Homework 8 **October 28, 2005** **Math 521**

Direction: **This homework is due on November 4, 2005. In order to receive full credit, answer each problem completely and must show all work.**

---

1. Show that the group $G \oplus H$ is abelian if and only if the groups G and H are abelian.

Answer: Suppose the groups G and H are abelian. We want to show that $G \oplus H$ is abelian. Since

$$\begin{aligned}(g_1,\, h_1) \star (g_2,\, h_2) &= (g_1 g_2,\ h_1 h_2) \\ &= (g_2 g_1,\ h_2 h_1) \qquad \text{(because G and H abelian)} \\ &= (g_2,\, h_2) \star (g_1,\, h_1),\end{aligned}$$

therefore $G \oplus H$ is abelian. Next, we prove the converse. Since

$$\begin{aligned}(g_1 g_2,\ h_1 h_2) &= (g_1,\, h_1) \star (g_2,\, h_2) \\ &= (g_2,\, h_2) \star (g_1,\, h_1) \qquad \text{(because G} \oplus \text{H is abelian)} \\ &= (g_2 g_1,\ h_2 h_1),\end{aligned}$$

therefore $g_1 g_2 = g_2 g_1$ and $h_1 h_2 = h_2 h_1$, and G and H are abelian groups.

2. If $G =< g >$ is a cyclic group of order n, then prove that g^k is a generator of G if and only if $gcd(k, n) = 1$.

Answer: Let $G =< g >$ be a cyclic group of order n. Then $|g| = n$. We know from Corollary 1 (Gallian, page 77) that $< g^i >=< g^j >$ if and only if $gcd(n, i) = gcd(n, j)$. Letting $i = k$ and $j = 1$, we get $< g^k > = < g > = G$ if and only if $gcd(n, k) = gcd(n, 1) = 1$.

3. Prove that the group of complex numbers $\mathbb{C}$ under addition is isomorphic to the group $\mathbb{R} \oplus \mathbb{R}$.

Answer: Define a mapping $\phi : \mathbb{R} \oplus \mathbb{R} \to \mathbb{C}$ by $\phi(x, y) = x + iy$, where i is the imaginary unit. Then it is easy to show that ϕ is well-defined, one-to-one, and onto. Further

$$\begin{aligned}\phi(x_1 + x_2,\, y_1 + y_2) &= (x_1 + x_2) + i(y_1 + y_2) \\ &= x_1 + iy_1 + x_2 + iy_2 \\ &= \phi(x_1,\, y_1) + \phi(x_2,\, y_2).\end{aligned}$$

Hence ϕ is an isomorphism and $\mathbb{R} \oplus \mathbb{R} \simeq \mathbb{C}$.

4. Show, by comparing orders of elements, that the group $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ is not isomorphic to the group $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

Answer: The order of the element $(1,1)$ in $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ is $|(1,1)| = lcm(8,2) = 8$. Similarly, the order of the element $(1,1)$ in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ is $|(1,1)| = lcm(4,4) = 4$. There is no element in $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ with order bigger than 4. Thus $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$

5. How many elements of order 9 does the group $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ have?

Answer: We are given that

$$9 = lcm(|a|,\ |b|)$$

where $a \in \mathbb{Z}_3$ and $b \in \mathbb{Z}_9$. Therefore we have the following table

$\lvert a\rvert$	$\lvert b\rvert$	Number of elements of order 9
1	9	$\phi(1)\phi(9) = (1)(6) = \ 6$
3	9	$\phi(3)\phi(9) = (2)(6) = 12$
	Total	$6 + 12 = 18$

Hence there are 18 elements in $\mathbb{Z}_3 \oplus \mathbb{Z}_9$ of order 9.

6. How many subgroups of order 4 does the group $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ have?

Answer: First compute the number of elements of order 4 in $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. We are given that

$$4 = lcm(|a|,\ |b|)$$

where $a \in \mathbb{Z}_4$ and $b \in \mathbb{Z}_2$. Therefore we have the following table

$\lvert a\rvert$	$\lvert b\rvert$	Number of elements of order 9
4	1	$\phi(4)\phi(1) = (2)(1) = 2$
4	2	$\phi(4)\phi(2) = (2)(1) = 2$
	Total	$2 + 2 = 4$

Hence there are 4 elements in $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ of order 4. Each cyclic subgroup of order 4 has 2 elements of order 4. Since no two cyclic subgroup can have an element of order 4 in common, therefore there are $\frac{4}{2} = 2$ cyclic subgroups of order 4. (Since the oder of this group is 8, this can also be done by directly computing the subgroups generated by each of the eight element.)

By inspection we see that $\{(0,0),\ (2,0),\ (0,1),\ (2,1)\}$ is a subgroup of $\mathbb{Z}_4 \oplus \mathbb{Z}_2$ and this subgroup is not cyclic. Hence one can find only 3 subgroups of order 4 in $\mathbb{Z}_4 \oplus \mathbb{Z}_2$.

7. What is the order of any nonidentity element of the group $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$?

Answer: The group $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ has order 27. Since 1, 3, 9 divide 27, the group $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ may have elements of order 1, 3, or 9. It can be shown (you should try to show!) that the group $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ has no element of order 9. The group $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ has one element, namely $(0,0,0)$ of order 1. Thus the order of all nonidentity elements must be 3.

8. How many elements of order 5 does the group $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ have?

Answer: We are given that

$$5 = lcm(|a|,\ |b|)$$

where $a \in \mathbb{Z}_5$ and $b \in \mathbb{Z}_{25}$. Therefore we have the following table

$\lvert a\rvert$	$\lvert b\rvert$	Number of elements of order 9
1	5	$\phi(1)\phi(5) = (1)(4) = 4$
5	1	$\phi(5)\phi(1) = (4)(1) = 4$
5	5	$\phi(5)\phi(5) = (4)(4) = 16$
	Total	$4 + 4 + 16 = 24$

Hence there are 24 elements in $\mathbb{Z}_5 \oplus \mathbb{Z}_{25}$ of order 5.

9. How many cyclic subgroups of order 10 does the group $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$ have?

Answer: First compute the number of elements of order 10 in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$. We are given that

$$10 = lcm(|a|,\ |b|)$$

where $a \in \mathbb{Z}_{100}$ and $b \in \mathbb{Z}_{25}$. Therefore we have the following table

$\lvert a\rvert$	$\lvert b\rvert$	Number of elements of order 9
10	1	$\phi(10)\phi(1) = (4)(1) = 4$
10	5	$\phi(10)\phi(5) = (4)(4) = 16$
2	5	$\phi(2)\phi(5) = (1)(4) = 4$
	Total	$4 + 16 + 4 = 24$

Hence there are 24 elements in $\mathbb{Z}_{100} \oplus \mathbb{Z}_{25}$ of order 10. Because $\phi(10) = 4$, each cyclic subgroup of order 10 has 4 elements of order 10. Since no two cyclic subgroup can have an element of order 10 in common, therefore there are $\frac{24}{4} = 6$ cyclic subgroups of order 10.

10. How many elements of the group $Aut\,(\mathbb{Z}_{720})$ have order 6?

Answer: Since $Aut(\mathbb{Z}_{720}) \simeq U(720)$, we will find the number of elements of order 6 in $U(720)$. Using Gauss's result, we see that

$$U(720) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4.$$

Therefore, we find the number of elements of order 6 in the group $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4$.

Now we compute the number of elements of order 6 in $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4$. We are given that

$$6 = lcm(\,|a|,\ |b|,\ |c|,\ |d|\,)$$

where $a \in \mathbb{Z}_2$, $b \in \mathbb{Z}_4$, $c \in \mathbb{Z}_6$ and $d \in \mathbb{Z}_4$. Therefore we have the following table

$\lvert a\rvert$	$\lvert b\rvert$	$\lvert c\rvert$	$\lvert d\rvert$	Number of elements of order 6
1	1	3	2	$\phi(1)\phi(1)\phi(3)\phi(2) = 2$
1	1	6	1	$\phi(1)\phi(1)\phi(6)\phi(1) = 2$
1	1	6	2	$\phi(1)\phi(1)\phi(6)\phi(2) = 2$
1	2	3	1	$\phi(1)\phi(2)\phi(3)\phi(1) = 2$
1	2	3	2	$\phi(1)\phi(2)\phi(3)\phi(2) = 2$
1	2	6	1	$\phi(1)\phi(2)\phi(6)\phi(1) = 2$
1	2	6	2	$\phi(1)\phi(2)\phi(6)\phi(2) = 2$
2	1	3	1	$\phi(2)\phi(1)\phi(3)\phi(1) = 2$
2	1	3	2	$\phi(2)\phi(1)\phi(3)\phi(2) = 2$
2	1	6	1	$\phi(2)\phi(1)\phi(6)\phi(1) = 2$
2	1	6	2	$\phi(2)\phi(1)\phi(6)\phi(2) = 2$
2	2	3	1	$\phi(2)\phi(2)\phi(3)\phi(1) = 2$
2	2	3	2	$\phi(2)\phi(2)\phi(3)\phi(2) = 2$
2	2	6	1	$\phi(2)\phi(2)\phi(6)\phi(1) = 2$
2	2	6	2	$\phi(2)\phi(2)\phi(6)\phi(2) = 2$
			Total	$2 \times 15 = 30$

Hence there are 30 elements in $Aut\,(\mathbb{Z}_{720})$ of order 6.