Theorem: $G, Z(G)$ be as before then $G/Z(G)$ is isomorphic to $Inn(G)$

Proof $T: G/Z(G) \to Inn(G)$
$gZ(G) \mapsto \phi_g : x \to gxg^{-1}$

Tutorial

1) H be a ~~g~~ subgroup of G.
i) if $a^2 \in H$ $\forall$ $a \in G$, Then prove that H is a Normal, Also show that G/H is commutative.

ii) if $[G:H] = 2$ Prove that H is Normal ~~in~~ in G. (use (i))

2) If every cyclic subgroup of G is Normal. Prove that any subgroup of G is normal.

Q $G = \mathbb{Z}_8$, $H = \{0, 4\}$
Prove H is normal subgroup of G.

Ans $\to$ [let $g \in G$, we need to prove $ghg^{-1} \in H$ $\forall g \in G$

Let $G = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$

as 0 is e (identity) $\therefore geg^{-1} = gg^{-1} = e \in H$

$((g+4) \% 8).g^{-1}$ As $g_1 . g_2 = g_2 . g_1$
i.e $(g_1+g_2) \% 8 = (g_2+g_1) \% 8$

$[ghg^{-1} = gg^{-1}h = eh = h \in H]$

* Q_8 $\{\pm 1, \pm i, \pm j, \pm k\}$ it has all subgroups normal, still not commutative (*)

ans2 given:

i) → H is subgroup of G
→ $a^2 \in H$
$\Rightarrow \langle a^2 \rangle \in H$ i.e

Q $\boxed{\text{Order of } |G/Z(G)| = 11}$ ⟺ prime ⟹ so this is cyclic ⟹ so abelian

Is it possible?

so $G = Z(G)$.

ans 1: i) $aha^{-1} = [(ah)^2 h^{-1} (a^{-1})^2 \in H]$
$= ahah\, h^{-1} a^{-1} a^{-1}$
$= ahaa^{-1}a^{-1}$
$= aha^{-1}$

we know $ah \in G$,
$\therefore (ah)^2 \in H$ as for every $a \in G$ $a^2 \in H$.

Commutative: $aH . bH = bH . aH$
To prove $abH = baH$.
i.e To prove $(ba)^{-1}ab \in H$.
⇓
$a^{-1}b^{-1}ab$.

Theorem
If 2 coset's
$aH = bH$
$\Rightarrow b^{-1}a \in H$

$(a^{-1}b^{-1})^2 (bab^{-1})^2 (b^2)$
$a^{-1}b^{-1}a^{-1}b^{-1}\, bab^{-1}\, bab^{-1}\, bb$
$a^{-1}b^{-1}b^{-1}b^{-1}b^{-1}b\, ab$
$a^{-1}b^{-1}ab$

$Z(G)$

ii) $[G:H]=2 \rightarrow$ exactly 2 cosets.

Let $\exists\, a \in G$, but $a^2 \notin H$ from (i)

$\Rightarrow a \notin H$

$\Rightarrow H, aH$ diff coset

$G = H \cup aH$

$a^2 \in aH$.

$a^2 = ah$

$\Rightarrow a = h_1 \in H$

contradicts

ans2 .. H be a subgroup.

$aha^{-1} \in H$

$aha^{-1} \in \langle h \rangle \subseteq H$.

Class

G abelian

$\therefore x$ normal subgroup

$\therefore$ $G/\langle x \rangle$ becomes group

$y \in G$

Proof: Mathematical Induction 2nd form of $|G| = 2$, then theorem is true, $|G| \geq 3$. Let x be an element of of order q for some prime q. Now consider the group $G/\langle x \rangle$, Now assume result to be true for all group of order $< |G|$.

we know $\left| G/\langle x \rangle \right| = \dfrac{|G|}{|\langle x \rangle|}$

Thus $G/\langle x \rangle$ has an element say $y\langle x \rangle$ of order p. Thus $(y\langle x \rangle)^p = y^p\langle x \rangle = \langle x \rangle$. That implies

$\Rightarrow$ i) $y^p = e$ ~~ii) $|y^p|$~~

ii) $y^p \in \langle x \rangle \Rightarrow |y^p| = q$

$\Rightarrow |y^q| = p$

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$Z(G)$

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