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Question 1

1. Let there be $n \geq 2$ people at the party. We can model this scenario as a simple graph where each vertex represents a person, and an edge between two vertices signifies that the two people know each other.
2. The number of people a person knows is the degree of their corresponding vertex. For a graph with n vertices, the possible degrees are the integers 0, 1, 2, ..., n-1.
3. Now, observe that the values 0 (knows no one) and n-1 (knows everyone) cannot both be present in the graph. If one person knows all n-1 others, then there cannot be a person who knows no one, as that person would be known by the first.
4. Therefore, the n vertices must collectively assume at most n-1 distinct degree values from the set {0, 1, ..., n-1}.
5. We now have n people (the pigeons) and at most n-1 possible degree values (the pigeonholes). By the Pigeonhole Principle, at least two people must have the same degree.
6. Therefore, at least two people know the same number of other people at the party.

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Question 2

Let a_n be the number of ternary strings of length n that contain the substring "00". We derive the recurrence by considering how to form such a string from a shorter one, based on its ending digits.

Case 1: Strings ending in '1' or '2'
If a valid string ends in '1' or '2', the "00" must be entirely contained in the first n-1 digits. We can append '1' or '2' to any of the a_{n-1} valid strings of length n-1.

Contribution: $2a_{n-1}$

Case 2: Strings ending in a single '0'
If a valid string ends in '0' but not in "00", then the (n-1)th digit must be '1' or '2' (to avoid creating "00" earlier), and the first n-2 digits must already contain "00". We append '10' or '20' to any of the a_{n-2} valid strings of length n-2.

Contribution: $2a_{n-2}$

Case 3: Strings ending in "00" for the first time
If the string ends with "00" and this is the first occurrence, then the first n-2 digits must be a string that does not contain "00". The number of such "00"-free strings is $3^{n-2} - a_{n-2}$. We simply append "00" to any of these.

Contribution: $3^{n-2} - a_{n-2}$

Summing all three cases gives the recurrence:

$a_n = 2a_{n-1} + 2a_{n-2} + (3^{n-2} - a_{n-2})$

Simplifying, we get the final recurrence relation:

$a_n = 2a_{n-1} + a_{n-2} + 3^{n-2}$

Initial Conditions:

$a_0 = 0$ (The empty string cannot contain "00")

$a_1 = 0$ (A single digit cannot contain "00")

$a_2 = 1$ (Only the string "00" satisfies the condition)

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question 3

A directed simple graph has no loops or multiple edges. Two graphs are isomorphic if one can be transformed into the other by relabeling the vertices.

(a) For n = 2 vertices (A and B):
The possible connections for the unordered pair {A, B} are:

- No edges.
- One edge: Either A→B or B→A. These two are isomorphic (simply swap A and B).
- Two edges: Both A→B and B→A (a pair of anti-parallel edges).

Thus, there are 3 non-isomorphic directed simple graphs on 2 vertices.

(b) For n = 3 vertices:
We use Burnside's Lemma, which is ideal for counting distinct objects under group symmetry. The group acting on the set of labeled digraphs is the symmetric group S_3.
Total number of labeled digraphs: 2^(3·2) = 2^6 = 64 (each of the 6 possible ordered pairs can be an edge or not).

We compute the number of digraphs fixed by each type of permutation in S_3:

- Identity permutation (1 permutation): All 64 digraphs are fixed. fix(identity) = 64
- Transposition (3 permutations): e.g., swapping vertices 1 and 2. The 6 ordered pairs form 3 orbits of size 2. fix(transposition) = 2^3 = 8
- 3-cycle (2 permutations): e.g., (1→2→3→1). The 6 ordered pairs form 2 orbits of size 3. fix(3-cycle) = 2^2 = 4

By Burnside's Lemma:
Number of orbits = $(1/|S_3|) \times \Sigma\, \text{fix}(g) = (1/6) \times (64 + 3\times8 + 2\times4) = (1/6) \times (64 + 24 + 8) = 96/6 = 16$

Thus, there are 16 non-isomorphic directed simple graphs on 3 vertices.
Final Answers:

- a) 3
- b) 16

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Question 4

The wheel graph W_n is defined for $n \geq 4$ as the graph formed by joining a single central vertex (the hub) to every vertex of a cycle C_{n-1} (the rim). The graph W_3 is typically defined as the complete graph K_4.

A fundamental theorem in graph theory states that a graph is bipartite if and only if it contains no odd cycles (cycles with an odd number of edges).

Let us analyze the cycle structure of W_n:

For $n = 3$:
W_3 is K_4. This graph contains multiple triangles (3-cycles), which are odd cycles. Therefore, W_3 is not bipartite.

For any $n \geq 4$:
Consider any two adjacent vertices u and v on the rim. The hub h is connected to both u and v. The edges (h, u), (u, v), and (v, h) form a cycle of length 3 (a triangle).

This is an odd cycle.

Since W_n contains an odd cycle for all $n \geq 3$, it cannot be bipartite.

There are no values of n for which the standard wheel graph W_n is bipartite.