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CS 30 - HW#6

① Claim: $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

Assume that $B = A_1 \cup A_2$.

① $|A_1 \cup A_2 \cup A_3| = |B \cup A_3|$, since $B = A_1 \cup A_2$

② $= |B| + |A_3| - |B \cap A_3|$, since $|X \cup Y| = |X| + |Y| - |X \cap Y|$

③ $= |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$, since $B = A_1 \cup A_2$

④ $= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$, since $|X \cup Y| = |X| + |Y| - |X \cap Y|$

⑤ $= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - |(A_1 \cap A_3) \cup (A_2 \cap A_3)|$, since $(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z)$

⑥ $= |A_1| + |A_2| - |A_1 \cap A_2| + |A_3| - (|A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_3 \cap A_2 \cap A_3|)$, since $|X \cup Y| = |X| + |Y| + |X \cap Y|$

⑦ $= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_3 \cap A_2 \cap A_3|$ rearranging

$= |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$ since $X \cap Y \cap Z \cap Y = X \cap Y \cap Z$

The claim thus follows from the above algebraic proof.

② a) $\boxed{10 \cdot 9 \cdot 8 \cdot 7 = \frac{10!}{6!}}$, because there 10 choices for the first digit, 9 for the second, 8 for the third and 7 for the fourth. We multiply these as described by the product principle.

b) $10 \cdot 10 \cdot 10 \cdot 5 = 10^3 \cdot 5$, because there are 10 choices for the first 3 digits but 5 for the 4th because there are 5 even digits

c) $\boxed{(4)(9)}$, because we must first pick which of the four digits is <u>not</u> going to be 9, and there are 4 ways to make that choice. Then, we must pick what digit will occupy that spot, and there are 9 ways to make that choice.

③ $(5)(2)(P(4,4)) = (5)(2)(4!)$, because the bride and groom can either be the first two, second two, third two, and so on. There are 5 ways to make that choice. Then we must decide what order the bride and groom are to be placed in - there are 2 ways to make that choice. Then we must decide how the other 4 are to be permuted - there are 4! ways to make that choice.

b) $6! - (5)(2)(4!)$, because we know that there are 6! total ways to arrange everyone, and $(5)(2)(4!)$ ways to arrange everyone so that the bride and groom are together. Subtracting this figure from the total will give the number of ways to arrange everyone so that the bride and groom are not together.

c) Assume that the six spots are numbered as follows: ① ② ③ ④ ⑤ ⑥. We know that the groom cannot take spot 1 (because then the bride cannot be to his left). If he takes spot 2, there is one way to position the bride, and 4! ways to position the rest. So there are 4! ways to arrange everyone so that the groom takes spot ②. Following this logic, and using the sum principle, $4! + 2(4!) + 3(4!) + 4(4!) + 5(4!)$ gives us all the ways that this can be done. This is equal to $\boxed{(1+2+3+4+5)(4!)}$

④ a) $\binom{10}{7}$, because we must choose the 7 spots that contain 1's.

b) There is exactly 1 string that contains no 1's. There are $\binom{10}{9}$ strings that contain only 1 "1". And there are $2^{10}$ strings in total. Subtracting from this the number of nonviable strings, we have $\boxed{2^{10} - (1 + \binom{10}{9})}$.

c) This means that the string must have 6, 7, 8, 9 or 10 "0"s. There is only 1 way for the string to have 10 "0"s, $\binom{10}{9}$ ways for the string to have 9 "0"s, $\binom{10}{8}$ ways for it to have 8 "0"s and so on. So, the number of ways for the string to have more 0's than 1's is $\left[1 + \binom{10}{9} + \binom{10}{8} + \binom{10}{7} + \binom{10}{6}\right]$.

⑤ Assume that 5 0's and 10 1's are arranged in a row such that every 0 is followed by at least 2 1's. There is only 1 way to do this. Now, we must calculate the number of ways to insert 4 more 1's into the string. Since it does not matter where in "11" you add an extra 1 this is similar to the stars and bars problem with ~~4~~ 5 stars and 4 bars. So this is $\binom{9}{4}$.

⑥ Assume at first that each kid receives 1 chocolate. There is one way to do this. Then, the ~~a~~ remaining 80 chocolates are distributed amongst the children. For each chocolate, we must give it to one kid - there are 5 choices here. So, this is the stars and bars problem with 80 stars and 19 bars. So, the answer is $\binom{99}{19}$.

⑦ There are 16 total spots. We must first pick 6 spots for the red balls, 5 for the green, and so on. So the answer is $\binom{16}{6} \cdot \binom{10}{5} \cdot \binom{5}{3} \cdot \binom{2}{1} \cdot \binom{1}{1}$

$= \boxed{\binom{16}{6} \cdot \binom{10}{5} \cdot \binom{5}{3} \cdot 2 \cdot 1}$ rule

(8) A $S(k)$ is the number of sequences of the elements of the set $\{1,2,3,4\}$ such that the sequence is of length k and consecutive elements of the ~~set~~ sequence are not the same;

$$S(k) = \begin{cases} 1, & \text{if } k=0 \\ 4(3)^{k-1}, & \text{if } k \geq 1 \end{cases}$$

This is because if ~~n~~ $k=0$, there is only one sequence possible of length 0. If $k \geq 1$, then there are 4 choices for the first spot in the sequence and 3 choices for each of the other $k-1$ spots of the sequence. So, the number of possible sequences is $(4)(3)^{k-1}$ by the product principle.

(9) Assume that there are $n$ elements in set A. We know that every element in A must map to an element in B. We also know that both elements of B must be mapped to for it to be an onto function. Let us first consider the total number of functions from $A \to B$. For each element in A, we have 2 choices in B. So there are $2^n$ functions from $A \to B$. There are only 2 choices for functions that are not onto — either all the elements of A map to one element of B, or the other. So, subtracting, we have, $\boxed{2^n - 2}$ if $n \geq 2$, and $\boxed{0 \text{ if } n < 2}$

(10) $R \subseteq A \times A$. So we know that in the pair $(a_1, a_2)$ where $a_1, a_2 \in A$, if the size of A is $n$, then we have $n$ picks for the first "coordinate" and $n$ for the second. So, by the product principle, $|R| \leq n^2$. Since we are told to exclude all non-reflexive relations, assume that all relations already contain all of the $n$ possible reflexive pairs of the form $(a,a), (b,b) \ldots$ are already elements of all the possible relations. As such there are $n^2 - n$ optional pairs. For each pair, we may either add the pair, or choose not to. So there are $2^{n^2-n}$ possible combinations of pairs that can be added. So the answer is $\boxed{2^{n^2-n}}$

(11) $(3^8)(2^9)\binom{17}{9}$ from the binomial theorem, which states that $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

12 We know from the binomial theorem that the coefficient of $(x^2)^i\left(\frac{-1}{x}\right)^{100-i}$ is $\binom{100}{i}$. So;

$$\binom{100}{i}(x^2)^i\left(-\frac{1}{x}\right)^{100-i} = (-1)^{100-i}\binom{100}{i}(x^{2i})\left(\frac{1}{x}\right)^{100-i}$$
$$= (-1)^{100-i}\binom{100}{i}(x^{2i})(x)^{i-100}$$
$$= (-1)^{100-i}\binom{100}{i}(x)^{3i-100}$$

Setting $3i-100=k$, we know that $i=\frac{k+100}{3}$. The above is thus expressed as $(-1)^{100-\frac{k+100}{3}}\binom{100}{\frac{k+100}{3}}x^k$, so the coefficient of $x^k$ is:

$$\boxed{(-1)^{100-\frac{k+100}{3}}\cdot\binom{100}{\left(\frac{k+100}{3}\right)}}$$, when $\frac{k+100}{3}$ is a whole number.

13) a) Since $x_1$ is 11 at minimum, $X_2$ is 16 at minimum and $X_3$ is 21 at minimum, if we think of this situation in terms of 100 stars and 2 bars, then 48 of the stars are already accounted for. So, using 52 stars and 2 bars, we have $\binom{54}{2}$.

b) Since $x_1 \geq 11$ and $x_2 \geq 16$, $x_1+x_2 \geq 27$, and so 27 "stars" are accounted for and 73 are remaing. Of the remainder, $X_3$ may assume values from 0 to 39. So, for each of these values we are left with a different "stars and bars" scenario. The numbers of ways is therefore $\sum_{i=0}^{39}\binom{74-i}{1} = \sum_{i=0}^{39} 74-i$

14) a) $\binom{n}{k} = \frac{n!}{(n-k)!k!}$

$\binom{k}{j} = \frac{k!}{(k-j)!j!}$

$\binom{n}{k}\binom{k}{j} = \frac{n!k!}{(n-k)!\,k!\,(k-j)!\,j!} = \boxed{\frac{n!}{(n-k)!(k-j)!\,j!}}$

$\binom{n}{j} = \frac{n!}{(n-j)!\,j!}$ ; $\binom{n-j}{k-j} = \frac{(n-j)!}{(n-j-(k-j))!\,(k-j)!} = \frac{(n-j)!}{(n-k)!(k-j)!}$

$\binom{n}{j}\binom{n-j}{k-j} = \frac{n!\,(n-j)!}{(n-j)!\,j!\,(n-k)!\,(k-j)!} = \boxed{\frac{n!}{(n-k)!\,(k-j)!\,j!}}$

Since both expressions reduce to the same quantity, they are equivalent.

b) Assume that we are to choose $k$ people (among $n$ total people) to be professors at a university, and among those $k$ professors, $j$ will be department chairs. We want to calculate the number of ways this can be done.

- Of the $n$ people, we may first choose $k$ people to be professors at the University. There are $\binom{n}{k}$ ways to do this. We must then pick $j$ professors among the $k$ total to be department chairs. As such, there are $\binom{k}{j}$ ways to make this choice. Applying the product principle, the total number of ways to perform both tasks is $\binom{n}{k}\binom{k}{j}$.

- Another way to approach the problem is to first pick the $j$ department chairs first among the $n$ total people first: there are $\binom{n}{j}$ ways to do this. Among the remaining $n-j$ people then, we must choose $k-j$ professors who are not department chairs. There are $\binom{n-j}{k-j}$ ways to make this choice. By the product principle, the number of ways to perform both tasks is $\binom{n}{j}\binom{n-j}{k-j}$.

Since both expressions are counting the same thing, they must be equivalent. Thus, it follows that $\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$.

15) $B$ can take on any size from $0$ to $n$. For each size $i$ that it takes on, there are $\binom{n}{i}$ number of ways to populate it with $i$ elements. Since the size of $A$ is atmost $i$, there are $2^i$ possible subsets of $B$ that can be possible candidates for $A$. By the product principle, for each size of $i$, there are $\binom{n}{i}(2^i)$ possible ways to pick elements for $B$ and make subsets of $B$. By the sum principle then, the answer is $\sum_{i=0}^{n}\left[\binom{n}{i}2^i\right]$.

16) We must first pick 2 aces among 4; since order does not matter there are $\binom{4}{2}$ ways to do this. Then we must pick 3 different suits from 4 total; these will be the suits of the next 3 cards. There are $\binom{4}{3}$ ways to do this. Then, we must pick 3 non-Aces such that they are of different kinds; here order matters because the set {3 of spades, 2 of hearts and 4 of diamonds} is different from {4 of spades, 2 of hearts and 3 of diamonds}. As such, since there are 12 possible values excluding the aces, there are $12 \cdot 11 \cdot 10$ ways to do this. By the

product principle then, the ansrer is $\binom{4}{2}\binom{4}{3}\cdot 12\cdot 11\cdot 10$.