

# Circles

## 10<sup>th</sup> Maths - Chapter 10

This is Problem-3 from Exercise 1

1. A tangent **PQ** at a point **P** of a circle of radius 5 cm meets a line through the centre **O** at a point **Q** so that **OQ** = 12 cm. Length **PQ** is

**Solution:**

The input parameters for this problem are available in Table (1)

Symbol	Value	Description
$r$	5	Radius
<b>O</b>	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre <b>O</b>
$d_1$	5	Length of $OP$
$d_2$	12	Length of $OQ$
$d_3$	?	Length of $PQ$

Table 1

The points of intersection of the line is given by

$$L : \quad \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad \mu \in \mathbb{R} \quad (1)$$

with the conic section

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \quad (2)$$

is given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (3)$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left( -\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f) (\mathbf{m}^\top \mathbf{V} \mathbf{m})} \right) \quad (4)$$

If  $L$  in (1) touches (2) at exactly one point  $\mathbf{q}$ ,

$$\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) = 0 \quad (5)$$

In this case, conic intersection has exactly one root. Hence, in (4)

$$[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{m}^\top \mathbf{V} \mathbf{m}) (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f) = 0 \quad (6)$$

So, the equation of conic can be written in the form of (4) as,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} - 25 = 0 \quad (7)$$

Let direction vector  $\mathbf{m}$  be,

$$\mathbf{m} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \quad (8)$$

and let  $\mathbf{q}$  be the point  $\mathbf{Q}$ ,

$$\mathbf{q} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (9)$$

Substituting (7),(8) and (9) in (6) gives,

$$[\mathbf{m}^\top (\mathbf{V} \mathbf{q})]^2 - (\mathbf{m}^\top \mathbf{V} \mathbf{m}) (\mathbf{q}^\top \mathbf{V} \mathbf{q} + (-25)) = 0 \quad (10)$$

$$\implies \left[ (1 \ \lambda) \begin{pmatrix} 12 \\ 0 \end{pmatrix} \right]^2 - \left( (1 \ \lambda) \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right) \left( (12 \ 0) \begin{pmatrix} 12 \\ 0 \end{pmatrix} - 25 \right) = 0 \quad (11)$$

$$\implies (12)^2 - (1 + \lambda^2)(144 - 25) = 0 \quad (12)$$

$$\implies 144 - (1 + \lambda^2)(119) = 0 \quad (13)$$

$$\implies (1 + \lambda^2)(119) = -144 \quad (14)$$

$$\implies \lambda = \pm \frac{5}{\sqrt{119}} \quad (15)$$

Then

$$\mathbf{m} = \begin{pmatrix} 1 \\ \pm \frac{5}{\sqrt{119}} \end{pmatrix} \quad (16)$$

$$= \begin{pmatrix} 1 \\ \pm 0.4583 \end{pmatrix} \quad (17)$$

From (4) and (6)

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} (-\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})) \quad (18)$$

$$= \frac{1}{\begin{pmatrix} 1 & 0.4583 \end{pmatrix} \mathbf{I} \begin{pmatrix} 1 \\ 0.4583 \end{pmatrix}} \begin{pmatrix} 1 \\ 0.4583 \end{pmatrix} \left( - \begin{pmatrix} 1 & 0.4583 \end{pmatrix} \left( \mathbf{I} \begin{pmatrix} 12 \\ 0 \end{pmatrix} \right) \right) \quad (19)$$

$$= -\frac{119}{12} \quad (20)$$

$$= -9.916 \quad (21)$$

Now (3) becomes,

$$\mathbf{x}_i = \begin{pmatrix} 12 \\ 0 \end{pmatrix} + (-9.916) \begin{pmatrix} 1 \\ \pm \frac{5}{\sqrt{119}} \end{pmatrix} \quad (22)$$

$$\mathbf{x}_i = \begin{pmatrix} 12 \\ 0 \end{pmatrix} + \begin{pmatrix} -9.916 \\ \pm 4.545 \end{pmatrix} \quad (23)$$

Then,

$$\mathbf{x}_i = \begin{pmatrix} 2.083 \\ \pm 4.545 \end{pmatrix} \quad (24)$$

Therefore,

$$\mathbf{P}_1 = \begin{pmatrix} 2.083 \\ 4.545 \end{pmatrix} \quad (25)$$

$$\mathbf{P}_2 = \begin{pmatrix} 2.083 \\ -4.545 \end{pmatrix} \quad (26)$$

the length of tangent  $\mathbf{P}$  and  $\mathbf{Q}$  is given by

$$d = \|\mathbf{P} - \mathbf{Q}\| \quad (27)$$

$$d^2 = (\mathbf{P} - \mathbf{Q})(\mathbf{P} - \mathbf{Q})^\top \quad (28)$$

Then

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} 2.083 \\ 4.545 \end{pmatrix} - \begin{pmatrix} 12 \\ 0 \end{pmatrix} \quad (29)$$

$$= \begin{pmatrix} -9.916 \\ 4.545 \end{pmatrix} \quad (30)$$

Then substituting (30) in (28) gives

$$d^2 = \frac{14161}{144} + \frac{2975}{144} \quad (31)$$

$$d = \sqrt{119} \quad (32)$$

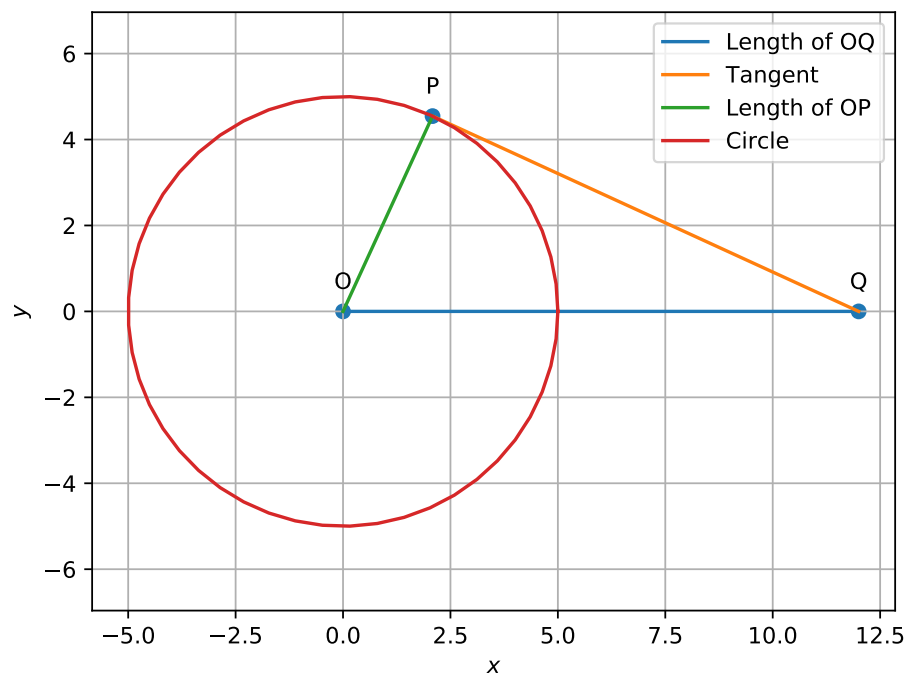


Figure 1