MACHINE LEARNING

Through Practise

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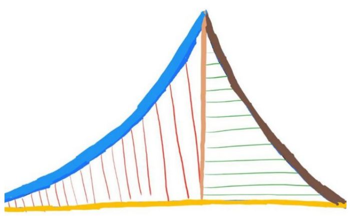

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Introduction

This book introduces machine learning through simple examples

Chapter 1

Least Squares

1.0.1 Find the shortest distance between the lines

$\overrightarrow{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and

$\overrightarrow{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$

1.0.2 Find the shortest distance between the lines

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$ **Solution:** The given lines can be written as

$$\mathbf{x} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix} \tag{1.1}$$

$$\mathbf{x} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{1.2}$$

$$\mathbf{x_1} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 7 \\ -6 \\ 1 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \tag{1.3}$$

We first check whether the given lines are skew. The lines

$$\mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1}, \ \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{1.4}$$

intersect if

$$\mathbf{M}\lambda = \mathbf{x_2} - \mathbf{x_1} \tag{1.5}$$

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{1.6}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{1.7}$$

$$\tag{1.8}$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{1.9}$$

We check whether the equation (1.10) has a solution

$$\begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{1.10}$$

the augmented matrix is given by,

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ -6 & -2 & 6 \\ 1 & 1 & 8 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{7}R_1]{R_2 \leftarrow R_2 + \frac{6}{7}R_1} \tag{1.11}$$

$$\left(\begin{array}{cc|c} 7 & 1 & 4 \\ 0 & -\frac{8}{7} & \frac{66}{7} \\ 0 & \frac{6}{7} & -\frac{52}{7} \end{array}\right) \xleftrightarrow{R_3 \leftarrow R_3 + \frac{3}{4}R_2} \tag{1.12}$$

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{5}{14} \end{array}\right) \tag{1.13}$$

The rank of the matrix is 3. So the given lines are skew. The closest points on two skew lines defined by (1.4) are given by

$$\mathbf{M}^\top \mathbf{M} \boldsymbol{\lambda} = \mathbf{M}^\top \left(\mathbf{x_2} - \mathbf{x_1}\right) \tag{1.14}$$

$$\implies \begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 1 \\ -6 & -2 \\ 1 & 1 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 7 & -6 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \tag{1.15}$$

$$\implies \begin{pmatrix} 86 & 20 \\ 20 & 6 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.16}$$

The augmented matrix of the above equation (1.16) is given by,

$$\left(\begin{array}{cc|c} 86 & 20 & 0 \\ 20 & 6 & 0 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - \frac{10}{43} R_1} \left(\begin{array}{cc|c} 86 & 20 & 0 \\ \\ 0 & \frac{58}{43} & 0 \end{array}\right) \xleftrightarrow[R_2 \leftarrow \frac{43}{58} R_2]{R_1 \leftarrow \frac{1}{86}\left(R_1 - \frac{430}{29} R_2\right)} \tag{1.17}$$

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ \\ 0 & 1 & 0 \end{array}\right) \tag{1.18}$$

yielding

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.19}$$

The closest points $\mathbf{A}$ on line l_1 and $\mathbf{B}$ on line l_2 are given by,

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \tag{1.20}$$

$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \tag{1.21}$$

The minimum distance between the lines is given by

$$\|\mathbf{B}-\mathbf{A}\| = \left\|\begin{pmatrix}4\\6\\8\end{pmatrix}\right\| = 2\sqrt{29} \tag{1.22}$$

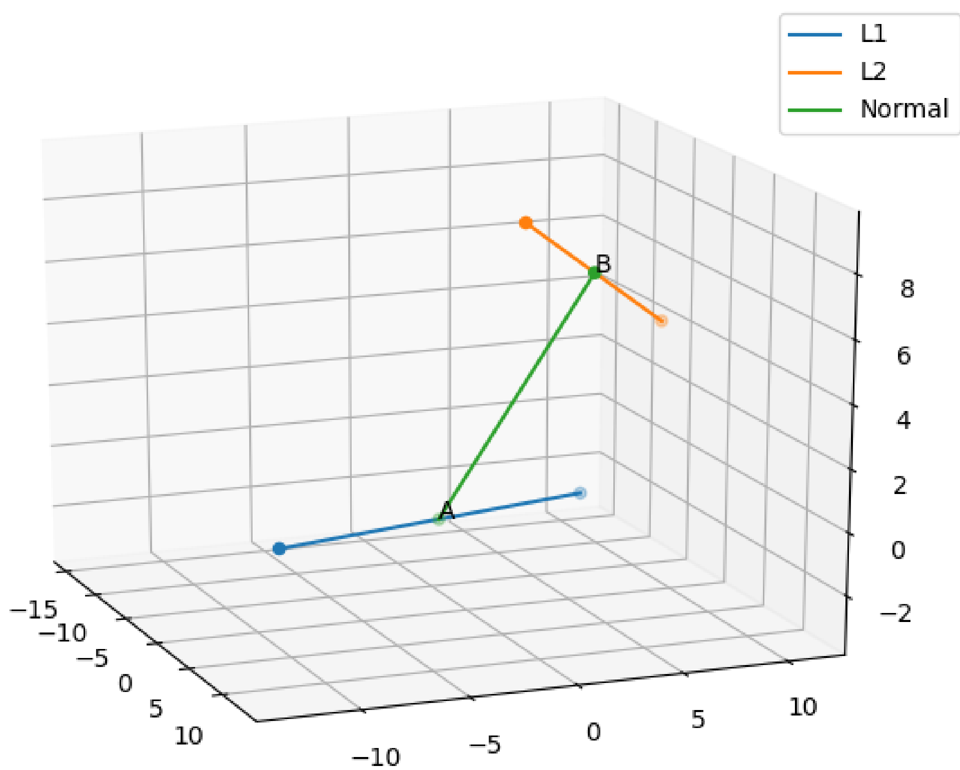

Figure 1.1:

1.0.3 Find the shortest distance between the lines whose vector equations are

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \tag{1.23}$$

and

$$\mathbf{x} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{1.24}$$

Solution: In this case,

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \mathbf{x_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \quad \mathbf{m_1} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \quad \mathbf{m_2} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \tag{1.25}$$

To check whether (A.3) has a solution in λ, we use the augmented matrix.

$$\begin{pmatrix} 1 & 2 & 3 \\ -3 & 3 & 3 \\ 2 & 1 & 3 \end{pmatrix} \xleftrightarrow{R_2 \leftarrow R_2 + 3R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 2 & 1 & 3 \end{pmatrix} \tag{1.26}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & -3 & -3 \end{pmatrix} \tag{1.27}$$

$$\xleftrightarrow{R_3 \leftarrow 3R_3 + R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 9 & 12 \\ 0 & 0 & 3 \end{pmatrix} \tag{1.28}$$

Clearly, the rank of this matrix is 3, and therefore, the lines are skew. Substituting from (1.25) in (A.6) and forming the augmented matrix,

$$\begin{pmatrix} 14 & -5 & 0 \\ -5 & 14 & 18 \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_1 + R_2} \begin{pmatrix} 9 & 9 & 18 \\ -5 & 14 & 18 \end{pmatrix} \tag{1.29}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{9}} \begin{pmatrix} 1 & 1 & 2 \\ -5 & 14 & 18 \end{pmatrix} \tag{1.30}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 + 5R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 19 & 28 \end{pmatrix} \tag{1.31}$$

$$\xleftrightarrow{R_1 \leftarrow 19R_1 - R_2} \begin{pmatrix} 19 & 0 & 10 \\ 0 & 19 & 28 \end{pmatrix} \tag{1.32}$$

$$\xleftrightarrow[R_2 \leftarrow \frac{R_2}{9}]{R_1 \leftarrow \frac{R_1}{19}} \begin{pmatrix} 1 & 0 & \frac{10}{19} \\ 0 & 1 & \frac{28}{19} \end{pmatrix} \tag{1.33}$$

$$\implies \lambda = \frac{1}{19} \begin{pmatrix} 10 \\ 28 \end{pmatrix} \tag{1.34}$$

Hence, using (A.5) and substituing into (A.7) and (A.8),

$$\mathbf{A} = \frac{1}{19} \begin{pmatrix} 29 \\ 8 \\ 77 \end{pmatrix} \quad \mathbf{B} = \frac{1}{19} \begin{pmatrix} 20 \\ 11 \\ 86 \end{pmatrix} \tag{1.35}$$

Thus, the required distance is

$$\|\mathbf{B} - \mathbf{A}\| = \frac{\sqrt{9^2 + 3^2 + (-9)^2}}{19} = \frac{3}{\sqrt{19}} \tag{1.36}$$

The situation is depicted in Fig. 1.2.

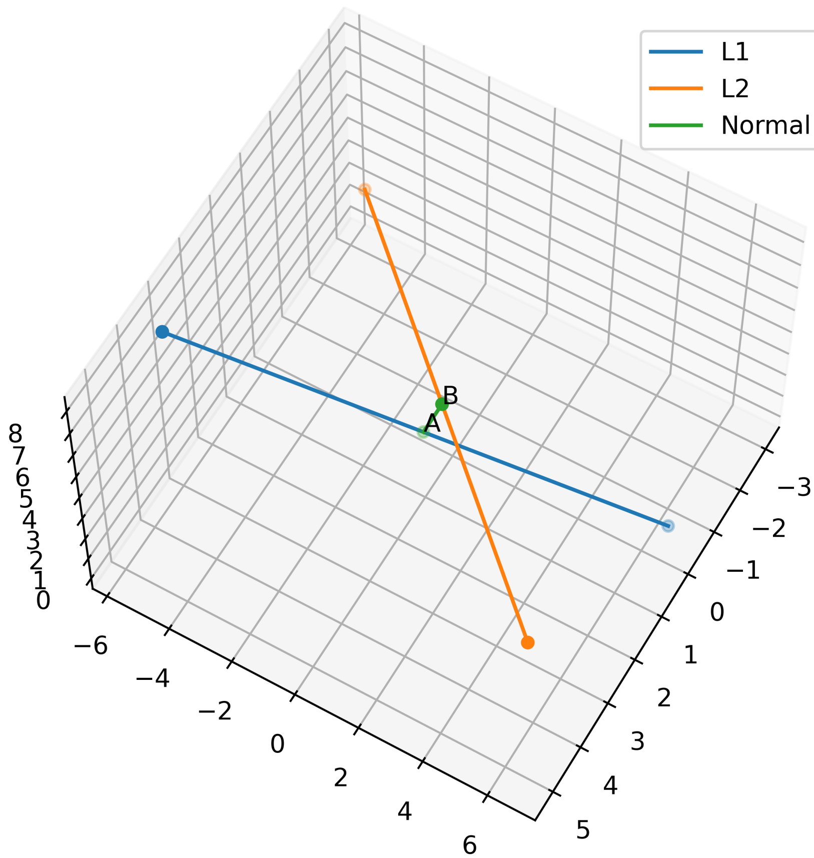

Figure 1.2: AB is the required shortest distance.

1.0.4

1.0.5 Find the shortest distance between the lines l_1 and l_2 whose vector equations are $\overrightarrow{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\overrightarrow{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$. **Solution:** The givne lines can be written in vector form as

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.37}$$

$$\implies \mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \mathbf{m_1} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \mathbf{m_2} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} \tag{1.38}$$

We first check whether the given lines are skew. The lines

$$\mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1}, \mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{1.39}$$

intersect if

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x_2} - \mathbf{x_1} \tag{1.40}$$

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{1.41}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{1.42}$$

$$\tag{1.43}$$

Here we have,

$$\mathbf{M} = \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix}, \mathbf{x_2} - \mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{1.44}$$

We check whether the equation (1.45) has a solution

$$\begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{1.45}$$

The augmented matrix is given by,

$$\left(\begin{array}{cc|c} 2 & 3 & 1 \\ -1 & -5 & 0 \\ 1 & 2 & -1 \end{array}\right) \xleftrightarrow[R_3 \leftarrow R_3 - \frac{1}{2}R_1]{R_2 \leftarrow R_2 + \frac{1}{2}R_1} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \end{array}\right) \tag{1.46}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 7R_2} \left(\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & -\frac{7}{2} & \frac{1}{2} \\ 0 & 0 & -10 \end{array}\right) \tag{1.47}$$

The rank of the matrix is 3. So the given lines are skew. The closest points on two skew lines defined by (1.39) are given by

$$\mathbf{M}^{\top}\mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^{\top}\left(\mathbf{x_2} - \mathbf{x_1}\right) \tag{1.48}$$

$$\implies \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -5 \\ 1 & 2 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 2 & -1 & 1 \\ 3 & -5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{1.49}$$

$$\implies \begin{pmatrix} 6 & 13 \\ 13 & 38 \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{1.50}$$

The augmented matrix of the above equation (1.50) is given by,

$$\left(\begin{array}{cc|c} 6 & 13 & 1 \\ 13 & 38 & 1 \end{array}\right) \xleftrightarrow{R_2 \leftarrow R_2 - \frac{13}{6} R_1} \left(\begin{array}{cc|c} 6 & 13 & 1 \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{array}\right) \tag{1.51}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{78}{59} R_2} \left(\begin{array}{cc|c} 6 & 0 & \frac{150}{59} \\ 0 & \frac{59}{6} & -\frac{7}{6} \end{array}\right) \tag{1.52}$$

So, we get

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{25}{59} \\ -\frac{7}{59} \end{pmatrix} \tag{1.53}$$

The closest points $\mathbf{A}$ on line l_1 and $\mathbf{B}$ on line l_2 are given by,

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} = \frac{1}{59}\begin{pmatrix} 109 \\ 34 \\ 25 \end{pmatrix} \tag{1.54}$$

$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} = \frac{1}{59}\begin{pmatrix} 139 \\ 24 \\ -45 \end{pmatrix} \tag{1.55}$$

The minimum distance between the lines is given by,

$$\|\mathbf{B} - \mathbf{A}\| = \left\| \frac{1}{59}\begin{pmatrix} 30 \\ -10 \\ -70 \end{pmatrix} \right\| = \frac{10}{\sqrt{59}} \tag{1.56}$$

See Fig. 1.3.

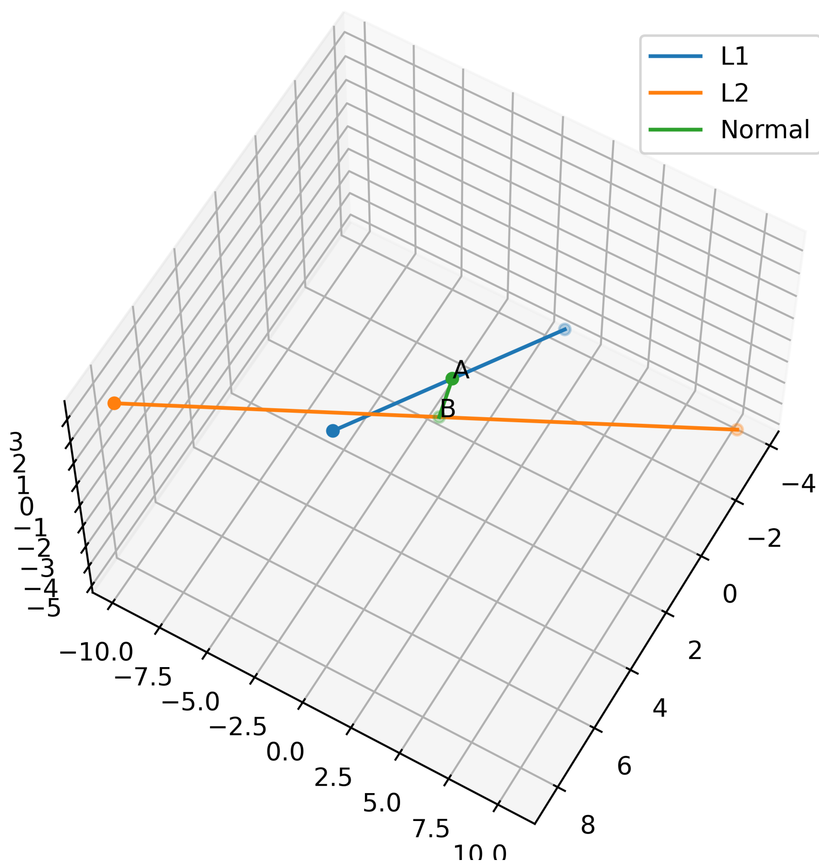

Figure 1.3:

Appendix A

Three Dimensions

A.1. The lines

$$\mathbf{x} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{A.1}$$

$$\mathbf{x} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{A.2}$$

intersect if

$$\mathbf{M}\boldsymbol{\lambda} = \mathbf{x_2} - \mathbf{x_1} \tag{A.3}$$

where

$$\mathbf{M} \triangleq \begin{pmatrix} \mathbf{m_1} & \mathbf{m_2} \end{pmatrix} \tag{A.4}$$

$$\boldsymbol{\lambda} \triangleq \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} \tag{A.5}$$

A.2. The closest points on two skew lines are given by

$$\mathbf{M}^\top \mathbf{M}\boldsymbol{\lambda} = \mathbf{M}^\top \left(\mathbf{x_2} - \mathbf{x_1}\right) \tag{A.6}$$

Solution: For the lines defined in (A.1) and (A.2), Suppose the closest points on both lines are

$$\mathbf{A} = \mathbf{x_1} + \lambda_1 \mathbf{m_1} \tag{A.7}$$

$$\mathbf{B} = \mathbf{x_2} + \lambda_2 \mathbf{m_2} \tag{A.8}$$

Then, AB is perpendicular to both lines, hence

$$\mathbf{m_1}^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{A.9}$$

$$\mathbf{m_2}^\top (\mathbf{A} - \mathbf{B}) = 0 \tag{A.10}$$

$$\implies \mathbf{M}^\top (\mathbf{A} - \mathbf{B}) = \mathbf{O} \tag{A.11}$$

Using (A.7) and (A.8) in (A.11),

$$\mathbf{M}^\top (\mathbf{x_1} - \mathbf{x_2} + \mathbf{M}\boldsymbol{\lambda}) = \mathbf{0} \tag{A.12}$$

$$\tag{A.13}$$

yielding A.6.