6.045

Lecture 5:
Minimizing DFAs,
Myhill-Nerode Theorem

DFA Minimization Theorem:

For every regular language A, there is a unique (up to re-labeling of the states) minimal-state DFA M* such that A = L(M*).

Furthermore, there is an *efficient algorithm* which, given any DFA M, will output this unique M*.

Extending transition function δ to strings

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$, we extend δ to a function $\Delta : Q \times \Sigma^* \rightarrow Q$ as follows:

$$\Delta(q, \varepsilon) = q$$

$$\Delta(q, \sigma) = \delta(q, \sigma)$$

$$\Delta(q, \sigma_1 \ldots \sigma_{k+1}) = \delta(\Delta(q, \sigma_1 \ldots \sigma_k), \sigma_{k+1})$$

$\Delta(q, w)$ = *the state of M reached after reading in w, starting from state q*

Note: $\Delta(q_0, w) \in F \Leftrightarrow$ M accepts w

Def. $w \in \Sigma^*$ *distinguishes* states q_1 and q_2 iff *exactly one* of $\Delta(q_1, w)$, $\Delta(q_2, w)$ is a final state

Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q \in Q$

Definition:

State p is *distinguishable* from state q
iff there is $w \in \Sigma^*$ that distinguishes p and q
iff there is $w \in \Sigma^*$ so that
exactly *one* of $\Delta(p, w)$, $\Delta(q, w)$ is a final state

State p is *indistinguishable* from state q
iff p is not distinguishable from q
iff for all $w \in \Sigma^*$, $\Delta(p, w) \in F \Leftrightarrow \Delta(q, w) \in F$

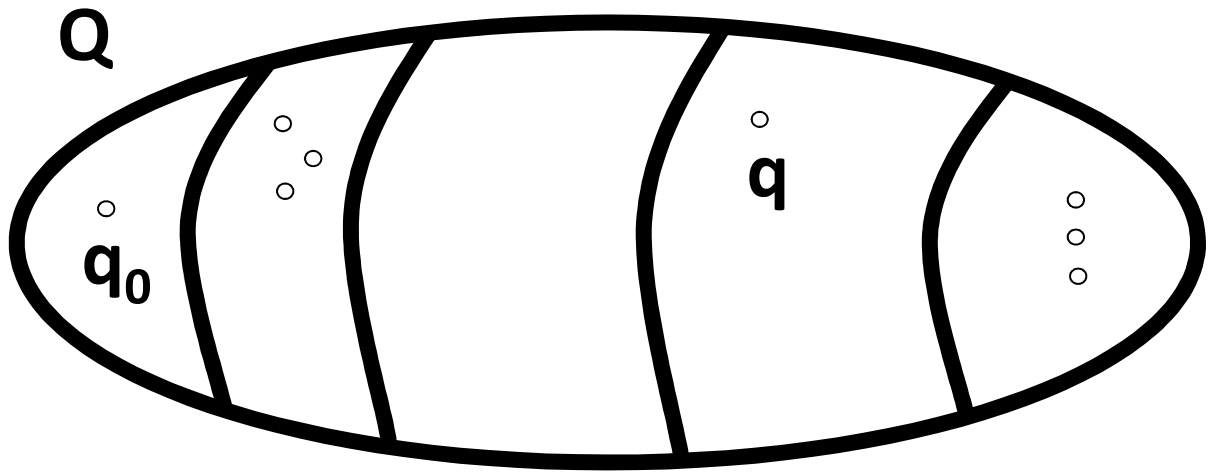
(EITHER both $\Delta(p, w)$, $\Delta(q, w)$ are in F, OR both are not in F)

Pairs of indistinguishable states are redundant…

Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

$p \sim q$ iff p is indistinguishable from q

Proposition: $\sim$ is an equivalence relation



States of M_{MIN} = *Equivalence classes* of states of M

Algorithm: MINIMIZE-DFA

Input: DFA M

Output: DFA M_{MIN} such that:

$$L(M) = L(M_{MIN})$$

M_{MIN} has no *inaccessible* states

M_{MIN} is *irreducible*

||

For all states $p \neq q$ of M_{MIN}, p and q are distinguishable

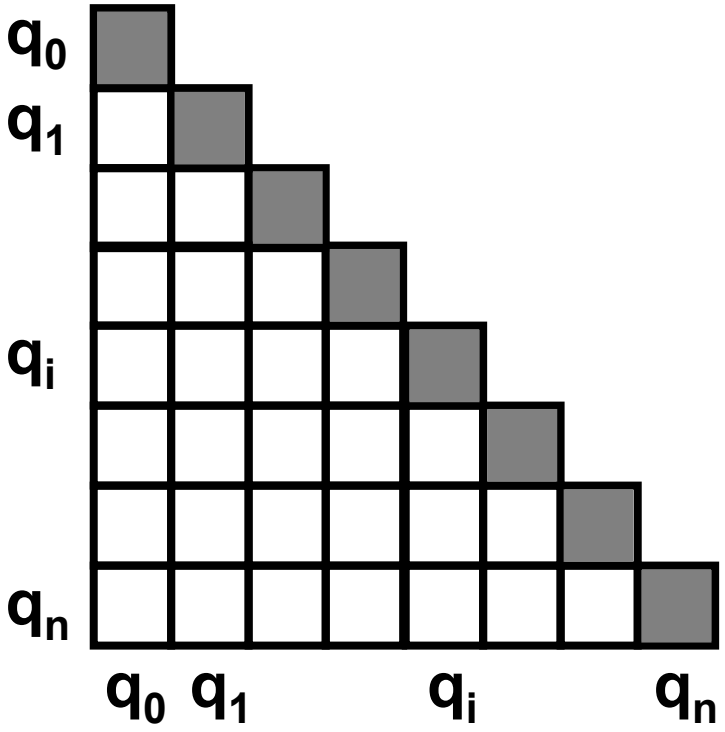
Theorem: M_{MIN} is the unique minimal DFA that is equivalent to M

The Table-Filling Algorithm

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p, q) \mid p, q \in Q \text{ and } p \nsim q \}$

(2) $EQUIV_M = \{ [q] \mid q \in Q \}$

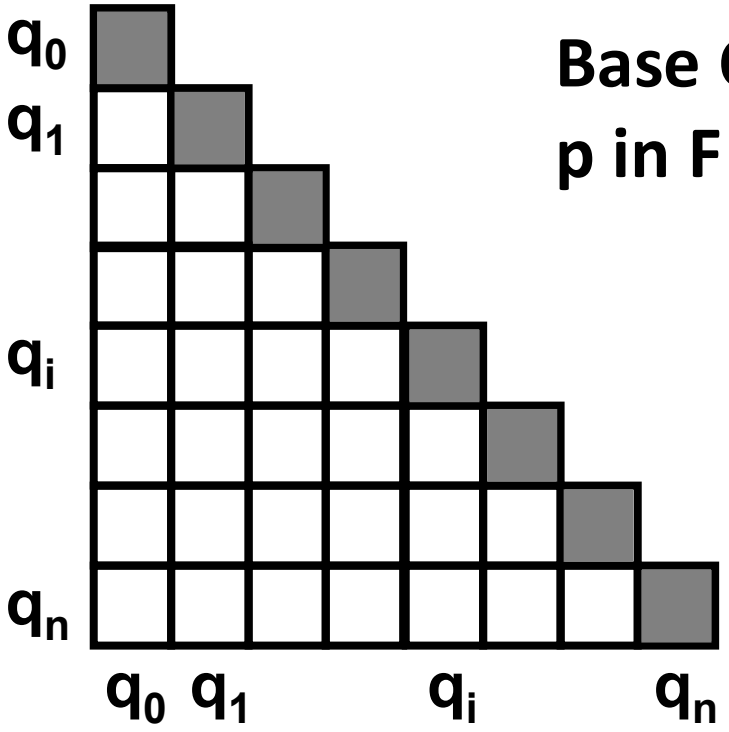


The Table-Filling Algorithm

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p, q) \mid p, q \in Q \text{ and } p \not\sim q \}$

(2) $EQUIV_M = \{ [q] \mid q \in Q \}$



Base Case: For all (p, q) such that p in F and q not in F $\Rightarrow$ mark $p \not\sim q$

The Table-Filling Algorithm

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p, q) \mid p, q \in Q \text{ and } p \not\sim q \}$

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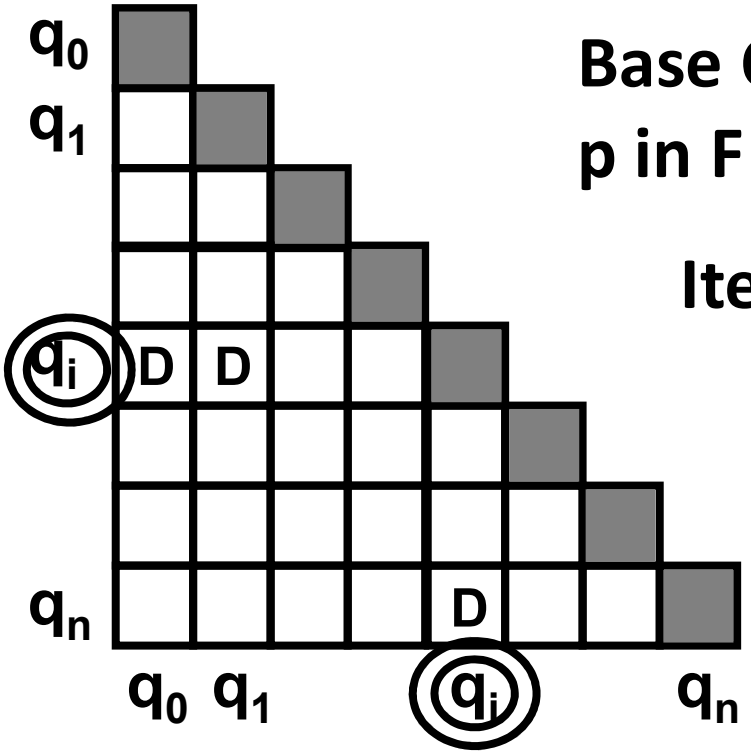
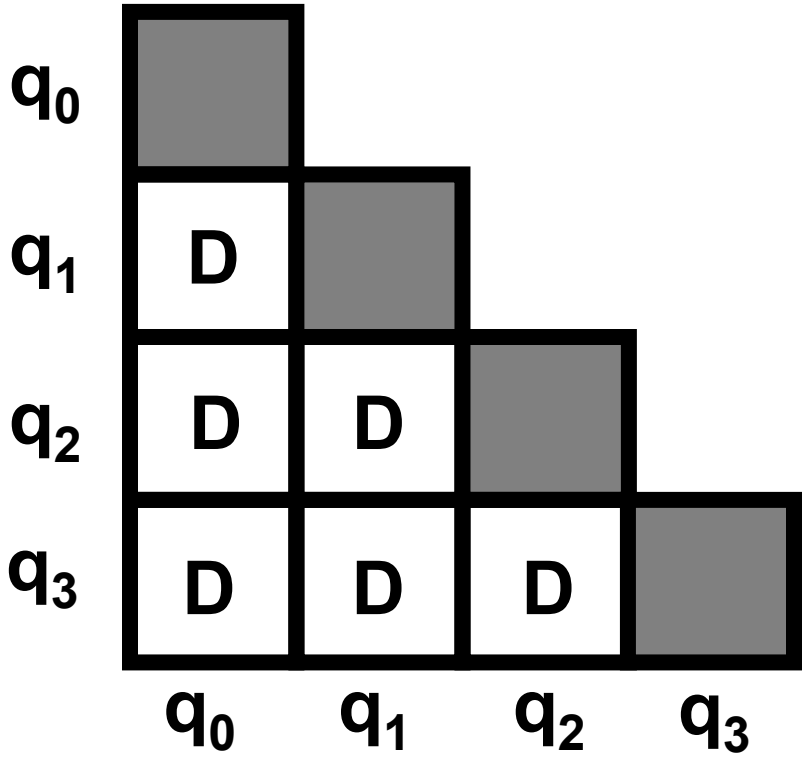


Base Case: For all (p, q) such that p in F and q not in F $\Rightarrow$ mark $p \not\sim q$

Iterate: If there are states p, q and symbol $\sigma \in \Sigma$ satisfying:

$$\delta(p, \sigma) = p' \qquad \not\sim \Rightarrow \text{mark } p \not\sim q$$
$$\delta(q, \sigma) = q'$$

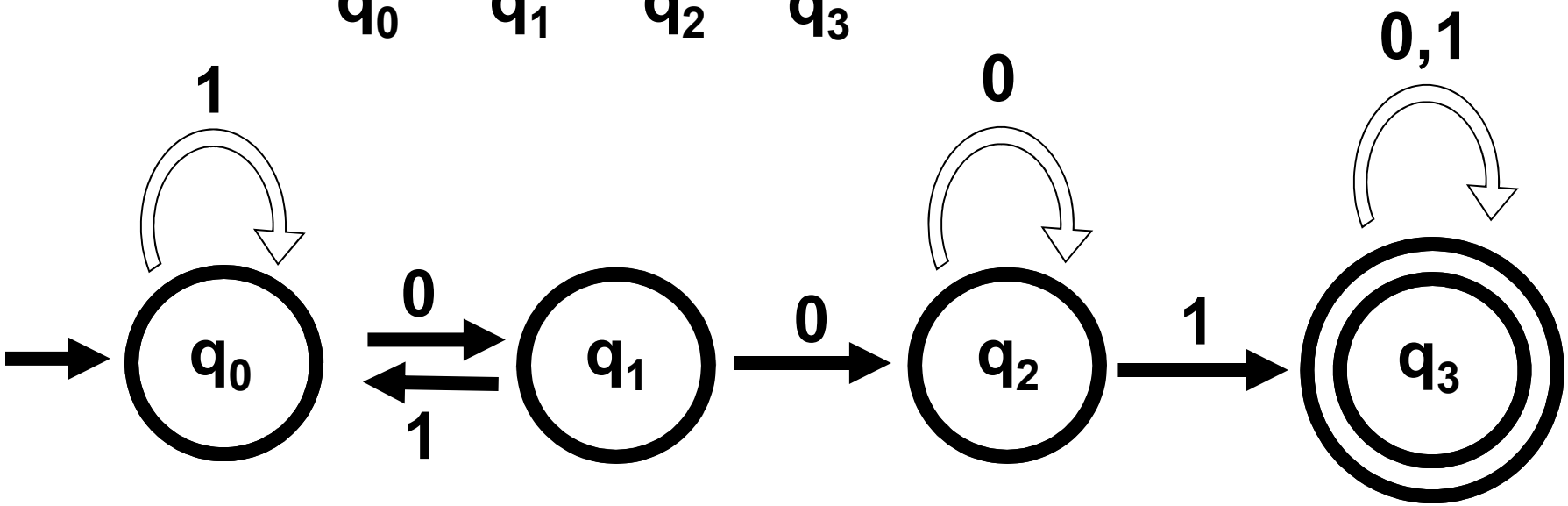
Repeat until no more D's can be added

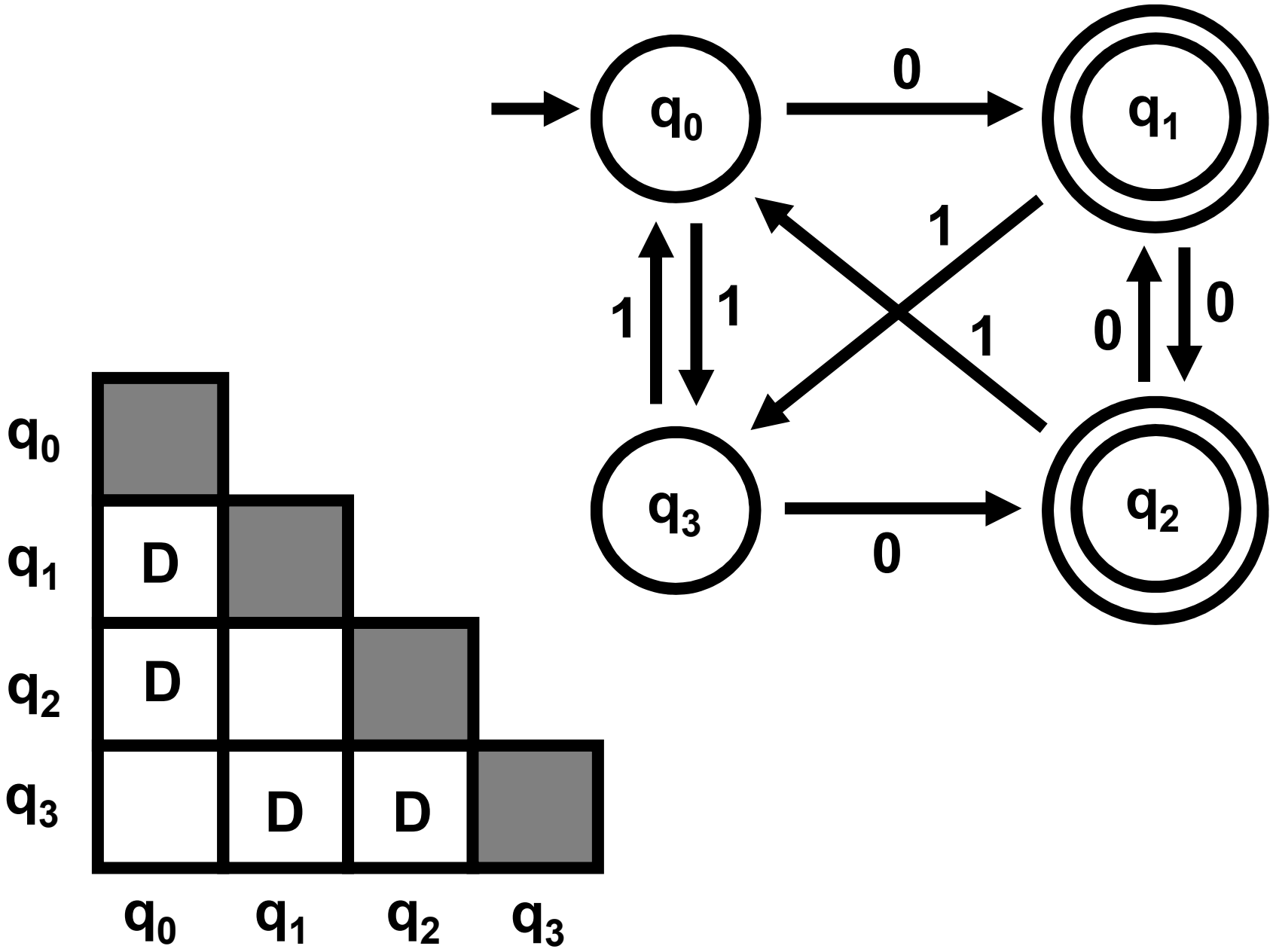


Can we mark (q_1,q_2) as distinguishable?

Are q_0 and q_1 distinguishable?

Are q_0 and q_2 distinguishable?

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Claim: If (p, q) is marked D by the Table-Filling algorithm, then $p \not\sim q$

Proof: By induction on the number of iterations in the algorithm before (p, q) is marked D

If (p, q) is marked D in the base case, then one state's in F and the other isn't, so ε distinguishes p and q

Suppose (p, q) is marked D in a later iteration.

Then there are states p′, q′ such that:

1. (p′, q′) is marked D $\Rightarrow$ $p' \not\sim q'$ (by induction)

So there's a string w s.t. $\Delta(p', w) \in F \Leftrightarrow \Delta(q', w) \notin F$

2. $p' = \delta(p,\sigma)$ and $q' = \delta(q,\sigma)$, for some $\sigma \in \Sigma$

Then the string σw distinguishes p and q!

Claim: If (p, q) is not marked D by the Table-Filling algorithm, then $p \sim q$

Proof (by contradiction):

Suppose the pair (p, q) is not marked D by the algorithm, yet $p \not\sim q$ (call this a "bad pair")

Then there is a string w such that $|w| > 0$ and:

$$\Delta(p, w) \in F \Leftrightarrow \Delta(q, w) \notin F$$ **(Why is $|w| > 0$?)**

Of all such bad pairs, let (p, q) be a bad pair with a *minimum-length* distinguishing string w

Claim: If (p, q) is not marked D by the Table-Filling algorithm, then $p \sim q$

Proof (by contradiction):

Suppose the pair (p, q) is not marked D by the algorithm, yet $p \not\sim q$ (call this a "bad pair")

Of all such bad pairs, let (p, q) be a bad pair with a *minimum-length* distinguishing string w

$$\Delta(p, w) \in F \Leftrightarrow \Delta(q, w) \notin F$$

We have $w = \sigma w'$, for some string w' and some $\sigma \in \Sigma$

Let $p' = \delta(p,\sigma)$ and $q' = \delta(q,\sigma)$

Then (p', q') is also a bad pair!

But then (p', q') has a SHORTER distinguishing string, w'

Contradiction!

Algorithm MINIMIZE

Input: DFA M

Output: Equivalent minimal-state DFA M_{MIN}

1. **Remove all inaccessible states from M**

2. **Run Table-Filling algorithm on M to get:**
   $EQUIV_M = \{ [q] \mid q$ is an accessible state of M $\}$

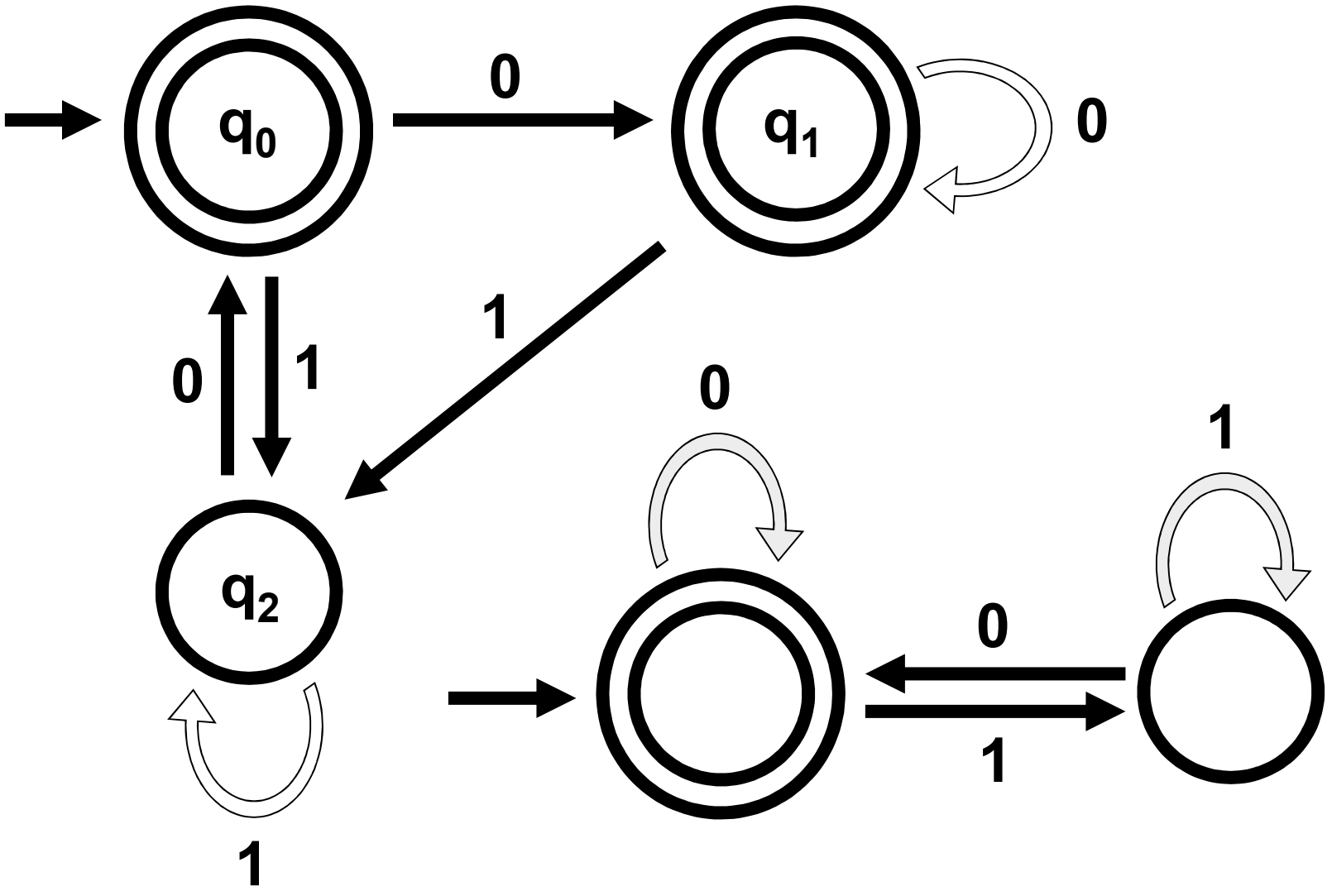
3. **Define: $M_{MIN} = (Q_{MIN}, \Sigma, \delta_{MIN}, q_{0\,MIN}, F_{MIN})$**

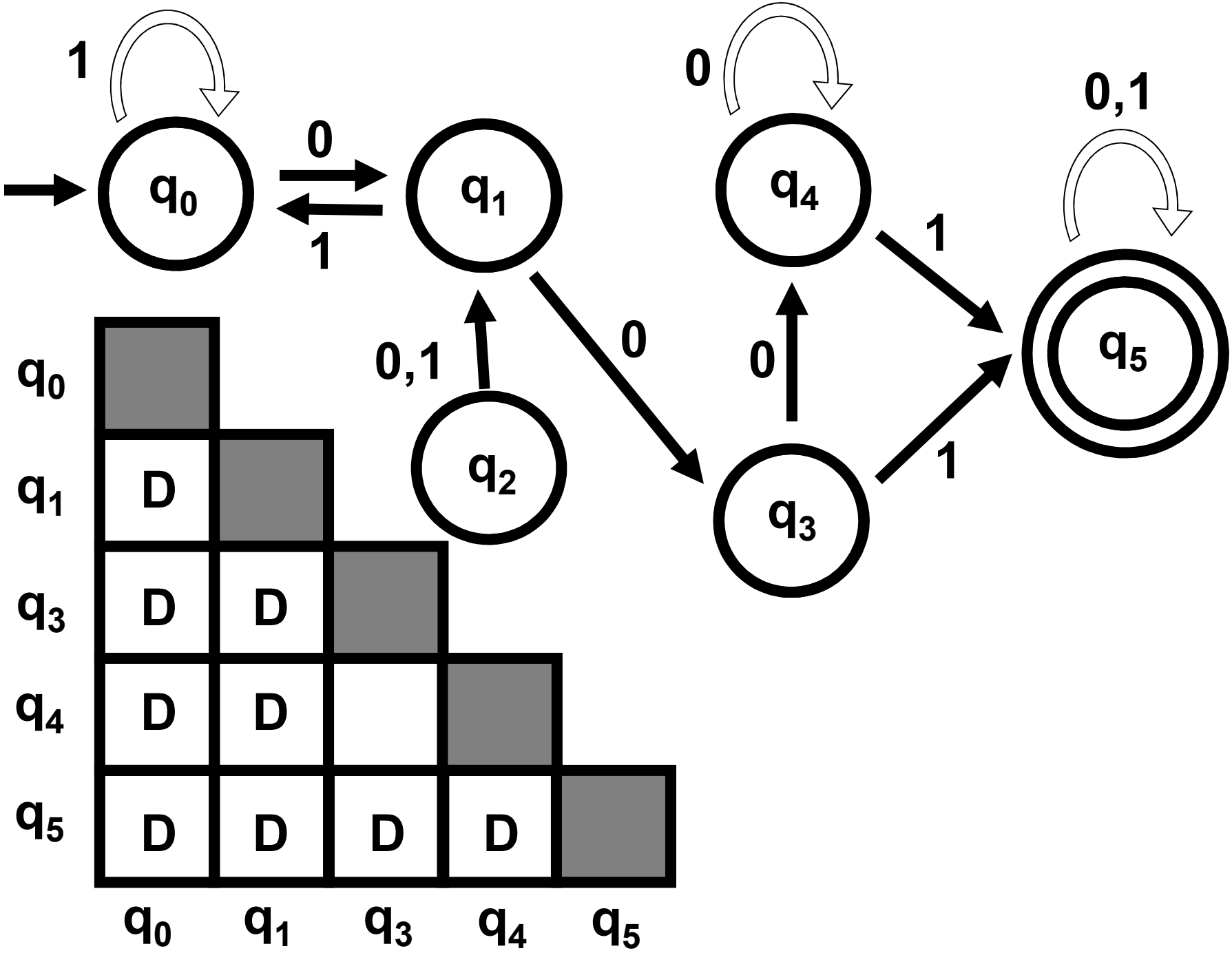
$$Q_{MIN} = EQUIV_M,\ \ q_{0\,MIN} = [q_0],\ \ F_{MIN} = \{ [q] \mid q \in F \}$$

$$\delta_{MIN}([q], \sigma) = [\delta(q, \sigma)]$$

Claim: $L(M_{MIN}) = L(M)$

MINIMIZE

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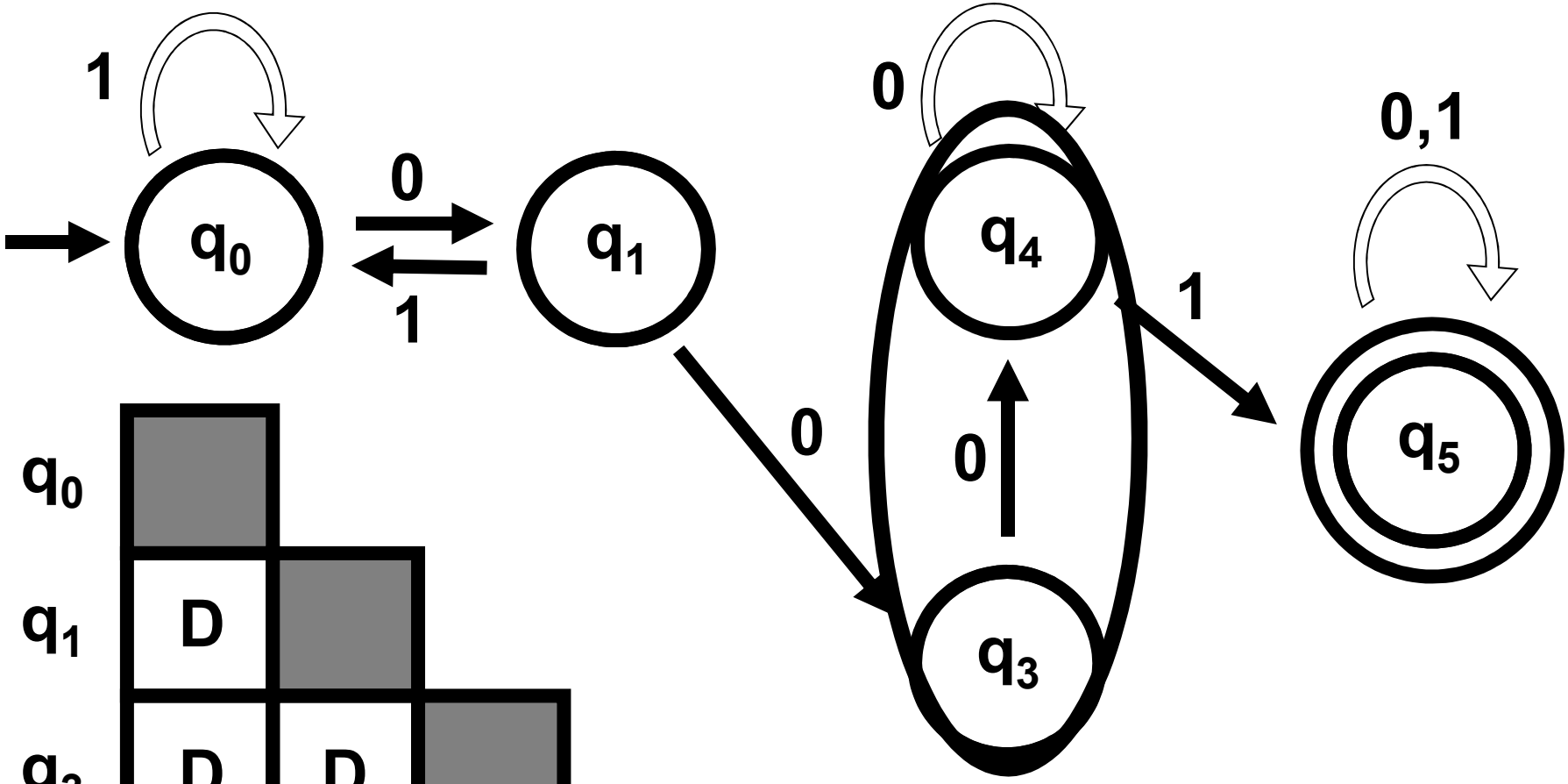


q_0					
q_1	D				
q_3	D	D			
q_4	D	D			
q_5	D	D	D	D	
	q_0	q_1	q_3	q_4	q_5

Thm: M_{MIN} is the unique minimal DFA equivalent to M

Claim: Suppose $L(M') = L(M_{MIN})$ and M' has no inaccessible states and M' is irreducible.
Then there is an *isomorphism* between M' and M_{MIN}

If M' is a minimal DFA, then M' has no inaccessible states and is irreducible. So the Claim implies:

If M' is a minimal DFA for M, then there is an isomorphism between M' and M_{MIN}. So the Thm holds!

Corollary: If M has no inaccessible states and is irreducible, then M is minimal.

Proof: Let M^{min} be minimal for M. Then $L(M) = L(M^{min})$, no inaccessible states in M, and M is irreducible. By Claim, both M^{min} and M are isomorphic to M_{MIN}!

Thm: M_{MIN} is the *unique* minimal DFA equivalent to M

Claim: Let M′ be a DFA where L(M′)=L(M_{MIN}) and M′ has no inaccessible states and M′ is irreducible. Then there is an *isomorphism* between M′ and M_{MIN}

Suppose we have proved the Claim is true.
Assuming the Claim we can prove the Thm:

Proof of Thm: Let M′ be any minimal DFA for M.
Since M′ is minimal, M′ has no inaccessible states and is irreducible *(why?)*
By the Claim, there is an isomorphism between M' and the DFA M_{MIN} that is output by MINIMIZE(M).
That is, M_{MIN} is isomorphic to every minimal M'.

Thm: M_{MIN} is the *unique* minimal DFA equivalent to M

Claim: Let M′ be a DFA where L(M′)=L(M_{MIN}) and M′ has no inaccessible states and M′ is irreducible. Then there is an *isomorphism* between M′ and M_{MIN}

Proof: We recursively construct a map from the states of M_{MIN} to the states of M′

Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: **If $p \mapsto p'$**

$$\begin{array}{cc} \downarrow\sigma & \downarrow\sigma \\ q & q' \end{array}$$

Then $q \mapsto q'$

Base Case: $q_{0\ MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$

σ σ Then $q \mapsto q'$

q q'

Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$

$p \xrightarrow{\sigma} q$ and $p' \xrightarrow{\sigma} q'$

Then $q \mapsto q'$

Claim: Map is an isomorphism. Need to prove:

- **The map is defined everywhere**
- **The map is well defined**
- **The map is a bijection**
- **The map preserves all transitions:**
  If $p \mapsto p'$ then $\delta_{MIN}(p, \sigma) \mapsto \delta'(p', \sigma)$
  (this follows from the definition of the map!)

Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$

σ σ

q q'

Then $q \mapsto q'$

The map is defined everywhere

That is, for all states q of M_{MIN} there is a state q' of M' such that $q \mapsto q'$

If $q \in M_{MIN}$, there is a string w such that

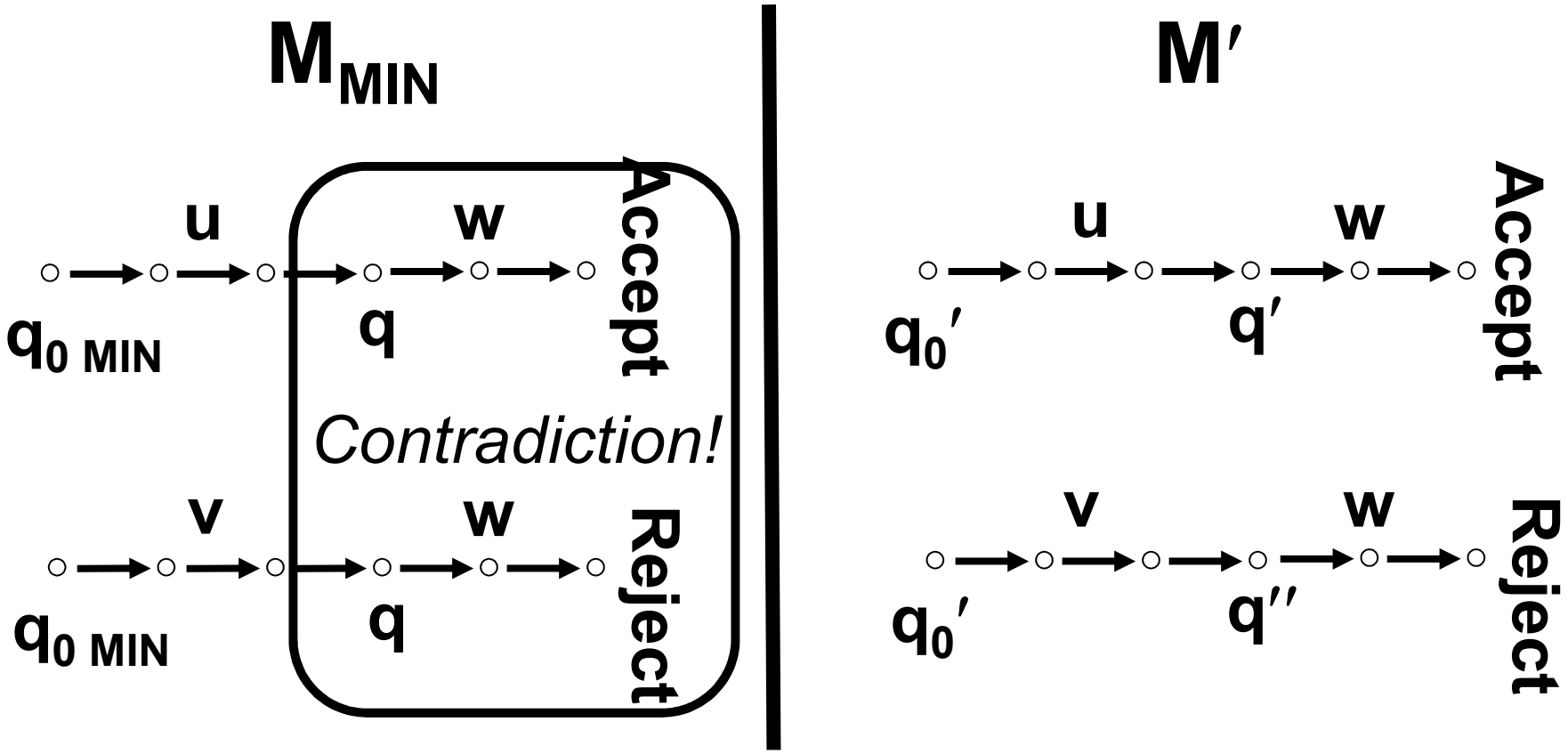
$$\Delta_{MIN}(q_{0\,MIN}, w) = q$$

Let $q' = \Delta'(q_0', w)$. Then we claim $q \mapsto q'$

(prove by induction on $|w|$)

Suppose there are states q' and q'' such that $q \mapsto q'$ and $q \mapsto q''$

Suppose q' and q'' are distinguishable



Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$

σ σ Then $q \mapsto q'$

q q'

The map is well defined

Proof by contradiction.

Suppose there are states q' and q'' such that $q \mapsto q'$ and $q \mapsto q''$

We show that q' and q'' are *indistinguishable*, so it must be that $q' = q''$ *(why?)*

Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$

$p \xrightarrow{\sigma} q$ and $p' \xrightarrow{\sigma} q'$ Then $q \mapsto q'$

The map is onto

Want to show: For all states q' of M' there is a state q of M_{MIN} such that $q \mapsto q'$

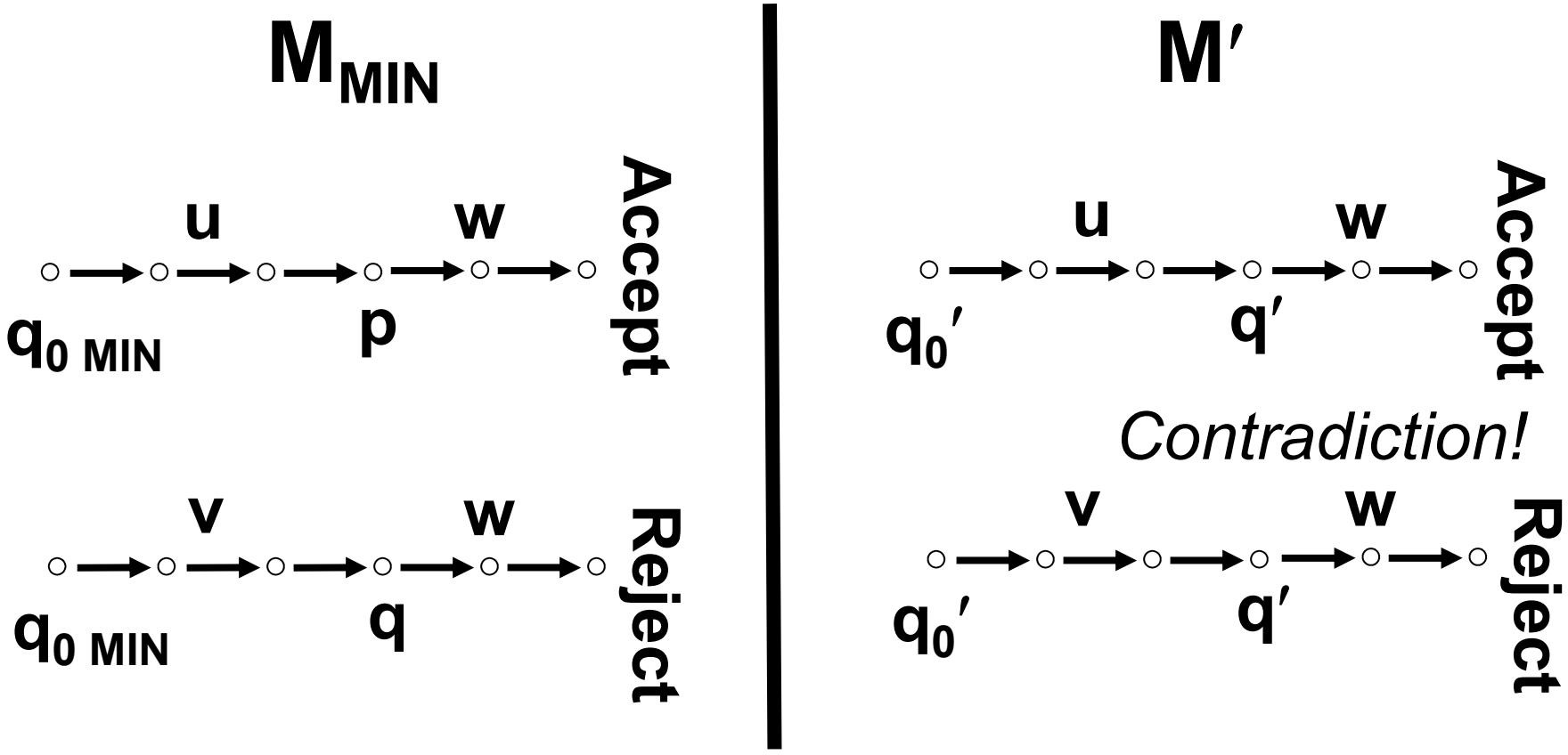
For every q' there is a string w such that M' reaches state q' after reading in w

Let q be the state of M_{MIN} after reading in w
Claim: $q \mapsto q'$ ***(prove by induction on $|w|$)***

The map is one-to-one

Proof by contradiction. **Suppose there are states $p \neq q$ such that $p \mapsto q'$ and $q \mapsto q'$**

If $p \neq q$, then p and q are distinguishable



How can we prove that two regular expressions are equivalent?

The Myhill-Nerode Theorem

In DFA Minimization, we defined an equivalence relation between states of a DFA.

We can also define a similar equivalence relation over *strings* for a *language*:

$$\text{Let } L \subseteq \Sigma^* \text{ and } x, y \in \Sigma^*$$

$$x \equiv_L y \quad \text{iff} \quad \text{for all } z \in \Sigma^*, xz \in L \Leftrightarrow yz \in L$$

Define: x and y are indistinguishable to L iff $x \equiv_L y$

Claim: $\equiv_L$ is an equivalence relation

Proof? Same as before!

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ iff for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

The Myhill-Nerode Theorem:
A language L is regular *if and only if* the number of equivalence classes of $\equiv_L$ is *finite*.

Proof ($\Rightarrow$) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for L.

Define the relation: $x \sim_M y \Leftrightarrow \Delta(q_0,x) = \Delta(q_0,y)$

Claim: $\sim_M$ is an equivalence relation with $|Q|$ classes

Claim: If $x \sim_M y$ then $x \equiv_L y$

Proof: $x \sim_M y$ implies for all $z \in \Sigma^*$, xz and yz reach the *same state* of M. So $xz \in L \Leftrightarrow yz \in L$, and $x \equiv_L y$

Corollary: Number of equiv. classes of $\equiv_L$ is *at most* the number of equiv. classes of $\sim_M$ (which is $|Q|$)

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ iff for all $z \in \Sigma^*$, $xz \in L \Leftrightarrow yz \in L$

($\Leftarrow$) If the number of equivalence classes of $\equiv_L$ is *k* then there is a DFA for L with *k* states

Idea: Build a DFA whose *states* are the *equiv classes* of $\equiv_L$

Define a DFA M where:

Q is the set of equivalence classes of $\equiv_L$

$q_0 = [\varepsilon] = \{y \mid y \equiv_L \varepsilon\}$

$\delta([x], \sigma) = [x\,\sigma]$

$F = \{[x] \mid x \in L\}$

Claim: M accepts x if and only if $x \in L$

The Myhill-Nerode Theorem gives us a *new* way to prove that a given language is not regular:

L is not regular

if and only if

there are infinitely many equiv. classes of $\equiv_L$

L is not regular

if and only if

Distinguishing set for L

There are infinitely many strings w_1, w_2, … so that for all $w_i \neq w_j$, w_i and w_j are distinguishable to L:

there is a $z \in \Sigma^*$ such that

***exactly one* of $w_i z$ and $w_j z$ is in L**

The Myhill-Nerode Theorem gives us a *new* way to prove that a given language is not regular:

Theorem: $L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Proof: Consider the infinite set of strings

$$S = \{0, 00, 000, \ldots, 0^n, \ldots\}$$

Take any pair $(0^m, 0^n)$ of distinct strings in S

Let $z = 1^m$

Then $0^m 1^m$ is in L, but $0^n 1^m$ is *not* in L

So all pairs of strings in S are distinguishable to L

Hence there are infinitely many equivalence classes of $\equiv_L$, and L is not regular.