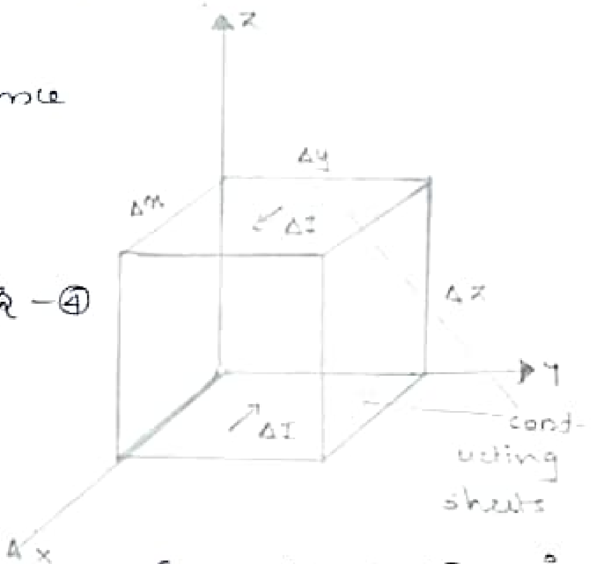
Q. Derive the expression for energy stored in Magnetic field

Ans: A simple expression for magnetic energy in the field of inductor: $W_m = \frac{1}{2} L I^2$ — ①

Consider a differential volume in a magnetic field let the volume be covered with conducting sheets at the top and bottom surfaces with current ΔI

→ Assuming the whole region is filled with such differential volume

Each volume has an inductance



$$\Delta L = \frac{\Delta \varphi}{\Delta I} \quad — ②$$

$$\Delta \varphi = \vec{B} \cdot \Delta S \quad — ③ \qquad \Delta S = \Delta x \cdot \Delta z \quad — ④$$

③ and ④ in ②:

$$\Delta L = \frac{\vec{B} \cdot \Delta x \cdot \Delta z}{\Delta I} \quad — ⑤$$

⑥ and ⑤ in ①:

$\left\{ \Delta I = H \cdot \Delta y \right.$ — ⑥ ⇒ By using Ampere's law $[\oint \vec{H} \cdot \vec{dl} = I_{enc}]$

$$\Delta W_m = \frac{1}{2} \frac{\vec{B} \cdot \Delta x \cdot \Delta z \cdot (\Delta I)^2}{\Delta I}$$

$$= \frac{1}{2} \vec{B} \cdot \Delta x \cdot \Delta z \cdot \Delta I$$

$$= \frac{1}{2} \vec{B} \cdot \Delta x \cdot \Delta z \cdot \vec{H} \cdot \Delta y \qquad \{\vec{B} = \mu \vec{H}\}$$

$$\Delta W_m = \frac{1}{2} \mu \vec{H} \, \Delta x \cdot \Delta z \cdot \vec{H} \cdot \Delta y = \frac{1}{2} \mu \vec{H}^2 \Delta x \cdot \Delta y \cdot \Delta z .$$

$$\Delta W_m = \frac{1}{2} \mu \vec{H}^2 \Delta V$$

The magnetostatic energy density w_m (in J/m^3) is:

$$w_m = \lim_{\Delta V \to 0} \frac{\Delta W_m}{\Delta V} = \frac{1}{2} \mu \vec{H}^2$$

$$w_m = \frac{1}{2} \mu \vec{H}^2 = \frac{1}{2} \vec{B} \cdot \vec{H} = \frac{\vec{B}^2}{2\mu} .$$

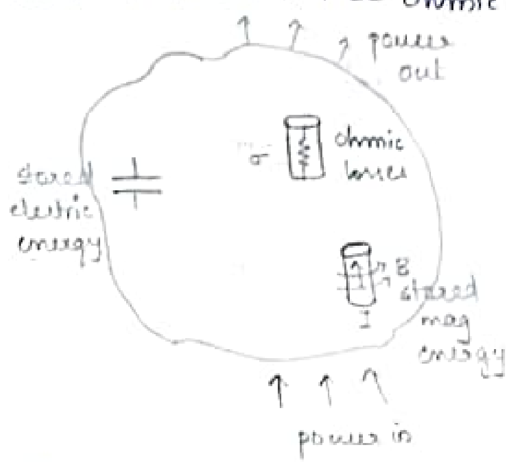
Thus the energy in magnetostatic field in linear medium:

$$W_m = \int w_m \, dv = \frac{1}{2}\int \vec{B}\cdot\vec{H}\, dv = \frac{1}{2}\int \mu \vec{H}^2 \, dv$$

Q. State Poyting theorem. Derive Poynting theorem starting from Maxwell's equations.

Ans: ~~From Maxwell's third and fourth equations;~~

Poynting theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the ohmic losses.



→ Poynting vector: The cross product of E and H at any point gives the power per unit area.

$$\underline{P = E \times H} \text{ watts}/m^2$$

From Maxwell's 3rd and 4th equations:

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad — (1)$$

$$\nabla \times \vec{H} = \sigma E + \epsilon \frac{\partial \vec{E}}{\partial t} \quad — (2)$$

Taking dot product with $\vec{E}$ on both sides of (2)

$$\vec{E} \cdot (\nabla \times \vec{H}) = \sigma E^2 + \epsilon \left[\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right] \quad — (3)$$

Consider a vector identity:

$$\nabla \cdot (\vec{A} \times \vec{B}) = B \cdot (\nabla \times \vec{A}) - A \cdot (\nabla \times \vec{B}) \quad — (4)$$

Let $\vec{A} = \vec{H}$ and $\vec{B} = \vec{E}$

$$(4) \Rightarrow \nabla \cdot (\vec{H} \times \vec{E}) = \vec{E} \cdot (\nabla \times \vec{H}) - \vec{H} \cdot (\nabla \times \vec{E}) \quad — (5)$$

$$\vec{E} \cdot (\nabla \times \vec{H}) = \nabla \cdot (\vec{H} \times \vec{E}) + \vec{H} \cdot (\nabla \times \vec{E}) = \sigma E^2 + \epsilon \left[\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right] \quad — (6)$$

$$\vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}) = \sigma E^2 + \epsilon \left(\vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \right)$$

$$-\nabla\cdot(\vec{E}\times\vec{H}) = \sigma E^2 + \epsilon\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right) - \vec{H}(\nabla\times\vec{E}) \quad \left\{\nabla\times E = -\frac{\mu\partial H}{\partial t}\right\}$$

$$-\nabla\cdot(\vec{E}\times\vec{H}) = \sigma E^2 + \epsilon\left(\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}\right) + H\mu\cdot\frac{\partial H}{\partial t} \quad -(7)$$

let's take;

$$\frac{\partial}{\partial t}(E\ E) = E\cdot\frac{\partial E}{\partial t} + E\frac{\partial E}{\partial t} = 2E\cdot\frac{\partial E}{\partial t}$$

$$\therefore \vec{E}\frac{2\vec{E}}{\partial t} = \frac{1}{2}\frac{\partial\vec{E}^2}{\partial t^2}$$

Similarly:

$$\vec{H}\cdot\frac{\partial\vec{H}}{\partial t} = \frac{1}{2}\frac{\partial\vec{H}^2}{\partial t^2}$$

$$\therefore -\nabla\cdot(\vec{E}\times\vec{H}) = \sigma E^2 + \frac{\epsilon}{2}\frac{\partial\vec{E}^2}{\partial t^2} + \frac{\mu}{2}\frac{\partial\vec{H}^2}{\partial t^2}$$

Take volume integral;

$$-\int_v \nabla\cdot(\vec{E}\times\vec{H})dv = \int_v \sigma E^2 dv + \int_v\left(\frac{\epsilon}{2}\frac{\partial\vec{E}^2}{\partial t^2} + \frac{\mu}{2}\frac{\partial\vec{H}^2}{\partial t^2}\right)dv.$$

Applying divergence theorem to LHS;

$$-\oint_s(\vec{E}\times\vec{H})ds = +\frac{\partial}{\partial t}\int_v \sigma E^2 dv + \int_v\frac{\partial}{\partial t}\int_v\left(\epsilon\frac{E^2}{2} + \frac{\mu H^2}{2}\right)dv.$$

multiplying by -ve sign:

$$\oint_s(\vec{E}\times\vec{H})ds = -\frac{\partial}{\partial t}\int_v\left[\frac{1}{2}\epsilon\vec{E}^2 + \frac{1}{2}\mu\vec{H}\right]dv - \int_v \sigma E^2 dv \quad -(9)$$

ie, Total power leaving the volume = Rate at which stored electromagnetic energy is decreasing in the volume + Power dissipated in the volume.

(9) is the referred to as Poynting theorem.

Q. Derive the Conductor-dielectric and Conductor-free space boundary conditions for electric field.

Ans:- **DIELECTRIC - CONDUCTOR BOUNDARY CONDITIONS:**

Under static conditions, the following assumptions can be made about perfect conductor.

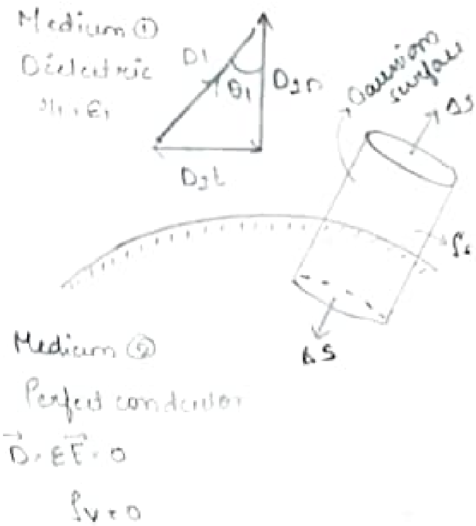
→ No electric field co-exists within a perfect conductor. ($E=0$)
→ Conductor is considered as equipotential surface.

Applying Gauss's law to the Gaussian surface:

$$\oint_S \vec{D}\cdot\vec{ds} = Q_{enc}$$

$$D_{1n}\,\Delta S - 0(\Delta S) = \rho_s\,\Delta S$$

$$\boxed{D_{1n} = \rho_s} \quad \text{— ①}$$

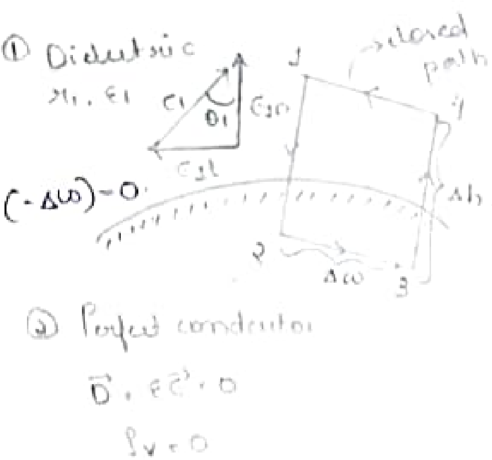


Applying $\oint_l \vec{E}\cdot\vec{dl} = 0$ to the path 1-2-3-4-1.

$$\int_1^2 \vec{E}\cdot\vec{dl} + \int_2^3 \vec{E}\cdot\vec{dl} + \int_3^4 \vec{E}\cdot\vec{dl} + \int_4^1 \vec{E}\cdot\vec{dl} = 0.$$

$$= E_{1n}\left(-\frac{\Delta h}{2}\right) + 0 + E_{1n}\left(\frac{\Delta h}{2}\right) + E_{1t}(-\Delta w) = 0.$$

$$-E_{1n}\cancel{\frac{\Delta h}{2}} + E_{1n}\left(\cancel{\frac{\Delta h}{2}}\right) + E_{1t}\,{-\Delta w} = 0$$



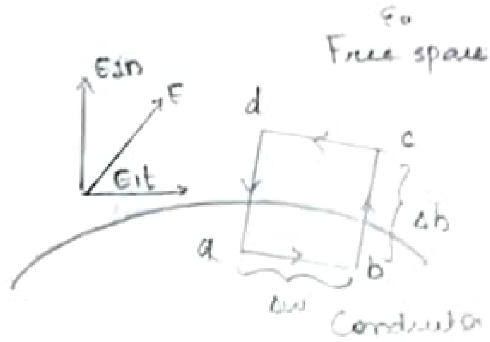
$$\boxed{E_{1t} = 0}.$$

No electric field exist within a conductor, under static conditions.

CONDUCTOR - FREE SPACE BOUNDARY CONDITIONS.

$$\oint_c \vec{E} \cdot \vec{dl} = 0.$$

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0.$$



$$\cancel{E_1 t\, \Delta w + \left(E_{1n_b}\frac{\Delta h}{2} + E_{1n_a}\frac{\Delta h}{2}\right) +}$$

$$\cancel{E_{1t}\, \Delta w + \left(E_{1n_d}\frac{\Delta h}{2} + E_{1n_a}\frac{\Delta h}{2}\right) = 0.}$$

$$E_{1t}\, \Delta w + 0 + 0 + 0 = 0.$$

$$E_{1t}\, \Delta w = 0$$

$$\boxed{E_{1t} = 0}$$

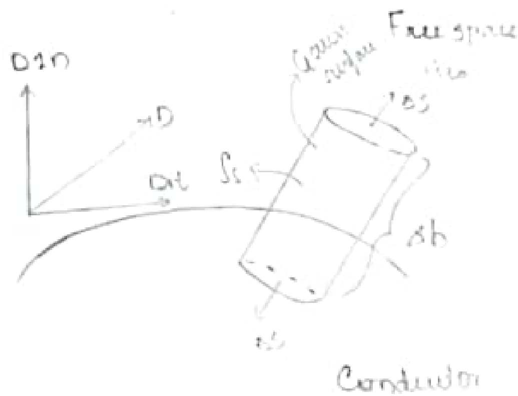
$$\oint \vec{D} \cdot \vec{ds} = Q_{enc}$$

$$\int_{top} + \int_{bott} + \int_{surface} = Q_{en}$$

$$D_{1n}\, \Delta s + 0 + 0 = \rho_s\, \Delta s$$

$$D_{1n}\, \cancel{\Delta s} = \rho_s\, \cancel{\Delta s}$$

$$\boxed{D_{1n} = \rho_s}$$



$\{\Delta h \to 0\}$

Q. Find $\nabla \cdot \vec{A}$ at $(2, 30^\circ, 90^\circ)$ for the field :

$$\vec{A} = 0.2r^3\phi\sin^2\theta\,\hat{a}_r + 0.2r^3\phi\sin^2\theta\,\hat{a}_\theta + 0.2r^3\phi\sin^2\theta\,\hat{a}_\phi$$

Ans: $\nabla \cdot \vec{A} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(A_\theta \sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\phi}{\partial \phi}$

$A_r = 0.2r^3\phi\sin^2\theta$

$A_\theta = 0.2r^3\phi\sin^2\theta$

$A_\phi = 0.2r^3\phi\sin^2\theta$

$$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \times 0.2r^3\phi\sin^2\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(0.2r^3\phi\sin^3\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(0.2r^3\phi\sin^2\theta)$$

$$= \frac{1}{r^2}\left[0.2r^4 \times 5 \times \phi\sin^2\theta\right] + \frac{1}{r\sin\theta}\left[0.2r^3\phi\,3\sin^2\theta \times \cos\theta\right] + \frac{1}{r\sin\theta}\left[0.2r^3\sin^2\theta\right]$$

Here, $r = 2$ $\theta = \dfrac{\pi}{6}$ $\phi = \dfrac{\pi}{2}$.

$$\nabla \cdot \vec{A} = 0.2r^2 \times 5 \times \phi \times \sin^2\theta + 0.2r^2\phi \times 3 \times \sin\theta \times \cos\theta + 0.2r^2\sin\theta$$

$$= 0.2 \times 4 \times 5 \times \frac{\pi}{2} \times \sin^2\frac{\pi}{6} + 0.2 \times 4 \times \frac{\pi}{2} \times 3 \times \sin\frac{\pi}{6}\cos\frac{\pi}{6} + 0.2 \times 4 \times \sin\frac{\pi}{6}$$

$= 1.5707 + 1.6324 + 0.4$

$= \underline{\underline{3.6031}}$

Q. A uniform line charge, infinite in extent, with $\rho_L = 20\,nC/m$ lies along z-axis. Find $\vec{E}$ at $(6, 8, 3)$?

Ans: $\vec{D} = \dfrac{\rho_L \hat{a_z}}{2\pi\rho}$

$\{\vec{D} = \epsilon\vec{E}\}$

$\rho = \sqrt{6^2 + 8^2}$

$= \underline{\underline{10}}$.

$\vec{E} = \dfrac{\rho_L \hat{a_z}}{2\pi\epsilon_0\rho}$

$= \dfrac{20 \times 10^{-9}}{6.28 \times 8.85 \times 10^{-12} \times 10} \hat{a_z}$

$= \dfrac{20 \times 10^{-9}}{5.5578 \times 10^{-10}} \hat{a_z}$

$= \underline{\underline{35.985 \hat{a_z}}}\ V/m$.