CS 30 - HW #4

Arun Hari Anand

① Claim: Successive splitting of n stones into n piles leads to a "sum of product" of $\frac{n(n-1)}{2}$ for $n \geq 1$.

Predicate $P(n) \equiv$ splitting a pile of n stones successively until there are n piles of 1 stone each leads to a "sum-of-product" calculation of $\frac{n(n-1)}{2}$

Base case: when $n=1$, you cannot split the pile any further, so $\frac{(0)(1)}{2} = 0$, which means that $P(1)$ holds. Similarly, when $n=2$, you can only split the pile into 2 piles of 1 stone each, so $(1)(1) = 1 = \frac{(2)(1)}{2}$, so $P(2)$ holds as well.

Inductive step: Assume that $P(1) \wedge P(2) \dots \wedge P(k)$ holds. That is, $\bigwedge_{i=1}^{k} P(i)$ holds. This is the induction hypothesis

We now attempt to show that $P(k+1)$ holds.

Assume that we first split $k+1$ stones into 2 piles of r and $k+1-r$ stones respectively. As such, the "sum of products" calculation for this step becomes $(r)(k+1-r)$. Assume that we now split these two piles successively until we get $k+1$ piles of 1 stone each. By the induction hypothesis, the sum of products for r stones is $\frac{r(r-1)}{2}$ and for $k+1-r$ stones it is $\frac{(k+1-r)(k-r)}{2}$.

The overall sum is thus:

$$(r)(k+1-r) + \frac{r(r-1)}{2} + \frac{(k+1-r)(k-r)}{2}$$

$$= \frac{2r(k+1-r) + r(r-1) + (k+1-r)(k-r)}{2}, \text{ combining}$$

$$= \frac{2rk + 2r - 2r^2 + r^2 - r + k^2 - rk + k - r - rk + r^2}{2}, \text{ expanding}$$

$$= \frac{(2rk + 2r - 2r^2) + (2r^2 - 2r - 2rk) + k^2 + k}{2}, \text{ by grouping}$$

$$= \frac{k^2 + k}{2}, \text{ by algebra}$$

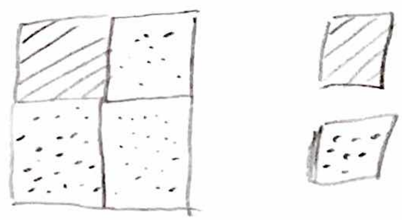
$$= \frac{(k+1)(k)}{2}$$

Thus, $P(k+1)$ holds. $P(n)$ thus holds for all $n \geq 1$ and so the claim follows from the principle of strong mathematical induction.

② Claim: the floor, leaving out the center goldplated tile, can be covered precisely using L-shaped wooden pieces for all square $2^n \times 2^n$ floors, such that $n \in \mathbb{N}$.

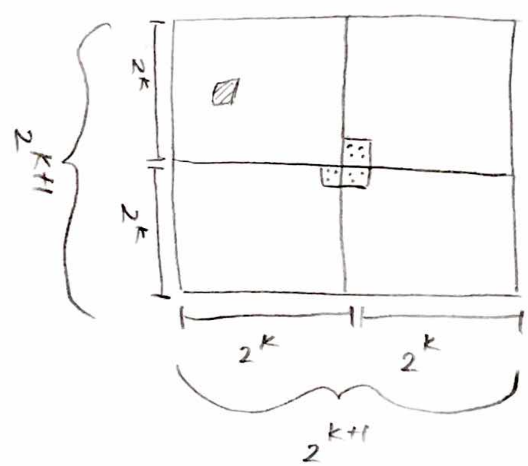
Predicate $P(n) \equiv$ For a square floor of length and width 2^n, with a golden tile anywhere on the floor, there exists a tiling of the remainder of the floor.

Base case: when $n=1$, the 2×2 floor is fully covered by the goldplated tile and one additional L-shaped tile in the following configuration:



Thus $P(1)$ holds and the base case is true.

Inductive step: Assume that a $2^k \times 2^k$ floor is adequately covered by 1 golden tile and a set of wooden pieces. That is $P(k)$ holds. We now attempt to show that $P(k+1)$ holds. Assume the following scenario, where the goldplated tile is placed in an arbitrary location on a floor of dimensions $2^{k+1} \times 2^{k+1}$:



Assume that the floor is divided into quadrants as pictured above. Since the quadrant that contains the golden tile is of dimensions $2^k \times 2^k$, there exists a tiling of that quadrant such that the whole quadrant is covered by wooden L-shaped pieces, by the induction hypothesis. We then place another L shaped tile in the center, such that each 1×1 units^2 section of the L-shaped tile falls into each of the remaining three quadrants. If we assume that this is similar to having a golden tile in each of these sections, we know by the induction hypothesis that there exists a tiling of the remainder of these $2^k \times 2^k$ quadrants of the floor. Thus we know that all four quadrants can be satisfactorily covered by wooden L-shaped pieces. Thus, $P(k+1)$ holds. The predicate $P(n)$ is thus true for all $k \geq 1$ by the principle of induction. Since the claim is a special case of the predicate P where the golden tile is one of the center four pieces, the claim thus follows from the principle of induction.

③ Claim: $f(n) > \alpha^{n-2}$ when $n \geq 3$, where $\alpha = \frac{\sqrt{5}+1}{2}$

Predicate $P(n) \equiv f(n) > \alpha^{n-2}$, $\alpha = \frac{\sqrt{5}+1}{2}$

Base case: when $n=3$, $f(3) = 2 > 1.61 > \frac{\sqrt{5}+1}{2} = (\alpha)^1 = \alpha^{n-2}$. Thus, $P(3)$ is true.
$f(4) = 3 > 2.6 > \left(\frac{\sqrt{5}+1}{2}\right)^2 = \alpha^2 = \alpha^{n-2}$, Thus these base cases ~~are~~ hold and $P(3)$ and $P(4)$ hold.

Inductive step: Assume that $\bigwedge_{i=1}^{k} P(i)$ is true for an arbitrary $k \geq 3$. We attempt to show now that $P(k+1)$ holds; that is $f(k+1) > \alpha^{k-1}$

(1) $f(k+1) = f(k) + f(k-1)$, by definition of the fibonacci function

(2) $f(k) > \alpha^{k-2}$, IH

(3) $f(k-1) + f(k) > \alpha^{k-2} + f(k-1)$, adding $k-1$ to both sides

(4) $> \alpha^{k-2} + \alpha^{k-3}$, by IH

(5) $= \alpha^{k-3}(\alpha+1)$ by factoring out α^{k-3}

(6) $= \alpha^{k-3}(\alpha^2)$, because $\alpha^2 = \alpha+1$: $\left(\frac{\sqrt{5}+1}{2}\right)^2 = \frac{5+1+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2} = 1 + \frac{\sqrt{5}+1}{2}$

(7) $= \alpha^{k-1}$

(8) $f(k+1) > \alpha^{k-1}$ Since (1) states that $f(k+1) = f(k) + f(k-1)$

The claim thus follows from the principle of strong mathematical induction.

④ The set B of balanced binary trees can be defined as follows:

Base cases: • λ, where λ is the empty string, is an element of B the string "()" is also an element of B

Inductive step: If x is an element of B, then $(x) \in B \wedge ()x \in B \wedge B() \in B$

⑤ Proof by strong induction shows the following to be true for a predicate $P(n)$.

~~I~~ $P(1)$ is true

$\bigwedge_{i\geq 1}^{k} P(i) \Rightarrow P(k+1)$

$\therefore \forall_{n\geq 1} P(n)$

Let us define predicate $Q(n) \equiv \bigwedge_{i=1}^{n} P(i)$

Basis Step : $Q(1)$ is true, because $Q(1) \equiv P(1)$ and we have already shown $P(1)$ to be true above.

Induction step: Assume that $Q(k)$ holds, for $k \geq 1$. This is our induction hypothesis. We now attempt to show $Q(k+1)$.

(1) $Q(k) \Rightarrow P(k+1)$, by definition of Q and induction hypothesis

(2) $Q(k) \Rightarrow Q(k)$, trivially

(3) $Q(k) \Rightarrow Q(k) \wedge P(k+1)$, by (1) and (2)

(4) $Q(k+1)$, by (3) and the definition of $Q(n)$

Thus $Q(k) \Rightarrow Q(k+1)$, and so $\forall_{k\geq 1} Q(k)$ by the principle of weak mathematical induction

$Q(k+1) \Rightarrow P(k+1)$, and so assuming that $P(1)$ is true we have shown that $P(k) \Rightarrow P(k+1)$, and so we can equivalently prove the claim using weak induction.

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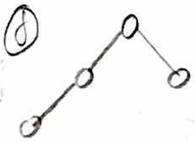
8 

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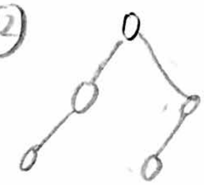
12 

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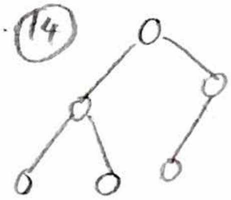
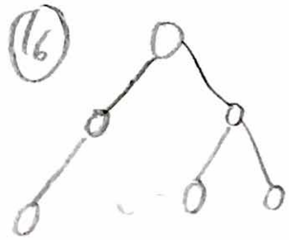
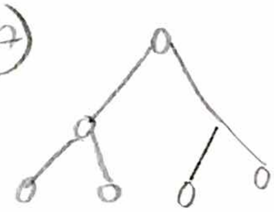
14 

16 

17 

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b) $h(T) \leq 2\log_2 n(T) \equiv 2^{\frac{h(T)}{2}} \leq n(T) \equiv n(T) \geq 2^{\frac{h(T)}{2}}$

Predicate $P(T) \equiv$ for $T \neq \lambda$, $2^{\frac{h(T)}{2}} \leq n(T)$ where $h(T)$ is the height of T and $n(T)$ is the number of nodes in T.

Base case: When $n(T) = 1$, that is, T only has one node, $h(T) = 0$. Thus $2^0 = 1 \leq 1 = n(T)$. Thus, the predicate $P(n)$ holds for the base case.

Induction Step: Assume that $T_1, T_2 \in BBT$, and assume that $2^{\frac{h(T_1)}{2}} \leq n(T_1)$ and $2^{\frac{h(T_2)}{2}} \leq n(T_2)$. We also assume that T_1 and T_2 have disjoint nodesets and that their heights differ by at most one. We now attempt to show that $P(T_3)$ holds for $T_3 = (x, T_1, T_2)$ where x is not a node of T_1 or T_2:

$$
\begin{aligned}
&(1)\quad n(T_3) &&= n(T_1) + n(T_2) + 1 &&\text{, by definition of } T_3 \text{ above} \\
&(2) &&> n(T_1) + n(T_2) &&\text{, since 1 is being added to (1)} \\
&(3) &&\geq 2^{\frac{h(T_1)}{2}} + 2^{\frac{h(T_2)}{2}} &&\text{, by the induction hypothesis} \\
&(4) &&\geq 2^{\frac{h(T_3)-1}{2}} + 2^{\frac{h(T_3)-2}{2}} &&\text{, since } h(T_3) = 1 + \max(h(T_1), h(T_2)) \text{ and } T_3 \text{ is a BBT} \\
&(5) &&= 2^{\frac{h(T_3)}{2}} \cdot 2^{-\frac{1}{2}} + 2^{\frac{h(T_3)}{2}} \cdot 2^{-1} &&\text{, expanding} \\
&(6) &&= 2^{\frac{h(T_3)}{2}} \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) &&\text{, factoring out } 2^{\frac{h(T_3)}{2}} \\
&(7) &&> 2^{\frac{h(T_3)}{2}} &&\text{, because } \frac{1}{\sqrt{2}} + \frac{1}{2} > 1
\end{aligned}
$$

We have thus shown that $P(T_3)$ holds. The claim thus follows from the principle of structural induction.

(7) a) Claim: for all $x \in S$, $[X] \neq \phi$.

Since R is an equivalence relation, if $x \in S$, (x,x) must surely be an element of R. This is because all equivalence relations must be reflexive. Then, by the definition of the equivalence class of x, x must be an element of $[x]$. Since x is always an element of $[x]$, $[x] \neq \phi$. The claim thus follows from the direct proof principle

b) Claim; for all $x,y \in S$, either $[X] = [Y]$ or $[X] \cap [Y] = \phi$

Assume that $[X] \cap [Y] \neq \phi$. Let us now assume that $a \in [X] \cap [Y]$. As such, $a \in [X] \wedge a \in [Y]$. Following from the definition of the equivalence class, $(X,a) \in R \wedge (Y,a) \in R$. By symmetry $(a,X) \in R$ and $(a,y) \in R$. By reflexivity $(X,X) \in R$ and $(Y,Y) \in R$. Since $(X,a) \in R$ and $(a,y) \in R$, by transitivity we have $(X,y) \in R$ and $(Y,X) \in R$. by symmetry. Let us now assume that an element $b \in [X]$. By the definition of the equivalence class, we have that $(X,b) \in R$, and that $(b,X) \in R$ by symmetry. By transitivity, we can also show that since $(Y,X) \in R$ and $(X,b) \in R$, $(y,b) \in R$. By the definition of equivalence class we know that $b \in [Y]$, followins from the above. Thus we have shown that $b \in [X] \Rightarrow b \in [Y]$ by the direct proof principle. $[X] \subseteq [Y]$. Employing a similar argument it can be shown that $[Y] \subseteq [X]$. Since $[X] \subseteq [Y] \wedge [Y] \subseteq [X]$, $[X] = [Y]$, We have thus shown that

$$\neg([X] \cap [Y] = \phi) \Rightarrow [X] = [Y]$$

which is logically equivalent to $[X] \cap [Y] = \phi \vee [X] = [Y]$. Thus, the claim follows.