5.5 Analog lowpass Butterworth Filter

The magnitude function of the Butterworth lowpass filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1+(\Omega/\Omega_c)^{2N}\right]^{1/2}} \quad N = 1, 2, 3, \ldots \tag{5.4}$$

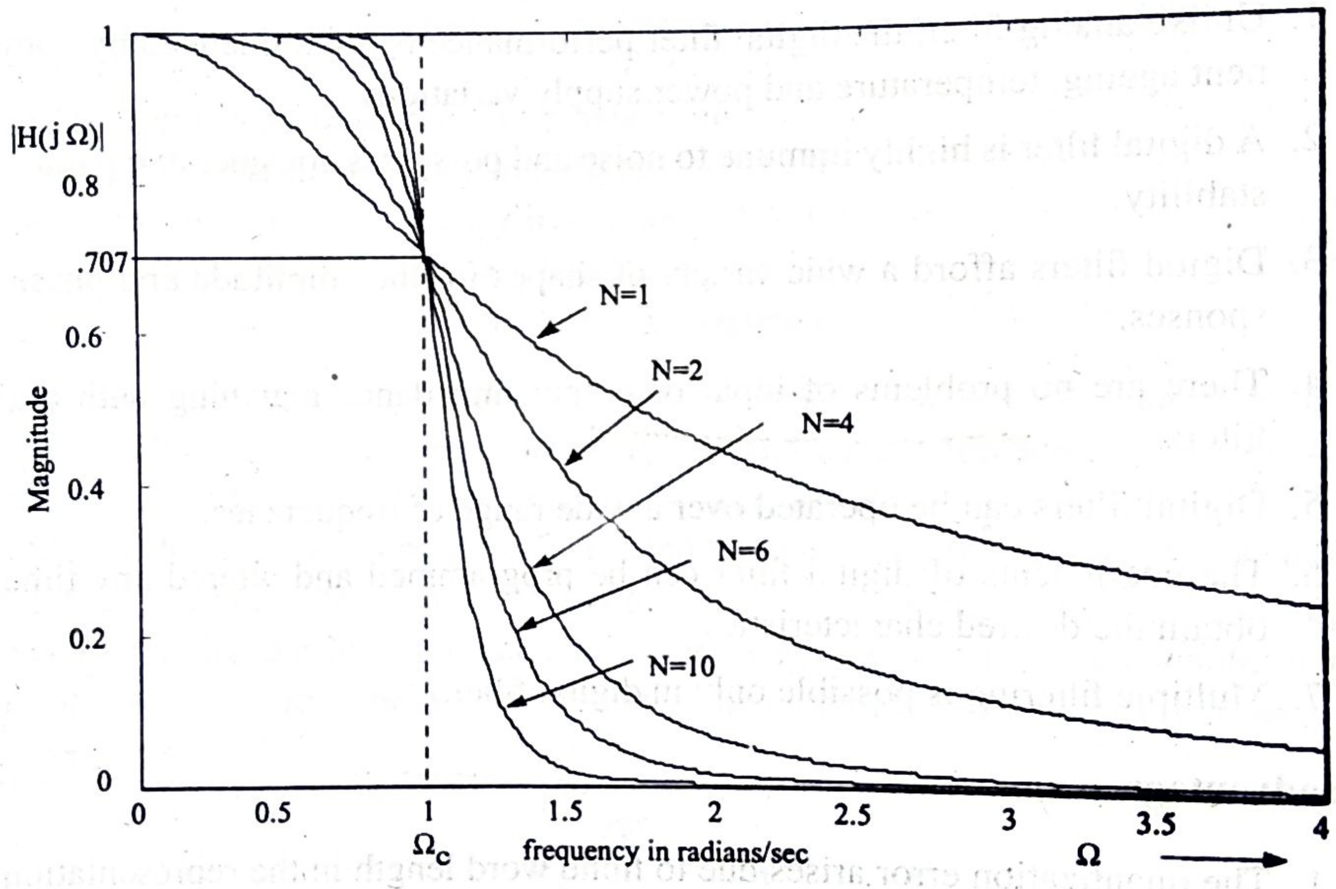


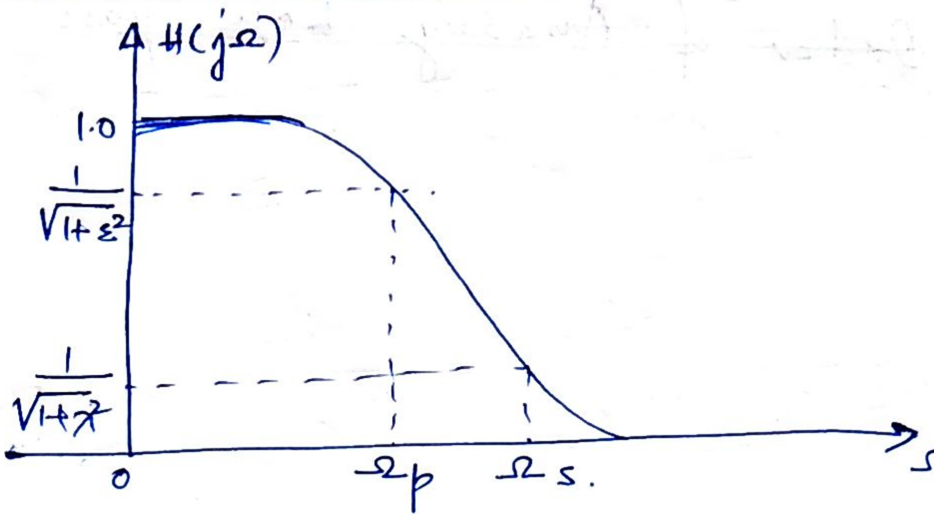
Fig. 5.5 Lowpass Butterworth magnitude response

where N is the order of the filter and Ω_c is the cutoff frequency. As shown in Fig. 5.5 the function is monotonically decreasing, where the maximum response is unity at $\Omega = 0$. The ideal response is shown by the dash line. It can be seen that the magnitude response approaches the ideal lowpass characteristics as the order N increases. For values $\Omega < \Omega_c$; $|H(j\Omega)| \approx 1$, for values $\Omega > \Omega_c$, the value of $|H(j\Omega)|$ decreases rapidly. At $\Omega = \Omega_c$, the curves pass through 0.707, which corresponds to - 3 dB point.

From Eq. (5.4), we can get magnitude square function of a normalized Butterworth filter (to 1 rad/sec cutoff frequency) as

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega)^{2N}} \quad N = 1, 2, 3, ... \tag{5.5}$$

Order of low pass Butterworth filter:



For a Butterworth filter.

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \quad \text{——— } ①$$

at $\Omega = \Omega_p \Rightarrow H(j\Omega) = \frac{1}{\sqrt{1+\varepsilon^2}}$

$$① \Rightarrow \left[\frac{1}{\sqrt{1+\varepsilon^2}}\right]^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} \quad \text{——— } ②$$

at $\Omega = \Omega_s \Rightarrow H(j\Omega) = \frac{1}{\sqrt{1+\lambda^2}}$.

$$\left[\frac{1}{\sqrt{1+\lambda^2}}\right]^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}} \quad \text{——— } ③$$

$$② \Rightarrow \frac{1}{\cancel{\sqrt{}}\,1+\varepsilon^2} = \frac{1}{1+\left[(\Omega_p/\Omega_c)^N\right]^2}$$

$$\Rightarrow \varepsilon = (\Omega_p/\Omega_c)^N \quad \text{——— } \cancel{④}$$

$$\Rightarrow \Omega_p/\Omega_c = \varepsilon^{1/N} \quad \text{——— } ④$$

$$③ \Rightarrow \frac{1}{1+\lambda^2} = \frac{1}{1+\left[\left(\frac{\Omega_s}{\Omega_c}\right)^N\right]^2}$$

$$\Rightarrow \lambda = \left(\frac{\Omega_s}{\Omega_c}\right)^N$$

$$\Rightarrow \frac{\Omega_s}{\Omega_c} = \lambda^{1/N} \quad \text{---} ⑤$$

$$⑤/④ \Rightarrow \frac{\Omega_s}{\Omega_p} = \left(\frac{\lambda}{\varepsilon}\right)^{1/N}$$

$$\log\left(\frac{\Omega_s}{\Omega_p}\right) = \frac{1}{N}\log(\lambda/\varepsilon)$$

$$\cancel{N = \frac{\log(\Omega_s/\Omega_p)}{\log(\lambda/\varepsilon)}}$$

$$\boxed{N = \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)}}$$

but $\lambda = \sqrt{10^{0.1\alpha_s} - 1}$

$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$.

$$\boxed{N = \frac{\log_{10}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log_{10}(\Omega_s/\Omega_p)}}$$

Q) Proove that

$$\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}} = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}}.$$

and

$$\Omega_c = \frac{\Omega_s}{\lambda^{1/N}} = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}.$$

proof: we have the magnitude square function of a Butterworth analog lowpass filter

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}. \quad \text{——— ①}$$

Now if α_p is maximum passband attenuation in positive dB (ε parameter specifying passband attenuation) at passband edge frequency Ω_p, the magnitude function can also be written as:

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 (\Omega/\Omega_p)^{2N}}. \quad \text{——— ②}$$

Comparing above equations ① and ②.

$$\frac{1}{\Omega_c^{2N}} = \frac{\varepsilon^2}{\Omega_p^{2N}}$$

$$\frac{1}{\Omega_c^N} = \frac{\varepsilon}{\Omega_p^N}.$$

$$\Longrightarrow \boxed{\Omega_c = \frac{\Omega_p}{\varepsilon^{1/N}}.} \quad \text{——— (3)}$$

but $\varepsilon = \left(10^{0.1\alpha_p} - 1\right)^{1/2}$.

$$(3) \Longrightarrow \Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)^{1/2N}}.$$

In similar way if α_s is minimum stop band attenuation and (λ parameter specifying stop band attenuation) at stop band edge frequency Ω_s.

$$\boxed{\Omega_c = \frac{\Omega_s}{\lambda^{1/N}}}$$

(Home work do as).

$$\Omega_c = \frac{\Omega_s}{\left(10^{0.1\alpha_s} - 1\right)^{1/2N}}.$$

Q) Given the specification $\alpha_p = 1dB$, $\alpha_s = 30dB$

$\Omega_p = 200 rad/sec$. $\Omega_s = 600 rad/sec$. Determine the

order of the filter.

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = \sqrt{10^{0.1 \times 30} - 1} =$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = \sqrt{10^{0.1 \times 1} - 1} =$$

$$N \geq \frac{\log \lambda/\varepsilon}{\log \Omega_s/\Omega_p} =$$

$$\geq 3.758$$

Round of N to the ~~se~~ next higher

integer $\underline{\underline{N = 4}}$