Vector Algebra

12^{th} Maths - Chapter 10

This is Problem-3 from Exercise 10.4

1. If unit vector $\overrightarrow{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\overrightarrow{a}$.

   Solution: Let $\mathbf{A}$ be the given vector and $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ be the unit vectors representing the unit vectors $\hat{i}, \hat{j}, \hat{k}$ respectively

   $$\mathbf{A} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1}$$

   The magnitudes for vectors $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ are

   $$\|\mathbf{e_1}\| = 1, \|\mathbf{e_2}\| = 1, \|\mathbf{e_3}\| = 1 \tag{2}$$

   Let

   $$\cos\theta_i = 1, 2, 3 \tag{3}$$

So for different values of $\cos\theta_i$ the angles of vector $\mathbf{A}$ are

$$\cos\theta_1 = \frac{\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{1} = a_1 \tag{4}$$

$$\cos\theta_2 = \frac{\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{1} = a_2 \tag{5}$$

$$\cos\theta_3 = \frac{\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{1} = a_3 \tag{6}$$

Then,

$$a_1 = \cos\theta_1 = \cos\frac{\pi}{3} = \frac{1}{2} \tag{7}$$

$$a_2 = \cos\theta_2 = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \tag{8}$$

$$a_3 = \cos\theta_3 \tag{9}$$

As $\mathbf{A}$ is unit vector then

$$\|\mathbf{A}\| = 1 \tag{10}$$

$$\sqrt{a_1^2 + a_2^2 + a_3^2} = 1 \tag{11}$$

$$\sqrt{\frac{1}{2}^2 + \frac{1}{\sqrt{2}}^2 + \cos^2\theta_3} = 1 \tag{12}$$

$$\cos\theta_3 = \pm\frac{1}{2} \tag{13}$$

As θ_3 is an acute angle

$$\theta_3 = 60^\circ, a_3 = \cos 60^\circ = \frac{1}{2} \tag{14}$$

$$\text{Hence } \mathbf{A} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \tag{15}$$