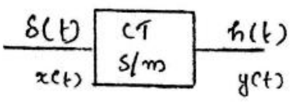
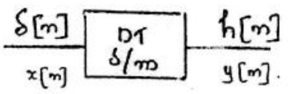
MODULE-2:

LINEAR TIME INVARIANT SYSTEM

→ Unit impulse fnⁿ $(\delta(t))$ = dirac delta fnⁿ.

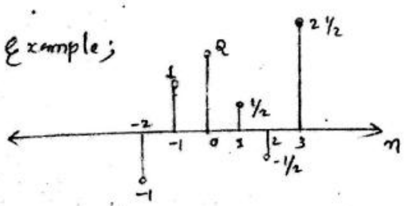


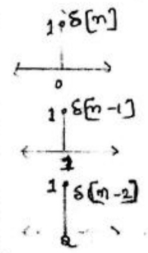


→ In a LTI systems, if we get a $h(t)$ for an impulse signal $\delta(t)$, then the $h(t)$ is called impulse response.

→ In discrete case, we get $h[n]$ for $\delta[n]$. It is also called impulse response.

✱ Example;





$x[n] = -1\cdot\delta[n+2] + 1\cdot\delta[n+1]$

$2\,\delta[n] + \tfrac{1}{2}\,\delta[n-1]$ ... $\tfrac{1}{2}\,\delta[n-2] + 2\tfrac{1}{2}\,\delta[n-3]$

✱ $\delta[n] \leftrightarrow h[k]$

$\delta[n-k] \leftrightarrow h[n-k]$

$x[k]\,\delta[n-k] \leftrightarrow x[k]\,h[n-k]$

ie, $\underbrace{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]}_{x[n]} \leftrightarrow \underbrace{\sum_{k=-\infty}^{\infty} x[k]\,h[n-k]}_{y[n]}$

✱ $y[n] = x[n] * h[n]$ (In a LTI system)

↖ Convolution

ie, $y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$

→ If we know the impulse response & i/p of an LTI system, we can determine the o/p. That is the importance of impulse response.

? Calculate the o/p $y[n]$ for an LTI system given $x[n] = \{-1, \underset{\uparrow}{\tfrac{1}{2}}, 1, -\tfrac{1}{2}\}$

$h[n] = \{\underset{\uparrow}{\tfrac{1}{2}}, 1, -\tfrac{1}{2}, \tfrac{1}{2}\}$

Let the no. of samples of

$x[n] = M = 4$

no. of samples of $h[n] = K = 4$.

$\therefore M + K - 1 = 4 + 4 - 1 = 7$.

$y[n]$ has 7 limits. or $y[n] = -1$ to 5.

(lower limit of $x[n]$ + lower limit of $h[n]$) To (upper limit of $x[n]$ + upper limit of $h[n]$)

Let $x[k] = \{-1, \underset{\uparrow}{1/2}, 1, -1/2\}$

Now, $h[k] = \{\underset{\uparrow}{1/2}, 1, -1/2, 1/2\}$

Now, $y[-1] = \sum_{k=-\infty}^{\infty} x[k]\, h[-1-k]$

(limit is from -1 to 5).

$= x[-1]h[0] + x[0]h[-1] + x[1]h[-2] + x[2]h[-3] + \ldots$

(Since, $h[-1]$ & $h[-2]$.... have no Values. $\therefore$ it is not needed)

$= x[-1]\, h[0]$

$y[-1] = -1 \times 1/2 = -1/2$ //

$y[0] = \sum_{k=-\infty}^{\infty} x[k]\, h[-k]$

$= x[-1]\, h[1] + x[-0]\, h[0] + x[1]\, h[-1]$

$= \ldots 1/2 \quad 1/2 \quad 1 \times 1/4$

$y[0] = -3/4$

$y[1] = \sum_{k=-\infty}^{\infty} x[k]\, h[1-k]$

$= x[-1]h[2] + x[0]h[1] + x[1]h[0] + x[2]h[-1]$

$= -1 \times -1/2 + 1/2 \times 1 + 1 \times 1/2$

$y[1] = 1/2 + 1/2 + 1/2 = 3/2$

$y[2] = \sum_{k=-\infty}^{\infty} x[k]\, h[2-k]$

$= x[-1]h[3] + x[0]h[2] + x[1]h[1] + x[2]h[0]$

$= -1 \times 1/2 + 1/2 \times -1/2 + 1 \times 1 + -1/2 \times 1/2$

$= -1/2 - 1/4 + 1 - 1/4 = -1/2 - 1/2 + 1 = -1 + 1 = 0$

$y[2] = 0$

$y[3] = \sum_{k=-\infty}^{\infty} x[k]\, h[3-k]$

$= x[-1]h[4] + x[0]h[3] + x[1]h[2] + x[2]h[1] + x[3]h[0]$

$= 1/2 \times 1/2 + 1 \times -1/2 + -1/2 \times 1 + 0$

$= 1/4 - 1/2 - 1/2 = 1/4 - 2/2 = 1/4 - 1 = -3/4$ //

$y[4] = \sum_{k=-\infty} x[k]h[4-k]$

$= x[-1]h[5] + x[0]h[4] + x[1]h[3] + x[2]h[2] + x[3]h[1]$

$= 1 \times 1/2 + -1/2 \times -1/2$

$= 1/2 + 1/4 = 3/4$

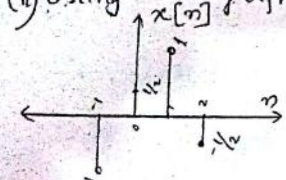
$y[5] = \sum_{k=-\infty}^{\infty} x[k]h[5-k]$

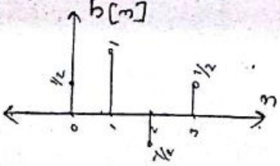
$= x[-1]h[6] + x[0]h[5] + x[1]h[4] + x[2] \times h[3]$

$= -1/2 \times 1/2$

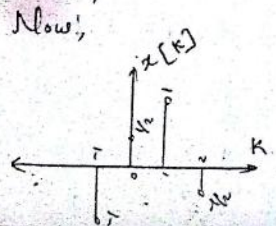
$y[5] = -1/4$

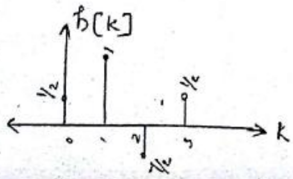
(ii) Using graph method.

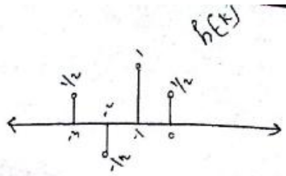




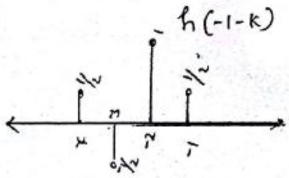
Now,

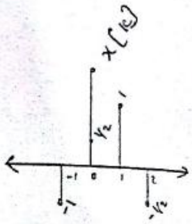






Now, we've n-values from -1 to 5. then, we have to findout h[-1-k], 0-k, 1-k, ... h[5-k].



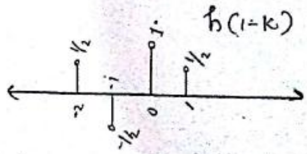


$\therefore y[-1] = -1 \times 1/2 = -1/2$.

$y[0] = x[k]\, h[0-k] = x[k]\, h[-k]$

$= -1 \times 1 + 1/2 \times 1/2 = -3/4$

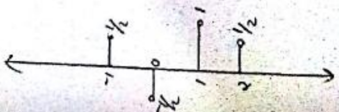
Now, h(1-k).



$\Rightarrow y[1] = -1 \cdot -1/2 + 1 \cdot 1$

$= 3/2$

Now, h(2-k)



$\Rightarrow y[2] = 1/2 \cdot 1/2 + \ldots$

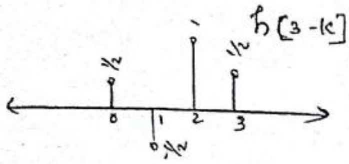
$= 1 + \ldots$

$\frac{1}{4} \cdots \frac{1}{4} \quad \frac{1}{2}$

$= \frac{1}{2} - \frac{2}{4} = \frac{1}{2} - \frac{1}{2} = 0$

$y[2] = 0$

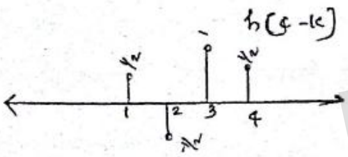
$h[3-k]$;



$\Rightarrow y[3] = \frac{1}{2} \times \frac{1}{2} + 1 \times -\frac{1}{2} + -\frac{1}{2} \times 1$

$= \frac{1}{4} - \frac{1}{2} - \frac{1}{2} = \frac{1}{4} - 1 = -\frac{3}{4}$

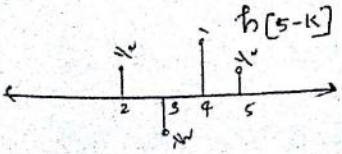
Now, $h[4-k]$;



$y[4] = 1 \times \frac{1}{2} + -\frac{1}{2} \times -\frac{1}{2}$

$= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Now, $h[5-k]$;



$y[5] = \frac{1}{2} \times -\frac{1}{2} = -\frac{1}{4}$

(iii)

$x[n] = \{-1, \underset{\uparrow}{\frac{1}{2}}, 1, -\frac{1}{2}, -\frac{1}{2}\}$ $\quad -1$ to 2

$h[n] = \{\underset{\uparrow}{\frac{1}{2}}, 1, -\frac{1}{2}, \frac{1}{2}\}$

$x[n] \downarrow$ / $h[n] \rightarrow$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$
-1	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{4}$

$y[n] = \{-\frac{1}{2}, \underset{\uparrow}{-\frac{3}{4}}, \frac{3}{2}, 0, -\frac{3}{4}, \frac{3}{4}, -\frac{1}{4}\}$

$\{-1$ to $5\}$

HW

? $x[n] = \{1, 2, 3, 4\}$

$h[n] = \{1, 1, 1, 1\}$. Calculate $y[n]$?

? $x[n] = \alpha^n u[n]$, where $0 < \alpha < 1$.

$h[n] = u[n]$. find $y[n]$.

$x[n] = i/p$

$h[n] =$ impulse response

Ans. $y[n] = x[n] * h[n]$

$y[n] = \sum_{k=-\infty}^{\infty} x[k] \, h[n-k]$

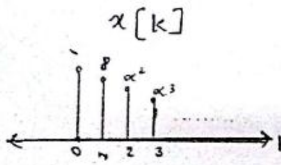
Now, $x[k] = \alpha^k u[k]$

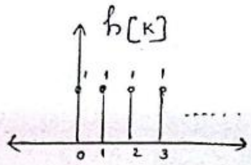
where, $x[k] = \begin{cases} \alpha^k, & k \geq 0 \\ 0, & \text{else} \end{cases}$ $\quad \left(\because u[k] = \begin{cases} 1, & k \geq 0 \\ 0, & \text{else} \end{cases}\right)$

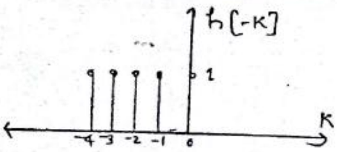
$h[k] = u[k]$

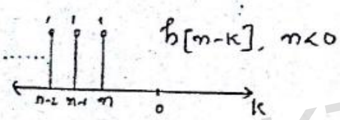
$h[k] = 1 \;,\; k \geq 0$

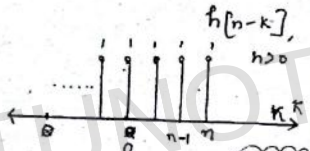
$h[-k] = 1, \quad k < 0$











(i) $y[n] = x[k] * h[n-k]$, where $n<0$

$y[n] = 0$; for $n<0$

for $n<0$, there is no common points.

0 com. zero, no common

(ii) $n > 0$,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\; h[n-k]$$

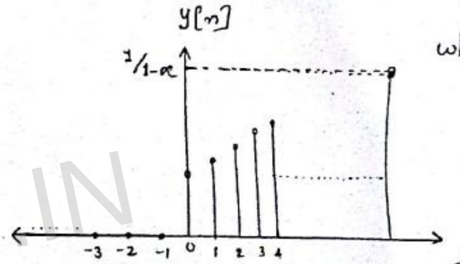
$$= \sum_{k=0}^{n} \alpha^k \cdot 1 .$$

n to ∞ → no common points
0 to −∞ → no common points
∴ from 0 to n

$$\sum_{k=0}^{N} \alpha^k = \frac{1-\alpha^{N+1}}{1-\alpha}$$

$$\therefore y[n] = \frac{1-\alpha^{n+1}}{1-\alpha}$$

$$\therefore y[n] = \begin{cases} \dfrac{1-\alpha^{n+1}}{1-\alpha} & , n>0 \\ 0 & , n<0 \end{cases}$$



when $n=0$, $y[n] = 1$

$n=1$, $y[n] = 1+\alpha$ $=>1$

$n=2$, $y[n] = 1+\alpha+\alpha^2$

$\alpha = 1/2$

$y[n] = \frac{1-(1/2)^{\infty}}{1-1/2} = \frac{1}{1-1/2} = \frac{1}{1-\alpha}$

~~Ans.~~

Ques. Given, ~~x[n]~~

? $x[n] = 2^n u[-n]$

$h[n] = u[n]$

Ans $y[n] = x[n] * h[n]$

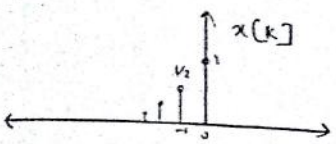
$$= \sum_{k=-\infty}^{\infty} x[k]\; h[n-k]$$

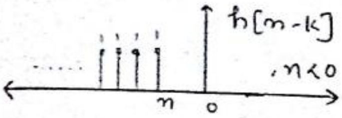
Now, $x[k] = 2^k\, u[-k]$

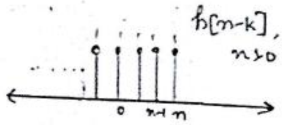
$h[k] = u[k]$

$h[-k] = u[-k]$

$h[n-k] = u[n-k]$







(i) $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \quad , n<0$

$= \sum_{k=-\infty}^{n} 2^k$

$\sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha} \Rightarrow |\alpha|<1$

Now, put $l = \cancel{\ } \; n-k$

when, $k = -\infty$,

then $l = \infty$

when $k = n$, $l = 0$

$\sum_{k=0}^{N} \alpha^k = \frac{1-\alpha^{N+1}}{1-\alpha}$

$y[n] = \sum_{l=0}^{\infty} 2^{n-l}$

$= 2^n \sum_{l=0}^{\infty} 2^{-l} = 2^n \sum_{l=0}^{\infty} (1/2)^l$

$\alpha = 1/2$, $|\alpha| < 1$

$= 2^n \times \frac{1}{1 - 1/2} = 2^n \times \frac{1}{1/2}$

$y[n] = 2^{n+1}$ //

(ii) $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \quad , n \geq 0$

$= \sum_{k=-\infty}^{0} 2^k$

~~Put $l = n-k$~~

~~when $k=-\infty$, $l=\infty$~~

~~when $k=0$, $l=n$~~

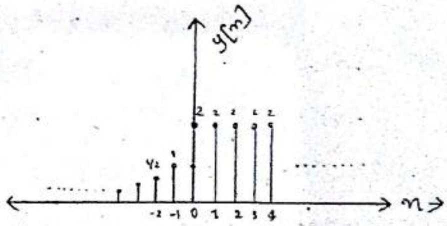
Put $r = -k$,

when $k = -\infty$, $r = \infty$

when $k = 0$, $r = 0$

$\therefore \sum_{k=0}^{\infty} 2^{-r} = \sum_{k=0}^{\infty} (1/2)^r = \frac{1}{1-1/2} = 2$ //

$\therefore y[n] = \begin{cases} 2^{n+1} & , n<0 \\ 2 & , n\geq 0 \end{cases}$



$2^{-1+1} = 2^0 = 1$

$2^{-2+1} = 2^{-1} = 1/2$

{ Discrete case Convolution is called "Convolution sum" }

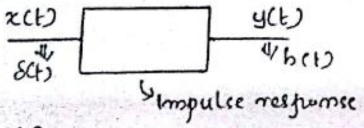
Convolution sum =

Commutative property

$y[n] = x[n] * h[n]$
$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$
$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$
$y[n] = h[n] * x[n]$
$\rightarrow (a+b = b+a)$

✱ Convolution Integral:

→ Continuous signal case convolution is called convolution integral.



$\delta(t) \rightarrow h(t)$

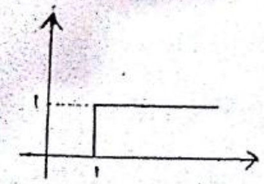
$\delta(t-\tau) \rightarrow h(t-\tau)$

$x(\tau)\,\delta(t-\tau) \rightarrow x(\tau)\,h(t-\tau)$

$$\underbrace{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau}_{x(t)} \longrightarrow \underbrace{\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau}_{y(t)}$$

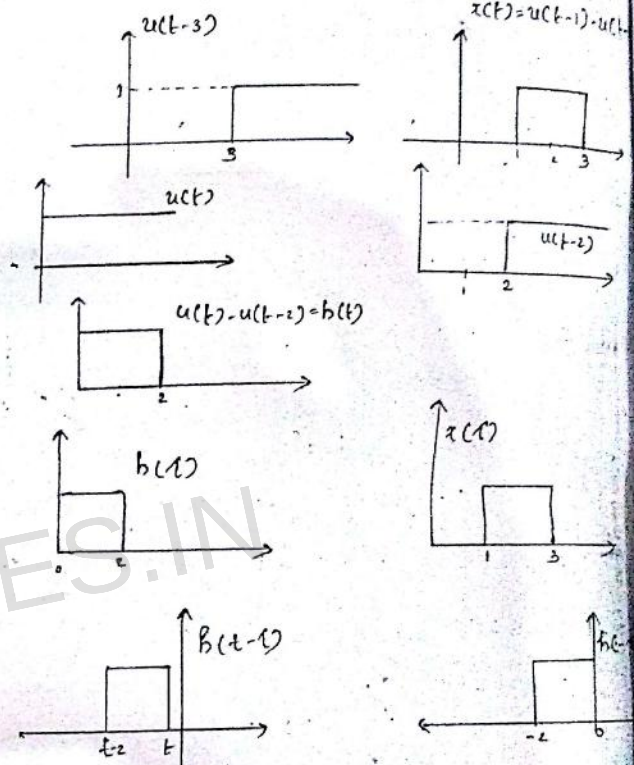
? For an LTI system the i/p $x(t)$ is given by $x(t) = u(t-1) - u(t-3)$ and the impulse response $h(t)$ is given by, $h(t) = u(t) - u(t-2)$. Calculate the o/p $y(t)$.

Ans: {Calculate $x(\tau)$, $h(\tau)$, $h(-\tau)$, $h(t-\tau)$ }.



$u(t) = 0$ to ∞ graph

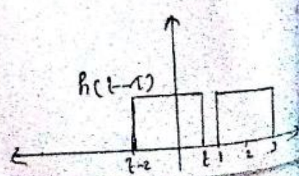
$u(t-1) = 1$ to ∞ graph



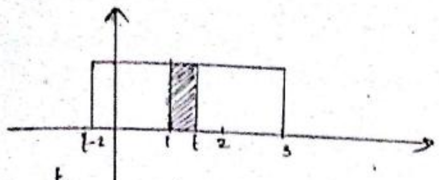
when $h(t-\tau)$ is moves right it meets at the points of $x(\tau)$ (common points).

(i) when $t < 1$, $y(t) = 0$

(there is no common points)



(ii) when $t > 1$ or $1 \le t < 3$.

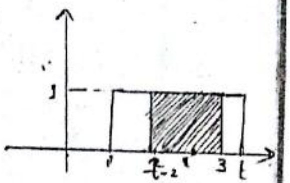


$$y(t) = \int_1^t x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_1^t 1 \cdot 1 \cdot d\tau = t - 1$$

(iii) when $1 \le t-2 < 3$

$\Rightarrow 3 \le t < 5$



$$y(t) = \int_{t-2}^{3} x(\tau)\, h(t-\tau)\, d\tau$$

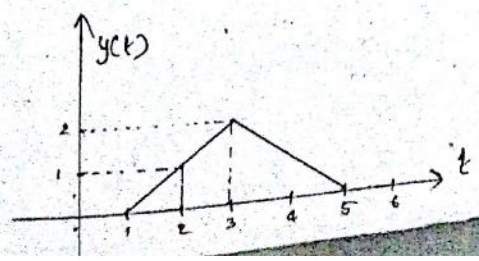
$$= [\tau]_{t-2}^{3} = 3 - (t-2) = 5 - t$$

(iv) when $t - 2 \ge 3 \Rightarrow t \ge 5$

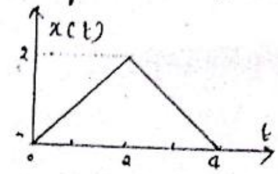
$\therefore y(t) = 0$.

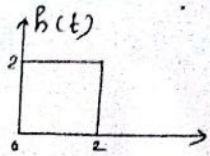
Now, $y(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 3 \\ 5-t, & 3 \le t < 5 \\ 0, & t \ge 5 \end{cases}$

then,



? Find the o/p of a Continuous time LTI system whose i/p $x(t)$ and impulse response $h(t)$ are given as follows.





Ans. Now,

$x(t)$ has two intervals, $0 \le t < 2$, $2 \le t < 4$

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

for $(0,0)$ & $(2,2)$, $\frac{y-0}{2} = \frac{x-0}{2} \Rightarrow y = x$

$x(t) = t$, for $0 \le t < 2$

Now, for $(2,2)$ & $(4,0)$

$$\frac{y-2}{0-2} = \frac{x-2}{4-2} \Rightarrow \frac{y-2}{-2} = \frac{x-2}{2}$$

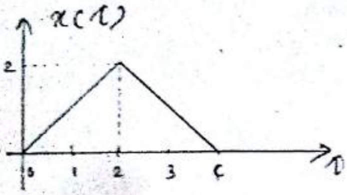
$y - 2 = -x + 2 \Rightarrow x + y = 4$

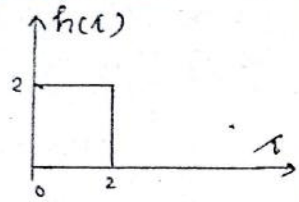
$\therefore x(t) = 4 - t$, for $2 \le t < 4$.

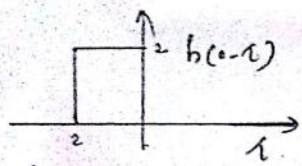
ie, $x(t) = \begin{cases} t, & 0 \le t < 2 \\ 4-t, & 2 \le t < 4. \end{cases}$

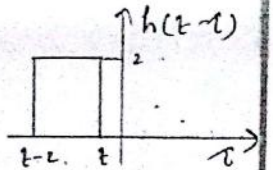
$h(t) = 2$, $0 \le t < 2$

Now $x(\tau)$ become as follows.









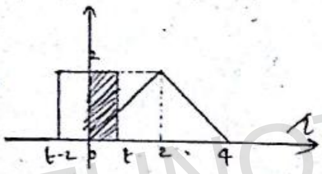
(i) Now, for $t<0$, $y(t)=0$

(ii) when $t>0$,

$y(t)=\int_0^t x(\tau)h(t-\tau)$

$=\int_0^t \tau * 2\, d\tau$

$=2\left[\tau^2/2\right]_0^t = t^2$



(iii) when $2<t\leq 4$

$y(t)=\int_{t-2}^{2} \tau \cdot 2\, d\tau + \int_2^4 (4-\tau)\cdot 2 \cdot d\tau$

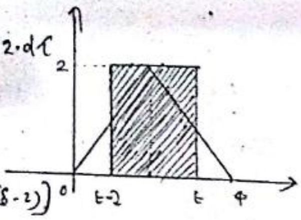
$=\left[\tau^2\right]_{t-2}^{2} + 2\left[4\tau - \tau^2/2\right]_2^t$

$=4-(t-2)^2 + 2\left[(4t - t^2/2) - (8-2)\right]$

$=4 - t^2 + 4t - 4 + 2(4t - t^2/2 - 8 + 2)$

$=4t - t^2 + 8t - t^2 - 12$

$=-2t^2+12t-12$



$y(t) = -2t^2+12t-12$

(iv) $2<t-2<4$

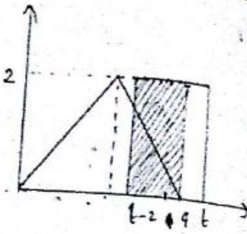
$\Rightarrow 4<t<6$

$y(t)=\int_{t-2}^{4}(4-\tau)2\,d\tau$

$=\int_{t-2}^{4} 8-2\tau = \left[8\tau - \tau^2\right]_{t-2}^{4} = \left[(32-16)-(8t-16-(t-2)^2)\right]$

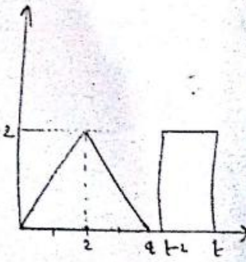
$=16-8t+16+(t-2)^2 = 16-8t+16+t^2-4t+4$

$y(t)=t^2-12t+36$



(v) when $t>6$,

$y(t)=0$.



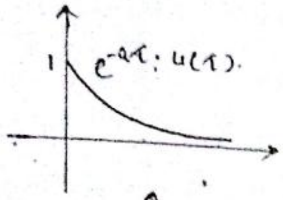
Then,

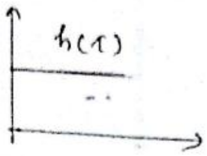
$$y(t)=\begin{cases} 0, & t<0 \\ t^2, & 0\leq t<2 \\ -2t^2+12t-12, & 2\leq t<4 \\ t^2-12t+36, & 4\leq t<6 \\ 0, & t\geq 6. \end{cases}$$

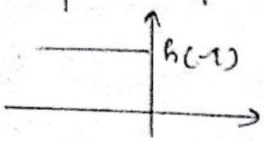
Q. Find the o/p of a continuous time LTI system for which the i/p $x(t)=e^{-at}\cdot u(t)$ and impulse response $h(t)=u(t)$, where, $a>0$

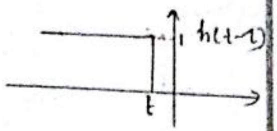
Qn. $x(t) = e^{-at} \cdot u(t)$

$h(t) = u(t)$



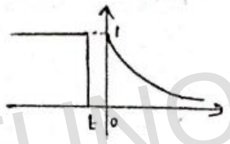




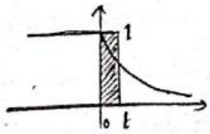


(i) when $t<0$,

$y(t) = 0$



(ii) when $t>0$,



$$y(t) = \int_0^t x(\tau) h(t-\tau) d\tau$$

$$= \int_0^t e^{-a\tau} \cdot u(\tau) \cdot 1 \cdot d\tau$$

$$= \int_0^t e^{-a\tau} \cdot d\tau = \left[e^{-a\tau}/{-a} \right]_0^t$$

$$= -1/a \left(e^{-at} - 1 \right)$$

$$y(t) = 1/a - e^{-at}/a$$

Q $x(t) = e^{at}\ u(-t)$, $a>0$

$$\therefore y(t) = \begin{cases} 0, & t<0 \\ \frac{1-e^{-at}}{a}, & t>0 \end{cases}$$

☆ CAUSALITY OF LTI SYSTEM.

→ The o/p $y[n]$ depends on i/p $x[k]$, for $k \leq n$, then it is causal.

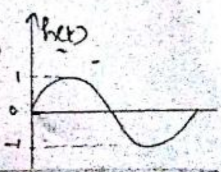
→ Let an LTI system be defined with input $x[n]$ impulse response $h[n]$ and o/p $y[n]$, for the system be causal, the o/p $y[n]$ depends only on paste and present values of i/p, $x[n]$, not future values. ie, $y[n]$ depends only on $x[k]$ for $k \leq n$. The impulse response $h[n]$ for a causal LTI system is given by $h[n]=0$, $n<0$

→ In continuous case, $y(t) \to$ o/p, $x(t) \to$ i/p, $h(t) \to$ impulse response.

$y(t)$ depends only on $x(\tau)$ for $\tau \leq t$.

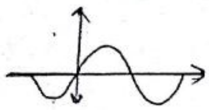
$h(t) = 0$, $t<0$

? Given an impulse response $h(t)$ of LTI system as below. Check whether the system is causal



Ans. the given system $h(t)=0$, for $t<0$, then the system is causal.

? $h(t) =$



Ans. $h(t) \neq 0$, for $t<0$ ⇒ Not causal system.

✱ STABILITY:

→ The impulse response of a stable LTI system always absolutely summable.

Proof:

For a discrete time LTI system,

we have $y[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$

$= \sum_{k=-\infty}^{\infty} x[n-k]\, h[k]$ → Commutative Property.

let $|x[n]| < B$.

then $|x[n-k]| < B$

then $|y[n]| = \sum_{k=-\infty}^{\infty} x[n-k]\, h[k] \leq \sum_{k=-\infty}^{\infty} |x[n-k]|\,|h[k]|$

$|y[n]| \leq \sum_{k=-\infty}^{\infty} B\, |h[k]| = B \sum_{k=-\infty}^{\infty} |h[k]| < \infty$

$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

→ For a system to be stable bounded input should results in bounded output. For a discrete time LTI system to be stable, the impulse response should be absolutely summable.

→ In the case of continuous signal,

$y(t) = \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} x(t-\tau)\cdot h(\tau)\, d\tau$.

let $|x(t)| < B \Rightarrow |x(t-\tau)| < B$; then

$|y(t)| = \left|\int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau\right| \leq \int_{-\infty}^{\infty} |x(t-\tau)|\,|h(\tau)|\, d\tau$

$\leq \int_{-\infty}^{\infty} B\,|h(\tau)|\, d\tau \leq B \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$.

$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$ → absolutely integrable.

For a continuous time LTI system to be stable impulse response should be absolutely integrable.

? Check whether the given LTI system is stable or not. $h[n] = (1/2)^n\, u[n]$.

Ans. We have $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$h[n] = (1/2)^n u[n]$

$h[k] = (1/2)^n u[k]$

$$h[k] = \begin{cases} (1/2)^k & , k \geq 0 \\ 0 & , \text{else} \end{cases}$$

$$\sum_{k=0}^{\infty} (1/2)^k = \frac{1}{1-1/2} = 2 < \infty$$

So, the system is stable

CORRELATION :

→ The similarities between two signals is called correlation.

→ There are two types. Auto correlation and cross correlation.

→ If the comparing two signals are same then it is autocorrelation, otherwise it is cross correlation.

Auto correlation :

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, d\tau$$

$x x \rightarrow$ to show autocorrelation

Similarities b/w signal and delayed version of signal is called autocorrelation

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, x(t)\, dt$$

→ These two are the auto correlation of energy signal.

→ All continuous time sinusoidal signals are periodic and power signals.

Also, $R_{xx}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, x(t-\tau)\, d\tau$

Cross Correlation :

$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, dt$ } in energy signal case

$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t+\tau)\, y(t)\, dt$

$R_{xy}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, y(t-\tau)\, dt$ } in the case of power

? Determine the auto correlation function of the signal given by $x(t) = \sin\omega_0 t$.

Ans $R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$

$$R_{xx}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \sin\omega_0 t \, \sin\omega_0 (t-\tau)\, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \frac{1}{2}\left[\cos(\omega_0 \tau) - \cos(2\omega_0 t - \omega_0 \tau)\right] dt$$

$$= \frac{1}{2T_0}\left[(\cos\omega_0 \tau)\,[t]_{-T_0/2}^{T_0/2}\right]$$

$$= \frac{1}{2T_0}\left(\cos\omega_0 \tau \times T_0\right) - \int_{-T_0/2}^{T_0/2} \cos(2\omega_0 t - \omega_0 \tau)\, dt$$

$$= \frac{1}{2 T_0} \times T_0 \cos\omega_0 \tau - 0$$

$$= \frac{1}{2} \cos\omega_0 \tau$$

? Find the autocorrelation function of a signal $x(t) = e^{-at} u(t)$.

Ans. Here the signal is non-periodic and always an energy signal.

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\, x(t-\tau)\, dt$$

$$= \int_{-\infty}^{\infty} \left(e^{-at} u(t) \cdot e^{-a(t-\tau)} u(t-\tau)\right) dt$$

integral of cos over a period

$\int$ = full area here, period is full. i.e $T_0/2$ to $-T_0/2$. Hence, the $\int = 0$

$u(t) = 0$ to ∞

$u(t-\tau) = \tau$ to ∞

$$= \int_{\tau}^{\infty} e^{-at} \cdot e^{-at}\, e^{a\tau}\, dt$$

$$= e^{a\tau} \int_{\tau}^{\infty} e^{-2at}\, dt = e^{a\tau} \left[\frac{e^{-2at}}{-2a}\right]_{\tau}^{\infty}$$

$$= \frac{e^{a\tau}}{-2a} \left[e^{-2a\infty} - e^{-2a\tau}\right]$$

$$= \frac{e^{a\tau}}{-2a} \left[0 - e^{-2a\tau}\right]$$

$$= e^{a\tau} \cdot \frac{e^{-2a\tau}}{2a}$$

$$= \frac{e^{(a-2a)\tau}}{2a} = \frac{e^{-a\tau}}{2a}$$

$$\therefore R_{xx}(\tau) = \frac{e^{-a\tau}}{2a}$$

✯ ORTHOGONALITY OF SIGNALS:

→ In this, we check the mutual independence of signals. If two signals are mutually independent then it is orthog

→ For two signals $x(t)$, $y(t)$

then $\int_a^b x(t)\, y^*(t)\, dt = 0 \Rightarrow$ Orthogonal

or $\int_0^T \phi_K(t)\, \phi_m^*(t)\, dt = 0$

$= 0$, for $K \neq m$

A_K, for $K = m$

→ In discrete case;

$$\sum_{n=N_1}^{N_2} \phi_K[n]\, \phi_m^*[n] = \begin{cases} A_K, & K = m \\ 0, & K \neq m \end{cases}$$

? Show that the functions $\phi_K(t) = e^{jK\omega_0 t}$ are orthogonal over any interval of length $T = \frac{2\pi}{\omega_0}$

Ans. $\phi_K(t) = e^{jK\omega_0 t}$

$\phi_m(t) = e^{jm\omega_0 t}$

$\phi_m^*(t) = + e^{-jm\omega_0 t}$

$$\therefore \int_0^T \phi_K(t)\, \phi_m^*(t)\, dt = 0$$

$$\Rightarrow \int_0 e^{jK\omega_0 t} \cdot e^{-jm\omega_0 t}\, dt$$

$$= \int_0^T e^{j\omega_0 t(k-m)}\, dt$$

$$= \int_0^T \cos[\omega_0 t(k-m)] + j\sin[\omega_0 t(k-m)]\, dt$$

$$= \frac{1}{j\omega_0(k-m)}\left[e^{j\omega_0 T(k-m)} - 1\right]$$

$$= \frac{1}{j\omega_0(k-m)}\left[e^{j2\pi(k-m)} - 1\right]$$

$$= \frac{1}{j\omega_0(k-m)}\left[\cos 2\pi(k-m) + j\sin 2\pi(k-m) - 1\right]$$

$$= 1/j\omega_0(k-m)\,[1+0-1]$$

$$= 1/j\omega_0(k-m) \times 0 = 0.$$

Hence, proved the given signal is orthogonal.

MODULE - 3

CONTINOUS TIME FOURIER SERIES

→ Fourier transformation is used to convert time domain to frequency domain.

→ If the signal is periodic signal, we use fourier series and if it is aperiodic signal, we use fourier transform.

→ CTFS, CTFT, DTFS, DTFT

- CTFS ↓ Continous with periodic
- CTFT ↓ Continous with aperiodic
- DTFS ↘ discrete with periodic
- DTFT ↘ Discrete with aperiodic

→ In Continous case;

$x(t+T) = x(t)$

$T = 2\pi/\omega_0$ → fundamental period

$\cos\omega_0 t$ and $e^{j\omega_0 t}$ are periodic signals.

The eqn of Continous time fourier series, is, CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

→ eqn is called synthesis eqn.