Coordinate Geometry

10^{th} Maths - Chapter 7

This is Problem-6 from Exercise 7.4

1. The vertices of $\triangle ABC$ are $\begin{pmatrix}4\\6\end{pmatrix}, \begin{pmatrix}1\\5\end{pmatrix}, \begin{pmatrix}7\\2\end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of the $\triangle ADE$ and compare it with the area of $\triangle ABC$.
Solution: The input parameters for this problem are available in Table (1)

Symbol	**Value**	**Description**
A	$\begin{pmatrix}4\\6\end{pmatrix}$	First point
B	$\begin{pmatrix}1\\5\end{pmatrix}$	Second point
C	$\begin{pmatrix}7\\2\end{pmatrix}$	Third point
D	?	Desired point
E	?	Desired point

Table 1

Given,

$$\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4} \tag{1}$$

From (1),

$$\frac{AD}{AB} = \frac{1}{4} \tag{2}$$

$$\frac{AD}{BD} = \frac{1}{3} \tag{3}$$

Point $\mathbf{D}$ divides $\mathbf{AB}$ in the ratio of $n = 1 : 3$. Using Section formula,

$$\mathbf{D} = \frac{\mathbf{A} + n\mathbf{B}}{1 + n} \tag{4}$$

Substituting the values $\mathbf{A}, \mathbf{B}$ and n in (4),

$$\mathbf{D} = \frac{\begin{pmatrix} 4 \\ 6 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 5 \end{pmatrix}}{1 + \frac{1}{3}} \tag{5}$$

$$= \frac{1}{1 + \frac{1}{3}} \left(\begin{pmatrix} 4 \\ 6 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 5 \end{pmatrix} \right) \tag{6}$$

$$= \frac{3}{4} \begin{pmatrix} 4 + \frac{1}{3} \\ 6 + \frac{5}{3} \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} \frac{13}{4} \\ \frac{23}{4} \end{pmatrix} \tag{8}$$

From (1),

$$\frac{AE}{AC} = \frac{1}{4} \tag{9}$$

$$\frac{AE}{CE} = \frac{1}{3} \tag{10}$$

Point $\mathbf{E}$ divides $\mathbf{AC}$ in the ratio of $n = \frac{1}{3}$. Using Section formula,

$$\mathbf{E} = \frac{\mathbf{A} + n\mathbf{C}}{1 + n} \tag{11}$$

Substituting the values $\mathbf{A}, \mathbf{C}$ and n in (11),

$$\mathbf{E} = \frac{\begin{pmatrix} 4 \\ 6 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 7 \\ 2 \end{pmatrix}}{1 + \frac{1}{3}} \tag{12}$$

$$= \frac{1}{1 + \frac{1}{3}}\left(\begin{pmatrix} 4 \\ 6 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 7 \\ 2 \end{pmatrix}\right) \tag{13}$$

$$= \frac{3}{4}\begin{pmatrix} 4 + \frac{7}{3} \\ 6 + \frac{2}{3} \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} \frac{19}{4} \\ \frac{20}{4} \end{pmatrix} \tag{15}$$

The ar(ADE) can be expressed as

$$ar(ADE) = \frac{1}{2}\left\|(\mathbf{A} - \mathbf{D}) \times (\mathbf{A} - \mathbf{E})\right\| \tag{16}$$

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} \frac{13}{4} \\ \frac{23}{4} \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \end{pmatrix} \tag{17}$$

$$\mathbf{A} - \mathbf{E} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} \frac{19}{4} \\ \frac{20}{4} \end{pmatrix} = \begin{pmatrix} \frac{-3}{4} \\ 1 \end{pmatrix} \tag{18}$$

Substituting the values of (17) and (18) in (16),

$$ar(ADE) = \frac{1}{2}\begin{vmatrix} \frac{3}{4} & \frac{-3}{4} \\ \frac{1}{4} & 1 \end{vmatrix} = \frac{15}{32} \tag{19}$$

The ar(ABC) can be expressed as

$$ar(ABC) = \frac{1}{2}\left\|(\mathbf{A} - \mathbf{B}) \times (\mathbf{B} - \mathbf{C})\right\| \tag{20}$$

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{21}$$

$$\mathbf{B} - \mathbf{C} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} - \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} \tag{22}$$

Substituting the values of (21) and (22) in (20),

$$ar(ABC) = \frac{1}{2}\begin{vmatrix} 3 & -6 \\ 1 & 3 \end{vmatrix} = \frac{15}{2} \tag{23}$$

Comparing ar(ADE) with ar(ABC),

$$\frac{ar\left(ADE\right)}{ar\left(ABC\right)} = \frac{\frac{15}{32}}{\frac{15}{2}} \tag{24}$$

$$= \frac{15}{32} \times \frac{2}{15} = \frac{1}{16} \tag{25}$$

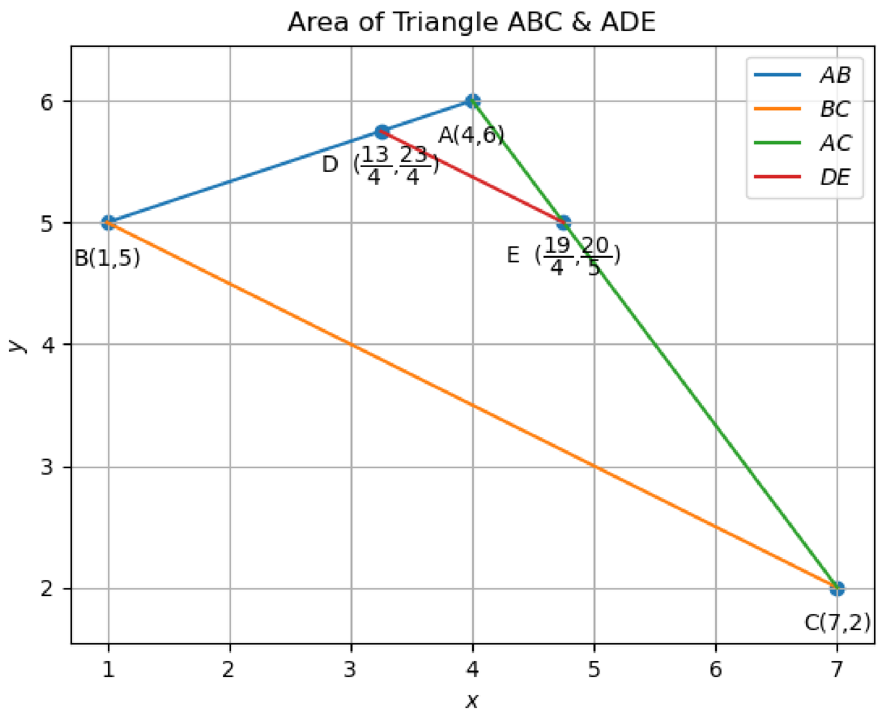

Figure 1