Discrete Math Chapter 3 (Attempt) Solutions

Question 1. (Nearly Exercise 3.3 parts (b), (c), (j), and (k)) Rewrite each of the following sets by listing their elements between braces. If the set is infinite, list enough elements to show the pattern, followed by ellipses.
Solution:
(b) $\{n \in \mathbb{N} : -7 \leq n < 6\} = \{1, 2, 3, 4, 5\}$
(c) $\{4n : n \in \mathbb{Z}, |2n| < 7\} = \{-12, -8, -4, 0, 4, 8, 12\}$
(j) $\left\{\frac{m}{n} \in \mathbb{Q} : \left|\frac{m}{n}\right| < 1 \text{ and } 1 \leq n \leq 4\right\} = \left\{0, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}\right\}$
(k) $\{x \in \mathbb{R} : x^2 - 1 = 0\} = \{-1, 1\}$

Question 2. (Nearly Exercise 3.5) Rewrite each of the following sets in set-builder notation.
Solution:
(a) $\{2, 5, 8, 11, 14, \dots\} = \{3n - 1 : n \in \mathbb{N}\}$
(b) $\left\{\dots, -\frac{3\pi}{2}, -\pi, -\frac{\pi}{2}, 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots\right\} = \left\{k \cdot \frac{\pi}{2} : k \in \mathbb{Z}\right\}$
(c) $\{-4, -3, -2, -1, 0, 1\} = \{n \in \mathbb{Z} : -4 \leq n \leq 1\}$
(d) $\left\{\dots, -\frac{3125}{243}, -\frac{125}{27}, -\frac{5}{3}, 1, \frac{25}{9}, \frac{625}{81}, \dots\right\} = \left\{\frac{(-5)^n}{3^n} : n \in \mathbb{Z}\right\}$

Question 3. (Exercise 3.8) Determine the truth value of each statement.
Solution:
(a) True. $1 \in \{1, \{1\}\}$.
(b) False. 1 is not a set, so $1 \not\subseteq \{1, \{1\}\}$.
(c) False. $1 \notin \mathcal{P}(\{1, \{1\}\})$ since it is not a subset.
(d) True. $\{1\} \in \{1, \{1\}\}$
(e) True. $\{1\} \subseteq \{1, \{1\}\}$
(f) True. $\{1\} \in \mathcal{P}(\{1, \{1\}\})$ since it is a subset.
(g) False. $\{\{1\}\}$ is not in $\{1, \{1\}\}$
(h) True. $\{\{1\}\} \subseteq \{1, \{1\}\}$
(i) True. $\{\{1\}\} \in \mathcal{P}(\{1, \{1\}\})$
(j) False. $\emptyset \notin \mathbb{N}$
(k) True. $\emptyset \subseteq \mathbb{N}$
(l) True. $\emptyset \in \mathcal{P}(\mathbb{N})$
(m) True. $\mathbb{Q} \times \mathbb{Q} \subseteq \mathbb{R} \times \mathbb{R}$
(n) False. $\mathbb{R}^2 \not\subseteq \mathbb{R}^3$ because pairs are not triples.
(o) True. $\emptyset$ is a subset of every set.

Question 4. (Nearly Exercise 3.10) Determine the familiar set equal to $\{7a + 2b : a, b \in \mathbb{Z}\}$.
Solution:

We want the set of all integer combinations of 7 and 2. Since $\gcd(7,2)=1$, the set $\{7a+2b\}$ covers all integers:

$$\{7a+2b : a,b\in\mathbb{Z}\}=\mathbb{Z}.$$

This follows from the fact that $\gcd(m,n)=1 \Rightarrow \exists$ integers a,b such that $am+bn=k$ for any $k\in\mathbb{Z}$.

Question 5. (Exercise 3.15) Suppose A and B are sets. Prove that $\mathcal{P}(A)\cup\mathcal{P}(B)\subseteq\mathcal{P}(A\cup B)$.
Example: Let $A=\{1\}$, $B=\{2\}$. Then:

- $\mathcal{P}(A)=\{\emptyset,\{1\}\}$
- $\mathcal{P}(B)=\{\emptyset,\{2\}\}$
- $A\cup B=\{1,2\}$ so $\mathcal{P}(A\cup B)$ includes both $\mathcal{P}(A)$ and $\mathcal{P}(B)$.

Proof:
Let $X\in\mathcal{P}(A)\cup\mathcal{P}(B)$. Then $X\in\mathcal{P}(A)$ or $X\in\mathcal{P}(B)$. So:

$$X\subseteq A \text{ or } X\subseteq B \Rightarrow X\subseteq A\cup B \Rightarrow X\in\mathcal{P}(A\cup B).$$

Thus, $\mathcal{P}(A)\cup\mathcal{P}(B)\subseteq\mathcal{P}(A\cup B)$.

Question 6. (Exercise 3.23) Prove that $\{n\in\mathbb{Z} : n\equiv 1 \mod 4\}\nsubseteq\{n\in\mathbb{Z} : n\equiv 1 \mod 8\}$.
Solution:
Consider $5\in\mathbb{Z}$. Then:

$$5\equiv 1 \mod 4 \Rightarrow 5\in\{n\in\mathbb{Z} : n\equiv 1 \mod 4\},$$

but:

$$5\not\equiv 1 \mod 8 \Rightarrow 5\notin\{n\in\mathbb{Z} : n\equiv 1 \mod 8\}.$$

Thus, the former set is not a subset of the latter.

Question 7. (Exercise 3.26) Disprove the identity $A\cup(B\cap C)=(A\cup B)\cap C$.
Solution:
Let $A=\{1,3\}$, $B=\{\pi,3,5\}$, $C=\{2,4,3,5\}$. Then:

$$B\cap C=\{3,5\},\quad A\cup(B\cap C)=\{1,3,5\}$$

$$(A\cup B)=\{1,3,\pi,5\}\Rightarrow(A\cup B)\cap C=\{3,5\}.$$

So:

$$A\cup(B\cap C)\neq(A\cup B)\cap C.$$

Thus, the identity fails.

Question 8. (Exercise 3.27) Disprove the claim $|A\cup B|=|A|+|B|$ for all finite sets.
Solution:
Let $A=\{1,2,3\}$, $B=\{3,4\}$. Then:

$$|A|=3,\quad |B|=2\Rightarrow|A|+|B|=5,\quad |A\cup B|=4.$$

So the claim is false.
Correct formula:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Question 9. (Exercise 3.31(a)) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
Solution:
Direction 1: Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$.

- If $x \in A$, then clearly $x \in A \cup B$ and $x \in A \cup C \Rightarrow$x $\in (A \cup B) \cap (A \cup C)$. - If $x \in B \cap C$, then $x \in B$ and $x \in C \Rightarrow$x $\in A \cup B$ and $x \in A \cup C$.

So in either case, $x \in (A \cup B) \cap (A \cup C)$.

Direction 2: Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$.

- If $x \in A$, then $x \in A \cup (B \cap C)$. - If $x \notin A$, then $x \in B$ and $x \in C \Rightarrow$x $\in B \cap C \Rightarrow$x $\in A \cup (B \cap C)$.

Hence, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Question 10. (Exercise 3.40(a)) Prove that there is a unique set $A \in \mathcal{P}(C)$ such that $A \cup B = B$ for all $B \in \mathcal{P}(C)$.
Solution:
Let $A = \emptyset$. Then for any $B \in \mathcal{P}(C)$, we have:

$$A \cup B = \emptyset \cup B = B.$$

So such an A exists.

Suppose there are A, A' such that $A \cup B = B$ and $A' \cup B = B$ for all $B \in \mathcal{P}(C)$. Let $B = A'$. Then:

$$A \cup A' = A' \Rightarrow A \subseteq A'.$$

Now let $B = A$. Then:

$$A' \cup A = A \Rightarrow A' \subseteq A.$$

Hence, $A = A'$. So the set is unique, and it must be $\emptyset$.