Coordinate Geometry

10^{th} Maths - Chapter 7

This is Problem-6 from Exercise 7.4

1. The vertices of $\triangle ABC$ are $\begin{pmatrix} 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. A line is drawn to intersect sides AB and AC at D and E respectively, such that $\frac{AD}{AB} = \frac{AE}{AC} = \frac{1}{4}$. Calculate the area of the $\triangle ADE$ and compare it with the area of $\triangle ABC$.
Solution: The input parameters for this problem are available in Table (1)

Symbol	**Value**	**Description**
A	$\begin{pmatrix} 4 \\ 6 \end{pmatrix}$	First point
B	$\begin{pmatrix} 1 \\ 5 \end{pmatrix}$	Second point
C	$\begin{pmatrix} 7 \\ 2 \end{pmatrix}$	Third point
D	?	Desired point
E	?	Desired point

Table 1

Given,

$$\frac{\mathbf{AD}}{\mathbf{AB}} = \frac{\mathbf{AE}}{\mathbf{AC}} = \frac{1}{4} \tag{1}$$

From (1),

$$\frac{\mathbf{AD}}{\mathbf{AB}} = \frac{1}{4} \tag{2}$$
$$4\mathbf{AD} = \mathbf{AD} + \mathbf{BD} \tag{3}$$
$$4\mathbf{AD} - \mathbf{AD} = \mathbf{BD} \tag{4}$$
$$3\mathbf{AD} = \mathbf{BD} \tag{5}$$
$$\frac{\mathbf{AD}}{\mathbf{BD}} = \frac{1}{3} \tag{6}$$

Point $\mathbf{D}$ divides $\mathbf{AB}$ in the ratio of $n = 1 : 3$.
using Section formula,

$$\mathbf{D} = \frac{\mathbf{B} + n\mathbf{A}}{1+n} \tag{7}$$

Substituting the values $\mathbf{A}$ and $\mathbf{B}$ in (7),

$$\mathbf{D} = \frac{\begin{pmatrix}1\\5\end{pmatrix} + n\begin{pmatrix}4\\6\end{pmatrix}}{1+n} \tag{8}$$
$$= \frac{1}{1+n}\left(\begin{pmatrix}1\\5\end{pmatrix} + n\begin{pmatrix}4\\6\end{pmatrix}\right) \tag{9}$$
$$= \frac{1}{1+n}\begin{pmatrix}4+n\\6+5n\end{pmatrix} \tag{10}$$

Substituting the value of n in (10),

$$\mathbf{D} = \frac{3}{4}\begin{pmatrix}4+\frac{1}{3}\\6+\frac{5}{3}\end{pmatrix} \tag{11}$$
$$= \frac{3}{4}\begin{pmatrix}\frac{13}{3}\\\frac{23}{4}\end{pmatrix} \tag{12}$$
$$= \begin{pmatrix}\frac{13}{4}\\\frac{23}{4}\end{pmatrix} \tag{13}$$

From (1),

$$\frac{\mathbf{AE}}{\mathbf{AC}} = \frac{1}{4} \tag{14}$$

$$4\mathbf{AE} = \mathbf{AE} + \mathbf{CE} \tag{15}$$

$$4\mathbf{AE} - \mathbf{AE} = \mathbf{CE} \tag{16}$$

$$3\mathbf{AE} = \mathbf{CE} \tag{17}$$

$$\frac{\mathbf{AE}}{\mathbf{CE}} = \frac{1}{3} \tag{18}$$

Point $\mathbf{E}$ divides $\mathbf{AC}$ in the ratio of $n = \frac{1}{3}$.
using Section formula,

$$\mathbf{E} = \frac{\mathbf{C} + n\mathbf{A}}{1+n} \tag{19}$$

Substituting the values $\mathbf{A}$ and $\mathbf{C}$ in (19),

$$\mathbf{E} = \frac{\begin{pmatrix} 7 \\ 2 \end{pmatrix} + n \begin{pmatrix} 4 \\ 6 \end{pmatrix}}{1+n} \tag{20}$$

$$= \frac{1}{1+n} \left(\begin{pmatrix} 7 \\ 2 \end{pmatrix} + n \begin{pmatrix} 4 \\ 6 \end{pmatrix} \right) \tag{21}$$

$$= \frac{1}{1+n} \begin{pmatrix} 4 + 7n \\ 6 + 2n \end{pmatrix} \tag{22}$$

Substituting the value of n in (22),

$$\mathbf{E} = \frac{3}{4} \begin{pmatrix} 4 + \frac{7}{3} \\ 6 + \frac{2}{3} \end{pmatrix} \tag{23}$$

$$= \frac{3}{4} \begin{pmatrix} \frac{19}{3} \\ \frac{20}{3} \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} \frac{19}{4} \\ \frac{20}{4} \end{pmatrix} \tag{25}$$

Now,
The ar(ADE) can be expressed as

$$ar(ABD) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{D})\times(\mathbf{A}-\mathbf{E})\right\| \tag{26}$$

$$\mathbf{A}-\mathbf{D} = \begin{pmatrix}4\\6\end{pmatrix} - \begin{pmatrix}\frac{13}{4}\\\frac{23}{4}\end{pmatrix} = \begin{pmatrix}\frac{3}{4}\\\frac{1}{4}\end{pmatrix} \tag{27}$$

$$\mathbf{A}-\mathbf{E} = \begin{pmatrix}4\\6\end{pmatrix} - \begin{pmatrix}\frac{19}{4}\\\frac{20}{4}\end{pmatrix} = \begin{pmatrix}\frac{-3}{4}\\1\end{pmatrix} \tag{28}$$

Substituting the values of (27) and (28) in (26),

$$ar(ADE) = \frac{1}{2}\begin{vmatrix}\frac{3}{4} & \frac{-3}{4}\\\frac{1}{4} & 1\end{vmatrix} = \frac{15}{32} \tag{29}$$

Now, The ar(ABC) can be expressed as

$$ar(ABC) = \frac{1}{2}\left\|(\mathbf{A}-\mathbf{B})\times(\mathbf{B}-\mathbf{C})\right\| \tag{30}$$

$$\mathbf{A}-\mathbf{B} = \begin{pmatrix}4\\6\end{pmatrix} - \begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix} \tag{31}$$

$$\mathbf{B}-\mathbf{C} = \begin{pmatrix}1\\5\end{pmatrix} - \begin{pmatrix}7\\2\end{pmatrix} = \begin{pmatrix}-6\\3\end{pmatrix} \tag{32}$$

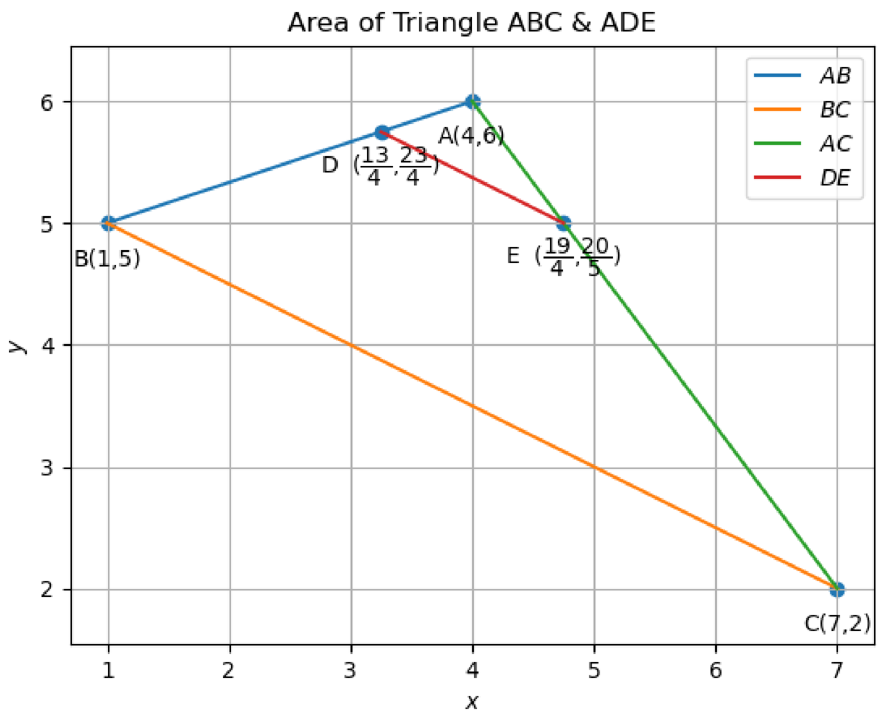
Substituting the values of (31) and (32) in (30),

$$ar(ABC) = \frac{1}{2}\begin{vmatrix}3 & -6\\1 & 3\end{vmatrix} = \frac{15}{2} \tag{33}$$

Thus,

$$\frac{ar\left(ADE\right)}{ar\left(ABC\right)} = \frac{\frac{15}{32}}{\frac{15}{2}} \tag{34}$$

$$= \frac{15}{32}\times\frac{2}{15} = \frac{1}{16} \tag{35}$$



Area of Triangle ABC & ADE
AB
BC
AC
DE
A(4,6)
D $(\frac{13}{4},\frac{23}{4})$
B(1,5)
E $(\frac{19}{4},\frac{20}{5})$
C(7,2)
y
x

Figure 1