Introduction to Game Theory

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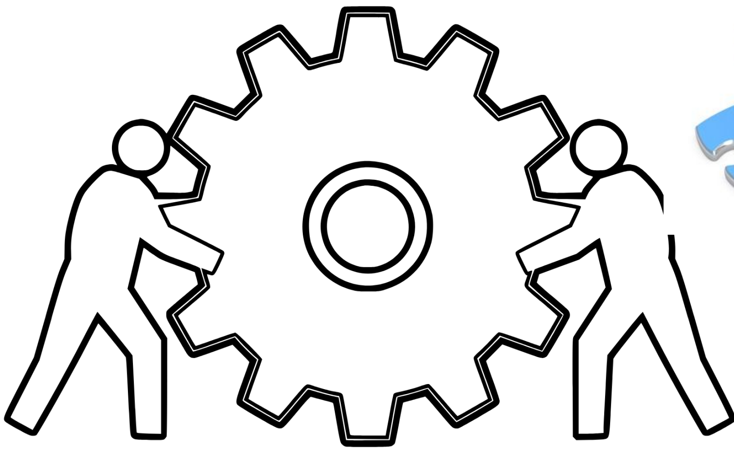
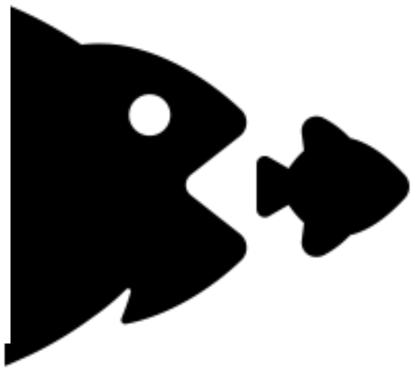
1. What is Game Theory?





What is Game Theory? (1/7)

– Game Theory does not refer to theories about board games such as chess, bridge, or monopoly;

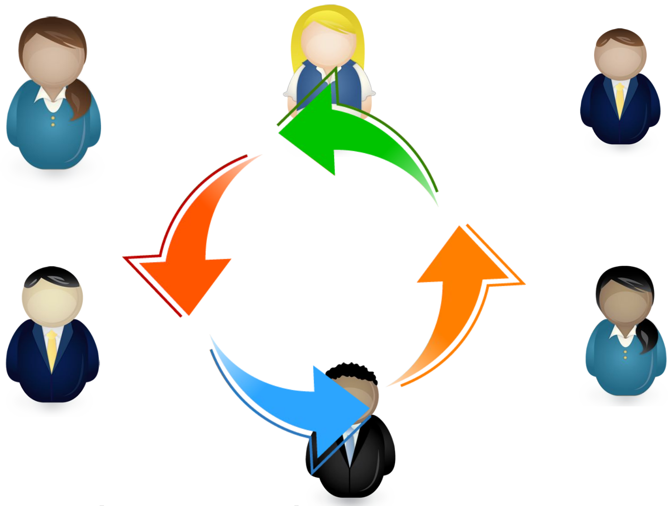
– Game Theory is not about computer games, or sports, either.

– What then is Game Theory?

Game Theory today is an umbrella term for

The science of logical decision making in humans, animals, and computers.

What is Game Theory? (2/7)



Game Theory addresses the following questions:

– How do self-interested people do strategic interactions and make strategic decisions?

– How to mathematically models conflict and cooperation between intelligent rational decision-makers?

– How should the strategic interactions be structured, e.g., by a government or a system designer so that we have *good outcomes*?

Game Theory has many applications and is especially used in economics, political science, and psychology, as well as logic, computer science, and biology.

What is Game Theory? (3/7)

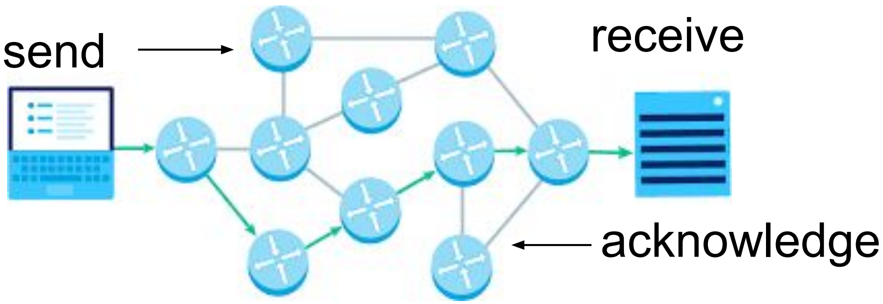
What is a **Game**?

*A **game** in general is any interaction between two or more people where the outcomes of the interaction depend on what everybody does and everybody has different levels of happiness for the different outcomes.*

A two-player version of the interactions is called a **two-player game**.

What is Game Theory? (4/7)

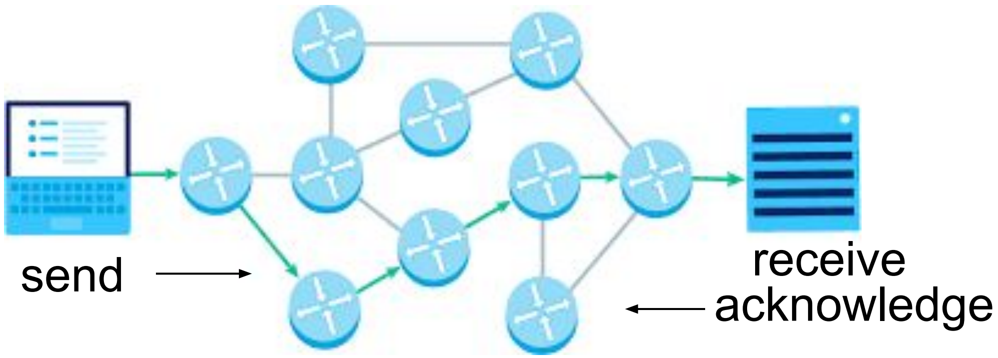
Consider an example from internet networking.



What happens when there is network congestion? The computer just throws some messages away without telling anyone! And it continues passing messages on appropriately.

Then how is reliable communication possible?

What is Game Theory? (5/7)



How reliable communication happens

– Your computer waits for some time for an acknowledgment msg; if it does not get it, it sends the message again.

– Your computer slows down the speed assuming congestion somewhere in the network.

And because all computers do this, throughput is pretty reasonable.

This is called the **backoff mechanism** in the TCP protocol.

What is Game Theory? (6/7)

Consider a two-player game in an Internet network. Two computers are sending messages to the internet. And they are deploying the TCP protocol for sending messages.

Let us assume the following

– Both use a correct implementation: both get 1 ms delay

– One correct, one defective: 4 ms for correct, 0 ms for defective

– Both defective: both get a 3 ms delay

Now, if you wanted to minimize the amount of delay that you experienced, how would you play this game?

What is Game Theory? (7/7)

Two computers are sending messages to the internet:

– Both use a correct implementation: both get 1 ms delay

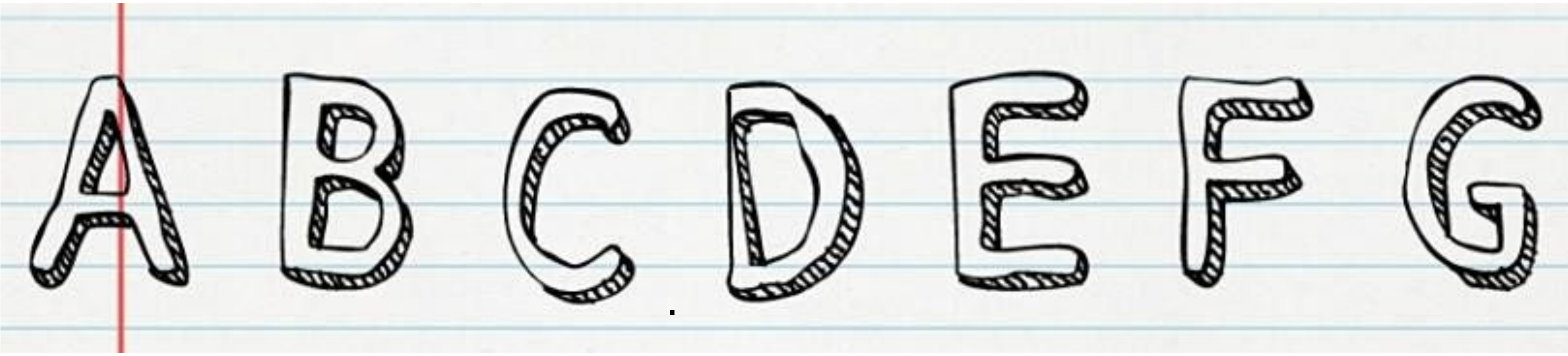
– One correct, one defective: 4 ms for correct, 0 ms for defective

– Both defective: both get a 3 ms delay

Computer 1 → Computer 2 ↓	**Correct**	**Defective**
Correct	1 ms, 1 ms	0 ms, 4 ms
Defective	4 ms, 0 ms	3 ms, 3 ms

Note: this is equivalent to the **Prisoner's Dilemma**, a famous problem.

2. Basic Ideas



Basic Ideas (1/6)

Key Ingredients of Games

1. **Players**: the central decision makers in the game, for example, people, governments, companies, etc.

2. **Actions**: What actions can players actually take? For example, bids, bargaining, etc.

3. **Payoffs**: What motivates the players? For example, profit, social benefit?

Basic Ideas (2/6)

A game in general is any interaction between two or more **self-interested** agents. Games can be simultaneous (some auctions) or sequential (e. g., chess). Agents can be invisible to each other (Internet) or have knowledge of others' decisions (poker).

Self-interest: Self-interest DOES NOT MEAN that players

(1) only care about themselves

(2) want to harm others

Self-interest MEANS that a player has its own description of the states of the world, and it acts "rationally" based on this description.

Basic Ideas (3/6)

Utility Function: Each player has a utility function. A utility function is a mathematical measure that tells you how much the player likes or does not like a given situation. It can be describe a preferences towards a distribution of such outcomes. How much does the player prefer outcome 1 to outcome 2 or 3 or 4?

Rationality: Is the idea that players **maximize their payoffs or utility function** (expected or average utility or payoffs).

Basic Ideas (4/6)

Types of Games: **Games of Pure Competition**

Constant-sum games where the players have exactly opposed interests.

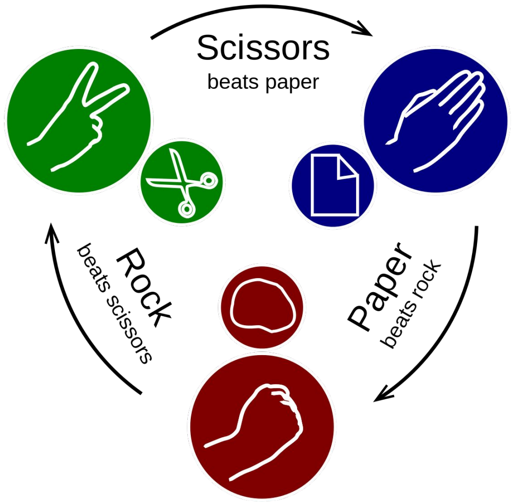
– Exactly two players

– For all action profiles $a \in A$, $\mathbf{u_1(a) + u_2(a) = c}$ where c is some constant and $u_1(a)$ and $u_1(a)$ are the utilities of players 1 and 2.

– IMPORTANT SPECIAL CASE: c = 0, which are called **zero-sum games**

Example: Rock, Paper, Scissors

Player 1 → Player 2 ↓	**Rock**	**Paper**	**Scissors**
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0



Basic Ideas (5/6)

Types of Games: **Games of Pure Cooperation/Coordination**

All the players have exactly the same interests.

– Multiple players

– The utility for player *i* is always the same as the utility for player *j* for every action vector that they choose

For all action profiles a ϵ A, and for all *i* and j, $\mathbf{u_i(a) = u_j(a)}$

Example: Road Traffic

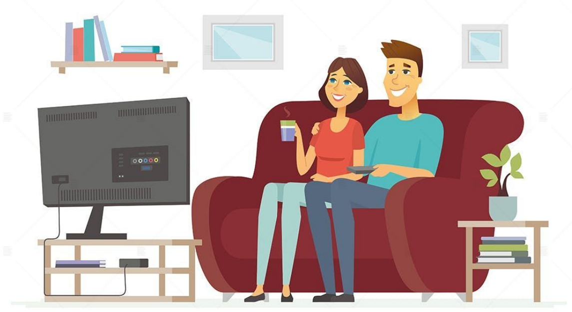
	Left	**Right**
Left	0, 0	1, 1
Right	1, 1	0, 0

Basic Ideas (6/6)

Types of Games: **General Games**

Combination of co-operation and competition

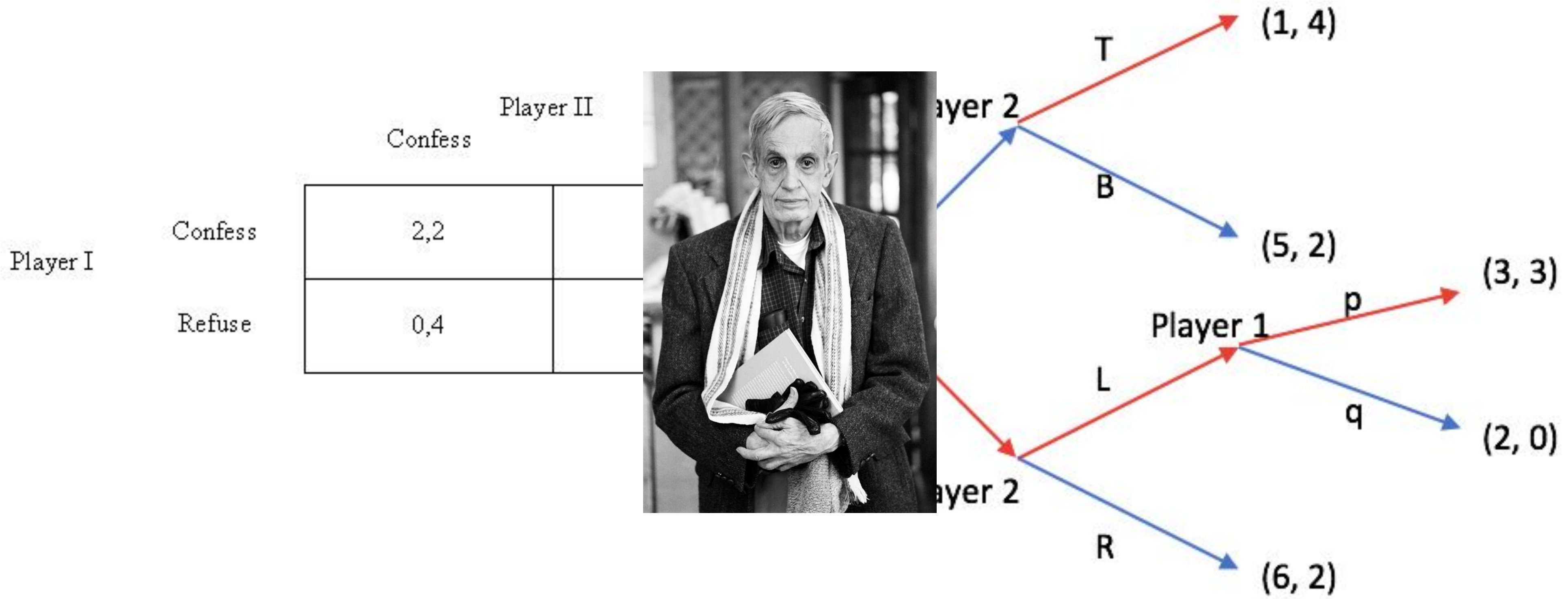
For example, consider a situation where a husband and a wife wh want to watch something on TV; above all, they want to watch *togethe* The wife wants to watch a live sports game while the husband wants t watch a romantic comedy. If they do this separately, then they are equally unhappy. They want to watch together, but they have conflictin preferences: thus, there is both cooperation and competition in thi game.



Example: Movie Watching Game

	Wife		
Husband		RomCom	Sports
	RomCom	2, 1	0, 0
	Sports	0, 0	1, 2

3. Nash Equilibrium



Nash Equilibrium (1/10)

Consider the following game

(1) Each person gets to name a real number between 0 and 100.

(2) Players move simultaneously.

(3) The player who names the real number that is closest to 2/3 of the average real number wins a prize.

(4) The other players get nothing.

(5) Ties are broken uniformly at random.

So, to win this game, you have to guess the average and then its two-third. How would you play this game?

Nash Equilibrium (2/10)

Let us now reason a solution:
(1) Suppose a player believes that the average play will be x (including his or her own integer).

(2) That player's optimal strategy is to name the closest integer to (2/3) * x.

(3) Since x < 100 , the optimal strategy of any (rational) player has to be no more than 66.67.

(4) If x < 66.67 , then the optimal strategy of any (rational) player has to be no more than (2/3) * 66.67.

(5) Now, every player wants to be a little bit lower than everybody else's guess.

(6) What is the only number which everybody can be consistently choosing as the best response they have to the average guess? The unique Nash equilibrium of this game is for every player to announce 0!

(7) Therefore, all the players announce 0 leading to a tie, and somebody wins at random.

(For more, search for "p-beauty contest game".)

Nash Equilibrium (3/10)

We had all the key ingredients of a Nash equilibrium—
(1) What are other players going to do?
(2) And what should you do in response?

Assume that everybody chooses their optimal response on the basis of what the other players will do. Then, informally, Nash equilibrium is

a set of strategies in which no player can do better by unilaterally changing their strategy.

– *Nobody has an incentive to deviate* from their action if a profile of actions form a Nash equilibrium.

– *Someone has an incentive to deviate* from their action if a profile of actions do not form a Nash equilibrium.

Nash Equilibrium (4/10)

What is the Nash Equilibrium in the Computer Networking problem?

– The only dominant strategy outcome is of both making defective implementations

– It is the best response irrespective of the actions of the other player.

– That is the only Nash equilibrium in this game.

What is the Nash Equilibrium in the Road Traffic problem?

– There are two Nash equilibria here!

Similarly, the Movie watching game also has two Nash equilibria.

Nash Equilibrium (5/10)

Some games have an important property—dominant strategies.

Let us consider strategy to mean choosing an action ("pure strategy")

Let s_i and s_i' be two strategies for a player i, and let S_{-i} be the set of all possible strategy profiles for the other player. What does it mean to say that s_i dominates s_i'? **Strict dominance.** s_i strictly dominates s_i' if

$$\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

s_i strictly dominates s_i' if for every other strategy profile of the other players, the utility obtained by the player i when playing s_i is more than the utility obtained by i when playing s_i'.

In other words, *i* is happier when playing play s_i than s_i'

Nash Equilibrium (6/10)

A strategy profile consisting of dominant strategies for every player must be a Nash equilibrium.

No player wants to change what they are doing; since the strategy is dominant, there is nothing better.

Nash Equilibrium (7/10)

Pareto Optimality

Consider a situation where
– An outcome *o* is at least as good for every player as another outcome *o′*.

– There is some player who strictly prefers *o* to *o′*.

– In such a case, it seems reasonable to say that *o* is better than *o′*.

– In such a situation, *o Pareto-dominates o′*.

– For example,
  * if *o* = *(7,8)* and *o′* = *(7,2)*, *o* is at least as good for everybody and it is strictly better for player 2.
  * It seems reasonable that for an outside observer, *o* is better than *o′*.
  * Technically, outcome *o Pareto-dominates o′*.

Nash Equilibrium (8/10)

An outcome o is Pareto-optimal if there is no other outcome that Pareto-dominates it.

This is a hard definition because it's defined in negative terms!

Basically, is there another solution where everyone gets at least the same, but someone (or multiple people) gets a better outcome? If so, then the current solution is not Pareto optimal.

Nash Equilibrium (9/10)

– *Is it possible for a game to have more than one Pareto-optimal outcome?*
It is, because its possible for two outcomes to not Pareto-dominate each other; for example, in a game with identical payoffs such as 1, nothing dominates anything else, because domination requires somebody to strictly prefer something to something else.

– *Does every game have at least one Pareto-optimal outcome?*
Yes, every game has to have at least one Pareto-optimal outcome.

Examples of Pareto Optimality:
(1) In the movie watching game, the outcomes (1,2) and (2,1) are Pareto-optimal; the change in payouts does not make a difference.

(2) In the computer networking game, all the outcomes except (defective, defective) are Pareto-optimal. Note the dilemma: the Nash Equilibrium is not Pareto-optimal!

Nash Equilibrium (10/10)

In summary, informally,

Nash equilibrium: No player benefits by changing their decision: no single player would be better off by deviating from the prescribed strategy, taking the other's as given.

Dominant strategy: It is always the best to choose this, no matter what the opponent will do.

Pareto optimal: There is no outcome that is better (or the same) for everybody.

4. Concluding Remarks

Concluding Remarks (1/5)

Mixed Strategy Nash Equilibrium

– A mixed strategy is an assignment of a probability to each pure strategy; this allows for a player to randomly select a pure strategy.

– Since probabilities are continuous, there are *infinitely many mixed strategies* available to a player.

– A pure strategy is a degenerate case of a mixed strategy, in which that particular pure strategy is selected with probability 1 and every other strategy with probability 0.

– Pure strategy: a special case where only one action is played with positive probability

– Mixed strategy: more than one action is played with positive probability

Concluding Remarks (2/5)

Nash's Theorem

Every finite game has a Nash Equilibrium!

This is a deep result!

There will always be some stable action that all of the players can do, which has the property that, if they knew what everyone was doing, none of them would want to change their strategy.

No matter what the game is, we can find a Nash equilibrium and reason about it! And that is why Nash Equilibrium is so powerful and a major reason why John Nash won a Nobel Prize in Economics.

Note that this theorem holds true only for mixed strategy Nash equilibrium.

Concluding Remarks (3/5)

Extensive form games: Sequential games in which time plays an important role.

– Perfect Information Extensive Form games: All actors have perfect information about the actions of other actors

– Imperfect Information Extensive Form games: All actors do not have perfect information about the actions of other actors

Repeated games: Sequential games in which time plays an important role.

– Infinitely Repeated games

– Finitely Repeated games

Concluding Remarks (4/5)

Bayesian games: Models uncertainty about the utility functions, for example, in an auction, a player is not quite sure what the good is worth to all of the other players in the auction.

Coalitional games: Coalitional games, unlike non-cooperative games, do not model individual agents taking actions; instead, they model groups of agents acting together.

- Not necessary that interests are aligned
- Models competition and cooperation
- Main difference: now, the basic modeling unit is the *group*, not the *individual*

Concluding Remarks (5/5)

A Quick Recap

– **Game Theory**: The science of logical decision making in humans, animals, and computers. Game Theory models co-operation and conflict between self-interested rational players.

– **Self-interested**: Players act based on their understanding of the world.

– **Utility function**: Players make an estimation of the value of an outcome of a game.

– **Rational**: Players are interested in maximizing the utility function.

– **Nash equilibrium**: No player benefits by changing their decision, assuming the other's decisions as given.

– **Dominant strategy**: It is always the best to choose this, no matter what the others will do.

– **Pareto optimal**: There is no outcome that is better (or the same) for everybody.