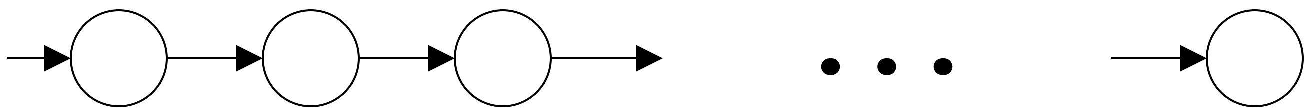
Time Complexity

Consider a <u>deterministic</u> Turing Machine M
which <u>decides</u> a language L

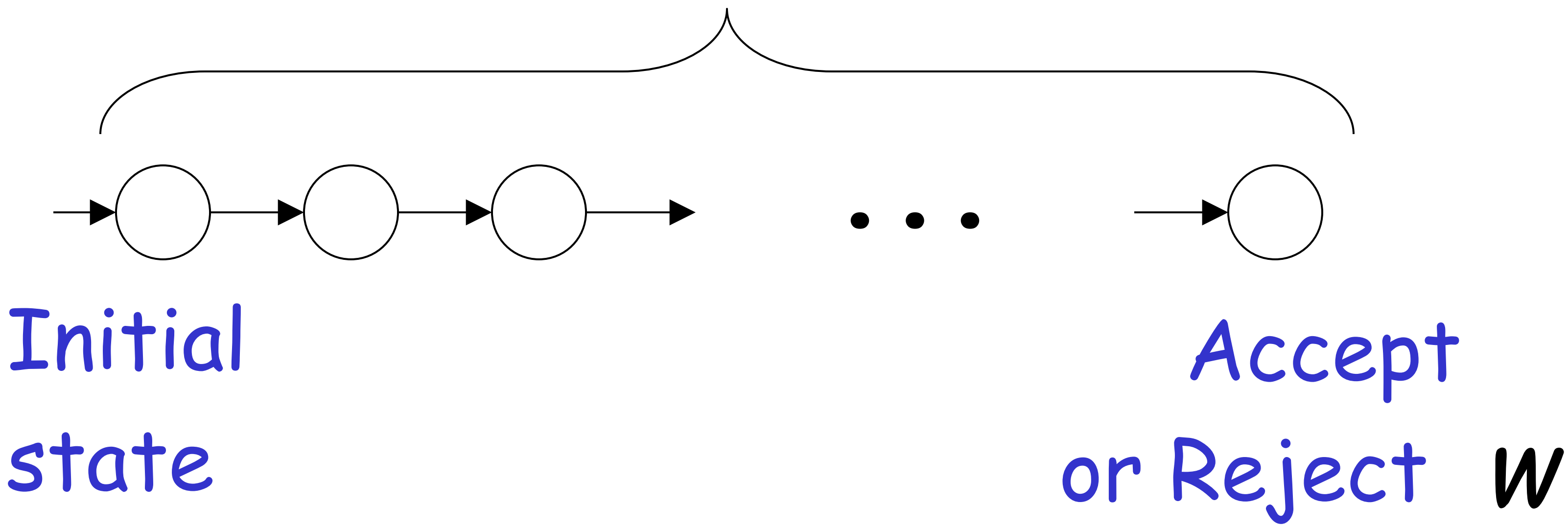
For any string w the computation of M terminates in a finite amount of transitions



Initial state

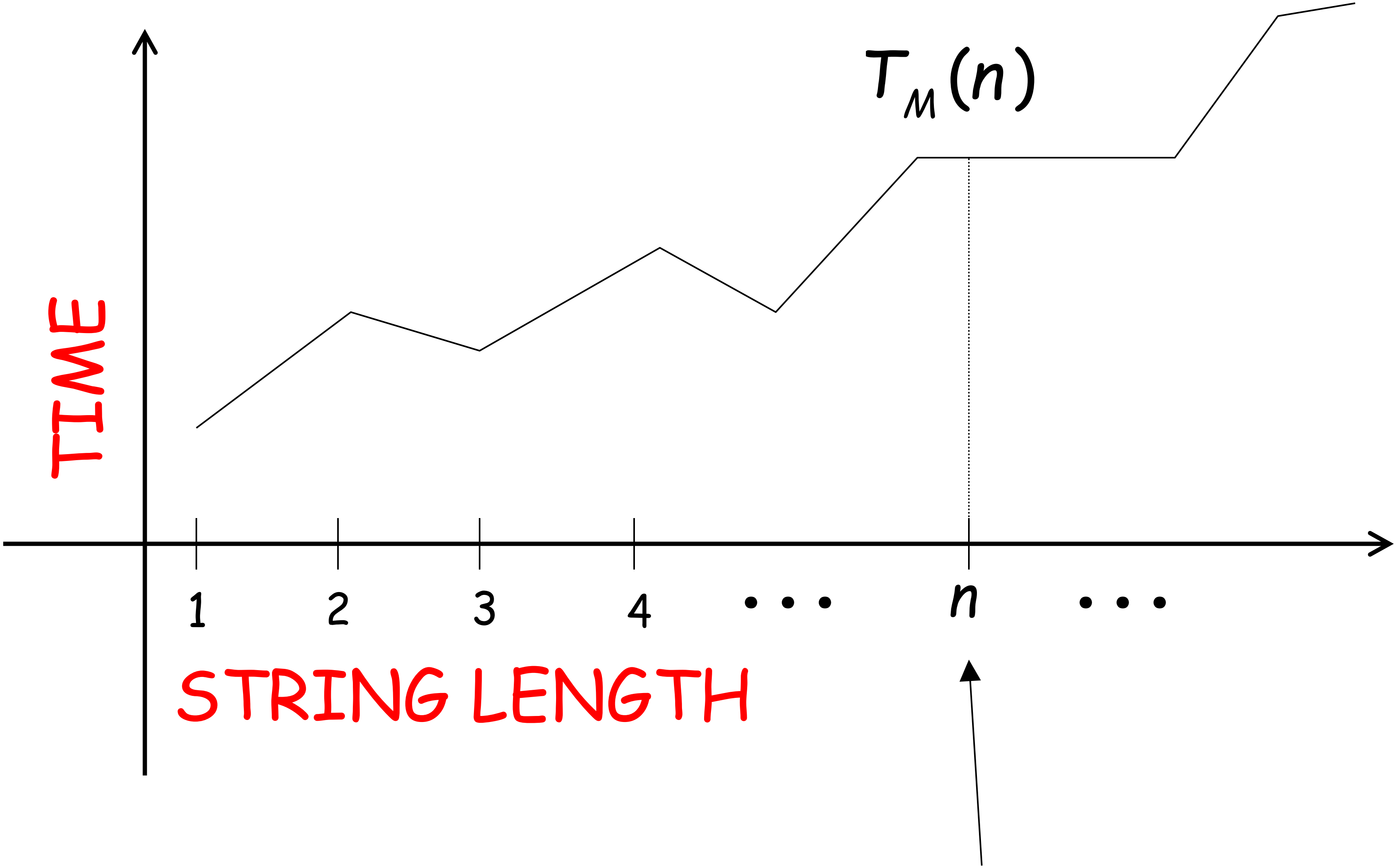
Accept or Reject w

Decision Time = #transitions



Consider now all strings of length n

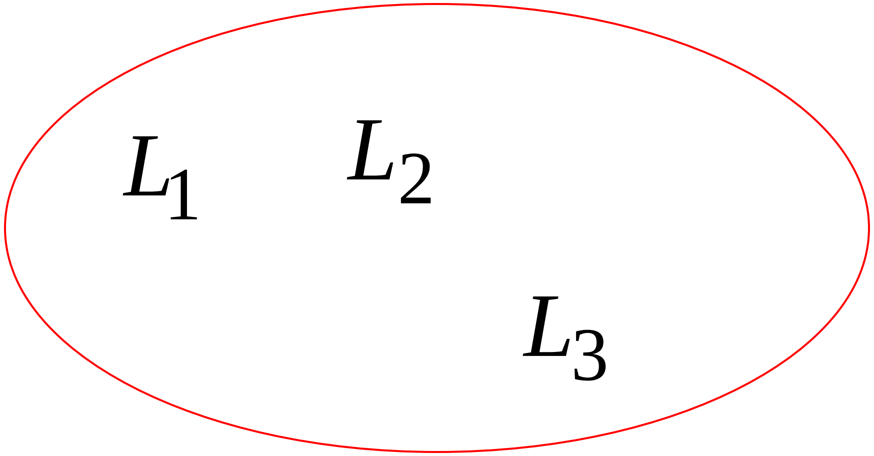
$T_M(n)$ = maximum time required to decide any string of length n



Max time to accept a string of length n

Time Complexity Class: $TIME(T(n))$

All Languages decidable by a deterministic Turing Machine in time $O(T(n))$



Example: $L_1 = \{a^n b : n \geq 0\}$

This can be decided in $O(n)$ time

$TIME(n)$

$L_1 = \{a^n b : n \geq 0\}$

Other example problems in the same class

$$TIME(n)$$

$$L_1 = \{a^n b : n \geq 0\}$$

$$\{ab^n aba : n, k \geq 0\}$$

$$\{b^n : n \text{ is even}\}$$

$$\{b^n : n = 3k\}$$

Examples in class:

$$TIME(n^2)$$

$$\{a^n b^n : n \geq 0\}$$

$$\{ww^R : w \in \{a,b\}\}$$

$$\{ww : w \in \{a,b\}\}$$

Examples in class:

$TIME(n^3)$

CYK algorithm

$L_2 = \{\langle G, w \rangle : w$ is generated by context - free grammar $G\}$

Matrix multiplication

$L_3 = \{\langle M_1, M_2, M_3 \rangle : n \times n$ matrices and $M_1 \times M_2 = M_3\}$

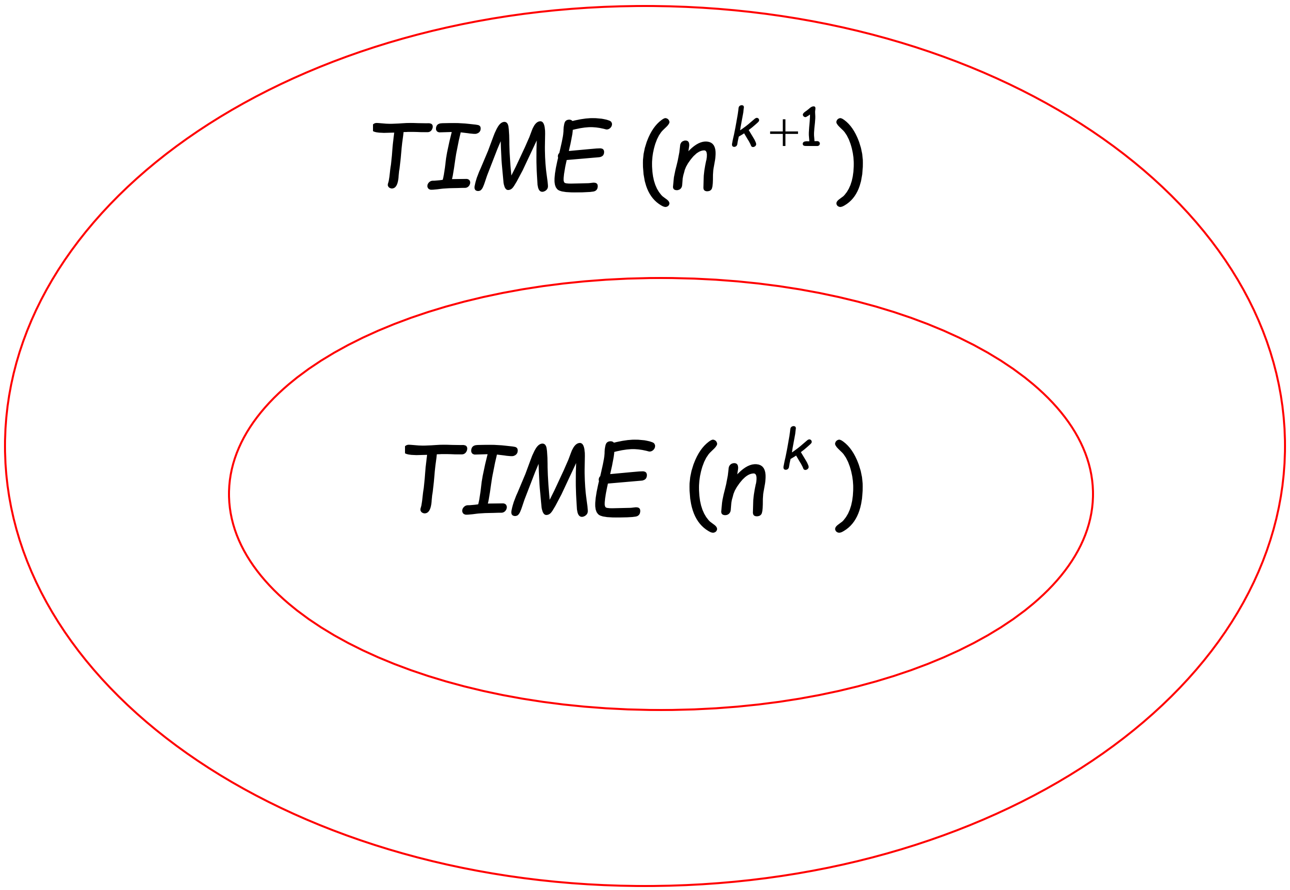
Polynomial time algorithms: $TIME(n^k)$

constant $k > 0$

Represents tractable algorithms:

for small k we can decide the result fast

It can be shown: $TIME(n^{k+1}) \subset TIME(n^{k})$



The Time Complexity Class P

$$P = \bigcup_{k>0} TIME(n^k)$$

Represents:

- polynomial time algorithms
- "tractable" problems

Class P

$\{a^n b\}$

$\{a^n b^n\}$ $\{ww\}$

CYK-algorithm

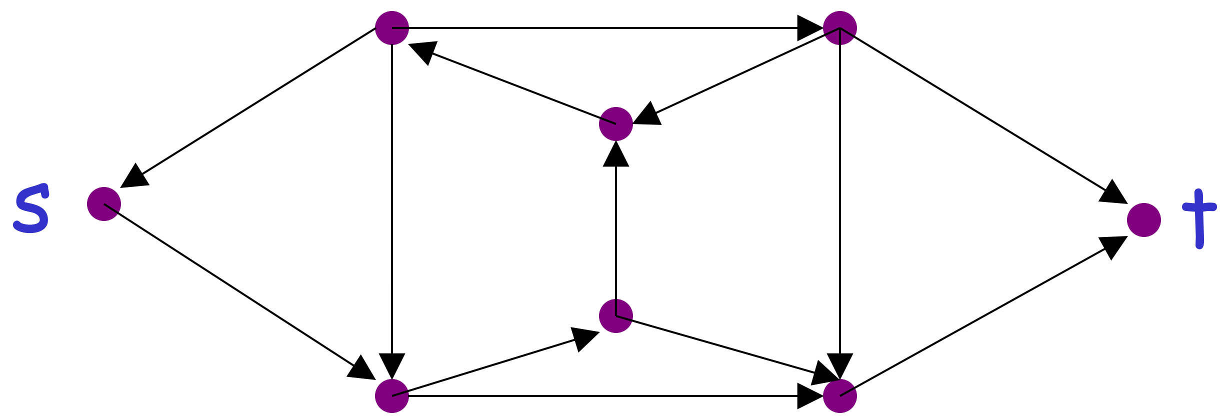
Matrix multiplication

Exponential time algorithms: $TIME(2^{n^k})$

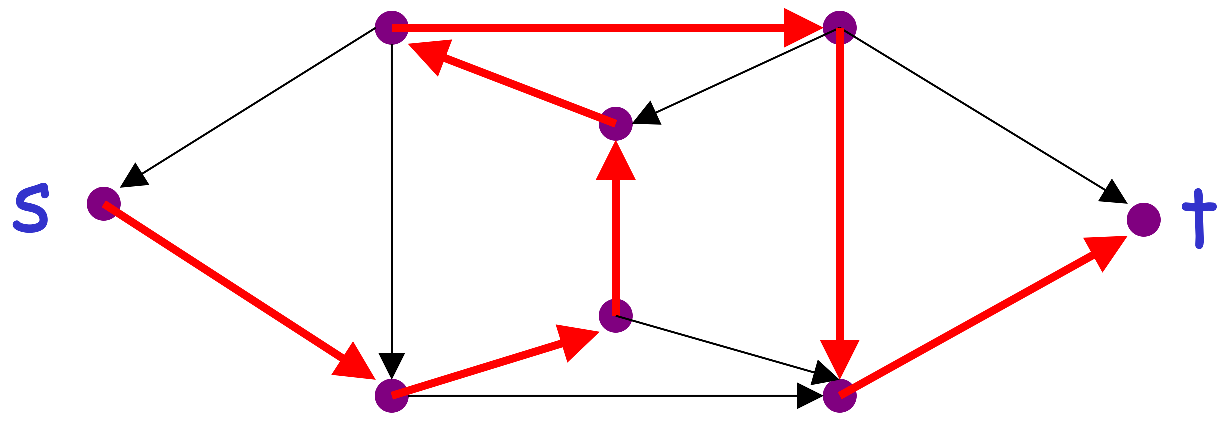
Represent intractable algorithms:

Some problem instances may take centuries to solve

Example: the Hamiltonian Path Problem



Question: is there a Hamiltonian path from s to t?



YES!

A solution: search exhaustively all paths

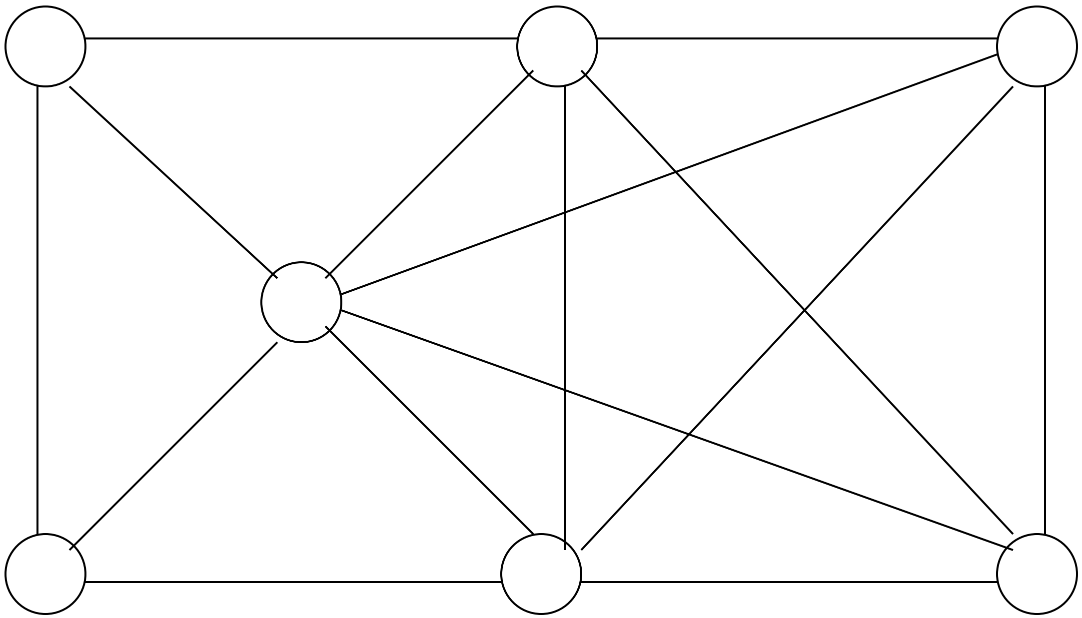
$L = \{\langle G,s,t \rangle$: there is a Hamiltonian path in G from s to t$\}$

$$L \in TIME(n!) \quad \approx \quad TIME(2^{n^k})$$

Exponential time

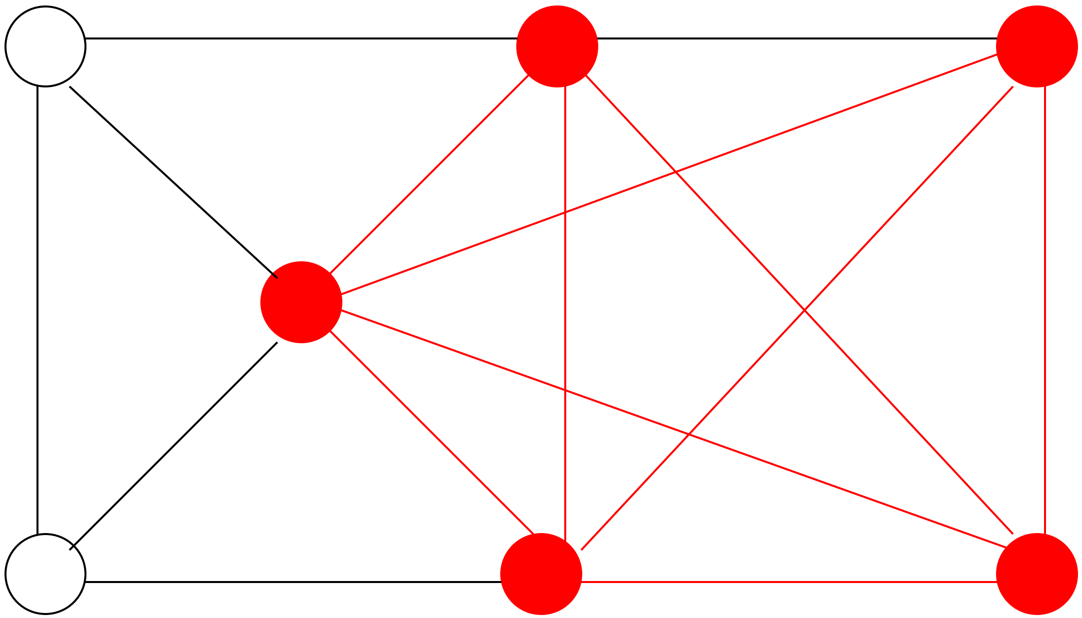
Intractable problem

The clique problem



Does there exist a clique of size 5?

The clique problem



Does there exist a clique of size 5?

Example: The Satisfiability Problem

Boolean expressions in
Conjunctive Normal Form:

$$t_1 \wedge t_2 \wedge t_3 \wedge \cdots \wedge t_k$$ clauses

$$t_i = x_1 \vee \bar{x}_2 \vee x_3 \vee \cdots \vee \bar{x}_p$$

Variables

Question: is the expression satisfiable?

Example: $(\overline{x}_1 \vee x_2) \wedge (x_1 \vee x_3)$

Satisfiable: $x_1 = 0, \; x_2 = 1, \; x_3 = 1$

$$(\overline{x}_1 \vee x_2) \wedge (x_1 \vee x_3) = 1$$

Example: $(x_1 \vee x_2) \wedge \overline{x}_1 \wedge \overline{x}_2$

Not satisfiable

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

$$L \in TIME(2^{n^k})$$

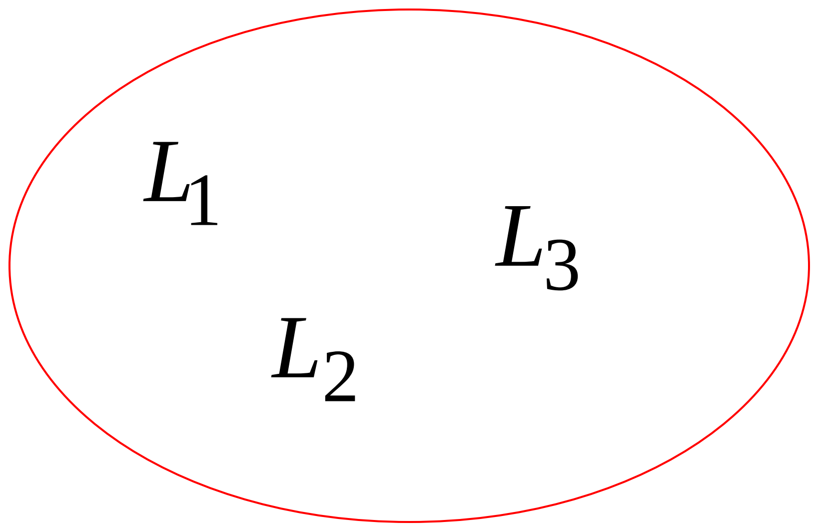
exponential

Algorithm:

search exhaustively all the possible binary values of the variables

Non-Determinism

Language class: $NTIME(T(n))$



A Non-Deterministic Turing Machine decides each string of length n in time $O(T(n))$

Non-Deterministic Polynomial time algorithms:

$$L \in NTIME(n^k)$$

The class NP

$$NP = \bigcup_{k>0} NTIME(n^k)$$

Non-Deterministic Polynomial time

Example: The satisfiability problem

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

Non-Deterministic algorithm:

- Guess an assignment of the variables

- Check if this is a satisfying assignment

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

Time for n variables:

- Guess an assignment of the variables $O(n)$

- Check if this is a satisfying assignment $O(n)$

Total time: $O(n)$

$$L = \{w : \text{expression } w \text{ is satisfiable}\}$$

$$L \in NP$$

The satisfiability problem is an NP - Problem

Observation:

$$P \subseteq NP$$

Deterministic Polynomial

Non-Deterministic Polynomial

Open Problem: $P = NP$?

WE DO NOT KNOW THE ANSWER

Open Problem: $P = NP$?

Example: Does the Satisfiability problem have a polynomial time deterministic algorithm?

WE DO NOT KNOW THE ANSWER