Circles

10^{th} Maths - Chapter 10

This is Problem-3 from Exercise 1

1. A tangent **PQ** at a point **P** of a circle of radius 5 cm meets a line through the centre **O** at a point **Q** so that **OQ** = 12 cm. Length **PQ** is

   Solution:
   The input parameters for this problem are available in Table The circle

Symbol	**Value**	**Description**
r	5	Radius
$\mathbf{O}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Centre **O**
d_1	5	Length of OP
d_2	12	Lenght of OQ
d_3	?	Length of PQ

   Table 1

   of radius 5cm and point **Q** at a distance 12cm from the centre. Tangent can be drawn from point **Q** on to the circle with point of contact **P**. The points of intersection of the line is given by

   $$L: \quad \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad \mu \in \mathbb{R} \tag{1}$$

   with the conic section

   $$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0 \tag{2}$$

is given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \tag{3}$$

where

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top \left(\mathbf{V} \mathbf{q} + \mathbf{u} \right) \right. \\ \left. \pm \sqrt{\left[\mathbf{m}^\top \left(\mathbf{V} \mathbf{q} + \mathbf{u} \right) \right]^2 - \left(\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f \right) \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right)} \right) \tag{4}$$

If L in (1) touches (2) at exactly one point $\mathbf{q}$,

$$\mathbf{m}^\top \left(\mathbf{V} \mathbf{q} + \mathbf{u} \right) = 0 \tag{5}$$

In this case, conic intersection has exactly one root. Hence, in (4)

$$\left[\mathbf{m}^\top \left(\mathbf{V} \mathbf{q} + \mathbf{u} \right) \right]^2 - \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right) \left(\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f \right) = 0 \tag{6}$$

So, the equation of conic can be written in the form of (4) as,

$$\mathbf{x}^\top \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x} + 2 \begin{pmatrix} 0 & 0 \end{pmatrix} \mathbf{x} - 25 = 0 \tag{7}$$

Let direction vector $\mathbf{m}$ be,

$$\mathbf{m} = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \tag{8}$$

and let $\mathbf{q}$ be the point $\mathbf{Q}$,

$$\mathbf{q} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{9}$$

Substituting (7),(8) and (9) in (6) gives,

$$\left[\mathbf{m}^\top \left(\mathbf{V} \mathbf{q} \right) \right]^2 - \left(\mathbf{m}^\top \mathbf{V} \mathbf{m} \right) \left(\mathbf{q}^\top \mathbf{V} \mathbf{q} + (-25) \right) = 0 \tag{10}$$

$$\implies \left[\begin{pmatrix} 1 & \lambda \end{pmatrix} \begin{pmatrix} 12 \\ 0 \end{pmatrix} \right]^2 - \left(\begin{pmatrix} 1 & \lambda \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \right) \left(\begin{pmatrix} 12 & 0 \end{pmatrix} \begin{pmatrix} 12 \\ 0 \end{pmatrix} - 25 \right) = 0 \tag{11}$$

$$\implies (12)^2 - \left(1+\lambda^2\right)(144-25) = 0 \tag{12}$$
$$\implies 144 - \left(1+\lambda^2\right)(119) = 0 \tag{13}$$
$$\implies \left(1+\lambda^2\right)(119) = -144 \tag{14}$$
$$\implies \lambda = \pm\frac{5}{\sqrt{119}} \tag{15}$$

Then

$$\mathbf{m} = \begin{pmatrix} 1 \\ \pm\frac{5}{\sqrt{119}} \end{pmatrix} \tag{16}$$
$$= \begin{pmatrix} 1 \\ \pm 0.4583 \end{pmatrix} \tag{17}$$

From (4) and (6)

$$\mu_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top \left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right) \tag{18}$$
$$= \frac{1}{\begin{pmatrix} 1 & 0.4583 \end{pmatrix}} \mathbf{I} \begin{pmatrix} 1 \\ 0.4583 \end{pmatrix} \left(-\begin{pmatrix} 1 & 0.4583 \end{pmatrix} \left(\mathbf{I} \begin{pmatrix} 12 \\ 0 \end{pmatrix}\right)\right) \tag{19}$$
$$= -\frac{119}{12} \tag{20}$$
$$= -9.916 \tag{21}$$

Now (3) becomes,

$$\mathbf{x_i} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} + (-9.916) \begin{pmatrix} 1 \\ \pm\frac{5}{\sqrt{119}} \end{pmatrix} \tag{22}$$
$$\mathbf{x_i} = \begin{pmatrix} 12 \\ 0 \end{pmatrix} + \begin{pmatrix} -9.916 \\ \pm 4.545 \end{pmatrix} \tag{23}$$

Then,

$$\mathbf{x_i} = \begin{pmatrix} 2.083 \\ \pm 4.545 \end{pmatrix} \tag{24}$$

Therefore,

$$\mathbf{P_1} = \begin{pmatrix} 2.083 \\ 4.545 \end{pmatrix} \tag{25}$$
$$\mathbf{P_2} = \begin{pmatrix} 2.083 \\ -4.545 \end{pmatrix} \tag{26}$$

the length of tangent $\mathbf{P}$ and $\mathbf{Q}$ is given by

$$d = \|\mathbf{P} - \mathbf{Q}\| \tag{27}$$
$$d^2 = (\mathbf{P} - \mathbf{Q})(\mathbf{P} - \mathbf{Q})^\top \tag{28}$$

Then

$$\mathbf{P} - \mathbf{Q} = \begin{pmatrix} 2.083 \\ 4.545 \end{pmatrix} - \begin{pmatrix} 12 \\ 0 \end{pmatrix} \tag{29}$$
$$= \begin{pmatrix} -9.916 \\ 4.545 \end{pmatrix} \tag{30}$$

Then substituting (30) in (28) gives

$$d^2 = \frac{14161}{144} + \frac{2975}{144} \tag{31}$$
$$d = \sqrt{119} \tag{32}$$

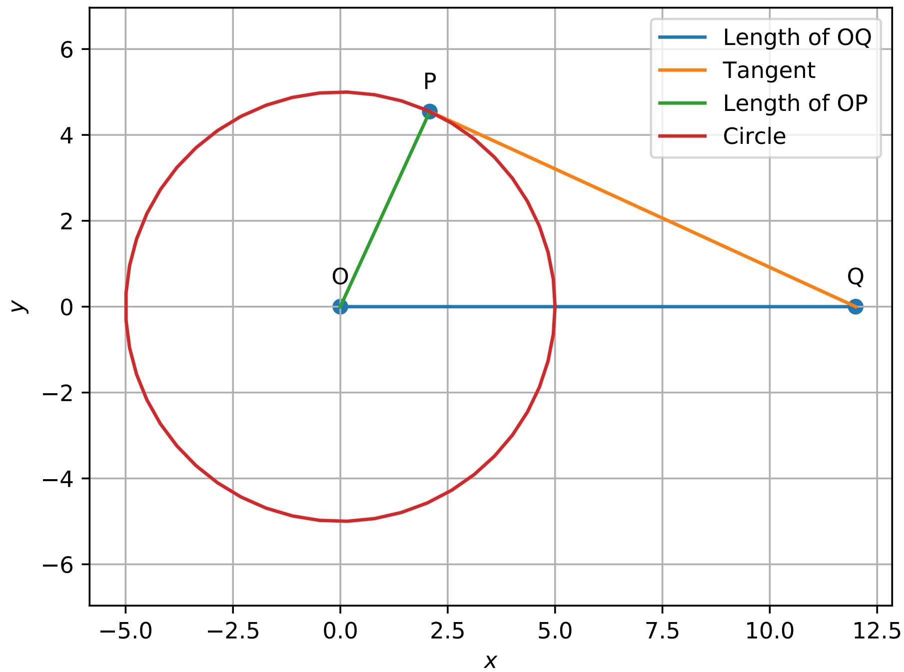
Length of OQ
Tangent
Length of OP
Circle
P
O
Q
6
4
2
0
−2
−4
−6
y
−5.0
−2.5
0.0
2.5
5.0
7.5
10.0
12.5
x

Figure 1