Derivation of Radix-2 DIT-FFT Algorithm.

Let $x(n)$ be an N point sequence, where N is assumed to be a power of 2 ie $N = 2^v$.

– Decimate or Break the sequence $x(n)$ into two sequences of length $N/2$ where one sequence consisting of the even indexed values of $x(n)$ and the other of odd indexed values of $x(n)$.

→ even indexed values → $x_e(n) = x(2n)$; $n = 0, 1 \ldots \frac{N}{2} - 1$

odd indexed values → $x_o(n) = x(2n+1)$; $n = 0, 1 \ldots N/2 - 1$

We have the N point DFT eqn of $x(n)$.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \quad ; k = 0, 1 \ldots N-1.$$

Seperating $x(n)$ into even and odd indexed values of $x(n)$, we get

$$X(k) = \sum_{\substack{n=0 \\ (\text{even})}}^{N-1} x(n)\, W_N^{nk} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x(n)\, W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_N^{(2n+1)k} \quad \text{--- (1)}$$

$$W_N^{2nk} = \left[e^{-\frac{j2\pi}{N}}\right]^{2nk} = \left[e^{-\frac{j2\pi}{N/2}}\right]^{nk} = W_{N/2}^{nk}$$

$$\text{ie } W_N^{2nk} = W_{N/2}^{nk}$$

Substituting this, eqn (1) can be rewritten a

$$X(k) = \sum_{n=0}^{N/2-1} x_e(n)\, W_{N/2}^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_o(n)\, W_{N/2}^{nk} \cdot W_N^{k}$$

$$= \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_e(n)\, W_{N/2}^{nk}}_{\substack{N/2 \text{ point DFT of} \\ \text{even indexed} \\ \text{sequence } x_e(n)}} + W_N^{k} \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_o(n)\, W_{N/2}^{nk}}_{\substack{N/2 \text{ point DFT of} \\ \text{odd indexed} \\ \text{sequence } x_o(n).}}$$

$$X(k) = X_e(k) + W_N^{k}\, X_o(k) \quad \text{--- (2)}$$

where $k = 0, 1 \ldots N-1$.

→ $X_e(k)$ and $X_o(k)$ are $N/2$ point DFT and are periodic with a period $N/2$.

Now let us take $N = 8$. Then $X_e(k)$ and $X_o(k)$ are 4 point ($N/2$ point) DFTs of $x_e(n)$ and ~~x_o~~ $x_o(n)$.

$x_e(n) \Rightarrow \quad x_e(0) = x(0)$

$x_e(1) = x(2)$

$x_e(2) = x(4)$

$x_e(3) = x(6)$

and $\quad x_o(n) \Longrightarrow \quad x_o(0) = x(1)$

$x_o(1) = x(3)$

$x_o(2) = x(5)$

$x_o(3) = x(7).$

we have $X(k) = X_e(k) + W_N^k X_o(k)$

for $N = 8$;

$$X(k) = X_e(k) + W_8^k X_o(k) \quad ; k = 0, 1 \ldots 7. \qquad \text{—— (3)}$$

Since $X_e(k)$ is periodic with period 4

$$\left.\begin{aligned} \cancel{X_e(4)} = \quad X_e(4) &= X_e(0) \\ X_e(5) &= X_e(1) \\ X_e(6) &= X_e(2) \\ X_e(7) &= X_e(3). \end{aligned}\right\} \text{—— (4)}$$

$\|^{ly}.$ $X_o(k)$ is also periodic with period 4.

$$\left.\begin{aligned} X_o(4) &= X_o(0) \\ X_o(5) &= X_o(1) \\ X_o(6) &= X_o(2) \\ X_o(7) &= X_o(3) \end{aligned}\right\} \text{—— (5)}$$

Applying eqn ④ and ⑤ in ~~eqn~~ and substituting $k=0,1 \cdots 7$ in eqn ③ we get.

$k=0 \Rightarrow \quad X(0) = X_e(0) + W_8^0 X_o(0)$

$k=1 \Rightarrow \quad X(1) = X_e(1) + W_8^1 X_o(1)$

$k=2 \Rightarrow \quad X(2) = X_e(2) + W_8^2 X_o(2)$

$k=3 \Rightarrow \quad X(3) = X_e(3) + W_8^3 X_o(3)$

$k=4 \Rightarrow \quad X(4) = X_e(4) + W_8^4 X_o(4)$

we have $X_e(4) = X_e(0)$

$X_o(4) = X_o(0)$

also $W_8^4 = W_8^{0+8/2} = -W_8^0$ (symmetry property).

$(W_N^{k+N/2} = -W_N^k)$

$\therefore X(4) = X_e(0) - W_8^0 X_o(0)$

$k=5 \Rightarrow \quad X(5) = X_e(5) + W_8^5 X_o(5) \quad (W_8^5 = W_8^{1+4} = -W_8^1$

$\therefore X(5) = X_e(1) - W_8^1 X_o(1)$

$k=6 \Rightarrow \quad X(6) = X_e(6) + W_8^6 X_o(6)$

$X(6) = X_e(2) - W_8^2 X_o(2)$

$k=7 \Rightarrow \quad X(7) = X_e(7) + W_8^7 X_o(7)$

$X(7) = X_e(3) - W_8^3 X_o(3)$

From the above set of equations we can find that

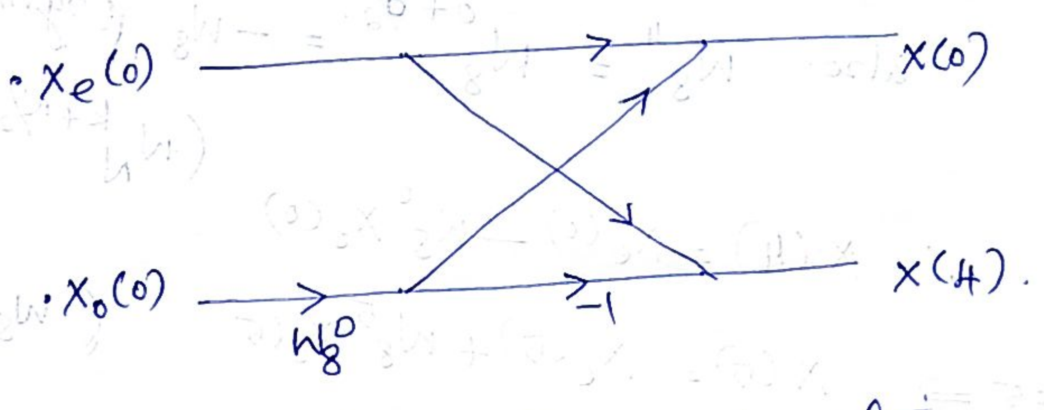
$X(0)$ and $X(4)$ has same inputs

||ly $X(1)$ and $X(5)$, $X(2)$ and $X(6)$, $X(3)$ and $X(7)$ has same inputs.

ie ~~X(0~~ $X(0) = X_e(0) + W_8^0 X_o(0)$

$X(4) = X_e(0) - W_8^0 X_o(0)$

This operation can be represented by a butterfly diagram as shown below.

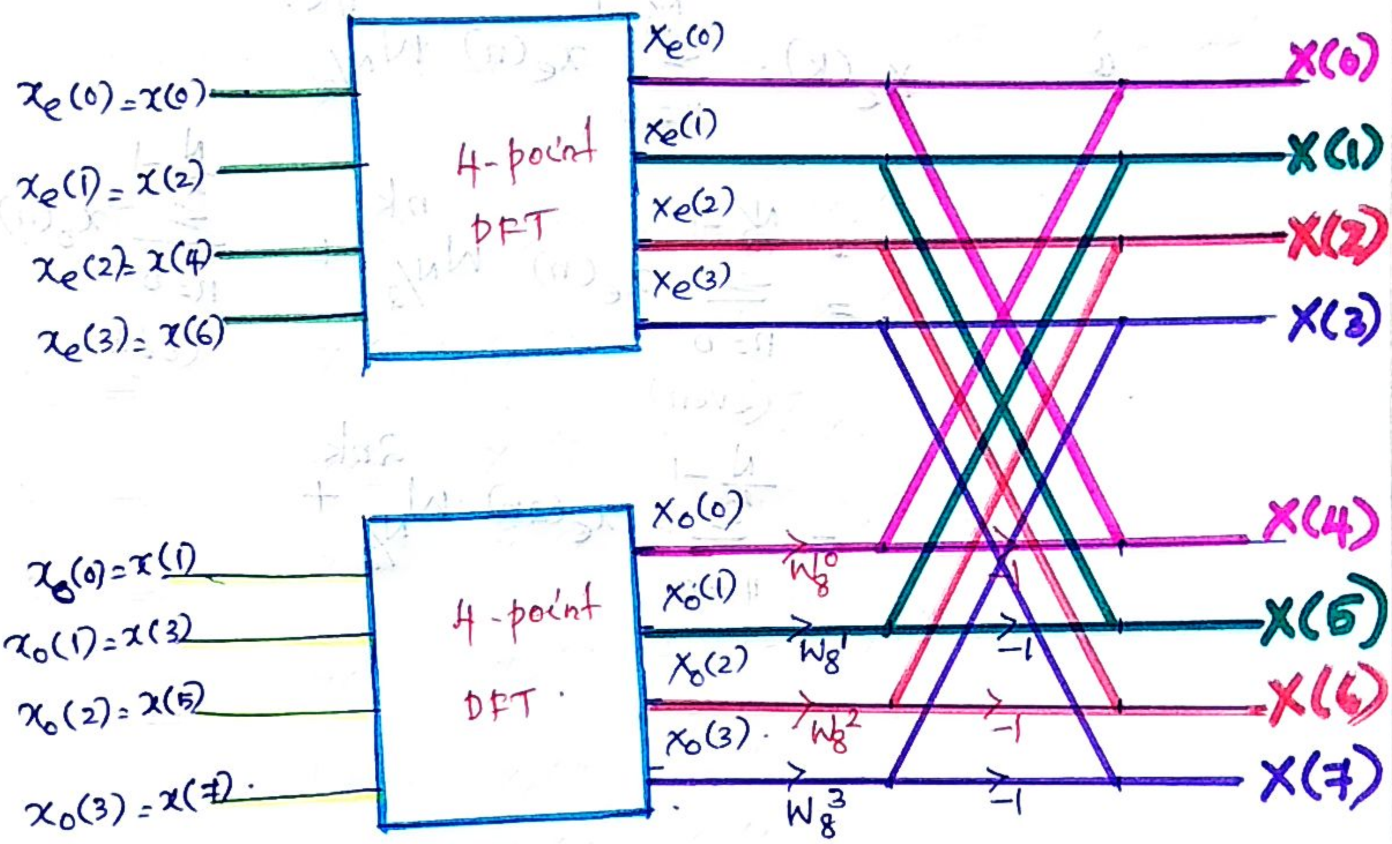


The same butterfly operation can be used to calculate $X(1)$ and $X(5)$

$X(2)$ and $X(6)$ and

$X(3)$ and $X(7)$.

By using two 4 point DFTs and four butterfly the 8-point DFT can be obtained as.



↑ Construction of an 8-point DFT from two 4-point DFTs.

– Now we apply the same approach to decompose each $N/2$ point DFTs. This can be done by dividing the sequence $x_e(n)$ and $x_o(n)$ into two sequence (even indexed and odd indexed.

$x_e(n)$ → $x_{ee}(n)$ (even), $x_{eo}(n)$ (odd)

$x_o(n)$ → $x_{oe}(n)$ (even), $x_{oo}(n)$ (odd)

We have the $N/2$ point DFT $X_e(k)$

i.e. $$X_e(k) = \sum_{n=0}^{N/2-1} x_e(n)\, W_{N/2}^{nk}.$$

$$= \underbrace{\sum_{n=0}^{N/2-1} x_e(n)\, W_{N/2}^{nk}}_{\text{(even)}} + \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_e(n)\, W_{N/2}^{nk}}_{\text{(odd)}}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} x_e(2n)\, W_{N/2}^{2nk} + \sum_{n=0}^{\frac{N}{4}-1} x_e(2n+1)\, W_{N/2}^{(2n+1)k}$$

$$X_e(k) = \sum_{n=0}^{\frac{N}{4}-1} x_{ee}(n)\, W_{N/4}^{nk} + W_{N/2}^{k} \sum_{n=0}^{\frac{N}{4}-1} x_{eo}(n)\, W_{N/4}^{nk}.$$

$$\therefore \boxed{X_e(k) = \underbrace{X_{ee}(k)}_{\substack{N/4 \text{ point DFT of } x_{ee}(n) \\ \text{(periodic with period} = N/4)}} + W_N^{2k} \underbrace{X_{eo}(k)}_{\substack{N/4 \text{ point DFT of } x_{eo}(n) \\ \text{(periodic with a period} = \frac{N}{4})}}} \quad \text{—— (8)}$$

$k = 0, 1 \cdots \frac{N}{2} - 1$

In the similar way the $N/2$ point DFT $X_o(k)$ can be rewritten as.

$$\boxed{X_o(k) = \underbrace{X_{oe}(k)}_{\substack{(N/4 \text{ point DFT of } x_{oe}(n) \\ \text{periodic with period} = N/4)}} + W_N^{2k} \underbrace{X_{oo}(k)}_{\substack{(N/4 \text{ point DFT of } x_{oo}(n) \\ \text{periodic with period} = N/4)}}} \quad \text{—— (9)}$$

$k = 0, 1 \cdots \frac{N}{2} - 1$

For $\underline{N = 8}$

$X_{ee}(k) \quad X_{oe}(k)$
$X_{eo}(k) \quad X_{oo}(k)$ } These $\frac{N}{4}$ point (2 point) DFT's are periodic with a period $\frac{N}{4} = 2$.

From eqn (8)
put $k = 0, 1, 2, 3$ we get.

$k=0 \Rightarrow X_e(0) = X_{ee}(0) + W_8^0 X_{eo}(0)$

$k=1 \Rightarrow X_e(1) = X_{ee}(1) + W_8^2 X_{eo}(1)$

$k=2 \Rightarrow X_e(2) = X_{ee}(2) + W_8^4 X_{eo}(2)$
$\quad = X_{ee}(0) - W_8^0 X_{eo}(0)$
$X_e(2) = X_{ee}(0) - W_8^0 X_{eo}(0)$

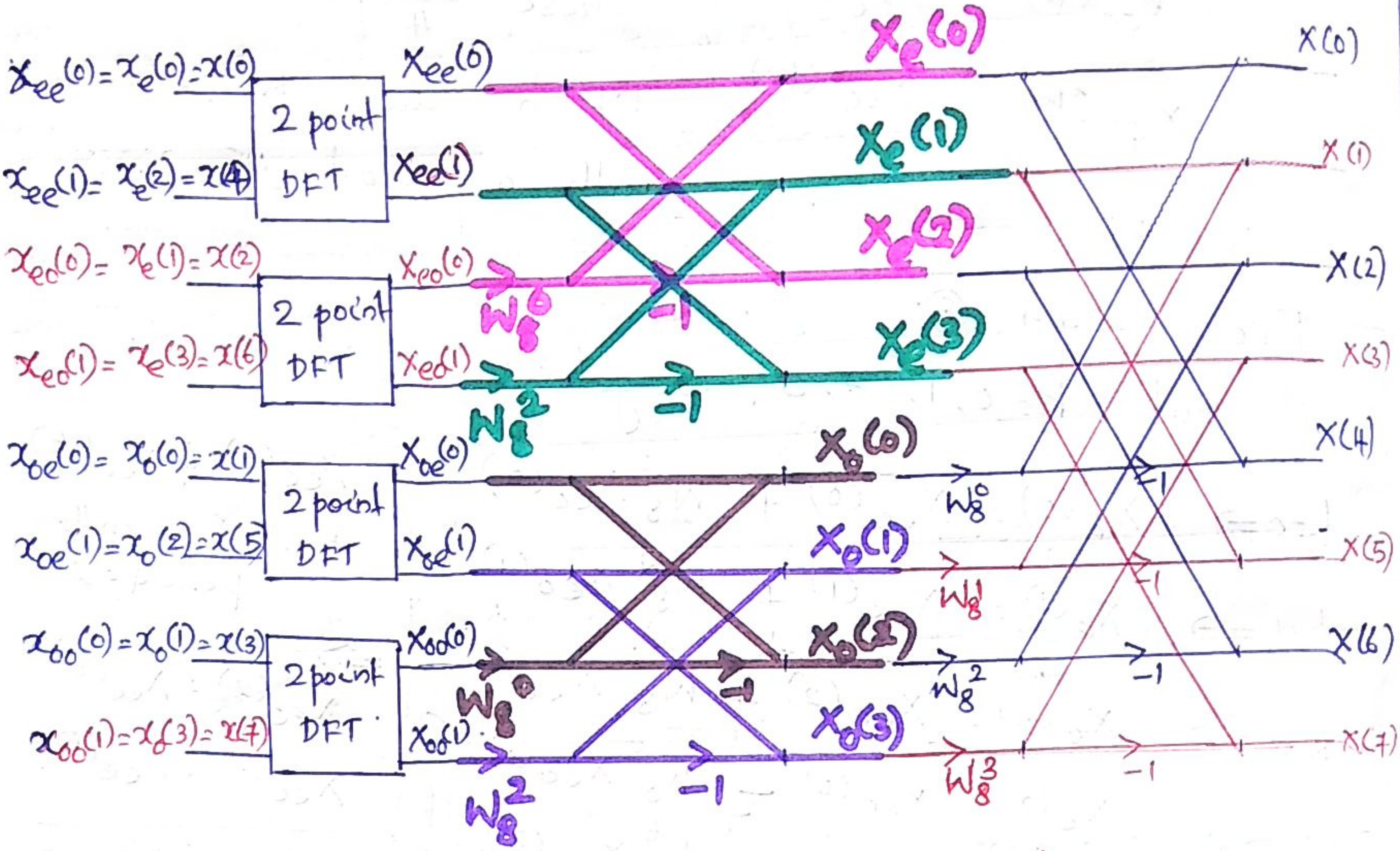
$k=3 \Rightarrow X_e(3) = X_{ee}(3) + W_8^6 X_{eo}(3)$
$\quad = X_{ee}(1) - W_8^2 X_{eo}(1)$ — (10)

$X_{ee}(k)$ & $X_{eo}(k)$ periodic with period $= 2$.
$\Rightarrow X_{ee}(2) = X_{ee}(0)$
$X_{ee}(3) = X_{ee}(1)$

$W_8^4 = W_8^{0+4} = -W_8^0$
(symmetry property)

||ly From eqn (9)
put $k = 0, 1, 2, 3$ we get.

$X_o(0) = X_{oe}(0) + W_8^0 X_{oo}(0)$
$X_o(1) = X_{oe}(1) + W_8^2 X_{oo}(1)$
$X_o(2) = X_{oe}(2) + W_8^4 X_{oo}(2)$
$\quad = X_{oe}(0) - W_8^0 X_{oo}(0)$
$X_o(3) = X_{oe}(3) + W_8^6 X_{oo}(3)$
$\quad = X_{oe}(1) - W_8^2 X_{oo}(1)$ — (11)



↑ Construction of 8-point DFT from two 4-point DFTs and four 2-point DFTs.

For decomposing stage 1.

Consider the 2 point DFT (Assuming $N=8$)

$$X_{ee}(k) = \sum_{n=0}^{\frac{N}{4}-1} x_{ee}(n)\, W_{N/4}^{nk}$$

$$= \sum_{n=0}^{1} x_{ee}(n)\, W_8^{4nk}$$

$$k=0 \Rightarrow X_{ee}(0) = x_{ee}(0)\, W_8^0 + x_{ee}(1)\, W_8^0$$

$$= x_{ee}(0) + x_{ee}(1)$$

$$= x(0) + x(4)$$

$k=1 \Rightarrow X_{ee}(1) = x_{ee}(0)\,W_8^0 + x_{ee}(1)\,W_8^4 \qquad (\because W_8^4 = -1)$

$= x_{ee}(0) - x_{ee}(1).$

i.e For stage 1 the 2-point DFTs can be easily found by adding and subtracting the input sequences.

Ily we get:-

$X_{eo}(0) = x_{eo}(0) + x_{eo}(1) = x(2) + x(6)$

$X_{eo}(1) = x_{eo}(0) - x_{eo}(1). = x(2) - x(6).$

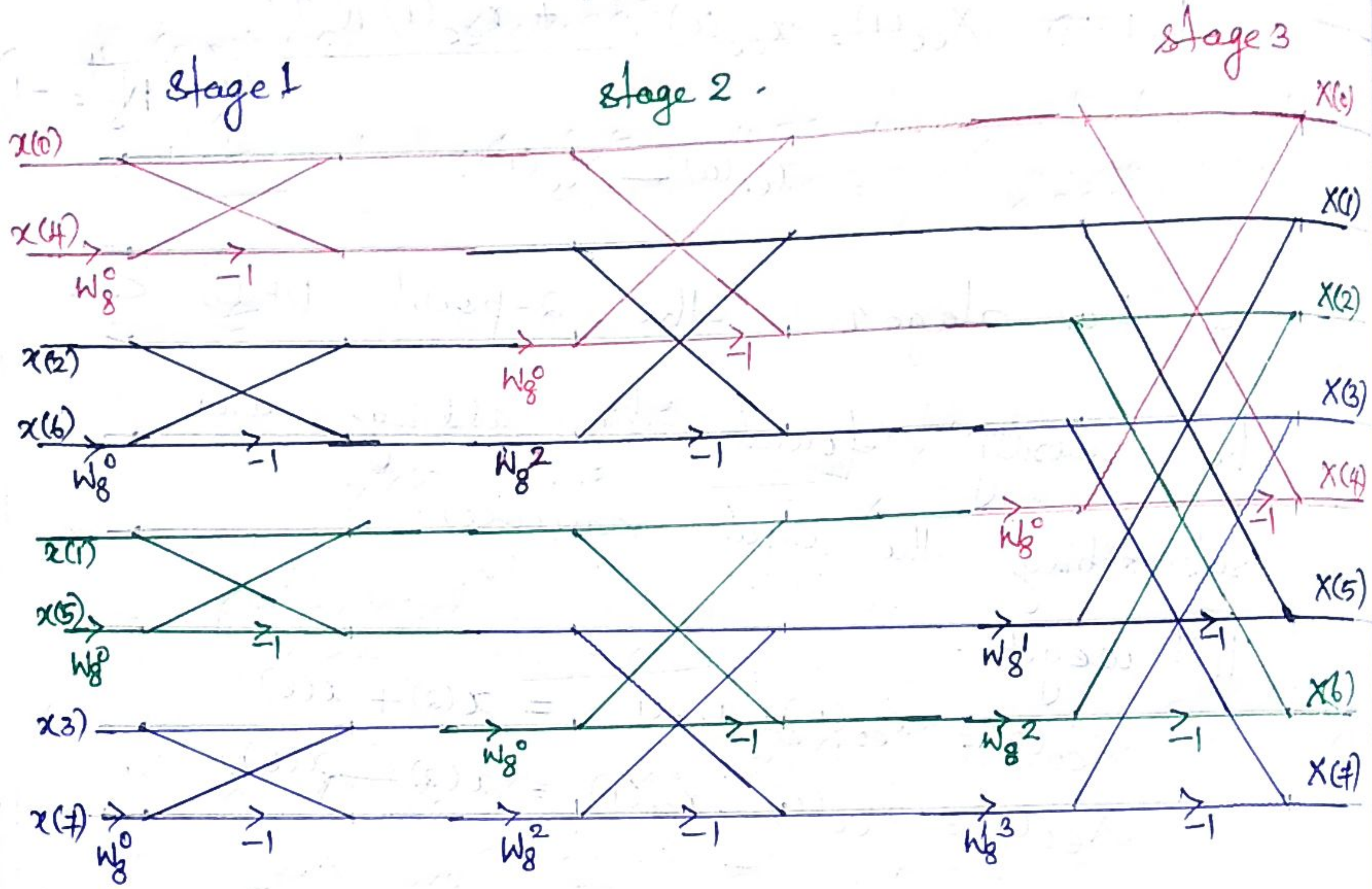
$X_{oe}(0) = x_{oe}(0) + x_{oe}(1). = x(1) + x(5)$

$X_{oe}(1) = x_{oe}(0) - x_{oe}(1) = x(1) - x(5)$

$X_{oo}(0) = x_{oo}(0) + x_{oo}(1) = x(3) + x(7)$

$X_{oo}(1) = x_{oo}(0) - x_{oo}(1) = x(3) - x(7).$

Replacing the 2-point DFT of ~~above~~ ~~equation~~ previous figure with above equations.



The above figure represents radix-2 DIT-FFT Algorithm flow.