Properties of DFT.. contd

Multiplication of two sequences.

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

then $x_1(n) . x_2(n) \xrightarrow[N]{DFT} \frac{1}{N} X_1(k) \textcircled{N} X_2(k)$

proof.

we have $x_1(n) = \frac{1}{N} \sum_{u=0}^{N-1} X_1(u) e^{j\frac{2\pi}{N} un}$.

$$x_2(n) = \frac{1}{N} \sum_{v=0}^{N-1} X_2(v) e^{j\frac{2\pi}{N} vn}.$$

$$DFT\left[x_1(n)\, x_2(n)\right] = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{u=0}^{N-1} X_1(u) e^{j\frac{2\pi}{N} un} \; \frac{1}{N} \sum_{v=0}^{N-1} X_2(v) e^{j\frac{2\pi}{N} vn}\right] e^{-j\frac{2\pi}{N} kn}$$

$$= \frac{1}{N^2} \sum_{u=0}^{N-1} X_1(u) \sum_{v=0}^{N-1} X_2(v) \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N} n(k-u-v)}$$

IIIrd term $= N$ if $(k-u-v) = 0 \Rightarrow$ ~~IIIrd term~~ N

$= 0$ else

$v = k - u \implies u = k - v$

$v = 0 \implies u = k.$

$v = N-1 \implies u = k - (N-1)$

$$= \frac{1}{N^2} \sum_{u=0}^{N-1} X_1(u) \sum_{u=k}^{k-(N-1)} X_2(k-u) \cdot N$$

$$= \frac{1}{N} \sum_{u=0}^{N-1} X_1(u) \sum_{u=0}^{N-1} X_2((k-u))_N.$$

$$= \frac{1}{N} \sum_{u=0}^{N-1} X_1(u)\, X_2(k-u)$$

$$= \frac{1}{N} X_1(k) \circledN X_2(k)$$

~~Parseval's Theorem~~ Circular Correlation property

For complex valued ~~seg~~ sequences in general if $x(n)$ and $y(n)$

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$y(n) \xleftrightarrow[N]{DFT} Y(k)$$

~~then $\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) Y^*(k).$~~

~~proof~~

$$\tilde{r}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(k) = X(k)\, Y^*(k)$$

where $\tilde{r}_{xy}(l)$ is the un normalized circular cross correlation of sequence defined as.

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N.$$

Proof.

We can write $\tilde{r}_{xy}(l)$ as the circular convolution of $x(n)$ and $y^*(-n)$ ie

$$\tilde{r}_{xy}(l) = x(l) \, (N) \, y^*(-l)$$

By complex conjugate property we have $y^*(-l)_N \xrightarrow[N]{DFT}$ ~~X(k)~~ $\cdot\, Y^*(k)$.

By applying circular convolution property

$$\tilde{r}_{xy}(l) \xrightarrow[N]{DFT} \tilde{R}_{xy}(k) = X(k) \cdot Y^*(k)$$

In the special case when $y(n) = x(n)$ we have the corresponding expression for circular auto correlation:

$$\tilde{r}_{xx}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xx}(k) = |X(k)|^2$$

$$= X(k) \cdot X^*(k).$$

Parseval's Theorem:

For complex valued sequences $x(n)$ and $y(n)$, if

$$x(n) \xleftrightarrow[N]{DFT} X(k)$$

$$y(n) \xleftrightarrow[N]{DFT} Y(k).$$

then $$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$$

Proof.

From circular correlation property we have.

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \tilde{r}_{xy}(0)$$

we have

$$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k).$$

$$\therefore r_{xy}(l) = \frac{1}{N} \sum_{k=0}^{N-1} R_{xy}(k)\, e^{j\frac{2\pi}{N}kl}.$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^{*}(k)\, e^{j\frac{2\pi}{N}kl}.$$

$$r_{xy}(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^{*}(k).$$

$$\Rightarrow \sum_{n=0}^{N-1} x(n)\, y^{*}(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^{*}(k).$$

In special case where $y(n) = x(n)$

above equ becomes

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

which expresses the energy in the finite duration sequence $x(n)$ in terms of the frequency components $X(k)$.

→ Energy of a signal $x(n)$

$$E = \sum_{n=0}^{N-1} |x(n)|^2 .$$

Q) Given $x(n) = \{1, -2, 3, -4, 5, -6\}$

without calculating DFT find the following quantities. (KTU Dec 2018)

(a) $X(0)$ (b) $\sum_{k=0}^{5} X(k)$ (c) $X(3)$

(d) $\sum_{k=0}^{5} |X(k)|^2$ (e) $\sum_{k=0}^{5} (-1)^k X(k)$.

Answer: Here $N=6$.

(a). we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}.$$

$N=6 \Rightarrow$

$$X(k) = \sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi}{6}kn}. \quad \text{--- (1)}$$

to find $X(0)$ put $k=0$ in eqn (1):

$$X(0) = \sum_{n=0}^{5} x(n)\, \underbrace{e^{-j\frac{2\pi}{6}\times 0 \times n}}_{e^{0}=1}$$

$$= \sum_{n=0}^{5} x(n)$$

$$= x(0) + x(1) + x(2) + x(3) + x(4) + x(5)$$

$$= 1 + -2 + 3 + -4 + 5 + -6 = -3$$

(b). we have IDFT equ.

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn}.$$

$$x(n) = \frac{1}{6} \sum_{k=0}^{5} X(k)\, e^{j\frac{2\pi}{6}kn} \quad \text{— ②}$$

to find $\sum_{k=0}^{5} X(k)$ put $n=0$ in equ ②

$$6 \times x(0) = \sum_{k=0}^{5} X(k)\, \underbrace{e^{j\cdot\frac{2\pi}{N}\cdot 0 \cdot n}}_{1}$$

$$\therefore \sum_{k=0}^{5} X(k) = 6 \times x(0)$$
$$= 6 \times 1 = \underline{\underline{6}}$$

(c) to find $X(3)$ put $k=3$ in equ ①.

$$X(3) = \sum_{n=0}^{5} x(n)\, e^{-j\frac{2\pi\, 3n}{6}}$$

$$= \sum_{n=0}^{5} x(n)\, e^{-j\pi n} \qquad (e^{-j\pi} = -1)$$

$$= \sum_{n=0}^{5} x(n)\, (-1)^n.$$

$$= x(0)(-1)^0 + x(1)(-1)^1 + x(2)(-1)^2$$
$$+ x(3)(-1)^3 + x(4)(-1)^4 + x(5)(-1)^5$$

$$= 1 + 2 + 3 + 4 + 5 + 6 = 21 //$$

(d) from Parseval's theorm

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \quad \text{——— (3)}$$

$\therefore$ $\cancel{\sum_{k=0}}$ Here $N=6$.

$$\sum_{n=0}^{5} |x(n)|^2 = \frac{1}{6} \sum_{k=0}^{5} |X(k)|^2$$

$$\therefore \sum_{k=0}^{5} |X(k)|^2 = 6 \times \sum_{n=0}^{5} |x(n)|^2$$

$$= 6\left[|x(0)|^2 + |x(1)|^2 + \cdots |x(5)|^2 \right]$$

$$= 6\left[1^2 + |-2|^2 + |3|^2 + |-4|^2 + |5|^2 + |-6|^2 \right]$$

$$= 6\left[1 + 4 + 9 + 16 + 25 + 36 \right]$$

$$\text{ie} \quad \sum_{k=0}^{5} |X(k)|^2 = 6 \times 9 = \underline{\underline{546}}$$

(e) to find $\sum_{k=0}^{5} (-1)^k X(k)$

Consider IDFT equation.

$$x(n) = \frac{1}{6} \sum_{k=0}^{5} X(k)\, e^{j\frac{2\pi}{6}kn}$$

put $n=3$ in above equation.

$$x(3) = \frac{1}{6} \sum_{k=0}^{5} X(k)\, e^{j\frac{2\pi}{6}k \cdot 3}$$

$$\therefore \sum_{k=0}^{5} X(k)\, e^{j\pi k} = 6 \times x(3)$$

$$(e^{j\pi} = -1)$$

$$\sum_{k=0}^{5} (-1)^k X(k) = 6 \times -4$$

$$= \underline{-24}$$

Q. State circular frequency shift property of DFT. 4 point DFT of the signal $x(n) = \{a, b, c, d\}$ is $X(k)$. Find the IDFT of $X(k-2)$? (KTU- Dec 2018)

Ans: Circular frequency shift property of DFT

if $x(n) \xrightarrow[N]{DFT} X(k)$

$$e^{j\frac{2\pi}{N}mn} x(n) \xrightarrow[N]{DFT} X((k-m))_N$$

Hence $x(n) = \{a, b, c, d\} \xrightarrow{DFT} X(k)$.

$$X(k-2) \xrightarrow{IDFT} e^{j\frac{2\pi}{4}2n} x(n)$$

$$= e^{j\pi n} x(n) = (-1)^n x(n).$$

$$= \{(-1)^0 \cdot a, (-1)^1 \cdot b, (-1)^2 \cdot c, (-1)^3 d\}.$$

$$= \{a, -b, c, -d\}$$