Arun Hari Anand

HW #7 CS30

① Sample space: $\{X \mid X \subseteq$ the standard deck of cards $\wedge |X| = 5\}$

a) Event E_1 = X contains the ace of diamonds and queen of spades.

Number of total poker hands: $\binom{52}{5}$

Number of hands that contain 2 specific hands: $\binom{50}{3}$, because there are $\binom{50}{3}$ ways to pick the other 3 cards.

$$P(E_1) = \frac{\binom{50}{3}}{\binom{52}{5}}$$

b) Event E_2 : X consists of 5 cards that contain consecutive kinds, and of the same suit:

Counting the number of values that the starting card can hold, we know that $\{A, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ represents all the ways that the first card's value can be decided. If the first card is J, for instance, the sequence cannot extend past J K Q A. As such, we know that there are 4 choices for the suit of these cards.

By the product principle, there are (4×10) ways to make a straight flush. So $P(E_2) = \frac{40}{\binom{52}{5}}$

1c) E_3: X contains three of a kind

There are 13 ways to pick the kind that are in common, and and $\binom{4}{3}$ ways to pick the values of these cards. There are then 48 cards remaining, of which we must pick 2; this is done in $\binom{48}{2}$ ways. As such, there are $(13)\binom{4}{3}\binom{48}{2}$ ways to get a "three of a kind" draw. $P(E_3) = \dfrac{(13)\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$

12^{2} Sample space : The set of all poverball draws consisting of (i) 5 distinct numbers between 1 and 55, and (ii) 1 povreball number betceen 1 and 42.

(2)

a) Event E_1: The powerball draw matches exactly the winning number: all the non-powerball numbers in the winning draw are in the draw and the powerball numbers match.

There are $\binom{55}{5}$ ways to pick 5 distinct numbers from 1 to 55. There are 42 ways to pick a powerball number. The total number of possible draws are thus $\binom{55}{5} \cdot 42$. Exactly 1 of these draws will match the winning combination. So $P(E_1) = \frac{1}{\binom{55}{5} \cdot 42}$

b) Event E_2: Exactly 5 of the 6 numbers in the draw match the winning draw.

The number of ways to match the non-powerball numbers is 41, because, for each combination of non-powerball numbers, there are exactly 41 powerball numbers that do not match the winning combination. Assuming that the powerball numbers match, there are $\binom{5}{4}$ ways to choose the 4 other numbers that match, and 50 ways to pick the number that doesnt match. By the sum and product principles, the number of ways to perform E_2 is $\binom{5}{4} \cdot 50 + 41$. $P(E_2)$ is thus

$$\frac{\binom{5}{4} \cdot 50 + 41}{\binom{55}{5} \cdot 42}$$

(3) Sample space: All possible permutations of $\{1, 2 \ldots 10\}$

a) Event E_1: 1 precedes 3 in the permutation. Assuming a uniform distribution each permutation is equally likely. Additionally, by symmetry we know that 1 may either precede or succede 3. Thus we know that $P(1 \text{ precedes } 3) = P(3 \text{ precedes } 1)$ and $P(1 \text{ precedes } 3) + P(3 \text{ precedes } 1) = 1$. As such, $P(E_1) = \frac{1}{2}$.

b) Event E_2: 3 precedes 1 in the permutation. By the same logic employed in part a) $P(3 \text{ precedes } 1) = \frac{1}{2}$

c) Event E_3: 3 precedes 1 in the permutation, and 3 also precedes 2 in the permutation. There are 6 ways for $\{1,2,3\}$ to be in the permutation:

i) 1 . . . 2 . . . 3
ii) 1 . . . 3 . . . 2
iii) 2 . . . 1 . . . 3
iv) 2 . . . 3 . . . 1
v) 3 . . . 1 . . . 2
vi) 3 . . . 2 . . . 1

All of these permutations have equal probability, due to uniformity of the distribution. As such, only 2 of the 6 permutations sho have 3 preceding both 1 and 2. So $\frac{2}{6} = \boxed{\frac{1}{3} = P(E_3)}$

4

4) Sample space: $\{B, G\} \times \{B, G\}$, where B denotes a boy and G denotes a girl child.

a) Event E_1: Both children are girls, Event E_2: One of the children is a girl.

$P(E_1|E_2)$ = Knowing that one of the children is a girl only allows you to eliminate the possibility that both children are boys. As such, the remaining scenarios are: i) B, G ii) G, B and iii) G, G with equal probabilities (where B = Boy and G = Girl). So P(2 girls | 1 child is a girl) = $\boxed{\frac{1}{3} = P(E_1|E_2)}$

b) Event E_3: The family has two boys Event E_4: The older child is a boy

P(

$P(E_3|E_4)$: Now that we are specifically referring to the older child, we must consider all possible permutations of B and G, where the first entry is the younger child and the second the older.

i) (B, B)
ii) (B, G)
iii) (G, B)
iv) (G, G)

Knowing that the elder child is a boy allows us to eliminate options ii) and iv). As such, the options i) and iii) carry equal probabilities, and so, P(family has two boys | older child is a boy) = $\frac{1}{2}$.

⑤ Sample space: $\{H, T\} \times \{H, T\} \times \{H, T\}$, where H = Heads and T = Tails

a) $P(E_1 \cap E_2)$ = P(both coins come up heads). The 4 possibilities for the first two coin tosses are

i) H, T
ii) T, H
iii) T, T
iv) H, H

With each permutation carrying equal probability. Since only 1 of these options represents the Event $E_1 \cap E_2$, $P(E_1 \cap E_2) = \frac{1}{4}$.
$P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$. $P(E_1) \cdot P(E_2) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$
Since $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$, the events are independent.

b) $P(E_1 \cap E_2)$ = P(first coin comes up T, and the next two tosses come up H)
There are 2 choices for each toss: $\{H, T\}$. So the number of ways this event may occur is 2^8. $P(E_1 \cap E_2) = P((T, H, H))$ which can only be done in one way. So, $P(E_1 \cap E_2) = \frac{1}{2^3}$

Number of ways to perform E_1 is $1 \cdot 2^2 = 4$, and the number of ways to perform E_2 is 2 (since either the first two are heads or the second two are). So $P(E_1) = \frac{4}{2^3}$ and $P(E_2) = \frac{2}{2^3}$. $P(E_1)P(E_2) =$

$\frac{8}{64} = \frac{1}{2^8}$. Since $\frac{1}{2^8} = \frac{1}{2^8}$, the events are independent.

c) $P(E_1 \cap E_2)$: $E_1 \cap E_2 = \emptyset$ $P(E_1 \cap E_2) = 0$.

$P(E_1)$: There are 2^2 ways for E_1 to occur, so $P(E_1) = \frac{2^2}{2^3}$
$P(E_2)$: There are 2 ways for E_2 to occur. So $P(E_2) = \frac{2}{2^3}$

$P(E_1)P(E_2) = 0 \neq \frac{2^3}{2^6} = P(E_1 \cap E_2)$, so the events are not independent

(6) E and F are independent events.

(1) $P(E \cap F) = P(E)\,P(F)$, definition of independence

(2) $P(\bar{E} \cap \bar{F}) = P(\overline{E \cup F})$, since $\bar{E} \cap \bar{F} = \overline{E \cup F}$ by De Morgan's law for sets

(3) $= 1 - P(E \cup F)$, by definition of complement

(4) $= 1 - [P(E) + P(F) - P(E \cap F)]$, by definition of $P(E \cup F)$

(5) $= 1 - [P(E) + P(F) - P(E) \cdot P(F)]$, from (1)

(6) $= 1 - P(E) - P(F) + P(E) \cdot P(F)$, distributing

(7) $= (1 - P(E))(1 - P(F))$, algebra, reverse foil

(8) $= P(\bar{E}) \cdot P(\bar{F})$, definition of complement.

(9) Since we have shown that $P(E \cap F) = P(E) \cdot P(F) \Leftrightarrow P(\bar{E} \cap \bar{F}) = P(\bar{E}) \cdot P(\bar{F})$

7) Sample space: $\{1,2,3,4,5,6\}^8$, where $\{1,2,3,4,5,6\} = A$

a) $\bigcup_{i\in A} E_i$ is the event that none of the numbers show up on any dice.

As such, the complement of this is the event that all of the numbers show up on some die. So the question is $P\left(\overline{\bigcup_{i\in A} E_i}\right)$.

b) $P(E_i \cap E_j)$ where $i \neq j$ is the probability that 2 of the numbers do not match. Since there are 4^8 ways for this to happen, and 6^8 total ways to roll 8 dice, $P(E_i \cap E_j) = \left(\frac{4}{6}\right)^8$

In general: $\bigcap_{i\in I} E_i$ for $I \subseteq \{1,2,3,4,5,6\}$

$$\bigcap_{i\in I} E_i = (\{1,2,3,4,5,6\} - I)^8$$

$$\left|\bigcap_{i=I} E_i\right| = (6 - |I|)^8$$

As such, $P\left(\bigcap_{i=I} E_i\right) = \frac{(6-|I|)^8}{6^8}$.

c) ① $P\left(\bigcup_{i=1}^{6} E_i\right) = P(E_1) + P(E_2) + P(E_3)\dots - P(E_1\cap E_2) - P(E_1 \cap E_3)\dots - P(E_5 \cap E_6) + P(E_1 \cap E_2 \cap E_3)\dots - P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5 \cap E_6)$

② $= \sum_{\substack{i=I \\ I\subseteq A, |I|=n}} (-1)^{|I|+1} P\left(\bigcap_{i=I} E_i\right) = \sum_{\substack{i=1 \\ I\subseteq A, |I|=i}}^{6} (-1)^{|I|+1} \binom{6}{|I|} \left(\frac{6-|I|}{6}\right)^8$ since there are $\binom{6}{|I|}$ ways to choose a subset of size i.

$$P\left(\overline{\bigcup_{i=1}^{6} E_i}\right) = 1 - P\left(\bigcup_{i=1}^{6} E_i\right) = 1 - \sum_{i=1}^{6} (-1)^{|I|+1} \binom{6}{|I|}\left(\frac{6-|I|}{6}\right)^8$$

This works, because even sized sets above are being subtracted and odd numbered sets are being added.

From part 6), we have:

$$1 - \sum_{\substack{i=1 \\ I \subseteq A, |I|=i}}^{6} (-1)^{|I|+1} \binom{6}{|I|} \left(\frac{6-|I|}{6} \right)^{8}$$

$$= 1 - \left[\binom{6}{1}\left(\frac{5}{6}\right)^{8} - \binom{6}{2}\left(\frac{4}{6}\right)^{8} + \binom{6}{3}\left(\frac{3}{6}\right)^{8} - \binom{6}{4}\left(\frac{2}{6}\right)^{8} + \binom{6}{5}\left(\frac{1}{6}\right)^{8} - \binom{6}{6}\left(0\right)^{8} \right]$$

⑧ Sample space: $\{S,F\}^n$, where S = success and F = Failure

Events: E_1 = probability of no failures, E_2 = probability of atleast one failure, E_3 = the probability of at most 1 failure, E_4 = Probability of atleast 2 failures.

a) $P(E_1) = p^n$, because the BI's are independent, and $P(\text{success}) = p$

b) $P(E_2) = 1 - p^n$ because p^n is the probability of all success, and so $1-p^n$ is the probability of 1 or more failures.

c) $P(E_3) = (n)(p^{n-1})(1-p) + p^n$, because q is the probability of failure, and there are n ways to choose the failed trial. We must also add the probability that there are no failures $= P(E_1) = p^n$.

d) Since p^n is the probability of all success and $n \cdot p^{n-1} q$ is the probabilit of one failure, and the event E_4 is $\overline{E_2 \cup E_3}$, $P(E_4) = 1 - p^n - n(p^{n-1})(1-p)$

⑨ The sample space is all possible permutations of 2 goats and a car, all possible choices made by the contestant, all possible choices for the door to be opened, and whether or not the contestant wins the car.

C_x = The event that the car is behind door x, for $1 \le i \le 3$.

In the beginning $P(C_i) = P(C_j) = P(C_k) = \frac{1}{3}$, where i, j, k is a permutation of 1,2,3

Additionally, let D_x be the event that Door x is opened and has a goat

Assume that the contestant picked door i, and door j was opened, $i \neq j$.

Calculate $P(C_i | D_j)$:

$$P(C_i | D_j) = P(D_j | C_i) \cdot \frac{P(C_i)}{P(D_j)}$$

$P(D_j | C_i) = \frac{1}{2}$, since the doors j and k both contain goats

$P(C_i) = \frac{1}{3}$, from the above

$$P(D_j) = P(D_j|C_i) \cdot P(C_i) + P(D_j|C_j) \cdot P(C_j) + P(D_j|C_k) \cdot P(C_k)$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) + 0 \cdot \frac{1}{3} + (1) \cdot \left(\frac{1}{3}\right)$$

Since we assume that we have chosen door i. As such,

$$P(D_i) = \left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right) + \tfrac{1}{3} = \tfrac{1}{2}$$

Thus $P(C_i \mid D_i) = \dfrac{\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{3}$

Since $P(C_j \mid D_i) = 0$, $P(C_k \mid D_i)$ must equal $\frac{2}{3}$, since $P(C_i \mid D_i) + P(C_j \mid D_i) + P(C_k \mid D_k) = 1$. So, since $P(C_k \mid D_i) > P(C_i \mid D_i)$, it is always better to switch doors.

(10) Sample space: Ramesh takes either the bicycle, bus, or car to work, and he either reaches on time or he is late. I.e $\{B, U, C\} \times \{L, \overline{L}\}$, where:

Event L = event that Ramesh is late

Event B = event that Ramesh took the bicycle to work

Event U = event that Ramesh took the bus to work

Event C = event that Ramesh took the car to work.

a) $P(L|C) = 0.5 = \frac{1}{2}$, $P(L|U) = 0.2 = \frac{1}{5}$, $P(L|B) = 0.05 = \frac{1}{20}$

a) $P(C|L)$, given that $P(B) = P(U) = P(C) = \frac{1}{3}$

$$P(C|L) = \frac{P(L|C) \cdot P(C)}{P(L|B) \cdot P(B) + P(L|U) \cdot P(U) + P(L|C) \cdot P(C)}$$

$$= \frac{(\frac{1}{2})(\frac{1}{3})}{(\frac{1}{20})(\frac{1}{3}) + (\frac{1}{5})(\frac{1}{3}) + (\frac{1}{2})(\frac{1}{3})} = \frac{2}{3}$$

b) $P(C|L)$, given that $P(C) = \frac{3}{10}$, $P(U) = \frac{1}{10}$, $P(B) = \frac{3}{5}$

$$= \frac{P(L|C) \cdot P(C)}{P(L|B) \cdot P(B) + P(L|U) \cdot P(U) + P(L|C) \cdot P(C)}$$

$$= \frac{(\frac{1}{2})(\frac{3}{10})}{(\frac{1}{20})(\frac{3}{5}) + (\frac{1}{5})(\frac{1}{10}) + (\frac{1}{2})(\frac{3}{10})} = \boxed{\frac{3}{4}}$$

(11) Sample space: A given burger might be well-cooked, burned on one side or burned on both sides.

Event W = A burger is well-cooked
O = A burger is burned on one side
B = A burger is burned on both sides
F = The side facing up is burned.

$$P(B|F) = \frac{P(F|B)\ P(B)}{P(F|B)\cdot P(B) + P(F|O)\cdot P(O) + P(F|W)\cdot P(W)}$$

$$\left\{\begin{array}{l} P(F|B) = 1 \\ P(B) = P(O) = P(W) = \frac{1}{3} \\ P(F|O) = \frac{1}{2} \\ P(F|W) = 0 \end{array}\right\}$$

$$= \frac{(1)(\frac{1}{3})}{(1)(\frac{1}{3}) + (\frac{1}{2})(\frac{1}{3}) + (0)(\frac{1}{3})}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \boxed{\frac{2}{3}}$$