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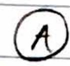
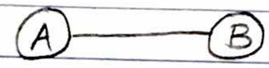
CS 30 HW #9

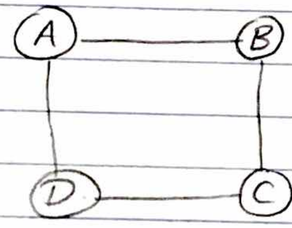
① a) Dimension 0 -



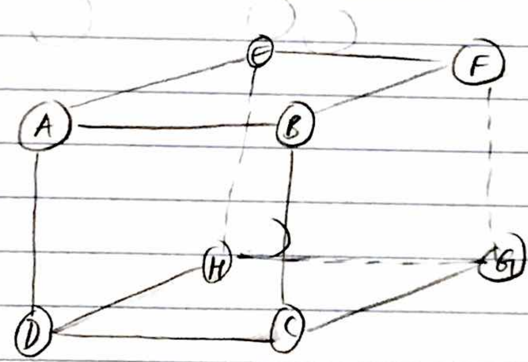
Dimension 1 -



Dimension 2 -



Dimension 3 -



b) Base case: An HC of dimension 0 is a single vertex with no edges.

Recursive: For $n > 0$; An HC of dimension n is defined as follows:

If A is an HC of dimension $n-1$ such that $A = (V, E)$, then A', a hypercube of dimension n, is defined as $A' = (V', E')$, where

$$V' = V \times \{0, 1\}$$

$$E' = \{\{(a,0),(b,0)\} \mid \{a,b\} \in E\} \cup \{\{(a,1),(b,1)\} \mid \{a,b\} \in E\} \cup \{\{(a,0),(a,1)\} \mid a \in V\}$$

As such, the hypercube of dimension n is the set of all graphs isomorphic to an HC of dimension n.

c) A hypercube of dimension n has 2^n vertices.
Since a hypercube of dimension k has vertices of degree k, the number of edges for a hypercube of dimension n will be $\boxed{n \cdot 2^{n-1}}$, because accounting for all the degrees of one "half" of a hypercube will give us the number of edges in the hypercube with doublecounting any edges.
This is because we know from the handshake theorem that $\sum_{u \in V(G)} \deg(u) = 2|E(G)|$, so $\frac{1}{2} \sum_{u \in V(G)} \deg(u) = |E(G)|$

(2) Assume that there are n people in a group, such that $n \geq 2$. Assume that the people in the friend group are represented by vertices in a $A=(V,E)$, $|V| \geq 2$, whose vertex set is $V = \{a_1, a_2, a_3 \ldots a_n\}$. Also assume that friendships among people in the group are represented by vertices: for instance if a_1 and a_2 are friends (without loss of generality), then the edge $\{a_1 a_2\}$ exists in E. This is valid since friendships are considered symmetric, which allows us to use an undirected graph.

Assume for a contradiction that all vertices of A have distinct degrees, which is analoguous to assuming that all members of the group have a different number of friends. As shown in class, since $\deg(a_i)$ has to be an integer between 0 and $|V|-1$, deg is a function from a_i to $[0 \ldots |V|-1]$. Since, by assumption, deg is one-to-one and onto (because the size of the domain and the codomain are the same), we may consider 2 edges in the graph a_i and a_j such that $\deg(a_i) = 0$ and $\deg(a_j) = |V|-1$. Since $|V| \geq 2$, $|V|-1 \geq 1$, so we know that $\deg(a_i) \neq \deg(a_j)$, and so $a_i \neq a_j$. Since $\deg(a_i) = 0$, there is no edge incident on a_i, which implies that a_i and a_j are not adjacent, which means that $\deg(a_j) < |V|-1$, which contradicts the assumption that $\deg(a_j) = |V|-1$. So, the vertices of A do not have distinct degrees, from the principle of proof by contradiction

This argument is analoguous to the following: assume that all n people have a distinct number of friends. As such, since the number of friends a given person can have ranges between 0 and $n-1$, for every number x such that $0 \leq x \leq n-1$, there exists exactly one person who has i friends. So there must exist 2 people a_i and a_j such that a_i has no friends and a_j has $n-1$ friends. Since this implies that a_j is friends with everyone but himself, and yet a_j is not friends with a_i, we have a contradiction. Thus, we know that all n people do not have a distinct number of friends, by the principle of proof by contradiction.

(3) i) Assume that R is a symmetric relation on S. We know that

$$\text{i)}\ (a,b) \in R \implies (b,a) \in R$$

By the definition of R^*, we know that

$$\text{ii)}\quad a \in S \Rightarrow (a,a) \in R^*$$

$$\text{iii)}\quad (a,b) \in R \Rightarrow (a,b) \in R^*$$

$$\text{iv)}\quad (a,b) \in R^* \wedge (b,c) \in R^* \Rightarrow (a,c) \in R^*$$

① We know from assumption that

$$a \in S \Rightarrow (a,a) \in R^*$$

$$(a,b) \in R^* \wedge (b,c) \in R^* \Rightarrow (a,c) \in R^*$$

So the only remaining fact that must be proved to show that R^* is an equivalence class is that $(a,b) \in R^* \Rightarrow (b,a) \in R^*$

Predicate $P(a,b) \equiv (a,b) \in R^* \Rightarrow (b,a) \in R^*$; Claim $\forall_{(a,b) \in R^*} P(a,b)$

Proof: structural induction with Predicate $P(a,b) \equiv (a,b) \in R^* \Rightarrow (b,a) \in R^*$

Base case:

Rule a) states that $\forall_{a \in S} (a,a) \in R^*$; since (a,a) is trivially symmetric with itself, $P(a,b)$ holds for all elements added based on rule a)

Rule b) states that $\forall_{(a,b) \in R} (a,b) \in R^*$; since $(a,b) \in R \Rightarrow (b,a) \in R$

if (a,b) is added to R^* based on this rule, (b,a) will also be added based on this rule. Hence the base case.

Induction step: Assume that all elements in R^*. We now attempt to show that any new element (a^*,b^*) constructed from the elements of R^* must satisfy $P(a^*,b^*)$ using the previous statement as our induction hypothesis

: Rule c), which states that $(a,b) \in R^* \wedge (b,c) \in R^* \Rightarrow (a,c) \in R^*$.

So, If (a^*,b^*) is in R^* due to this rule, there must exist an x such that $(a^*,x) \in R^* \wedge (x,b^*) \in R^*$. As such, from the induction hypothesis, we know that $(x,a^*) \in R^* \wedge (b^*,x) \in R^*$. So, from rule c) we know that $(b^*,a^*) \in R^*$. So $P(a^*,b^*)$ holds in this case.

Since $P(a^*, b^*)$ holds in all cases $\forall_{(a,b) \in R^*}$ $P(a,b)$ holds from the principle of structural induction. Thus, R^* is symmetrical, and so we know that it is an equivalence relation.

ii) Claim: $(u,v) \in R^*(G) \iff u \rightsquigarrow v$

First, we attempt to show that $u \rightsquigarrow v \Rightarrow (u,v) \in R^*(G)$

From the definition of the reachability relation, $u \rightsquigarrow v \Rightarrow$ there exists a path from u to v. Let us call this path P and its length ℓ.

$P(n) \equiv$ If $u \rightsquigarrow v$, and P is a path from u to v with length n, then $(u,v) \in R^*(G)$ for an arbitrary graph G

$\forall_{n \geq 0} P(n)$ is the claim.

Base case: when $n=0$ $P(0)$ states that $u \rightsquigarrow v \Rightarrow (u,v) \in R$ when the length of the path from u to v is 0. If the length is 0, that means that $u=v$. From the definition of $R^*(G)$ we know that $\forall_{u \in V(G)} (u,u) \in R^*(G)$. As such, when $u=v$ $u \rightsquigarrow v \Rightarrow (u,v) \in R^*(G)$. So $P(0)$ holds. Hence, the base case.

Induction step: Assume $P(0) \wedge P(1) \ldots P(k)$ for some arbitrary $k > 0$. We now attempt to show $P(k+1)$, using this as our induction hypothesis. $P(k+1)$ states that If $u \rightsquigarrow v$ and P is a path from u to v of length $k+1$, then $(u,v) \in R^*(G)$ for an arbitrary graph G.

Path $P = (u, u_1, u_2 \ldots u_{\ell}, v)$. We know that $u \rightsquigarrow u_\ell$ and that the length of this path is k. So we know that $(u, u_\ell) \in R^*(G)$. Since the length of the path (u_ℓ, v) is 1 and $u_\ell \rightsquigarrow v$, $(u_\ell, v) \in R^*(G)$

Since we know that $R^*(G)$ is an equivalence relation, and thus transitive, $(u, u_\ell) \in R^*(G) \wedge (u_\ell, v) \in R^*(G) \Rightarrow (u,v) \in R^*(G)$.

Thus $P(k+1)$ holds and the claim follows from principle of strong induction.

We now attempt to show that $u,v \in R^*(G) \Rightarrow u \rightsquigarrow v$

Proof by structural induction: $P(u,v) \equiv u,v \in R^*(G) \Rightarrow u \rightsquigarrow v$

Base case: We know any vertex $u \in E(G)$ has the following property: $u \rightsquigarrow u$. As such, since $\forall_{u \in V(G)} (u,u) \in R^*(G)$ by the definition of $R^*(G)$, all of these reflexive pairs satisfy $P(u,u)$.
Additionally, we know from the definition of R^* that $\forall_{(u,v) \in R} (u,v) \in R^*(G)$.
And we know from the definition of R we know that $(u,v) \in R \Rightarrow \{u,v\} \in E(G)$. Thus if (u,v) has been added to the set R^* because of this basis step, we know that the edge $\{u,v\} \in E(G)$ and so $u \rightsquigarrow v$ by definition of reachability. Hence we have the base case.

Inductive step: Assume that $P(u,v)$ holds for all (u,v) in the set $R^*(G)$. This is our induction hypothesis. WWTS $P(u,v)$ holds for all (u,v) that are produced from elements already added to the set.
We know that the inductive rule is that $(a,b) \in R^*(G) \wedge (b,c) \in R^*(G) \Rightarrow (a,c) \in R^*(G)$. $(a,b) \in R^*(G) \rightarrow a \rightsquigarrow b$ by induction hypothesis. $(b,c) \in R^*(G) \Rightarrow b \rightsquigarrow c$ by induction hypothesis. We also know that the reachability relation $\rightsquigarrow$ is transitive. So $a \rightsquigarrow b \wedge b \rightsquigarrow c \Rightarrow a \rightsquigarrow c$. So, $a \rightsquigarrow c$ and $P(u,v)$ holds for all (u,v) added to the set $R^*(G)$ based on the inductive rule. The claim thus follows from the principle of structural induction.

(9) Claim: If π and π' are longest paths in a connected graph, they have no common vertices.
Assume that π, expressed as $(u_0, e_1, u_1, e_2 \ldots e_\ell, u_\ell)$ is a longest path in a connected graph G.
Also assume that π', expressed as $(u'_0, e'_1, u'_1 \ldots e'_\ell, u'_\ell)$ is also a longest path in G, and the length of π and π' is K.

Assume for a contradiction that π and π' have no shared vertices. Since G is connected, there is a shortest path from π to π', between u_i and u'_j for some i, j (such that $1 \leq i \leq K$ and $1 \leq j \leq K$) such that the path has no common vertices with π or π' except for u_i and u'_j. This divides π and π' into sections each each; π is divided into $\pi_1 = (u_0 e_1, u_1 \ldots u_i)$ and $\pi_2 = (e_{i+1}, u_{i+1} \ldots u_\ell)$ and π' is divided into $\pi'_1 = (u'_0, e'_1, u'_1 \ldots u'_j)$ and $\pi'_2 = (e'_{j+1}, u'_{j+1} \ldots u'_\ell)$
Since the length of the paths π and π' is K by assumption, we know that the longer section of each of the paths π and π' must be atleast of length $\lceil \frac{K}{2} \rceil$. The length of the path between π and π' must be atleast 1, and since π and π' do not share vertices by assumption there exists a path P that traverses the longer section of π, the path between π and π', and the longer section of π'. Adding up the least possible lengths of each of these sections we know that the length of P is atleast $1 + \lceil \frac{K}{2} \rceil + \lceil \frac{K}{2} \rceil$ Which is atleast equal to $K+1$. Thus we have a path P that is longer than π and π', which contradicts the assumption that π and π' are 2 longest paths of G. Thus, the claim follows from the principle of proof by contradiction.

⑤ Claim: If G is a connected graph and u is a vertex of odd degree, there is another vertex $v \neq u$ such that $u \rightsquigarrow v$ and v is of odd degree.

Assume that G is a connected graph and u is a vertex of odd degree. By the handshake theorem, we know $\sum_{u \in V(G)} \deg(u) = 2|E(G)|$. As such

the sum of all degrees in a graph is an even number, and as such, it can be represented as $2k$ for some $k \in W$. Since one of the degrees is odd, $\deg(u)$ can be represented as $2m+1$ for some $m \in W$. So there must exist atleast 1 $v \neq u$ such that $\deg(v) = 2n+1$ for some $n \in W$, such that $\deg(v) + \deg(u) = 2n+1+2m+1 = 2(m+n)+2 = 2(m+n+1)$. This is because atleast one other odd degree vertex (v) is required to result in an even sum. Since G is connected, we also know that $u \rightsquigarrow v$ from the definition of connected. The claim thus follows from the direct proof principle.

(6) Claim: Graph $G = (V, E)$ is connected $\implies |E| \geq |V| - 1$

Proof: by induction

$P(n) \equiv$ for any graph $G = (V, E)$ such that $|E| = n$, $n \geq |V| - 1$
Claim $\equiv \forall_{n \geq 0} P(n)$

Base case: When $n = 0$, $|E| = 0$. This means that, for the graph to be connected, the graph can only have 1 vertex. So $|V| = 1$. Since $0 \geq 1 - 1$, $P(0)$ holds, and this is the base case.

Induction: Assume $P(0) \wedge P(1) \wedge P(2) \ldots P(k)$ for some arbitrary $k \geq 0$.
This is our induction hypothesis. We now attempt to show $P(k+1)$.

Consider an arbitrary $G = (V, E)$ such that $|E| = k+1$. We now attempt to show $k+1 \geq |E| - 1$.

Consider a set E' defined as follows: E' contains some edges which, when removed from E, lead to exactly 2 connected components in the graph. Let these connected components be named A and B. We know that $|V(A)| + |V(B)| = |V(G)|$ since no vertices were removed. We also know $|E(A)| + |E(B)| + |E'| = |E(G)|$. Also, $|E(A)| \leq k$ and $|E(B)| \leq k$. As such, from our induction hypothesis, we know that $|E(A)| \geq |V(A)| - 1$ and $|E(B)| \geq |V(B)| - 1$. Combining those inequalities we have $|E(A)| + |E(B)| \geq |V(A)| + |V(B)| - 2$. Since $|V(A)| + |V(B)| = |V(G)|$, so $|E(A)| + |E(B)| \geq |V(G)| - 2$. ~~As~~ Since atleast 1 edge must be removed from G to produce 2 connected components, $|E'| \geq 1$ since the graph G was initially connected. So, $|E(G)| - |E'| \geq |V(G)| - 2$. For all $|E'| \geq 1$, we thus know that $|E(G)| \geq |V(G)| - 1$. Since $|E(G)| = k+1$, $k+1 \geq |V(G)| - 1$. So, $P(k+1)$ holds. The claim thus follows from the principle of strong induction.

(7) $P(n) \equiv$ If a graph $G=(V,E)$ has $|E|=n$, and every cycle of G has even length, then G is bipartite.

Claim: $\forall_{n \geq 0}\ P(n)$

Base case: For $n=0$, $|E|=0$ so there are no edges in G. Assume that $L=V(G)$ and $R=\phi$. For every edge in $E(G)$ (of which there are none) one endpoint is in L and the other in R trivially. So the graph is bipartite and so $P(0)$ holds.

Induction step:
Assume that $P(0) \wedge P(1) \ldots \wedge P(k)$ holds for $k \geq 1$. This is our induction Hypothesis. WWTS $P(k+1)$. Let $G=(V,E)$ be a graph such that $|E|=k+1$, and that every cycle of G has even length.
Case 1: Assume that there exists a leaf in G, and let this leaf be named v.
We know that there exists a vertex u such that the edge $\{u,v\} \in E$. Consider the graph $G'=(V-\{v\}, E-\{u,v\})$ $|E-\{u,v\}|=k$, because only one edge was removed. Since no cycle was changed, all cycles in G' are of even length. From the induction hypothesis, since $P(k)$ holds, G' is a bipartite graph. Let L', R' be a partition of the vertices of G'. Assume that $u \in L'$: in this case L', $R' \cup \{v\}$ will be a valid partition of the vertices of G, since v is a leaf. Assume that $u \in R'$: In this case $L' \cup \{v\}$, R' will be a valid partition of the vertices of G. As such, in this case G is bipartite.

Case 2: Assume that there is no leaf in the graph G
Consider the longest path in the graph G, Π.
$\Pi=(V_1, V_2 \ldots V_{\ell-1}, V_\ell)$. Since there exists no leaf in the graph, we know that there exists a vertex $V_i \neq V_{\ell-1}$ such that the edge $\{V_i, V_\ell\}$ exists. Assume for a contradiction that $i \notin \{1,2 \ldots \ell-2\}$.
In this case, the path $\Pi \cdot V_i$ will be longer than Π, which contradicts

the assumption that π is the longest path. So $i \in \{1,2 \ldots l-2\}$. As such $(v_i, v_{i+1} \ldots v_l, v_i)$ is of length $l-i+1 \geq 3$. It is a cycle of even length, from the assumption

Let us consider removing the edge $\{v_l, v_i\}$. $G' = (V, E-\{v_l, v_i\})$ is thus a graph such that $|E-\{v_l, v_i\}|$ is k and all cycles are of even length. Since $P(k)$ holds from induction hypothesis, there exists a partition L', R' of the vertices of the graph G'. More importantly, since $(v_i, v_{i+1} \cdots v_l)$ is odd length (because the cycle was of even length and one edge was removed), v_i and v_l are in distinct partitions. As such, if the edge were added back in, L', R' would still be a valid partition of the vertices of G. So, G is bipartite in this case as well, because $P(n+1)$ holds.

The claim thus follows from the principle of strong induction.