Math 412. §3.2, 3.2: Examples of Rings and Homomorphisms

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DEFINITION: A **subring** of a ring R (with identity) is a subset S which is itself a ring (with identity) under the operations $+$ and $\times$ for R.

DEFINITION: An **integral domain** (or just **domain**) is a commutative ring R (with identity) satisfying the additional axiom: *if* $xy = 0$, *then* x *or* $y = 0$ *for all* $x, y \in R$.

DEFINITION: A **ring homomorphism** is a mapping $R \xrightarrow{\phi} S$ between two rings (with identity) which satisfies:

(1) $\phi(x + y) = \phi(x) + \phi(y)$ for all $x, y \in R$.
(2) $\phi(x \cdot y) = \phi(x) \cdot \phi(y)$ for all $x, y \in R$.
(3) $\phi(1) = 1$.

DEFINITION: A **ring isomorphism** is a bijective ring homomorphism. We say that two rings R and S are **isomorphic** if there is an isomorphism $R \to S$ between them.

You should think of a ring isomorphism as a renaming of the elements of a ring, so that two isomorphic rings are "the same ring" if you just change the names of the elements.

A. WARM-UP: Which of the following rings is an **integral domain**? Which is a field? Which is commutative? In each case, be sure you understand what the *implied ring structure* is: why is each below a ring? What is the identity in each?

(1) $\mathbb{Z}, \mathbb{Q}$.
(2) $\mathbb{Z}_n$ for $n \in \mathbb{Z}$.
(3) $\mathbb{R}[x]$, the ring of polynomials with $\mathbb{R}$-coefficients.
(4) $M_2(\mathbb{Z})$, the ring of 2×2 matrices with $\mathbb{Z}$ coefficients.
(5) The subring $D_2(\mathbb{R})$ of diagonal matrices in the ring $M_2(\mathbb{R})$.

B. SUBRINGS: Suppose that R is a ring[1] and S is a subset.

(1) Suppose $R = \mathbb{Z}$. Is the subset $\mathbb{N}$ closed under $+$ and $\times$. Is it is subring?
(2) Without writing out the proof, determine which of the following are subrings of $\mathbb{C}$: $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, the set of even integers.
(3) Without writing out the proof, determine which of the following are subrings of $M_2(\mathbb{R})$: $M_2(\mathbb{N})$, $D_2(\mathbb{Z})$, $O_2(\mathbb{R}) = \{A \mid AA^T = I_2\}$, $GL_2(\mathbb{R}) = \{A \mid \text{A is invertible}\}$,
(4) Which of the following are subrings of $\mathbb{R}[x]$: $\mathbb{R}$, $\mathbb{Z}$, the subset of $\mathbb{R}[x]$ consisting of polynomials with non-negative coefficients, the set of polynomials such that $f(0) = 0$.
(5) Define the phrase "the subset S of the ring R is **closed under addition, multiplication, and (additive) inverses.**" How can this idea be used to identify subrings?

C. ISOMORPHISM. Consider the set $S = \{a, b, c, d\}$, with the associative binary operations $\heartsuit$ and $\spadesuit$ listed below.

$\heartsuit$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	a	c
d	a	d	c	b

$\spadesuit$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

[1] with identity

(1) Prove that S has the structure of a commutative ring using these operations. Is it a domain? Is it is field?
(2) Write down the addition and multiplication tables for the ring $\mathbb{Z}_4$.
(3) Find an **explicit** isomorphism showing that $\mathbb{Z}_4$ is isomorphic to $(S, \spadesuit, \heartsuit)$.

D. PRODUCT RINGS. Let R_1 and R_2 be rings (with identity). Consider the set

$$R_1 \times R_2 := \{(r_1, r_2) \,|\, r_1 \in R_1, r_2 \in R_2\}.$$

(1) Define a binary operation called $+$ on $R_1 \times R_2$ by $(r_1, r_2) + (s_1, s_2) = (r_1 + s_1, r_2 + s_2)$. The three different plus signs in the preceding sentence have three different meanings; explain. Prove that this binary operation on $R_1 \times R_2$ is associative, commutative, and has an identity. Show finally that every $(r_1, r_2) \in R_1 \times R_2$ has an inverse under $+$.
(2) Define multiplication on $R_1 \times R_2$ similarly. Explain why $R_1 \times R_2$ is a ring and identify its additive and multiplicative identities.
(3) Let $R_1 = R_2 = \mathbb{Z}_2$. How many elements are in the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$? List them out explicitly.
(4) Make tables for the addition and multiplication in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Identify zero and one.
(5) Is $\mathbb{Z}_2 \times \mathbb{Z}_2$ a field? is it a domain?
(6) Is $\mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic to $\mathbb{Z}_4$? If so, give an explicit isomorphism. If not, explain why no isomorphism exists.
(7) Is $\mathbb{Z}_2 \times \mathbb{Z}_2$ isomorphic to the ring $(S, \oplus, \otimes)$, where $S = \{a, b, c, d\}$ and $\oplus, \otimes$ are defined by the charts below? If so, give an explicit isomorphism. If not, explain why no isomorphism exists. Is your isomorphism the only possible isomorphism?

$\oplus$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

$\otimes$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	a
d	a	d	a	d

E. CANONICAL HOMOMORPHISM FOR $\mathbb{Z}_n$. Fix $n \in \mathbb{Z}$. Consider the mapping

$$\pi_n : \mathbb{Z} \to \mathbb{Z}_n, \quad a \mapsto [a]_n.$$

(1) Prove that π_n is a **ring homomorphism.**
(2) Is π_n surjective? Is it injective? What is the kernel? By definition, the **kernel** of a ring homomorphism f is the set of elements in the source that are mapped to 0 under f.
(3) Is the map $\mathbb{Z}_6 \to \mathbb{Z}_2$ sending $[a]_6 \mapsto [a]_2$ a ring homomorpshism? Is it an isomorphism?

F. TRUE OR FALSE:

(1) If S is a subring of R, then the inclusion map $S \hookrightarrow R$ is a ring homomorphism.
(2) If R_1 and R_2 are rings, then the projection map $R_1 \times R_2 \to R_2 \to R_1$ sending $(r_1, r_2) \mapsto r_1$ is a ring homomorphism.
(3) If S is a subring of R, then the inclusion map $S \hookrightarrow R$ is a ring homomorphism.
(4) The map $\mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3$ sending $[a]_6 \mapsto ([a]_2, [a]_3)$ is a ring isomorphism.
(5) The map $\mathbb{Z}_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ sending $[a]_4 \mapsto ([a]_2, [a]_2)$ is a ring isomorphism.