7.16 Quantization errors in FFT algorithms

Let us consider the computation of DFT using radix-2 DIT FFT algorithm for $N = 8$ shown in Fig. 7.29.

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The DFT is computed in $M = \log_2 N = 3$ stages. At each stage a new array of N numbers are formed from the previous array by using the basic operation of DIT-FFT algorithm given by

$$\begin{aligned} X_{m+1}(p) &= X_m(p) + W_N^k X_m(q) \\ X_{m+1}(q) &= X_m(p) - W_N^k X_m(q) \end{aligned} \tag{7.140}$$

Here the subscripts m and $m+1$ refer to the mth array and $(m+1)$st array. p and q represents the location of numbers in each array. The butterfly diagram for Eq. (7.140) is shown in Fig. 7.27.

In Eq. (7.140) the product of $X_m(q)$ and W_N^k produce a roundoff error. Therefore, we shall model the roundoff noise by associating an additive noise generator. With this model the butterfly of Fig. 7.27 is replaced by that of Fig. 7.28. The notation $e(m, q)$ represents a complex error introduced in multiplication.

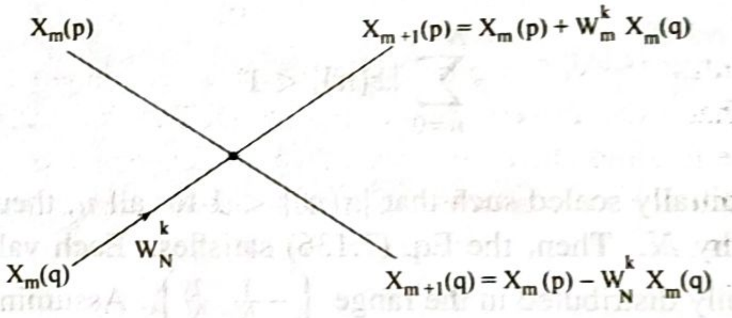

Fig. 7.27 Butterfly diagram for decimation-in-time.

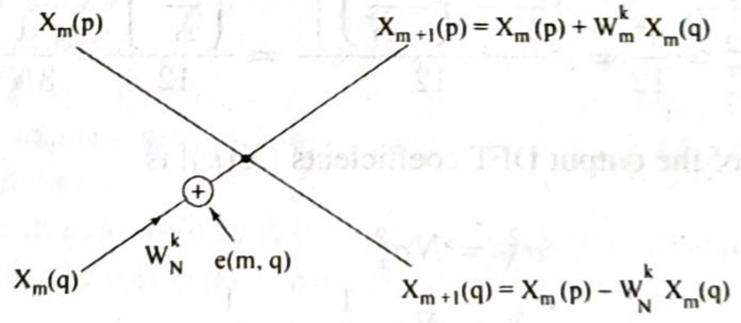

Fig. 7.28 Roundoff noise model in a decimation-in-time butterfly computation.

We make the following assumptions.

1. Roundoff noise due to each multiplication is uniformly distributed in amplitude between $-2^{-b}/2$ to $2^{-b}/2$.
2. All noise sources are uncorrelated with each other.
3. All noise sources are uncorrelated with input and output.

Since the input data and the twiddle factor are complex, four real multiplications are associated for each complex multiplication. Therefore, the variance of the roundoff error is given by

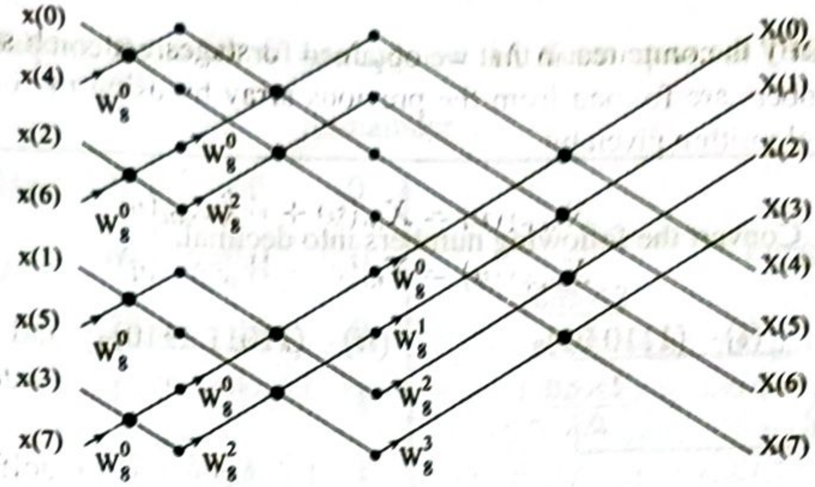

Fig. 7.29 Flow graph for DIT-FFT algorithm.

$$\sigma_\beta^2 = 4\sigma_e^2 = 4 \cdot \frac{2^{-2b}}{12} = \frac{2^{-2b}}{3} \tag{7.141}$$

To calculate the variance of output noise at any output node, we consider the butterflies that affect the computation of the DFT at that node. For example, from Fig. 7.30 we find that the output node $X(1)$ is connected to $N - 1 = 7$ butterflies. One noise source (not shown in Fig) is associated with each butterfly. Therefore, 7 noise sources affect the computation of $X(1)$.

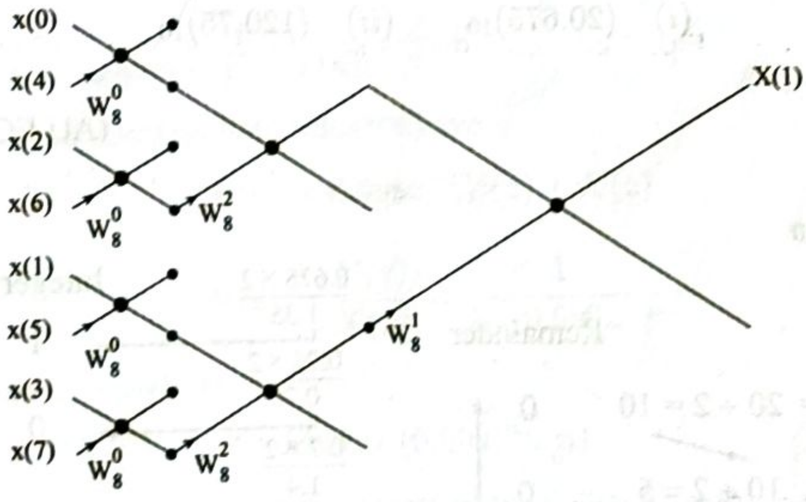

Fig. 7.30 Butterflies that effect the computation of $X(1)$.

This is applicable for all output nodes. In general we can say that $(N - 1)$ noise sources propagate to each output node, which results in an output noise variance given by

$$\sigma_\varepsilon^2 = (N-1)\sigma_\beta^2 = (N-1)\frac{2^{-2b}}{3} \tag{7.142}$$

For large value of N

$$\sigma_\varepsilon^2 \approx N\frac{2^{-2b}}{3} \tag{7.143}$$