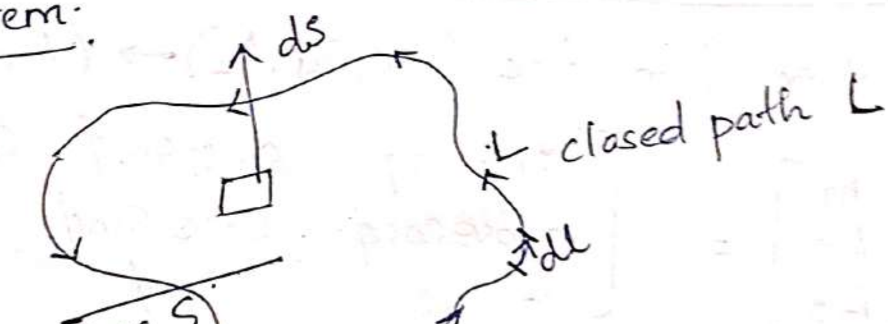
A Vector is said to be irrotational if its curl is zero.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{a_x} & \hat{a_y} & \hat{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$ Cartesian Co-ordinate

$$\nabla \times \vec{A} = \frac{1}{\rho} \begin{vmatrix} \hat{a_\rho} & \rho \hat{a_\phi} & \hat{a_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$ Cylindrical Co-ordinate

$$\nabla \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a_r} & r\hat{a_\theta} & r\sin\theta \, \hat{a_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta \, A_\phi \end{vmatrix}$$ Spherical Co-ordinate

Stoke's Theorem.



Line integral of a Vector $\vec{A}$ around a closed path L is equal to the surface integral of the curl of Vector $\vec{A}$ over any closed surface.

$$\boxed{\oint_L \vec{A} \cdot dl = \int_S (\nabla \times \vec{A}) ds}$$

or

It states that the circulation of the vector field $\vec{A}$ around a closed path L is equal to the surface integral of the curl of $\vec{A}$ over an open surface S bounded by L.

In matrix form, Vector from (A_x, A_y, A_z) to (A_ρ, A_ϕ, A_z) as,

$$\boxed{\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}}$$

The inverse transformation $(A_\rho, A_\phi, A_z) \rightarrow (A_x, A_y, A_z)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\boxed{\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}}$$

In matrix form, the $(A_x, A_y, A_z) \rightarrow (A_r, A_\theta, A_\phi)$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Inverse transformation $A_r, A_\theta, A_\phi \rightarrow A_x, A_y, A_z$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Static Electric field (ELECTROSTATICS)

Electrostatics - phenomenon associated with Static charge or electricity at rest.

An atom in normal condition is electrically neutral because, positive charges on protons and negative charges on electrons are equal.

If an atom accepts negatively charged electrons, then the atom is said to be negatively charged.

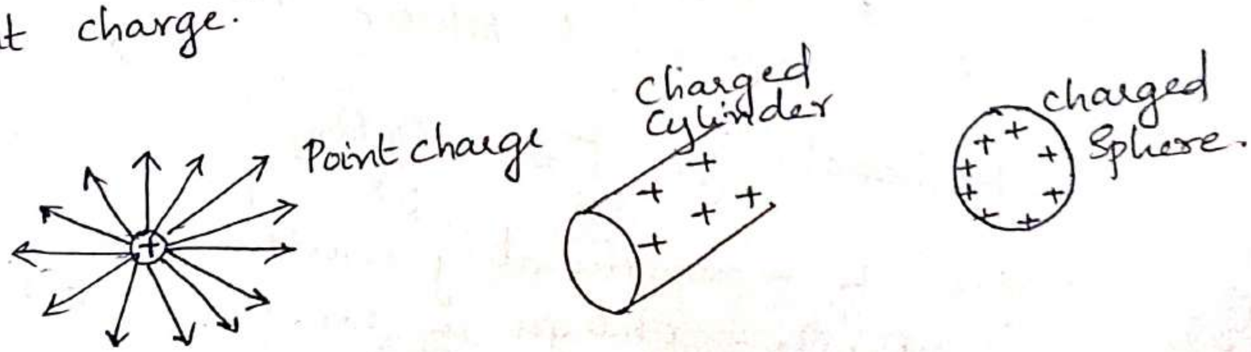
If electrons are removed from an atom, then the atom is said to be positively charged.

Thus negatively charged atom contains excess of electrons and positively charged atom is in deficiency of electrons and net excess or deficiency of electrons in an atom is known as charge.

Electric field:

Electric field is defined as the electric force per unit charge. The direction of the field is taken to be direction of the force it would exert on a positive test charge.

The electric field is radially outward from a positive charge and radially in toward a negative point charge.



Study of electrostatics deals with two fundamental laws governing electrostatic fields.

(i) Coulomb's law

(ii) Gauss law.

Coulomb's law

It deals with the force, a point charge exerts on another point charge.

By a point charge means, a charge that is located on a body whose dimensions are much smaller than other dimensions.

The polarity of charges can be positive or negative, like charges repel, while unlike charges attract.

one electron charge $e = -1.6019 \times 10^{-19}$ C

Coulomb's Law states that,

the force F between two point charges Q_1 and Q_2 is

1. Along the line joining them.
2. Directly proportional to the product $Q_1 Q_2$ of the charges.
3. Inversely proportional to the square of the distance R between them.

Mathematically, $\boxed{F = \frac{k Q_1 Q_2}{R^2}}$

where k - proportionality constant

Q_1, Q_2 - charges in Coulombs (C)

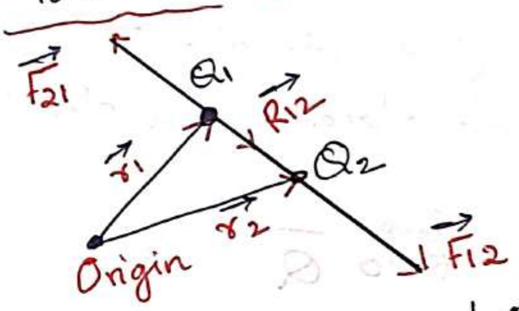
R — the distance in meters (m)

$k \to = \frac{1}{4\pi\varepsilon_0}$ ε_0, is known as permittivity in free space

$$\varepsilon_0 = 8.854 \times 10^{-12} F/m$$

$$\therefore k = \frac{1}{4\pi\varepsilon_0} \simeq 9\times10^{9} m/F$$

Coulomb's law in Vector form:



If point charges Q_1 and Q_2 are located at points having position vectors $\vec{r_1}$ and $\vec{r_2}$, then the force $\vec{F_{12}}$ on Q_2 due to Q_1 is shown figure above.

$$\vec{F_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2} \hat{a}_{R_{12}}$$

where $\vec{R_{12}} = r_2 - r_1$

$$R = |\vec{R_{12}}|$$

$$\therefore \hat{a}_{R_{12}} = \frac{\vec{R_{12}}}{|\vec{R_{12}}|} = \frac{\vec{R_{12}}}{R}$$

$$\therefore \vec{F_{12}} = \frac{Q_1 Q_2 \vec{R_{12}}}{4\pi\varepsilon_0 R^3}$$

or $$\vec{F_{12}} = \frac{Q_1 Q_2 \vec{R_{12}}}{4\pi\varepsilon_0 |r_2 - r_1|^3}$$

If we have more than two point charges, we can use the principle of superposition to determine force on a particular charge.

$$\vec{F} = \frac{Q}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{Q_k(\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^3}$$

Electric field Intensity (Electric field strength) $\vec{E}$

The electric field intensity $\vec{E}$ is the force per unit charge when placed in an electric field.

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q}$$

$$= \frac{\vec{F}}{Q}$$

The electric field intensity at point $\vec{r}$ due to a point charge located at $\vec{r}'$ is

$$E = \frac{Q}{4\pi\epsilon_0 R^2} \vec{a_R} = \frac{Q(\vec{r} - \vec{r}')}{4\pi\epsilon_0 |\vec{r} - \vec{r}'|^3}$$

For N point charges $Q_1, Q_2 \cdots Q_N$ located at $r_1, r_2 \cdots r_N$, the electric field intensity at point r

$$E = \frac{Q_1(\vec{r} - \vec{r}_1)}{4\pi\epsilon_0 |\vec{r} - \vec{r}_1|^3} + \frac{Q_2(\vec{r} - \vec{r}_2)}{4\pi\epsilon_0 |\vec{r} - \vec{r}_2|^3} + \cdots + \frac{Q_N(\vec{r} - \vec{r}_N)}{4\pi\epsilon_0 |\vec{r} - \vec{r}_N|^3}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{N} \frac{Q_k(r - r_k)}{|r - r_k|^3}$$

Q1. point charges 1mC and -2mC are located at (3,2,-1) and (-1,-1,4) respectively. Calculate the electric force on a 10nC charge located at (0,3,1) and the electric field intensity at that point.

$$F = \frac{Q}{4\pi\varepsilon_0}\sum_{k=1,2}^{N} \frac{Q_k(r-r_k)}{|r-r_k|^3}$$

Force on 10nC due to 1mC

$$F_1 = \frac{Q Q_1}{4\pi\varepsilon_0}\frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3}$$

$\vec{r} = (0,3,1)$

$\vec{r}_1 = (3,2,-1)$

$\therefore \vec{r}-\vec{r}_1 = (-3,1,2)$

$|\vec{r}-\vec{r}_1| = \sqrt{3^2+1^2+2^2}$

$= \sqrt{9+1+4}$

$= \underline{\underline{\sqrt{14}}}$

$Q = 10nC$

$Q_1 = 1mC$

$Q_2 = -2mC$

$\frac{1}{4\pi\varepsilon_0} = 9\times10^9$

$$= \frac{10\times10^{-9}\times1\times10^{-3}(-3,1,2)}{4\pi\varepsilon_0(\sqrt{14})^3}$$

$$= \frac{9\times\cancel{10^9}\times10\times\cancel{10^{-9}}\times1\times10^{-3}(-3,1,2)}{(\sqrt{14})^3}$$

$$= \frac{90\times10^{-3}(-3\hat{a_x}+\hat{a_y}+2\hat{a_z})}{(\sqrt{14})^3}$$

$$= (-5.15\hat{a_x}+1.7182\hat{a_y}+3.436\hat{a_z})10^{-3}N$$

Force on 10nC due to -2mC charge.

$$F_2 = \frac{Q Q_2(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0|(\vec{r}-\vec{r}_2)|^3}$$

$\vec{r} = (0,3,1)$

$\vec{r}_2 = (-1,-1,4)$

$\therefore \vec{r}-\vec{r}_2 = (1,4,-3)$

$|\vec{r}-\vec{r}_2| = \sqrt{1^2+4^2+3^2}$

$= \sqrt{1+16+9}$

$= \underline{\underline{\sqrt{26}}}$

$$= \frac{-9\times\cancel{10^9}\times10\times\cancel{10^{-9}}\times2\times10^{-3}(1,4,-3)}{(\sqrt{26})^3}$$

$$\frac{180(\hat{a_x}+4\hat{a_y}-3\hat{a_z})\times10^{-3}}{(\sqrt{26})^3}$$

$\left(-1.357\hat{a_x} + 5.428\hat{a_y} + 4.071\hat{a_z}\right) \times 10^{-N}$

$\therefore$ Total force on 10nC charge is

$$F = F_1 + F_2$$
$$= -6.511\hat{a_x} - 3.71\hat{a_y} + 7.51\hat{a_z} \text{ mN}$$

At that point,

$$E = \frac{F}{Q}$$
$$= \frac{\left(-6.511\hat{a_x} - 3.71\hat{a_y} + 7.51\hat{a_z}\right) \cdot 10^{-3}}{10 \times 10^{-9}}$$

$$E = \left(-651.1\hat{a_x} - 371.1\hat{a_y} + 751.1\hat{a_z}\right) \text{kV/m}$$

H.W Point charges 5nC and -2nC are located at (2,0,4) and (-3,0,5) respectively.

(a) Determine the force on a 1nC point charge located at (1,-3,7)

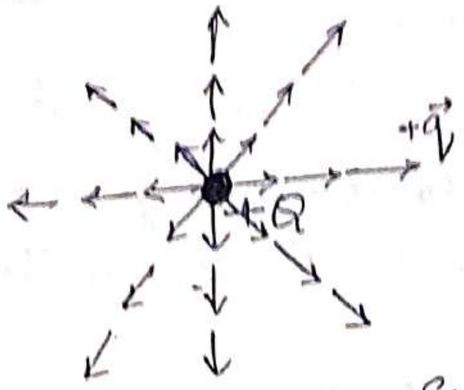
(b) Find the electric field E at (1,-3,7).

Electric flux Density (D)

It is the electric flux per unit area.

$$\vec{D} = \frac{\psi}{A} \text{ Coloumb/m}^2.$$

Electric field Intensity $\vec{E}$



If a small test charge q is placed at any point near a second fixed charge Q, the test charge q experiences a force. The magnitude and direction of q will depend upon its location with respect to Q.

Around Q, a 'field' is set up and consequently any charge brought into this field will be subjected to a force.

The strength of the field '$\vec{E}$' at any point is the force per unit charge at that point.

∴ The force on the test charge 'q'

$$\vec{F} = \frac{Qq}{4\pi\varepsilon_0 R^2}\hat{a_R}$$ where $\hat{a_R}$ is the Unit vector, radially outward

$$\vec{E} = \frac{\vec{F}}{q}$$

$$= \frac{Q\cancel{q}}{4\pi\varepsilon_0 \cancel{q} R^2}\hat{a_R}$$

$$\boxed{\therefore \vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a_R}\ N/C}$$

Electric flux Density ($\vec{D}$)

As described above, if we have a positive charge +Q, then a positive test charge 'q' brought anywhere near +Q will be repelled (out radially) in a radial direction.

The lines drawn to trace the direction in which the test charge will experience a force due to Q, are called lines of force.

These lines of force are designated as 'electric flux' and are equal to the charge itself.

Now if the expression for $\vec{E}$ is multiplied with ε_0

$$\varepsilon_0 \vec{E} = \frac{Q}{4\pi R^2} \hat{a_R}.$$

The expression on the right hand side has the dimension of charge in the numerator and surface area in denominator.

Hence $\varepsilon_0 \vec{E}$ is defined by a symbol $\vec{D}$ and has the dimension of charge-density.

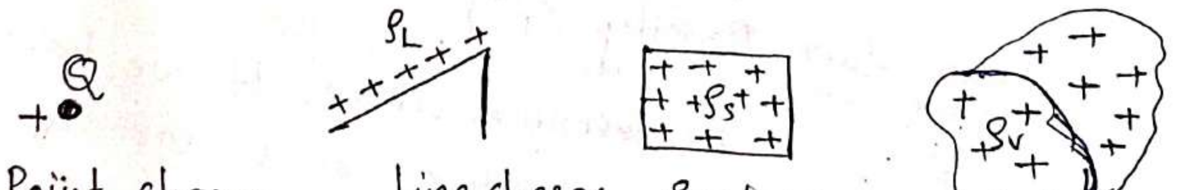
$\vec{D}$ is called electric flux density.

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} = \frac{Q}{4\pi R^2} \hat{a_R}.}$$

Unit $\vec{D} = C/m^2$

This expression is valid in free-space. In any other medium $\vec{D} = \varepsilon \vec{E}$ where $\varepsilon = \varepsilon_0 \varepsilon_r$, ε_r is the relative permittivity of the medium.

Electric fields due to continuous charge Distributions.



Point charge Line charge Surface charge Volume charge

Line charge density is denoted by ρ_L (in C/m)

Surface charge density " by ρ_S (in C/m^2)

Volume charge density " by ρ_V (in C/m^3)

The charge element dQ and the total charge Q due to these charge distributions are,

$$\boxed{\begin{aligned} dQ &= \rho_L dl \rightarrow Q = \int_L \rho_L dl \text{ (Line charge)} \\ dQ &= \rho_S ds \rightarrow Q = \int_S \rho_S ds \text{ (Surface charge)} \\ dQ &= \rho_V dV \rightarrow Q = \int_V \rho_V dV \text{ (volume charge)} \end{aligned}}$$

The Electric field Intensity $\vec{E}$ due to each of the charge distributions ρ_L, ρ_S and ρ_V is taken as the summation of the field contributed by numerous point charges making up the charge distribution.

Thus, replacing Q with charge element $dQ = \rho_L dl$, $\rho_S ds$, $\rho_V dV$ and integrating, we get

$$\boxed{\vec{E} = \int_L \frac{\rho_L dl}{4\pi\varepsilon_0 R^2} \hat{a_R} \text{ (Line charge)}}$$

$$\boxed{\vec{E} = \int_S \frac{\rho_S dS}{4\pi\varepsilon_0 R^2} \hat{a_R} \text{ (Surface charge)}}$$

$$\boxed{\vec{E} = \int_V \frac{\rho_V dv}{4\pi\varepsilon_0 R^2} \hat{a_R} \text{ (Volume Charge)}}$$

Gauss's law

Gauss's law states that the total electric flux 'ψ' through any closed surface is equal to the total charge 'Q' enclosed by that surface.

ie $\boxed{\psi = Q_{enclosed} = Q_{enc}}$

Since $\vec{D}$ is electric flux density, integrating it over the entire surface results in ψ

$$\therefore \psi = \oint_S \vec{D} \cdot ds$$

If ρ_v is the volume charge density,

$$Q_{enc} = \int_v \rho_v \, dv \quad \text{——— (1)}$$

$$\therefore \boxed{\oint_S \vec{D} \cdot ds = \int_V \rho_V \cdot dv} \quad \text{——— (1)}$$

Applying divergence theorem, to the LH·S

$$\oint_S \vec{D} \cdot ds = \int_V \nabla \cdot \vec{D} \, dv \quad \text{——— (2)}$$

Comparing (1) and (2)

$$\int_V \nabla \cdot \vec{D} \, dv = \int_V \rho_V \cdot dv$$

$$\therefore \boxed{\rho_V = \nabla \cdot \vec{D}} \quad \text{——— (3)}$$

Equation (1) is the integral form of Gauss's law, while (3) is the differential or point form.

$\boxed{\rho_V = \nabla \cdot \vec{D}}$ is called Maxwell's first Equation.

Electric Potential (V) or Electric Scalar Potential.

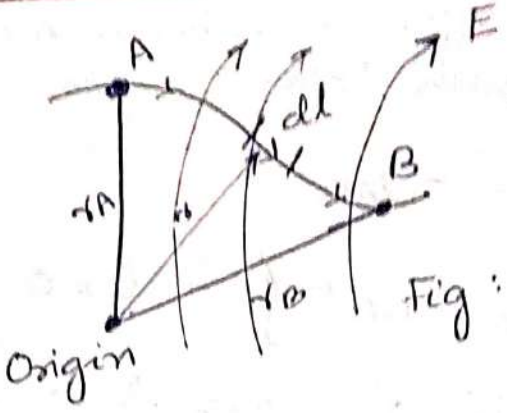


Fig : Displacement of point Charge Q in an electrostatic field E.

- Suppose we wish to move a point charge Q from point A to point B in an electric field $\vec{E}$.

From Coulomb's law, the force on Q is $\vec{F} = Q\vec{E}$

So the work done in displacing the charge by dl is

$$dW = -\vec{F} \cdot dl = -Q\vec{E} \cdot dl$$

The negative sign indicates that the work is being done by an external agent. Thus the total work done in moving Q from A to B is,

$$W = -Q\int_A^B \vec{E} \cdot dl$$

Dividing W by Q, gives the potential difference between A and B.

$$\boxed{V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E} \cdot dl}$$

This potential is a scalar quantity having only magnitude and no direction and is often called Scalar potential.

Assumptions :

(1) A is the initial points and B is the final point

(2) V_{AB} is independent of the path taken.

Relationship between E + V - Maxwell's equation

The potential difference between A and B is independent of path taken, Hence

$\therefore V_{BA} = -V_{AB}$

that is $V_{AB} + V_{BA} = \oint \vec{E} \cdot dl = 0$ —— (1)

$\oint_L \vec{E} \cdot dl = 0$, shows that the line integral of $\vec{E}$ along a closed path is zero. This implies that no net work is done in moving a charge along a closed path in an electrostatic field.

Applying stoke's theorem in equation (1)

$$\oint_L \vec{E} \cdot dl = \int_S (\nabla \times E) \cdot dS = 0$$

$\therefore \boxed{\nabla \times E = 0}$ This is called Maxwell's second equation for static electric field.

We have, $V = -\int \vec{E} \cdot dl$

ie $dV = -\vec{E} \cdot d\hat{l}$

$= -E_x dx - E_y dy - E_z dz$ —— (1)

But from calculus of multivariables, a total charge in $V(x,y,z)$ is the sum of partial charges with respect to x, y, z variables.

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \quad \text{—— (2)}$$

Comparing above two expressions, for dV,

$$\therefore E_x = -\frac{dV}{dx}, \quad E_y = -\frac{dV}{dy}, \quad E_z = -\frac{\partial V}{\partial z}$$

Thus

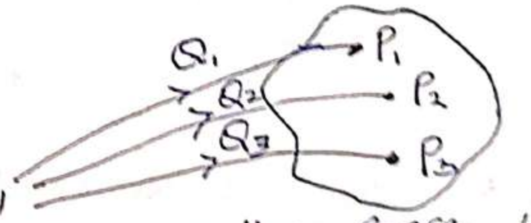
$$\boxed{E = -\nabla V}$$

ie Electric field intensity is the gradient of V. The negative sign shows that the direction of $\vec{E}$ is opposite to the direction in which V increases.

* Energy stored in Electrostatic fields.

Consider an electrostatic field created by the presence of an assembly of charges. To determine the energy present in the field, we first determine the amount of work done to assemble them.

Suppose we wish to position three point charges Q_1, Q_2 and Q_3 at points P_1, P_2, P_3 in an initially empty space.



* . No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field.

* Work done in transferring Q_2 from ∞ to P_2 = Product of Q_2 and the potential V_{21} at P_2 due to Q_1

* Similarly, the work done in placing Q_3 to P_3 = Product of Q_3 with the potential at P_3 due to Q_1 and Q_2.

Total work done in the electrostatic field,

$$W_E = W_1 + W_2 + W_3$$

$$= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \longrightarrow (1)$$

If the charges are placed in reverse order, (change Q_1 to Q_3 & Q_3 to Q_1 and hence potentials)

$$W_E = W_3 + W_2 + W_1$$

$$= 0 + Q_2 V_{23} + Q_1 (V_{13} + V_{12}) \longrightarrow (2)$$

Adding equation (1) and (2)

$$2W_E = Q_2 V_{21} + Q_3 V_{31} + Q_3 V_{32} + Q_2 V_{23} + Q_1 V_{13} + Q_1 V_{12}$$

$$2W_E = Q_1 (V_{13} + V_{12}) + Q_2 (V_{21} + V_{23}) + Q_3 (V_{31} + V_{32})$$

$$2W_E = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

$$W_E = \frac{1}{2} \left[Q_1 V_1 + Q_2 V_2 + Q_3 V_3 \right]$$

If there are n point charges,

$$\boxed{W_E = \frac{1}{2} \sum_{k=1}^{n} \left[Q_k V_k \right]}$$

Case 2: Continuous charge.

If instead of point charges, the region has a continuous charge distribution, summation becomes integration

The energy distributions are given as

$$W_E = \frac{1}{2} \int_L \rho_L \cdot V \, d\vec{l} \quad \text{(line charge)}$$

$$W_E = \frac{1}{2}\int_S \rho_s V d\vec{s} \quad (\text{Surface charge})$$

$$W_E = \frac{1}{2}\int_V \rho_v V dv \quad (\text{Volume charge})$$

Consider the Volume charge distribution, energy

$$W_E = \frac{1}{2}\int_V \rho_v V\, dv \quad \text{—— (1)}$$

From Gauss's Law,

$$\nabla \cdot D = \rho_v$$

$\therefore$ (1) becomes, $W_E = \frac{1}{2}\int_V (\nabla\cdot\vec{D}) V dv$ —— (2)

We have the relation

$$(\nabla\cdot\vec{A})V = (\nabla\cdot V)\vec{A} - \vec{A}\cdot\nabla V$$

Applying this result to (2)

$$\therefore W_E = \frac{1}{2}\int_V \left((\nabla\cdot V)\vec{D} - \vec{D}\cdot\nabla V\right) dv$$

$$= \frac{1}{2}\int_V (\nabla\cdot V)\vec{D}\, dv - \frac{1}{2}\int_V (\vec{D}\cdot\nabla V) dv \quad \text{—— (3)}$$

From divergence theorem,

$$\int_S \vec{A}\cdot ds = \int_V \nabla\cdot\vec{A}\, dv$$

Applying divergence theorem to I[st] term in equation (3).

$$\therefore W_E = \frac{1}{2}\oint_S V\vec{D}\cdot d\vec{s} - \frac{1}{2}\int_V (\vec{D}\cdot\nabla V)\, dv \quad \text{—— (4)}$$

We recall that V varies by $\frac{1}{r}$ and $\vec{D}$ Varies by $\frac{1}{r^2}$ for point charges and so on.

$\therefore$ the product $V\vec{D}$ must atleast Vary by $\frac{1}{r^3}$.

But the surface area $d\vec{s}$ Varies as r^2.

$\therefore$ when the surface becomes large and distance becomes infinity, the 1st integral term in equation (4) must tend to zero

$$\therefore W_E = -\frac{1}{2}\int_V \vec{D}\cdot\nabla V \, dv.$$

We have $-\nabla.V = \vec{E}$

$$\boxed{\therefore W_E = \frac{1}{2}\int_V \vec{D}\cdot\vec{E}\, dv} \quad \text{—— (5)}$$

Also $\vec{D} = \varepsilon\vec{E}$

$$\therefore W_E = \frac{1}{2}\int_V \varepsilon\vec{E}\cdot\vec{E}\, dv$$

$$\boxed{W_E = \frac{\varepsilon}{2}\int_V \vec{E}^2 dv}$$

Poisson's and Laplace equations:

From Gauss's Law,

$$\nabla\cdot\vec{D} = \rho_v \quad \text{—— (1)}$$

But $\vec{D} = \varepsilon\vec{E}$

$\therefore$ (1) becomes, $\nabla\cdot\varepsilon\vec{E} = \rho_v$

$$\varepsilon\nabla\vec{E} = \rho_v$$

$$\nabla\cdot\vec{E} = \frac{\rho_v}{\varepsilon} \quad \text{—— (2)}$$

But $E = -\nabla V$, substituting in (2)

$$\nabla\cdot(-\nabla V) = \frac{\rho_v}{\varepsilon}$$

$$\nabla\cdot\nabla V = -\frac{\rho_v}{\varepsilon}$$

$$\boxed{\nabla^2 V = \frac{-\rho_v}{\varepsilon}}$$ This is poisson's equation

For a charge free region, $\rho_v = 0$

Then, $\boxed{\nabla^2 V = 0}$ Laplace's equation

Recall that ∇^2 is called the laplacian operator. In the different Co-ordinate systems, the laplace's equation are

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{—— Cartesian}$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{—— Cylindrical}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

—— Spherical.

The poisson equation in each co-ordinate system can be found by replacing 0 on the RHS by $-\frac{\rho_v}{\varepsilon}$.

University Questions

Q1: Point charges 5nC and -2nC are located at (2,0,4) and (-3,0,5) respectively. (i) Determine the force on a 1nC point charge located at (1,-3,7).
(ii) Find the electric field E at (1,-3,7).

Q2. A charge of -0.3μC is located at A(25,-30,15) in cm and a second charge of 0.5μC at B(-10,8,12). Find E at (i) origin (ii) P(15,20,50) in cm.

Application of Gauss's Law:

Gauss's law is used to calculate the electric field, when symmetric charge distribution exists.

Once it has been found that symmetric ~~charge~~ charge distribution exists, we construct a mathematical closed surface (known as Gaussian Surface).

The surface is chosen such that $\vec{D}$ is normal or tangential to the Gaussian Surface.

When $\vec{D}$ is normal to the surface $\vec{D}\cdot d\vec{S} = D\,ds$ Because $\vec{D}$ is constant on the surface. When $\vec{D}$ is tangential to the surface $\vec{D}\cdot d\vec{s} = 0$.

(i) Point charge: [Derive the expression for electric field intensity at a distance 'r' from a point charge of Q, coulombs.] University Question, 2018

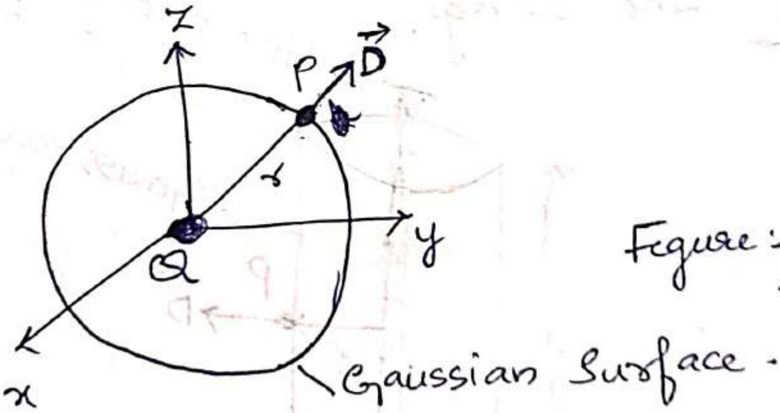


Figure: Gaussian Surface about a point charge.

Suppose a point charge Q is located at the origin.
To determine $\vec{D}$ at a point P, we choose a spherical surface containing P. Thus a spherical surface centered at the origin is the Gaussian surface.

Since $\vec{D}$ is everywhere normal to the Gaussian Surface, that is $\vec{D} = D_r \hat{a_r}$.

Applying Gauss's law, $(\psi = Q_{enclosed})$

$$Q = \oint_S \vec{D} \cdot ds = D_r \oint_S ds$$

$= D_r 4\pi r^2$, Since Surface area of the Gaussian Surface is $4\pi r^2$.

$$\boxed{\therefore \vec{D} = \frac{Q}{4\pi r^2} \hat{a_r}}$$

From the relation $\vec{D} = \varepsilon_0 \vec{E}$

$$\boxed{\therefore \text{Electric field Intensity of point charge } \vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{a_r}}$$

2) Infinite Line charge:

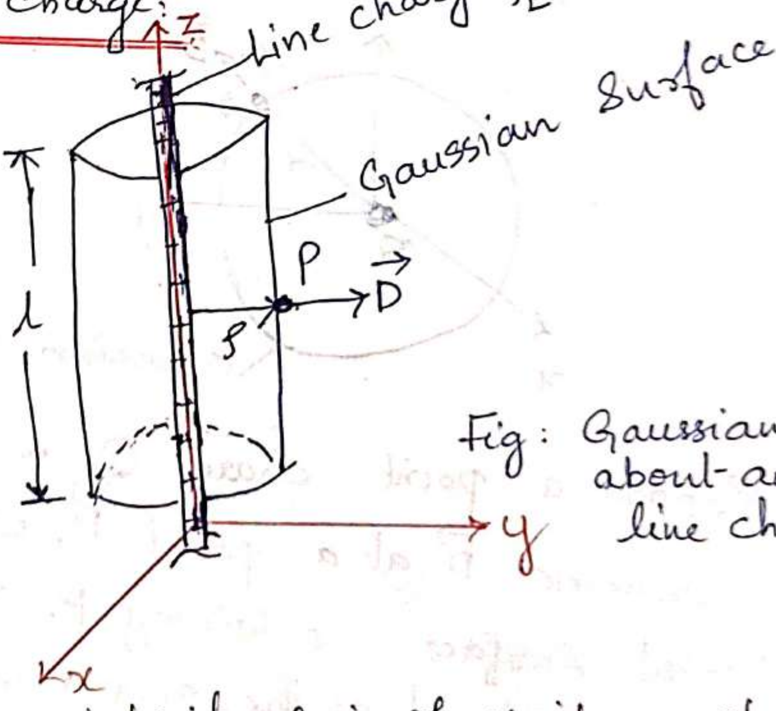


Fig: Gaussian Surface about an infinite line charge.

Suppose the infinite line of uniform charge ρ_L C/m lies along the z-axis. To determine $\vec{D}$ at a point P, we choose a cylindrical surface

(19)

containing Point P, Such that $\vec{D}$ (electric flux density) is constant on and normal to the cylindrical Gaussian Surface

that is $\vec{D} = D_\rho \hat{a}_\rho$

Applying Gauss's law along length 'l' of the line,

$$\rho_L l = Q = \oint_S \vec{D} \cdot d\vec{s} = D_\rho \oint_S ds = D_\rho 2\pi \rho l$$

$Q = \int \rho_L \cdot dl$

$= \rho_L \int dz$

$\rho_L \int_0^L dz$

where $\oint ds = 2\pi \rho l$ is the surface area of the Gaussian Surface.

$Q = \rho_L \cdot L$

$\oint \vec{D} \cdot ds =$ zero, on the top and bottom Surfaces, Since $\vec{D}$ has no z-component, that means $\vec{D}$ is tangential to those Surfaces.

$$\text{Thus } \vec{D_\rho} = \frac{\rho_L}{2\pi \rho} \hat{a}_\rho \cdot \text{ or } \frac{Q}{2\pi \rho l}$$

We know $\vec{D} = \varepsilon_0 \vec{E}$

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi \varepsilon_0 \rho} \hat{a}_\rho} \quad \text{Electric field Intensity of infinite line charge.}$$

$$\vec{E} = \frac{Q}{2\pi \varepsilon_0 \rho l}.$$

$\oint_S ds$

$\oint \rho \, d\phi \, dz = \hat{a}_\rho$

$\oint_{S} \int_0^{2\pi} \int_0^L \rho \, d\phi \, dz \, \hat{a}_\rho$

$\rho \, 2\pi L$

V density

Ampere's circuit law.

Just as Gauss's law is a special case of Coulomb's law, Ampere's law is a special case of Biot - Savart's law and is applicable in problem's involving symmetrical current distribution.

Statement: it states that the line integral of $\vec{H}$ (magnetic field intensity) around a closed path is same as the net current enclosed by the path.

$$\oint \vec{H} \cdot dl = I_{enc}.$$

By applying stoke's theorem to the L.H.S

$$\oint_L \vec{H} \cdot dl = \iint_S (\nabla \times \vec{H})\, ds .$$

If $\vec{J}$ is the current density,

$$I_{enc} = \iint_S \vec{J} \cdot ds$$

Comparing the two surface integrals,

$$\boxed{\nabla \times \vec{H} = \vec{J}}$$

Since $\nabla \times \vec{H} \neq 0$, magnetostatic field is not conservative.

Magnetic Flux Density ($\vec{B}$).

Just as electric flux density $\vec{D} = \varepsilon_0 \vec{E}$ in free space, magnetic flux density $\vec{B}$ is related to magnetic field intensity $\vec{H}$ as

$$\vec{B} = \mu_0 \vec{H}.$$

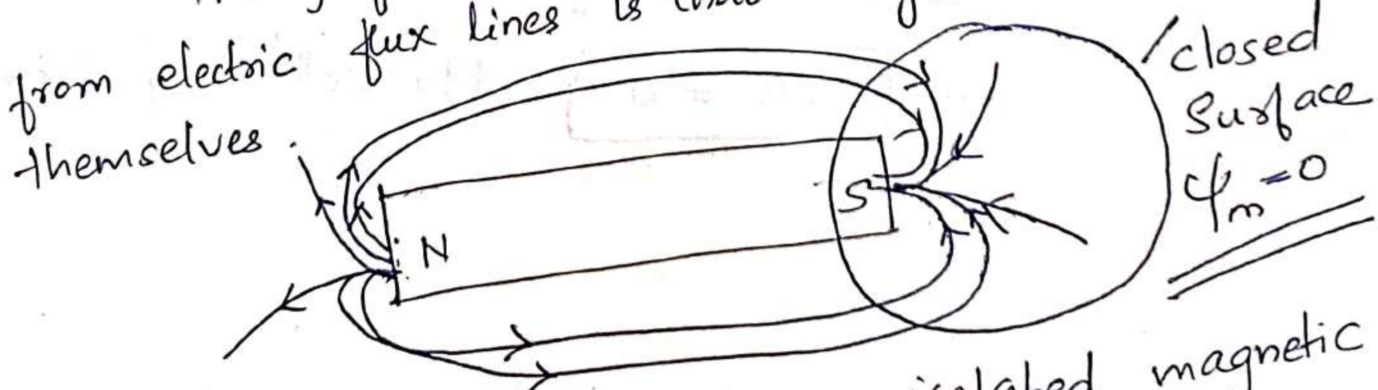
where μ_0 is called permeability of free space

$$\mu_0 = 4\pi \times 10^{-7}\ H/m.$$

The magnetic flux through a surface S is given by

$$\boxed{\psi_m = \int_S \vec{B} \cdot ds.}$$

A significant difference of magnetic flux lines from electric flux lines is that they always close upon themselves.



It is not possible to have isolated magnetic poles or magnetic charges.

If we try to split a magnetic bar to isolate a poles, we end up with pieces each having a north and south pole.

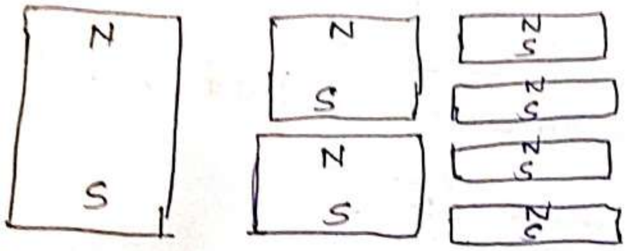


fig: Successive division of a bar magnetic results in pieces with north and South poles, showing that magnetic poles cannot be isolated.

Thus the total flux through a closed surface in a magnetic field will be zero.

$$\oint_S \vec{B} \cdot ds = 0$$

From divergence theorem,

$$\oint_S \vec{B} \cdot ds = \int_V \nabla \cdot \vec{B} \, dv$$

$$\therefore \boxed{\nabla \cdot B = 0}$$ Maxwell's equation

Divergence of a Vector:

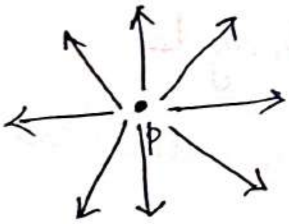
Divergence refers to spreading or converging of a quantity, such as fluid, gas or electric and magnetic flux lines from a point. It indicates the measure of difference between outflow and inflow (ie, the net flow) of a quantity through a closed surface enclosing certain volume.

Divergance is applicable on vectors only, and results into scalars.

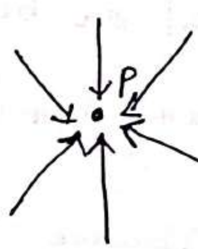
Definition: Consider a closed surface enclosing a volume ΔV, and P is a point on S. Then the divergence of a vector $\vec{A}$, denoted as div. $\vec{A}$ or $\nabla\cdot\vec{A}$ is defined as the net flow of the flux of $\vec{A}$ out of the volume ΔV through the closed surface S surrounding the volume as the volume tends to zero.

Mathematically, $$\text{div}\,\vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta V\to 0} \frac{\oint \vec{A}\cdot d\vec{s}}{\Delta V}$$

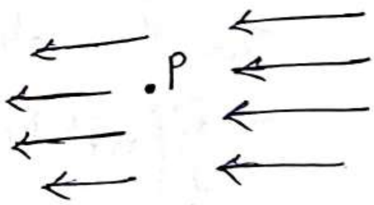
Here $\oint_S \vec{A}\cdot d\vec{s}$ denotes the flux of $\vec{A}$ through closed surface S



(a) Positive Divergence



(b) Negative Divergence



(c) Zero divergence.

In, figure (a), vector field spreads out, that is the net flow is outward, then the divergence of the vector at that point is positive.

Positive Divergence at a point implies, there is a Source of flow at that point.

On the other hand, If the vector converges at a point, that is, the net flow at that point is inward, the Vector is said to have a negative divergence at that point. The negative divergence at a point implies that there is a Sink at that point.

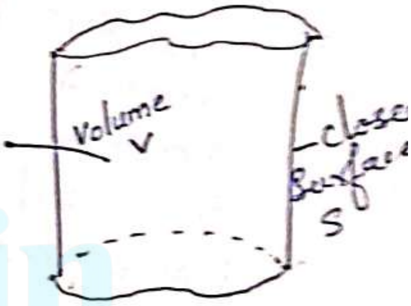
A Vector can also have Zero divergence at some points. Fig (c). Zero divergence at a point implies that is neither a Source nor a Sink at that point.

Divergence theorem: (Gauss Ostrogradsky Theorem).

Statement:

" The Divergence theorem states that the total outward flux of a Vector field $\vec{A}$ through the closed Surface S is same as the volume integral of the divergence of A.

$$\boxed{\oint_S \vec{A} \cdot ds = \int_v \nabla \cdot A \, dv}$$



Proof:

Divergence of any Vector $\vec{A}$

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad \text{where } dv = dx\,dy\,dz.$$

Take the Volume Integral on both Sides,

$$\iiint_V \nabla \cdot \vec{A} \, dv = \iiint_V \left[\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right] dx\,dy\,dz$$

Consider an elemental Volume in x, y, z direction

$$= \iiint_V \frac{\partial A_x}{\partial x} dx\,dy\,dz + \iiint_V \frac{\partial A_y}{\partial y} dx\,dy\,dz + \iiint_V \frac{\partial A_z}{\partial z} dx\,dy\,dz$$

$$= \int \frac{\partial A_x}{\partial x} dx = A_x$$

$$\therefore \iiint_V (\nabla \cdot \vec{A}) \, dv = \iint A_x \, dy\,dz + \iint A_y \, dx\,dz + \iint A_z \, dx\,dy$$

$ds_x = dydz \rightarrow$ Surface area of yz plane with Unit Vector $\hat{a_x}$

$ds_z = dxdy \rightarrow$ Surface area of xy plane with Unit Vector $\hat{a_z}$

$ds_y = dxdz \rightarrow$ Surface area of xz plane with Unit Vector $\hat{a_y}$.

$$\iiint \nabla \cdot \vec{A}\, dv = \iint_S \left[A_x ds_x + A_y ds_y + A_z ds_z \right]$$

$$\int_V \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot ds \quad \text{Hence proved.}$$

Curl :

The circulation of a vector field A around a closed path L is defined as the integral $\oint_L \vec{A} \cdot dl$.

Curl deals with rotation.

The curl of A is a rotational Vector whose magnitude is the maximum circulation of A per Unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

curl is mathematically defined as circulation per Unit area.

$$\boxed{\text{Curl } A = \frac{\text{Circulation}}{\text{Area.}}}$$

ie $$\text{Curl} A = \nabla \times A = \left(\lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot dl}{\Delta S} \right) a_{n_{max}}.$$

Where the area ΔS is bounded by the curve L and a_n is the Unit Vector normal to the Surface.

Properties of Curl

1. The curl of a Vector field is another Vector field
2. The Curl of a Scalar field V, $\nabla \times V$, make no sense.
3. $\nabla \times (A+B) = \nabla \times A + \nabla \times B$
4. $\nabla \times (A+B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla A - (A \cdot \nabla) B$
5. The divergence of the curl of a Vector field vanishes, that is $\nabla \cdot (\nabla \times A) = 0$.
6. The curl of the gradient of a scalar field Vanishes, that is $\nabla \times \nabla V = 0$.

Physical Significance of the curl.

The curl provides the maximum Value of the circulation of the field per unit area and indicates the direction along which this maximum value occurs.

$$\nabla \times A = \lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot dL}{\Delta S}$$

The curl of a Vector field $\vec{A}$ at a point P can be regarded as a measure of the circulation or how much the field curls around P.

eg:

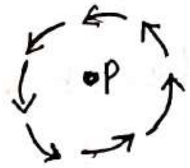


Fig: curl of a Vector field around P is directed out of the page

Fig: Curl at P is zero.