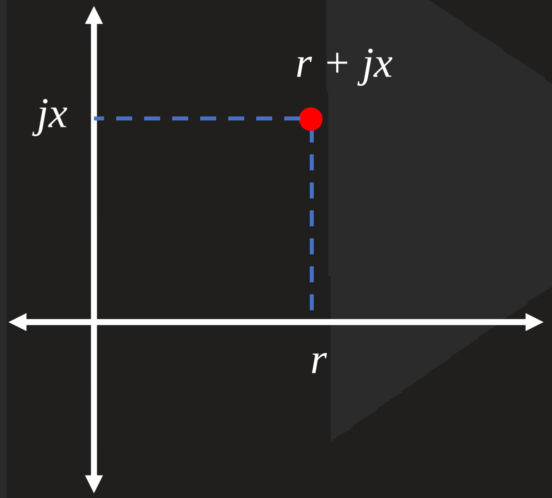
SMITH CHART

- It is a graphical method of solving the transmission line problems
- It is basically a graphical indication of impedance of transmission line as one move along the line
- Smith chart is constructed within a circle of unit radius and we represent the impedance value in real and imaginary



In a lossless transmission line

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Normalizing $\quad \overline{Z_L} = \dfrac{Z_L}{Z_0}$

$$\Gamma_L = \frac{\frac{1}{Z_0}\left(\frac{Z_L}{Z_0} - 1\right)}{\frac{1}{Z_0}\left(\frac{Z_L}{Z_0} + 1\right)}$$

$$\Gamma_L = \frac{\overline{Z_L} - 1}{\overline{Z_L} + 1}$$

$$\Gamma_L\left(\overline{Z_L} + 1\right) = \left(\overline{Z_L} - 1\right)$$

$$\overline{Z_L}\Gamma_L + \Gamma_L = \overline{Z_L} - 1$$

$$1 + \Gamma_L = \overline{Z_L}(1 - \Gamma_L)$$

$$\overline{Z_L} = \frac{1 + \Gamma_L}{1 - \Gamma_L} \qquad \text{----------- (1)}$$

Since Z_L is a complex term let us assume

$$\overline{Z_L} = r + jx \qquad \text{----------- (2)}$$

Equating (1) & (2)

$$\Gamma_L = \Gamma_r + j\Gamma_i$$

$$r + jx = \frac{1 + \Gamma_L}{1 - \Gamma_L}$$

$$r + jx = \frac{1 + (\Gamma_r + j\Gamma_i)}{1 - (\Gamma_r + j\Gamma_i)}$$

$$= \frac{(1 + \Gamma_r + j\Gamma_i)}{(1 - \Gamma_r - j\Gamma_i)} \times \frac{(1 - \Gamma_r + j\Gamma_i)}{(1 - \Gamma_r + j\Gamma_i)}$$

$$= \frac{1 - \Gamma_r^2 - \Gamma_i^2 + 2j\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

Comparing real and imaginary parts

$$r = \frac{1 - \Gamma_r^2 - \Gamma_i^2}{(1 - \Gamma_r)^2 + \Gamma_i^2} \qquad x = \frac{2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2}$$

REAL PART

$$r = \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2}$$

$$r(1-\Gamma_r)^2 + r\Gamma_i^2 = 1-\Gamma_r^2-\Gamma_i^2$$

$$r(1-\Gamma_r)^2 + r\Gamma_i^2 - 1 + \Gamma_r^2 + \Gamma_i^2 = 0$$

$$r(1-\Gamma_r)^2 - (1-\Gamma_r^2) + \Gamma_i^2(r+1) = 0$$

Multiplying (r+1) on both sides

$$r(r+1)(1-\Gamma_r)^2 - (r+1)(1-\Gamma_r^2) + \Gamma_i^2(r+1)^2 = 0$$

Adding $1/(r+1)^2$ on both sides

$$\frac{1 + r(r+1)(1-\Gamma_r)^2 - (r+1)(1-\Gamma_r^2) + \Gamma_i^2(r+1)^2}{(r+1)^2} = \frac{1}{(r+1)^2}$$

$$\frac{1 + r(r+1)(1-\Gamma_r)^2 - (r+1)(1-\Gamma_r^2) + \Gamma_i^2(r+1)^2}{(r+1)^2} = \frac{1}{(r+1)^2}$$

$$\frac{(r+1)[r(1-\Gamma_r)^2 - (1-\Gamma_r^2)] + 1}{(r+1)^2} + \Gamma_i^2 = \frac{1}{(r+1)^2}$$

$$\frac{(r+1)[r - 2r\Gamma_r + r\Gamma_r^2 - 1 + \Gamma_r^2] + 1}{(r+1)^2} + \Gamma_i^2 = \frac{1}{(r+1)^2}$$

$$\frac{r^2 - 2r^2\Gamma_r + r^2\Gamma_r^2 - r + r\Gamma_r^2 + r - 2r\Gamma_r + r\Gamma_r^2 - 1 + \Gamma_r^2 + 1}{(r+1)^2} + \Gamma_i^2 = \frac{1}{(r+1)^2}$$

$$\frac{r^2 - 2r^2\Gamma_r + r^2\Gamma_r^2 + 2r\Gamma_r^2 - 2r\Gamma_r + \Gamma_r^2}{(r+1)^2} + \Gamma_i^2 = \frac{1}{(r+1)^2}$$

$$\frac{\Gamma_r^2(r^2+2r+1) + r^2 - 2r\Gamma_r(r+1)}{(r+1)^2} + \Gamma_i^2 = \frac{1}{(r+1)^2}$$

$$\Gamma_r^2 + \frac{r^2}{(r+1)^2} - \frac{2r\Gamma_r}{(r+1)} + \Gamma_i^2 = \frac{1}{(r+1)^2}$$

$$\boxed{\left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 = \frac{1}{(r+1)^2}}$$

IMAGINARY PART

$$x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$(a-b)^2 = (b-a)^2$$

$$(1-\Gamma_r)^2 + \Gamma_i^2 = \frac{2\Gamma_i}{x}$$

$$(1-\Gamma_r)^2 + \Gamma_i^2 - \frac{2\Gamma_i}{x} = 0$$

Adding $1/x^2$ on both sides

$$(1-\Gamma_r)^2 + \Gamma_i^2 - \frac{2\Gamma_i}{x} + \frac{1}{x^2} = \frac{1}{x^2}$$

$$(1-\Gamma_r)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$

$$\boxed{(\boldsymbol{\Gamma_r} - \mathbf{1})^2 + \left(\boldsymbol{\Gamma_i} - \frac{\mathbf{1}}{\boldsymbol{x}}\right)^2 = \frac{\mathbf{1}}{\boldsymbol{x}^2}}$$

From real Part

$$\left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 = \frac{1}{(r+1)^2}$$

$Center \rightarrow \left(\frac{r}{r+1}, 0\right)$

$radius \rightarrow \frac{1}{r+1}$

Equation of circle

$$(x-h)^2 + (y-k)^2 = r^2$$

$Center \rightarrow (h, k)$

$radius \rightarrow r$

From imaginary Part

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \frac{1}{x^2}$$

$Center \rightarrow \left(1, \frac{1}{x}\right)$

$radius \rightarrow \frac{1}{x}$

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From real Part

$$\boxed{\left(\Gamma_r - \frac{r}{r+1}\right)^2 + \Gamma_i^2 = \frac{1}{(r+1)^2}}$$

Center $\rightarrow \left(\frac{r}{r+1}, 0\right)$ *radius* $\rightarrow \frac{1}{r+1}$

For r = 0 *Center* $\rightarrow (0, 0)$ *radius* $\rightarrow 1$

For r = 1 *Center* $\rightarrow \left(\frac{1}{2}, 0\right)$ *radius* $\rightarrow \frac{1}{2}$

For r = 2 *Center* $\rightarrow \left(\frac{2}{3}, 0\right)$ *radius* $\rightarrow \frac{1}{3}$

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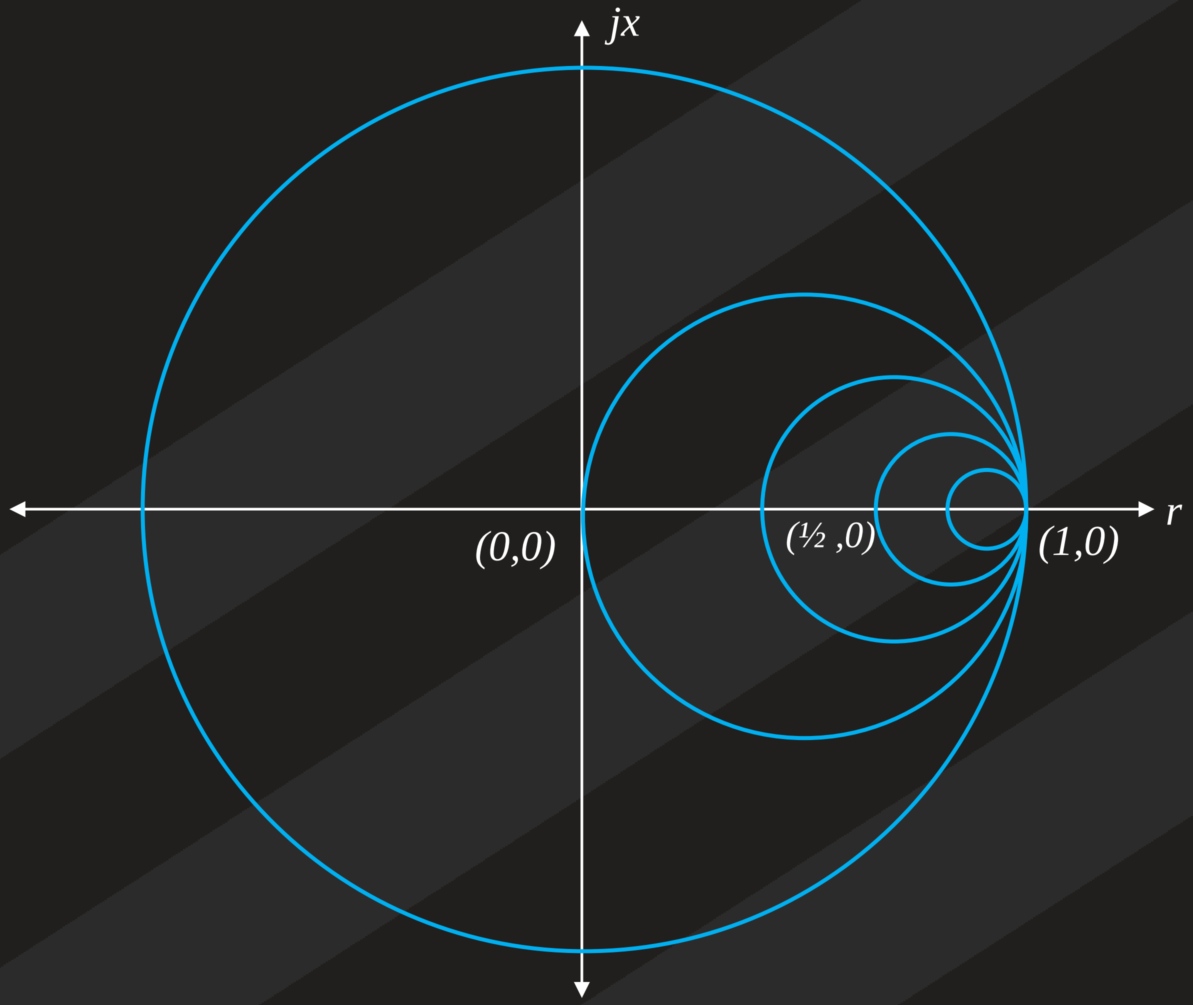
For r = ∞ *Center* $\rightarrow (1, 0)$ *radius* $\rightarrow 0$



From imaginary Part

$$\boxed{(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \frac{1}{x^2}}$$

$Center \rightarrow \left(1, \frac{1}{x}\right)$ $\quad radius \rightarrow \frac{1}{x}$

For x = 0 $\quad Center \rightarrow (1, \infty)$ $\quad radius \rightarrow \infty$

For x = 1 $\quad Center \rightarrow (1, 1)$ $\quad radius \rightarrow 1$

For x = 2 $\quad Center \rightarrow \left(1, \frac{1}{2}\right)$ $\quad radius \rightarrow \frac{1}{2}$

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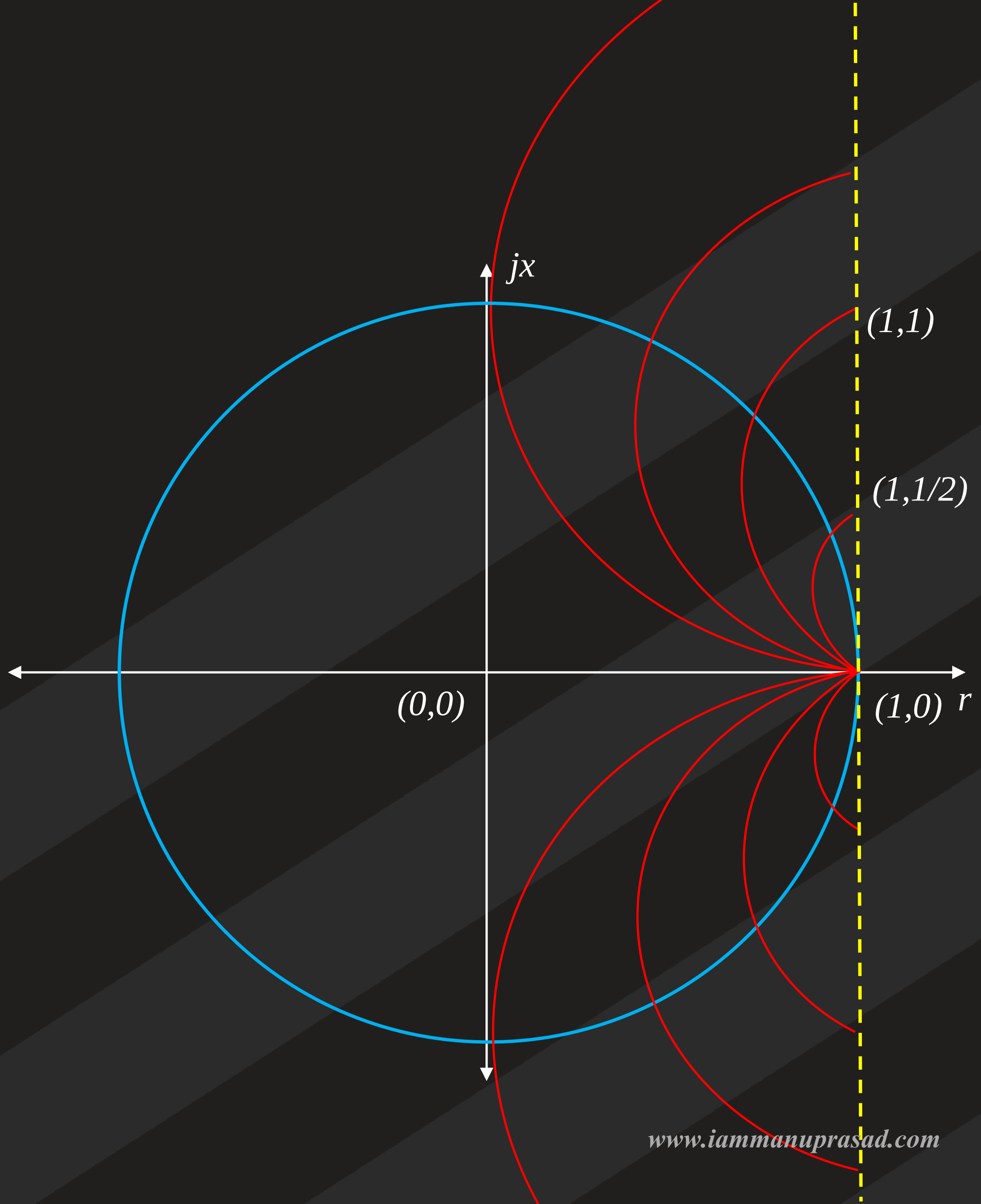
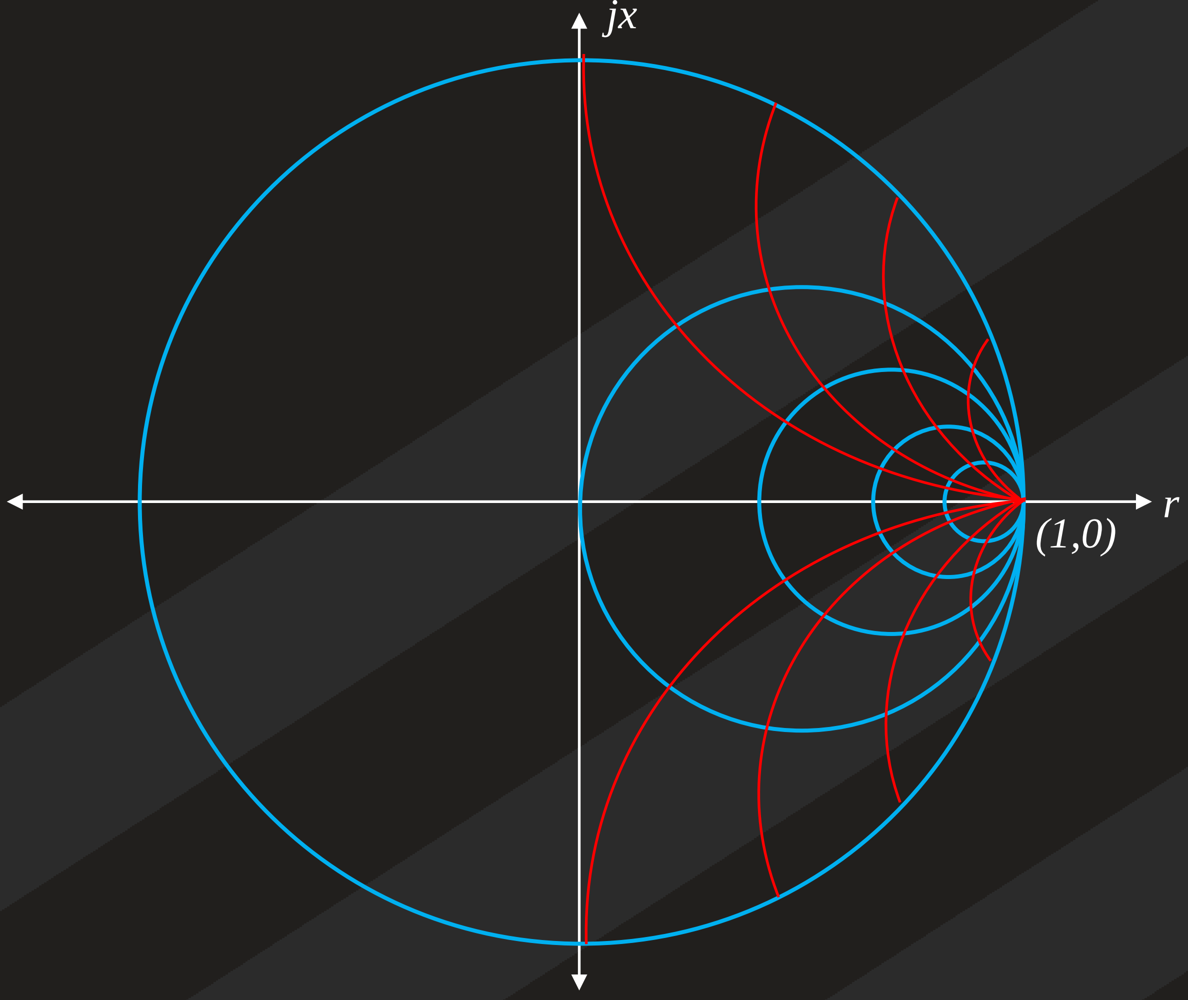
For r = ∞ $\quad Center \rightarrow (1, 0)$ $\quad radius \rightarrow 0$



- It can then be seen that all of the circles of one family will intersect all of the circles of the other family.
- Knowing the impedance, in the form of r + jx, the corresponding reflection coefficient can be determined.
- It is only necessary to find the intersection point of the two circles corresponding to the values r and x.
- The reverse operation is also possible, Knowing the reflection coefficient, find the two circles intersecting at that point and read the corresponding values r and × on the circles.
    - Determine the impedance as a spot on the Smith chart.
    - Find the reflection coefficient (Γ) for the impedance.
    - Having the characteristic impedance and Γ, find the impedance.
    - Convert the impedance to admittance.



Q) A lossless transmission line with $Z_0 = 50\Omega$ is $30m$ long and operates at $2MHz$. The line is terminated with a load $Z_L = 60+j40\ \Omega$. If u = 0.6c on the line find

(i) *Reflection coefficient*
(ii) *Standing wave ratio*
(iii) *Input impedance*

<u>Solution</u>

$Z_0 = 50 \qquad l = 30 \qquad F = 2MHz \qquad Z_L = 60+j40 \qquad u = 0.6C$

Normalizing

$$\overline{Z_L} = \frac{Z_L}{Z_0} = \frac{60+j40}{50} = 1.2 + j0.8$$

$$|\Gamma| = \frac{OP}{OQ} = \frac{3.2}{9.1} = 0.3516$$

$$\angle\Gamma = \angle SOP = 56^0$$

$$\boldsymbol{\Gamma = 0.3516\angle 56^0}$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{60+j40-50}{60+j40+50}$$

$$= 0.197 + j0.29$$

$$\boldsymbol{\Gamma = 0.35\angle 55.8}$$

$$SWR = 2.1$$

$$SWR = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.35}{1-0.35} = 2$$

$$\lambda = \frac{u}{f} = \frac{0.6(3\times10^8)}{2\times10^6} = 90m$$

$$l = 30m = \frac{30}{90}\lambda = \frac{1}{3}\lambda$$

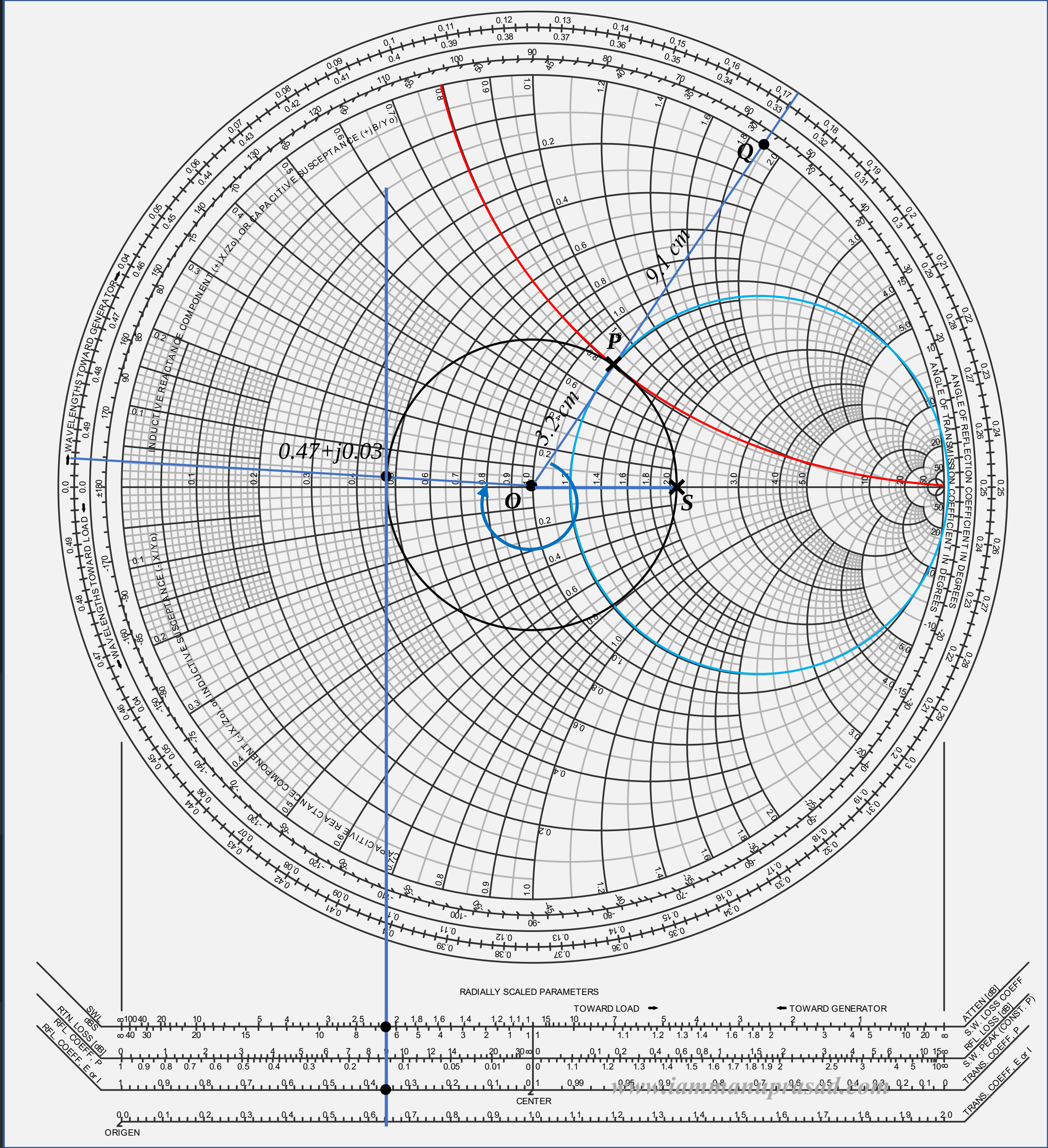
$$= \frac{720^0}{3} = 240^0$$

$$\boldsymbol{\overline{Z_{in}} = 0.47 + j0.03}$$

$$\boldsymbol{Z_{in} = \overline{Z_{in}}Z_0 = (0.47 + j0.03)50}$$

$$\boldsymbol{= 23.5 + j1.5}$$

$$\boldsymbol{Z_{in} = \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l}}$$



Q) The 0.1λ length line shown has a characteristic impedance of 50Ω and is terminated with a load impedance of $Z_L = 5 + j25\Omega$.

(i) What is the impedance at $l = 0.1\lambda$?

(ii) What is the VSWR on the line?

(iii) What is Γ_L?

(iv) What is Γ at $l = 0.1\lambda$ from the load?

Solution

$Z_0 = 50 \qquad l = 0.1\lambda \qquad Z_L = 5+j25$

Normalizing

$$\overline{Z_L} = \frac{Z_L}{Z_0} = \frac{5+j25}{50} = 0.1 + j0.5$$

$l = 0.1\lambda = 720 * 0.1 = 72^0$

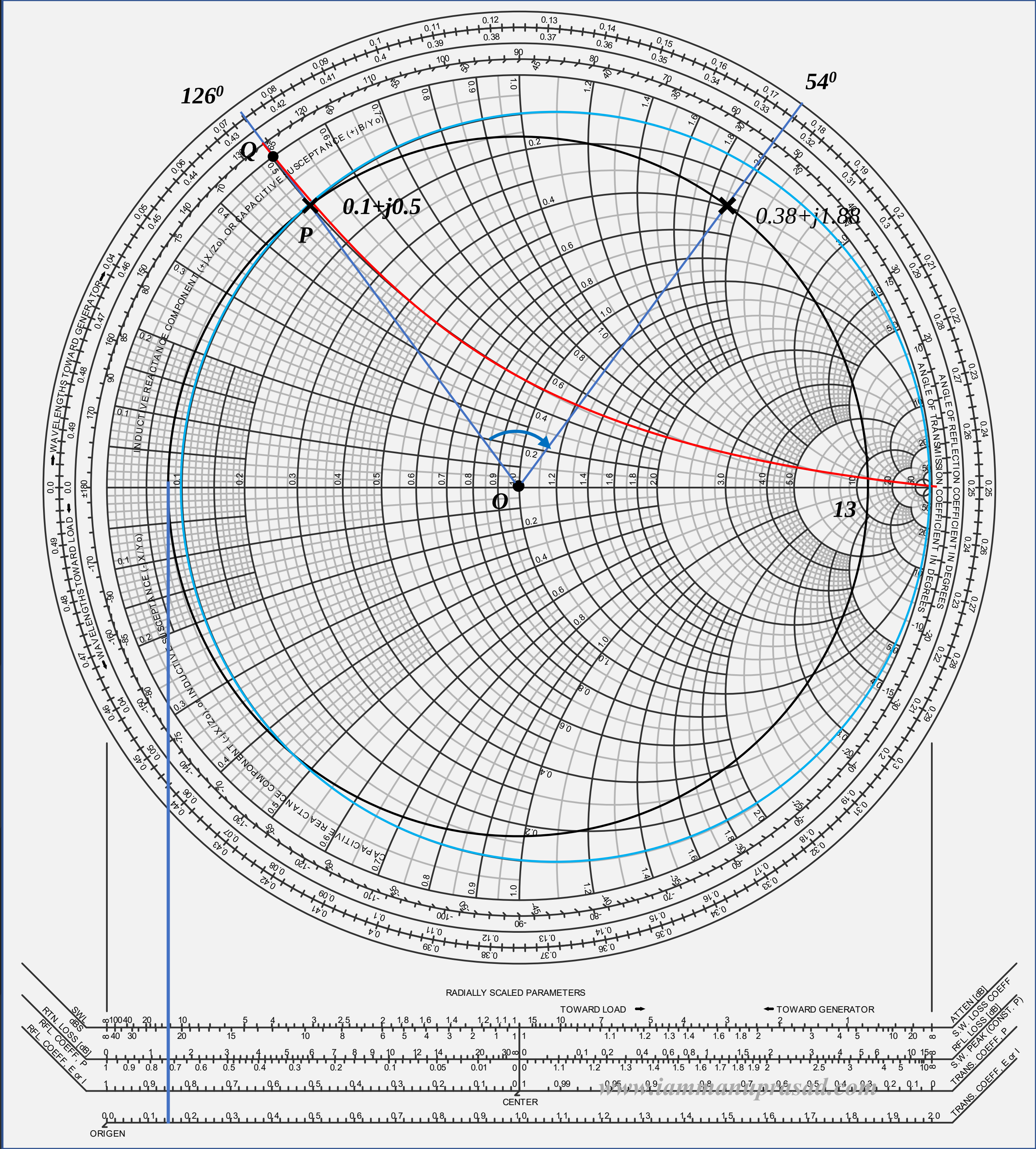
$\overline{Z_{in}} = \mathbf{0.38 + j1.88} \qquad Z_{in} = \overline{Z_{in}} Z_0 = (0.38 + j1.88)50 = 19 + j94$

$SWR = 13$

$$|\Gamma| = \frac{OP}{OQ} = \frac{7}{8.5} = 0.82 \qquad \angle\Gamma = 126^0 \qquad \Gamma = \mathbf{0.82\angle 126^0}$$

Γ at $l = 0.1\lambda$

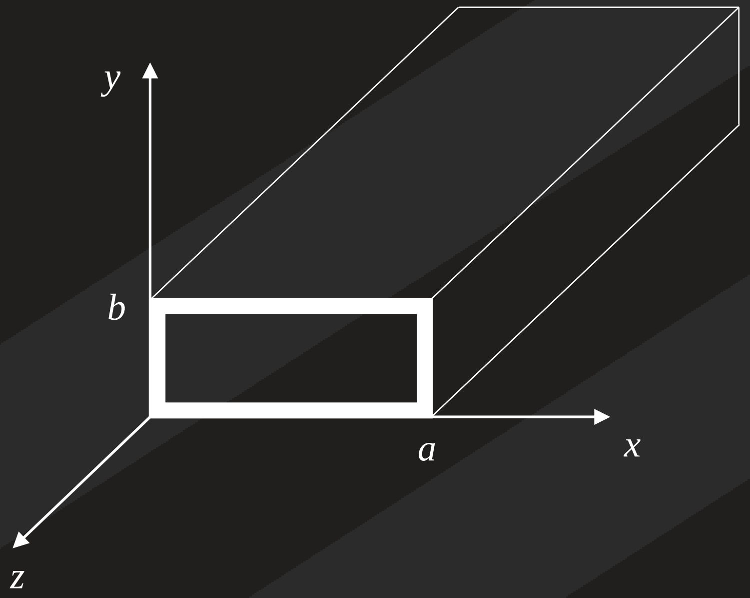
$|\Gamma| = 0.82 \qquad \angle\Gamma = 54^0 \qquad \Gamma = \mathbf{0.82\angle 54^0}$



RECTANGULAR WAVE GUIDE

- A waveguide is an electromagnetic feed line used in microwave communications, broadcasting, and radar installations. A waveguide consists of a rectangular or cylindrical metal tube or pipe. The electromagnetic field propagates lengthwise.

- Here we assume that inner surface is perfectly conducting and the region inside the guide is lossless dielectric, from Maxwell's equation



$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{----------- (1)}$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} \quad \text{----------- (2)}$$

The harmonic equation for the same (J = 0)

$$\nabla \times E = -j\omega\mu H \quad \text{----------- (3)}$$

$$\nabla \times H = j\omega\varepsilon E \quad \text{----------- (4)}$$

$$\boldsymbol{\nabla \times E = -j\omega\mu H} \quad \text{----------- (3)}$$

$$\nabla \times E = \begin{vmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu(H_x + H_y + H_z)$$

$$\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] a_x + \left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right] a_y + \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right] a_z = -j\omega\mu(H_x + H_y + H_z)$$

Equating x, y, z terms

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \quad \text{----------- (5)}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \quad \text{----------- (6)}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad \text{----------- (7)}$$

$$\boldsymbol{\nabla \times H = j\omega\varepsilon E} \quad \text{----------- (4)}$$

$$\nabla \times H = \begin{vmatrix} a_x & a_y & a_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = -j\omega\varepsilon(E_x + E_y + E_z)$$

$$\left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right] a_x + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right] a_y + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] a_z = -j\omega\varepsilon(E_x + E_y + E_z)$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \quad \text{----------- (8)}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad \text{----------- (9)}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \quad \text{----------- (10)}$$

Let us assume the filed varying along z-axis as $e^{-\gamma z}$

$$E_x = E_x^0 e^{-\gamma z}$$

$$E_y = E_y^0 e^{-\gamma z}$$

$$\frac{\partial E_x}{\partial z} = -\gamma E_x$$

$$\frac{\partial E_y}{\partial z} = -\gamma E_y$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x \quad \text{---------- (5)}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y \quad \text{---------- (6)}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad \text{---------- (7)}$$

(5) → $$\frac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x \quad \text{---------- (11)}$$

(6) → $$\gamma E_x + \frac{\partial E_z}{\partial x} = j\omega\mu H_y \quad \text{---------- (12)}$$

(7) → $$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z \quad \text{---------- (13)}$$

(12) → $$E_x = \frac{j\omega\mu H_y}{\gamma} - \frac{1}{\gamma}\frac{\partial E_z}{\partial x} \quad \text{---------- (17)}$$

$$H_x = H_x^0 e^{-\gamma z}$$

$$H_y = H_y^0 e^{-\gamma z}$$

$$\frac{\partial H_x}{\partial z} = -\gamma H_x$$

$$\frac{\partial H_y}{\partial z} = -\gamma H_y$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \quad \text{---------- (8)}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad \text{---------- (9)}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \quad \text{---------- (10)}$$

(8) → $$\frac{\partial H_z}{\partial y} + \gamma H_y = j\omega\varepsilon E_x \quad \text{---------- (14)}$$

(9) → $$-\gamma H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad \text{---------- (15)}$$

(10) → $$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \quad \text{---------- (16)}$$

$$H_y = \frac{j\omega\varepsilon E_x}{\gamma} - \frac{1}{\gamma}\frac{\partial H_z}{\partial y} \quad \text{---------- (18)}$$

Substitute (18) in (17)

$$E_x = \frac{j\omega\mu}{\gamma}\left[\frac{j\omega\varepsilon E_x}{\gamma} - \frac{1}{\gamma}\frac{\partial H_z}{\partial y}\right] - \frac{1}{\gamma}\frac{\partial E_z}{\partial x}$$

$$= \frac{\omega^2\mu\varepsilon E_x}{\gamma^2} - \frac{j\omega\mu}{\gamma^2}\frac{\partial H_z}{\partial y} - \frac{1}{\gamma}\frac{\partial E_z}{\partial x}$$

$$E_x\left[1 + \frac{\omega^2\mu\varepsilon}{\gamma^2}\right] = -\frac{j\omega\mu}{\gamma^2}\frac{\partial H_z}{\partial y} - \frac{1}{\gamma}\frac{\partial E_z}{\partial x}$$

$$E_x = \frac{\gamma^2}{\gamma^2 + \omega^2\mu\varepsilon}\left[-\frac{j\omega\mu}{\gamma^2}\frac{\partial H_z}{\partial y} - \frac{1}{\gamma}\frac{\partial E_z}{\partial x}\right]$$

$$E_x = \frac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left[-j\omega\mu\frac{\partial H_z}{\partial y} - \gamma\frac{\partial E_z}{\partial x}\right]$$

Substitute $h^2 = \gamma^2 + \omega^2\mu\varepsilon$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} \qquad H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} \qquad H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial y}$$

$$E_x = \frac{j\omega\mu H_y}{\gamma} - \frac{1}{\gamma}\frac{\partial E_z}{\partial x} \quad \text{----------- (17)}$$

$$H_y = \frac{j\omega\varepsilon E_x}{\gamma} - \frac{1}{\gamma}\frac{\partial H_z}{\partial y} \quad \text{----------- (18)}$$

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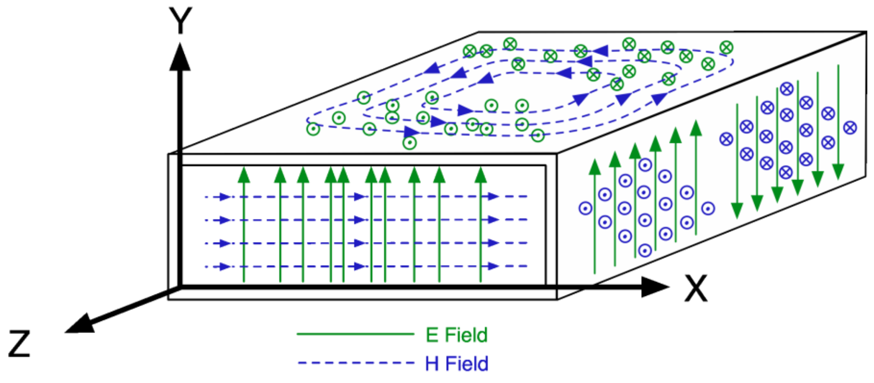
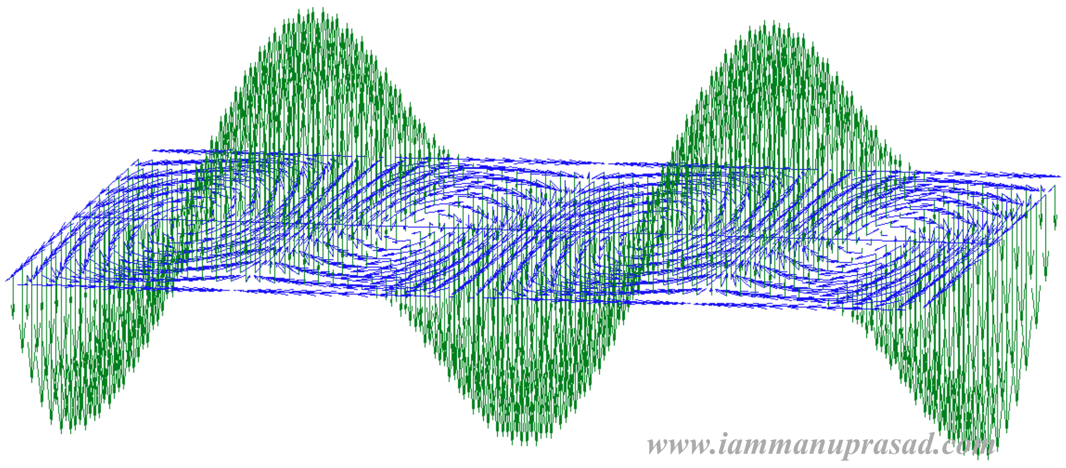
$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} \qquad H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} \qquad H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial y}$$

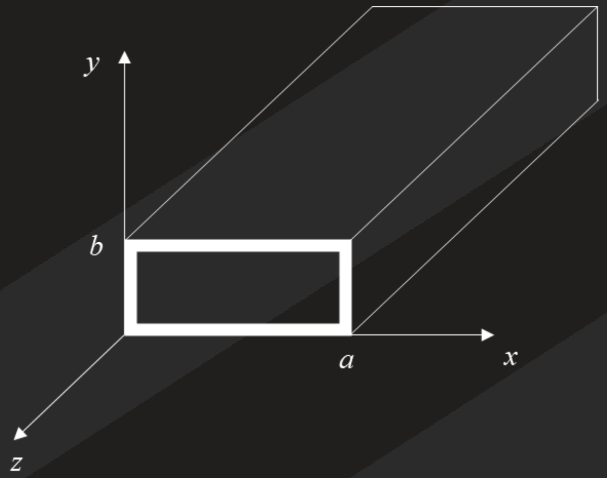
These equations gives the relationships among the fields within the guide

- *These four field components are in terms of E_z & H_z*
- *IF we assume that both E_z and H_z components are zero then all filed vanish or in other words* ***transverse electromagnetic (TEM) wave*** *cannot exist in a wave guide*
- *A wave pattern is possible if either $E_z \neq 0$ or $H_z \neq 0$.*
- *If $E_z = 0$ then the electric field is transverse, but there is a non zero H_z, this is called* ***Transverse Electric (TE) wave***
- *Similarly for* ***Transverse Magnetic (TM) wave*** *, $H_z = 0$ and $E_z \neq 0$*
- *Thus a rectangular wave guide can support TE or TM mode wave*





Transverse Magnetic (TM) Waves in Rectangular wave guide



In general wave equation is

$$\nabla^2 E = \gamma^2 E \quad \text{---------- (1)}$$

$$\nabla^2 H = \gamma^2 H \quad \text{---------- (2)}$$

$$\gamma = \sqrt{j\omega\mu\,(\sigma + j\omega\varepsilon)} \quad \text{---------- (3)}$$

For dielectric σ=0

$$\gamma = \sqrt{-\omega^2\mu\varepsilon}$$

$$\gamma^2 = -\omega^2\mu\varepsilon$$

$$(1) \rightarrow \quad \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = -\omega^2\mu\varepsilon E \quad \text{---------- (4)}$$

$$(2) \rightarrow \quad \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = -\omega^2\mu\varepsilon H \quad \text{---------- (5)}$$

We have to consider only z – component since the wave is travelling in z – direction

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2\mu\varepsilon E_z \quad \text{---------- (6)}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2\mu\varepsilon H_z \quad \text{---------- (7)}$$

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = -\omega^2 \mu\varepsilon E_z \quad \text{---------- (6)}$$

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} = -\omega^2 \mu\varepsilon H_z \quad \text{---------- (7)}$$

Let us assume

$$E_z(x, y) = E_z^0 e^{-\gamma z} \quad \text{---------- (8)}$$

Where

$$E_z^0 = XY$$

Substitute (8) in (6)

$$\frac{\partial^2}{\partial x^2}[XYe^{-\gamma z}] + \frac{\partial^2}{\partial y^2}[XYe^{-\gamma z}] + \frac{\partial^2}{\partial z^2}[XYe^{-\gamma z}] = -\omega^2 \mu\varepsilon[XYe^{-\gamma z}]$$

$$Ye^{-\gamma z}\frac{d^2 X}{dx^2} + Xe^{-\gamma z}\frac{d^2 Y}{dy^2} + \gamma^2 XYe^{-\gamma z} = -\omega^2 \mu\varepsilon XYe^{-\gamma z}$$

$$Y\frac{d^2 X}{dx^2} + X\frac{d^2 Y}{dy^2} + [\gamma^2 + \omega^2 \mu\varepsilon]XY = 0$$

Dividing both sides by $\frac{1}{XY}$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + [\gamma^2 + \omega^2 \mu\varepsilon] = 0$$

$$h^2 = \gamma^2 + \omega^2 \mu\varepsilon$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + h^2 = 0$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + h^2 = A^2 \qquad -\frac{1}{Y}\frac{d^2 Y}{dy^2} = A^2$$

$$\frac{1}{X}\frac{d^2 X}{dx^2} + B^2 = 0 \qquad B^2 = h^2 - A^2$$

Multiplying both sides by X

$$\frac{d^2 X}{dx^2} + B^2 X = 0 \quad \text{---------- (9)}$$

and

$$\frac{d^2 Y}{dy^2} + A^2 Y = 0 \quad \text{---------- (10)}$$

The above ordinary differential equation can be expressed as

$$X = C_1 \cos Bx + C_2 \sin Bx \quad \text{---------- (11)}$$

$$Y = C_3 \cos Ay + C_4 \sin Ay \quad \text{---------- (12)}$$

Power series solution of ordinary diff. equation

Series = $y'' + y = 0$ —— ①

Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad (n=0 \text{ is const})$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \quad (n=1 \text{ is const})$$

Substituting the value of y & y'' in ①

we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0 \quad \text{—— ②}$$

to equate the limit as identical we can rewrite the first term as

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \quad \text{—— ③}$$

Substitute ③ in ② we get

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n\right] x^n = 0 \quad \text{—— ④}$$

In the above condition $x^n \neq 0$, only coeff of x^n can be zero to satisfy the condition.

So $(n+2)(n+1) a_{n+2} + a_n = 0$.

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)} \quad \text{—— ⑤} \quad (\text{recursion relation})$$

Here we get two sequences.

for $n \to$ even nos $\Rightarrow a_0, a_2, a_4, a_6 \Rightarrow a_0 + a_2 x^2 + a_4 x^4 + \cdots$ —— ⑥

for $n \to$ odd nos $\Rightarrow a_1, a_3, a_5, a_7 \cdots \Rightarrow a_1 x + a_3 x^3 + a_5 x^5 + \cdots$ —— ⑦

Even series

$$a_0 = a_0, \quad a_2 = \frac{-a_0}{2!}, \quad a_4 = \frac{-a_0}{4!}, \quad a_6 = \frac{-a_0}{6!}, \cdots$$

Substitute the above values in ⑥

we get

$$a_0\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] = a_0 \cos x$$

Odd series

$$a_1 = a_1, \quad a_3 = \frac{-a_1}{3!}, \quad a_5 = \frac{a_1}{5!}, \quad a_7 = \frac{-a_1}{7!}, \cdots$$

Substitute the above value in ⑦

we get

$$a_1\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] = a_1 \sin x$$

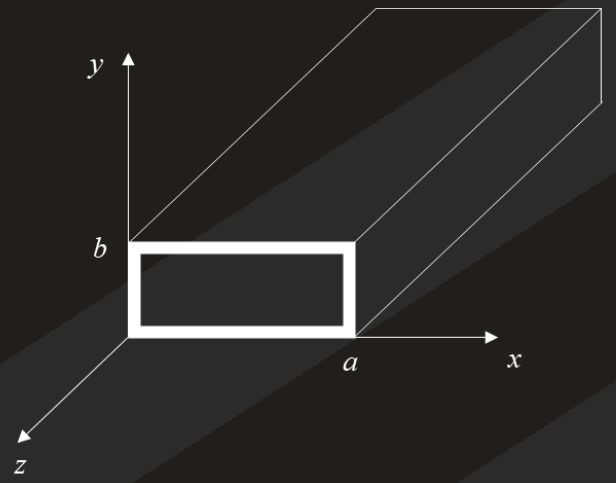
Now we can express the ode series ~~~~ in these two power series solution

$$\boxed{y(x) = a_0 \cos x + a_1 \sin x}$$

$X = C_1 \cos Bx + C_2 \sin Bx$ ---------- (11)

$Y = C_3 \cos Ay + C_4 \sin Ay$ ---------- (12)

$E_z(x,y) = E_z^0 e^{-\gamma z}$ ---------- (8)



From (8)

$$E_z^0 = XY$$

$$= (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay)$$

$$E_z^0 = C_1C_3 \cos Bx \cos Ay + C_1C_4 \cos Bx \sin Ay + C_2C_3 \sin Bx \cos Ay + C_2C_4 \sin Bx \sin Ay$$ ---------- (13)

$$\boxed{E_z^0 = C \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y}$$

The constants C1, C2, C3, C4, A & B can be calculated form the boundary conditions

x=0

(13) → $C_1C_3 \cos Ay + C_1C_4 \sin Ay = 0$

$C_1(C_3 \cos Ay + C_4 \sin Ay) = 0$

Which leads to $C_1 = 0$

$$E_z^0 = C_2C_3 \sin Bx \cos Ay + C_2C_4 \sin Bx \sin Ay$$ ------- (14)

y=0

(14) → $C_2C_3 \sin Bx = 0$

Which leads to $C_3 = 0$

$$E_z^0 = C_2C_4 \sin Bx \sin Ay$$ ------- (15)

x=a

(15) → $C_2C_4 \sin Bx \sin Ay = 0$

$C_2C_4 = C$

$C \sin Ba \sin Ay = 0$

Which leads to $\sin Ba = 0$

$\boxed{B = \frac{m\pi}{a}}$ *Where m=1,2,3,..*

y=b

$C \sin \frac{m\pi}{a} x \sin Ab = 0$

Which leads to $\sin Ab = 0$

$\boxed{A = \frac{n\pi}{b}}$ *When n=1,2,3,..*

In TM mode $H_z=0$

$$E_x = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} \qquad H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$E_y = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} \qquad H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} \qquad H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} \qquad H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial y}$$

Substitute the values of E_z we get

$$E_x = -\frac{\gamma C}{h^2}\frac{\partial}{\partial x}\left[\sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{b}y\right] \qquad H_x = \frac{j\omega\varepsilon C}{h^2}\frac{\partial}{\partial y}\left[\sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{b}y\right]$$

$$E_z(x,y) = E_z^0 e^{-\gamma z} \quad \text{---------- (8)}$$

$$E_y = -\frac{\gamma C}{h^2}\frac{\partial}{\partial y}\left[\sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{b}y\right] \qquad H_y = \frac{-j\omega\varepsilon C}{h^2}\frac{\partial}{\partial x}\left[\sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{b}y\right]$$

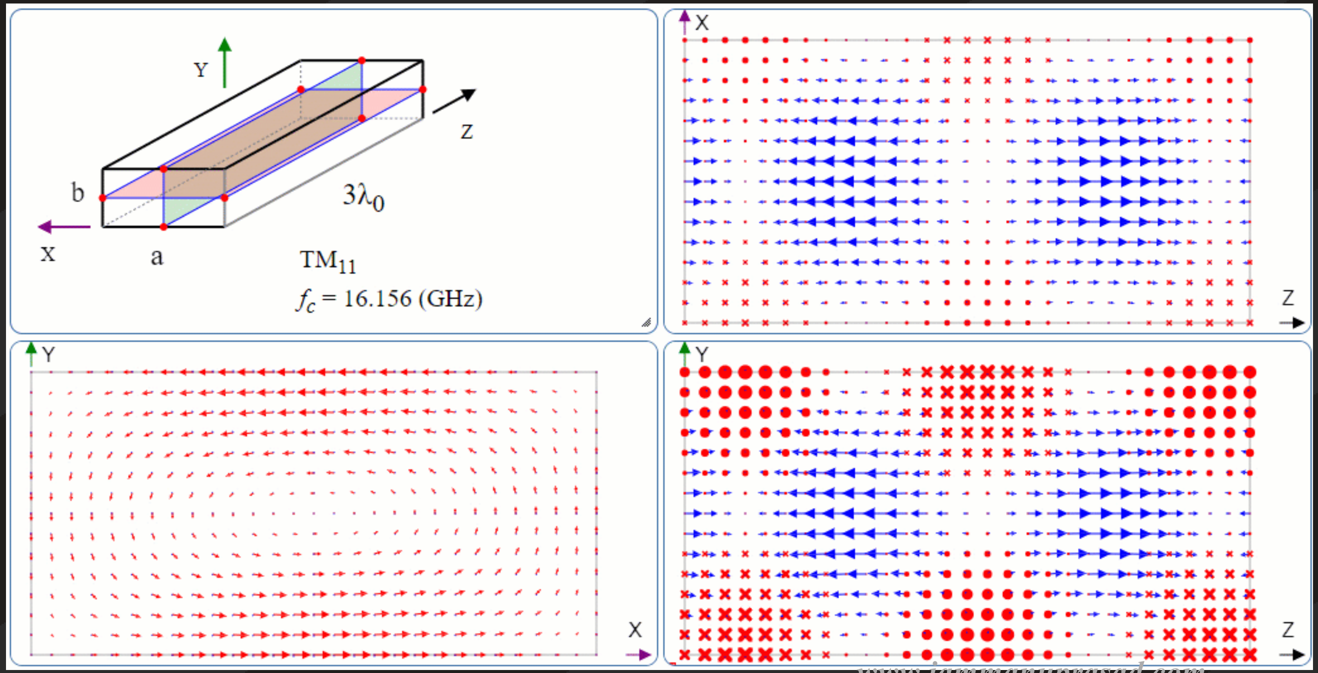
Assuming perfect conducting condition σ=0 and substitute the value of E^0 in (8)

$$E_x(x,y,z) = -\frac{j\beta}{h^2}\left(\frac{m\pi}{a}\right)C\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$E_y(x,y,z) = -\frac{j\beta}{h^2}\left(\frac{n\pi}{b}\right)C\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$H_x(x,y,z) = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{a}\right)C\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$

$$H_y(x,y,z) = \frac{-j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right)C\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-\gamma z}$$



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To find propagation const

$$h^2 = \gamma^2 + \omega^2\mu\varepsilon$$

$$\gamma^2 = h^2 - \omega^2\mu\varepsilon$$

$$\gamma = \sqrt{h^2 - \omega^2\mu\varepsilon}$$

$$B^2 = h^2 - A^2$$

$$h^2 = A^2 + B^2$$

$$\gamma = \sqrt{A^2 + B^2 - \omega^2\mu\varepsilon}$$

$$A = \frac{n\pi}{b} \qquad B = \frac{m\pi}{a}$$

$$\boxed{\gamma = \sqrt{\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon}}$$

We know propagation constant is a complex value

When $\sigma = 0$ $\qquad \gamma = j\beta$

$$\boxed{\beta = \sqrt{\omega^2\mu\varepsilon - \left[\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]}}$$

Consider

$$\omega_c^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\boxed{\omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$$

The lower limit of angular frequency (ω_c) is called **cut-off frequency**, below which wave propagation is absent

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Wavelength corresponding to the cut off frequency

$$\boxed{\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}}$$

velocity

$$v = \frac{\omega}{\beta}$$

$$\boxed{v = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}}}$$

Wavelength

$$\lambda = \frac{v}{f}$$

$$\boxed{\lambda = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}}}$$

Transverse Electric (TE) Waves in Rectangular wave guide

For a TE mode the component of electric field strength along the direction of propagation is zero ($E_z = 0$)

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial x} \qquad H_x = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} \qquad H_y = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial y}$$

Let us assume

$$H_z(x,y) = H_z^0 e^{-\gamma z}$$

Where

$$H_z^0 = XY$$

$$H_z^0 = C_1C_3 \cos Bx \cos Ay + C_1C_4 \cos Bx \sin Ay + C_2C_3 \sin Bx \cos Ay + C_2C_4 \sin Bx \sin Ay$$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} \qquad H_x = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} \qquad H_y = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial y}$$

$$H_z^0 = C_1C_3\cos Bx\cos Ay + C_1C_4\cos Bx\sin Ay + C_2C_3\sin Bx\cos Ay + C_2C_4\sin Bx\sin Ay \quad \text{---------- (1)}$$

Differentiating (1) with respect to x

$$\frac{\partial H_z}{\partial x} = -BC_1C_3\sin Bx\cos Ay - BC_1C_4\sin Bx\sin Ay + BC_2C_3\cos Bx\cos Ay + BC_2C_4\cos Bx\sin Ay$$

Differentiating (2) with respect to y

$$\frac{\partial H_z}{\partial y} = -AC_1C_3\cos Bx\sin Ay + AC_1C_4\cos Bx\cos Ay \quad \text{------- (3)}$$

The constants C1, C2, C3, C4, A & B can be calculated form the boundary conditions

x=0

(1) → $BC_2C_3\cos Ay + BC_2C_4\sin Ay = 0$

$C_2(BC_3\cos Ay + BC_4\sin Ay) = 0$

Which leads to $C_2 = 0$

$$H_z^0 = C_1C_3\cos Bx\cos Ay + C_1C_4\cos Bx\sin Ay \quad \text{------- (2)}$$

y=0

(3) → $AC_1C_4\cos Bx = 0$

Which leads to $C_4 = 0$

$$H_z^0 = C_1C_3\cos Bx\cos Ay \quad \text{------- (4)}$$

$C_1C_3 = C$

$$\boxed{H_z^0 = C\cos Bx\cos Ay} \quad \text{------- (5)}$$

Where

$$A = \frac{n\pi}{b} \qquad B = \frac{m\pi}{a}$$

$$H_z^0 = C\cos Bx \cos Ay \quad \text{-------} \; (5)$$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} \qquad H_x = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x} \qquad H_y = -\frac{\gamma}{h^2}\frac{\partial H_z}{\partial y}$$

$$H_z(x,y) = H_z^0 e^{-\gamma z} \quad \text{----------} \; (8)$$

$$E_x = \frac{-j\omega\mu}{h^2}\frac{\partial}{\partial y}(C\cos Bx\cos Ay) \qquad H_x = -\frac{\gamma}{h^2}\frac{\partial}{\partial x}(C\cos Bx\cos Ay)$$

$$E_y = \frac{j\omega\mu}{h^2}\frac{\partial}{\partial x}(C\cos Bx\cos Ay) \qquad H_y = -\frac{\gamma}{h^2}\frac{\partial}{\partial y}(C\cos Bx\cos Ay)$$

For TE wave the expression for β, fc, λc, vp are same as those of TM wave . However there is a difference for TE wave, it is possible to make either m or n but not both zero.
The lower order TE is possible than TM (TE_{10} mode ,m=1,n=0)

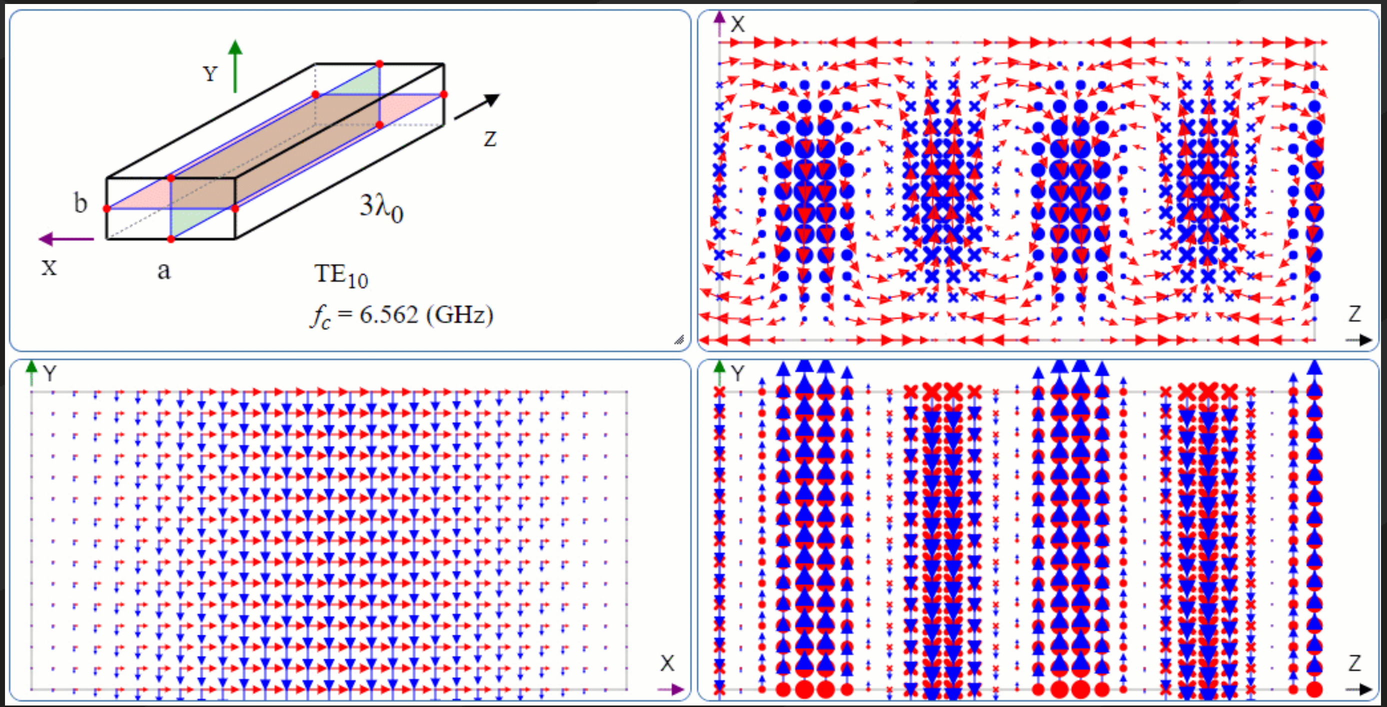
$$E_x = \frac{j\omega\mu}{h^2} CA\cos Bx \sin Ay\, e^{-\gamma z}$$

$$E_y = -\frac{j\omega\mu}{h^2} CB\sin Bx \cos Ay\, e^{-\gamma z}$$

$$H_x = \frac{\gamma}{h^2} CB\sin Bx \cos Ay\, e^{-\gamma z}$$

$$H_y = \frac{\gamma}{h^2} CA\cos Bx \sin Ay\, e^{-\gamma z}$$

$$A = \frac{n\pi}{b} \qquad B = \frac{m\pi}{a}$$



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For TE_{10} mode

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

Substituting m=1, n=0

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\pi}{a}$$

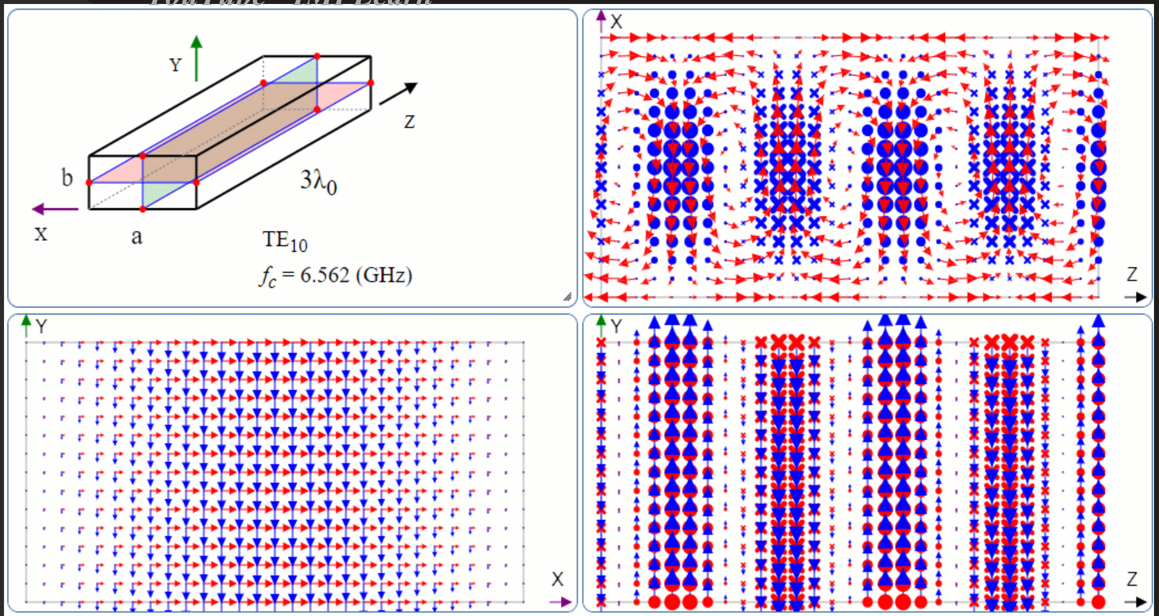
$$\boxed{f_c = \frac{v_0}{2a}}$$

The cut off frequency of TE_{10} mode is independent of the tube dimension b

$$\boxed{\lambda_c = \frac{v_0}{f_c} = 2a}$$

$$\boxed{\beta = \sqrt{\omega^2\mu\varepsilon - \left[\left(\frac{\pi}{a}\right)^2\right]}}$$

- *The mode with the lowest cut off frequency is called the* ***dominant mode***
- *In rectangular wave guide the dominant mode is* TE_{10}
- *For a given wave guide it is possible to operate only in the dominant mode over certain range of frequencies*
- *Most wave guide component are designed for operating in this mode*



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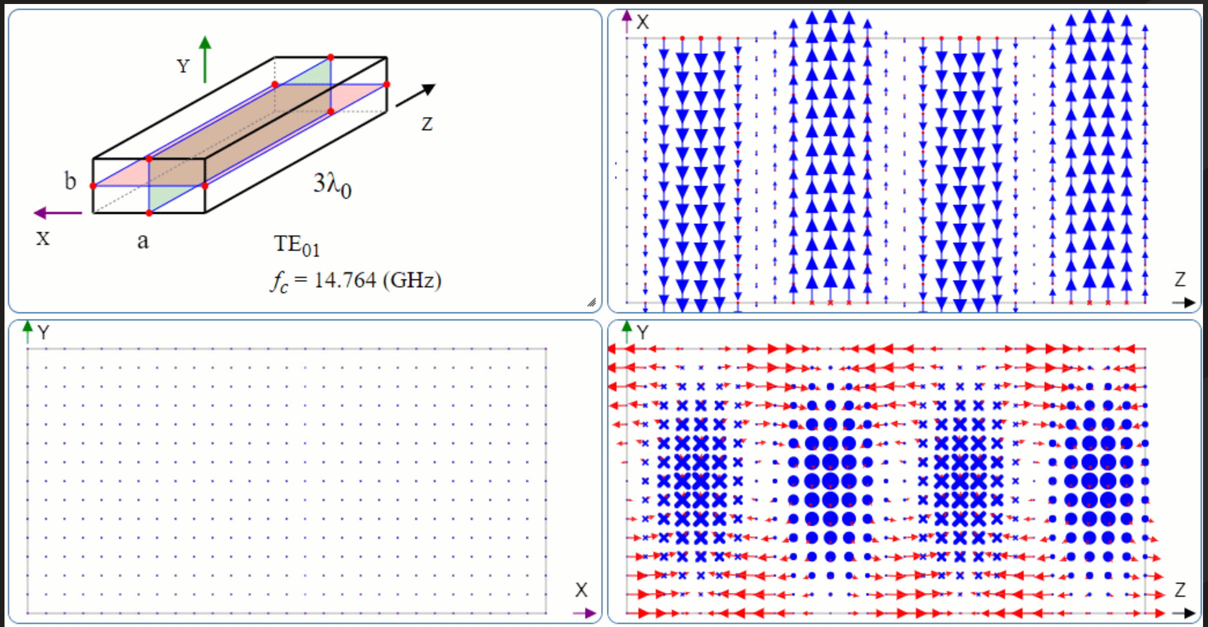

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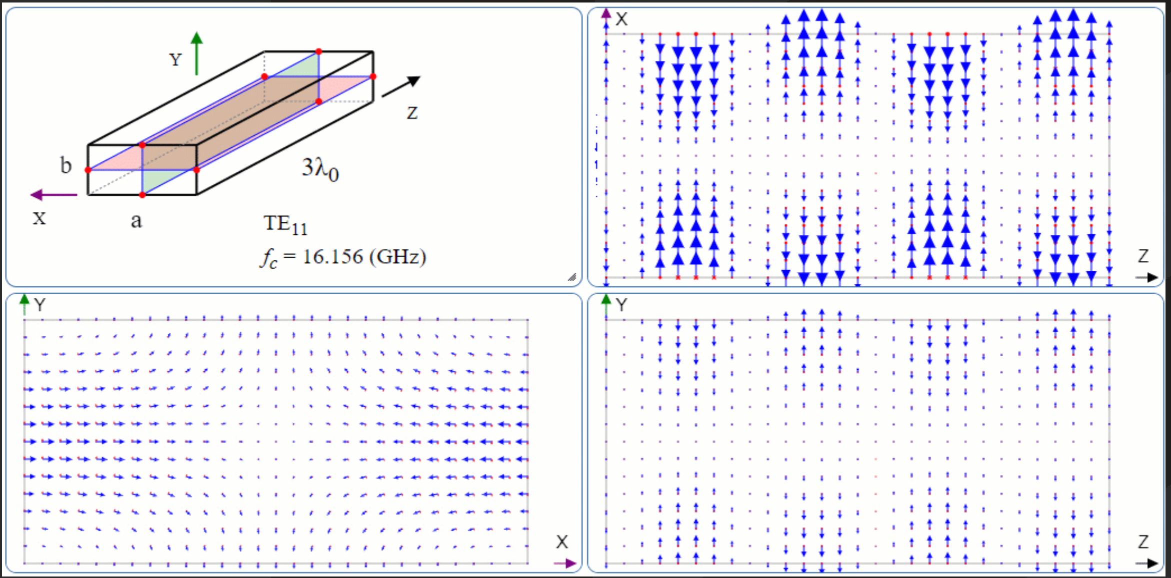

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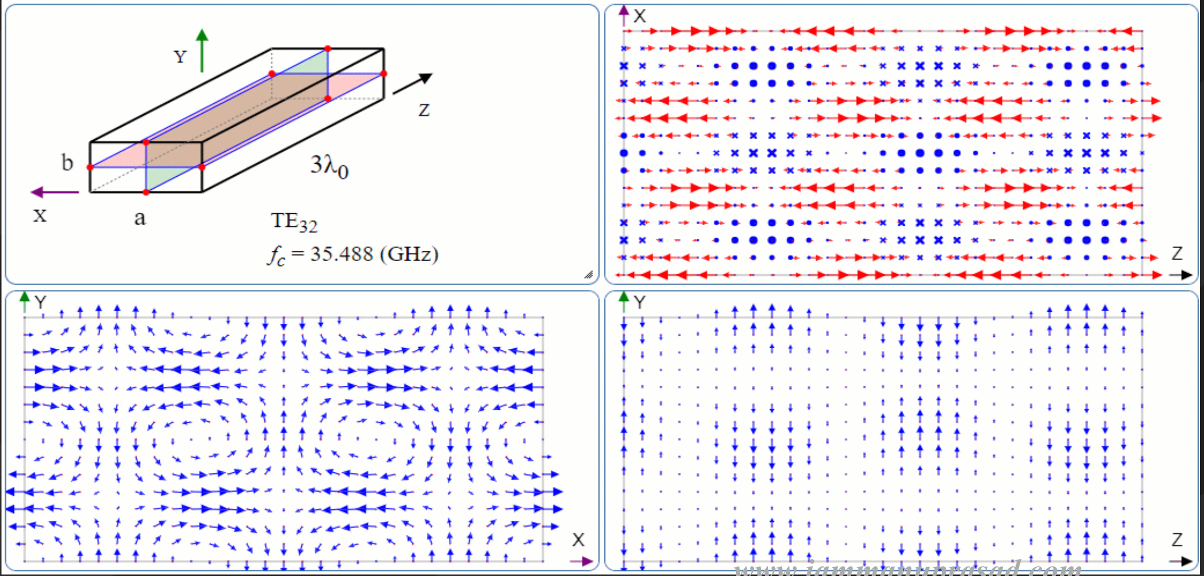

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