Chapter 5: Intro to Mathematical Induction
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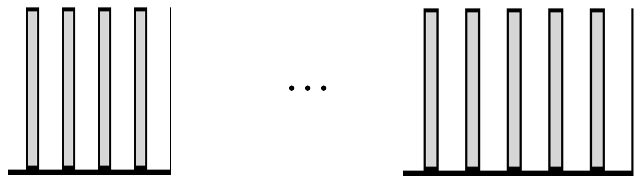
Lets consider a line of (perfectly) arranged dominoes, as shown in the diagram below.



Dominoes like this have the following properties:

- If you push the first domino, it will fall and in particular, it will fall into the second domino (knocking it over)
- *Every* domino, when knocked over, falls into the next one and knocks it over.

Principle of Induction: Consider a sequence of mathematical statements $\{S_1, S_2, S_3, \ldots\}$

- Suppose S_1 is true, and
- Suppose, for each $k \in \mathbb{N}$, if S_k is true then S_{k+1} is true.

Then, S_n is true for every $n \in \mathbb{N}$.

This can be modeled by the following picture.

We can use the following framework to prove a statement by induction.

> Proposition: S_1, S_2, S_3, … are all true.
>
> Proof: << General setup or assumption, if needed >>
>
> 1. Base Case: << Demonstration that S_1 is true >>
>
> 2. Inductive Hypothesis: Assume that S_k is true.
>
> 3. Induction Step: << Proof that S_k implies S_{k+1} >>
>
> 4. Conclusion: Therefore, by induction, all the S_n are true. $\square$

Starting in the next page page, we will prove a couple of propositions.

Proposition 5.1: For any $n \in \mathbb{N}$, $1+2+3+...+n=\frac{n(n+1)}{2}$. In other words, this is the sum of first n natural numbers.

Proof: Define $P(n)$: $1+2+3+...+n=\frac{n(n+1)}{2}$. We will proceed by induction.

<u>1. Base Case:</u> The base case is $P(1)$. In other words, lets work out the LHS[1] and RHS[2].

LHS: 1

RHS: $\frac{1(1+1)}{2}=\frac{2}{2}=1$

We have shown LHS = RHS, so the base case holds.

<u>2. Inductive Hypothesis:</u> Assume that $k \in \mathbb{N}$ such that $P(k)$: $1+2+3+...+k=\frac{k(k+1)}{2}$

<u>3. Induction Step:</u> We aim to prove the result holds for $k + 1$. That is, we want to show that

$$1+2+3+...+(k+1)=\frac{(k+1)((k+1)+1)}{2}$$, inserting $k \to k + 1$

Written slightly differently, we want to show $1+2+3+...+k+(k+1)=\frac{(k+1)(k+2)}{2}$

Now we just have to simplify the LHS and RHS respectively. So,

LHS: $\underbrace{1+2+3+...+k+...}_{\text{inductive hypothesis}}+(k+1)=\frac{k(k+1)}{2}+k+1$

From here, we just have to make common denominators by multiplying the top and bottom by 2. In other words,

$$\frac{k(k+1)}{2}+k+1=\frac{k(k+1)}{2}+\frac{(k+1)}{1}\cdot\frac{2}{2}=\frac{k^2+k+2k+2}{2}=\frac{k^2+3k+2}{2}=\frac{(k+1)(k+2)}{2}$$

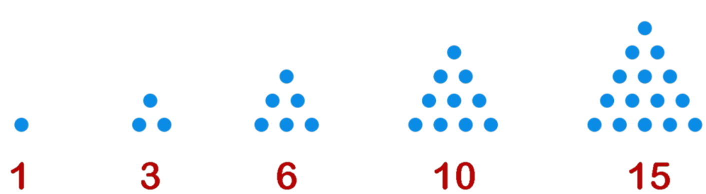
as desired.

<u>4. Conclusion:</u> We have shown that LHS = RHS and therefore by induction, $P(n)$ holds for all $n \in \mathbb{N}$. $\square$

Proposition 5.2: Let T_n be the sum of the first n *natru*al numbers. Then, for any $n \in \mathbb{N}$,

$$T_n + T_{n+1} = (n+1)^2.$$

Before we prove this, let's explore some of the triangular numbers (refer to diagram below).



1 LHS denotes "Left-Hand-Side."

2 RHS denotes "Right-Hand-Side."

Exploration of 5.2 Continued: Let’s construct a table for what we see in the diagram.

n	T_n	$T_n + T_{n+1}$
1	1	4
2	3	9
3	6	16
4	10	25
5	15	36
6	21	49

Notice that $T_n + T_{n+1}$ seems to generate perfect squares. So, our proposition checks out. However, in math it is not good enough to say “there’s a pattern, so I assume it’s true for all numbers *n*.” We must prove this. Unsurprisingly, we will use the method of mathematical induction.

Proof: Define $P(n)$: $T_n + T_{n+1} = (n+1)^2$. We will proceed by induction.

<u>1. Base Case:</u> The base case is $P(1)$. In other words, lets work out the LHS and RHS.

LHS: $T_1 + T_{1+1} = 1 + 3 = 4$

RHS: $(1+1)^2 = 2^2 = 4$

We have shown LHS = RHS, so the base case holds.

<u>2. Inductive Hypothesis:</u> Assume that $k \in \mathbb{N}$ such that $P(k)$: $T_k + T_{k+1} = (k+1)^2$

<u>3. Induction Step:</u> We aim to prove the result holds for $k + 1$. That is, we want to show that,

$$T_{k+1} + T_{(k+1)+1} = ((k+1)+1)^2 \Rightarrow T_{k+1} + T_{k+2} = (k+2)^2 \quad \text{inserting } k \to k+1$$

Since T_{k+1} is the sum of the first $k + 1$ natural numbers, we can write it as: $T_k + (k + 1)$ and similarly, $T_{k+2} = T_{k+1} + (k + 2)$.

Now we just have to simplify the LHS and RHS respectively. So,

LHS: $T_{k+1} + T_{k+2} = (T_k + (k+1)) + (T_{k+1} + (k+2)) = T_k + T_{k+1} + 2k + 3$

$$T_k + T_{k+1} + 2k + 3 = \underbrace{T_k + T_{k+1}}_{\text{inductive hypothesis}} + 2k + 3 = (k+1)^2 + 2k + 3$$

$$(k+1)^2 + 2k + 3 = (k^2 + 2k + 1) + 2k + 3 = k^2 + 4k + 4 = (k+2)^2$$

as desired.

<u>4. Conclusion:</u> We have shown that LHS = RHS and therefore by induction, $P(n)$ holds for all $n \in \mathbb{N}$. □

We have successfully proven Proposition 5.2! However, we could have also use the method of direct proof (not in this chapter). For fun, I will show direct proof (using the result from Proposition 5.1).

Second Proof of Proposition 5.2: From Proposition 5.1, it directly follows that

$$\begin{aligned} T_n + T_{n+1} &= \frac{n(n+1)}{2} + \frac{(n+1)((n+1)+1)}{2} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} \\ &= \frac{1}{2}(n^2+n+n^2+3n+2) = \frac{1}{2}(2n^2+4n+2) = n^2+2n+1 \\ &= (n+1)^2 \end{aligned}$$

Thus, our result is proven. □

Proposition 5.3: For every $n \in \mathbb{N}$, the product of the first n odd natural numbers equals $\frac{(2n)!}{2^n n!}$. In other words,

$$1 \cdot 3 \cdot 5 \cdot \ldots (2n-1) = \frac{(2n)!}{2^n n!}$$

Warning: This example is harder than the others, as it makes use of exponent properties and factorials, some of which you may have not encountered. I just think this a really cool example, but don't worry if you don't completely understand the solution.

Proof: Define $P(n)$: $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1) = \frac{(2n)!}{2^n n!}$. We will proceed by induction.

<u>1. Base Case:</u> The base case is $P(1)$. In other words, lets work out the LHS and RHS.

LHS: $(2(1) - 1) = 2 - 1 = 1$

RHS: $\frac{(2(1))!}{2^1 1!} = \frac{2}{2} = 1$

We have shown LHS = RHS, so the base case holds.

<u>2. Inductive Hypothesis:</u> Assume that $k \in \mathbb{N}$ such that $P(k)$: $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) = \frac{(2k)!}{2^k k!}$

<u>3. Induction Step:</u> We aim to prove the result holds for $k + 1$. That is, we want to show that,

$$1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2(k+1)-1) = \frac{(2(k+1))!}{2^{k+1} k+1!} \quad \text{inserting } k \to k+1$$

Writing it slightly differently, $1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1) \cdot (2k+1) = \frac{(2k+2)!}{2^{k+1} k+1!}$

Now we just have to simplify the LHS and RHS respectively.

3. Inductive Step Continued:
LHS:

$$\underbrace{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k-1)}_{\text{inductive hypothesis}}(2k+1) = \frac{(2k)!}{2^k k!} \cdot (2k+1)$$

Before we proceed, observe that $(k + 1)! = (k + 1) \times k \times \ldots 3 \times 2 \times 1$
$= (k + 1)\, k!$
and similarly $(2k + 2)! = (2k + 2) \times (2k + 1) \times (2k)!$

We can use this information to simplify the above and multiple the top and bottom by $(2k + 2)$, not changing the expression. In other words,

$$\frac{(2k)!}{2^k k!} \cdot (2k+1) = \frac{(2k)!(2k+1)}{2^k k!} \cdot \frac{(2k+2)}{(2k+2)} = \frac{(2k+2)!}{2^k k! 2(k+1)} = \frac{(2k+2)!}{2^{k+1}(k+1)!}$$

4. Conclusion: We have shown that LHS = RHS and therefore by induction, $P(n)$ holds for all $n \in \mathbb{N}$. □

Lets introduce some new notation: Define $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.

Proposition 5.4: For every $n \in \mathbb{N}_0$,. $1+2+4+8+\ldots+2^n = 2^{n+1} - 1$

Proof: Define $P(n)$: $1+2+4+8+\ldots+2^n = 2^{n+1} - 1$ We will proceed by induction.

1. Base Case: The base case is $P(0)$. In other words, lets work out the LHS and RHS.
LHS: $2^0 = 1$
RHS: $2^{0+1} - 1 = 1$
We have shown LHS = RHS, so the base case holds. Note: You can check other cases as well i.e. where $n = 1, 2, 3, \ldots$

2. Inductive Hypothesis: Assume that $k \in \mathbb{N}_0$ such that $P(k)$: $1+2+4+8+\ldots+2^k = 2^{k+1} - 1$

3. Induction Step: We aim to prove the result holds for $k + 1$. That is, we want to show that,
$1+2+4+8+\ldots+2^{k+1} = 2^{(k+1)+1} - 1$ inserting $k \to k + 1$
Writing it slightly differently, $1+2+4+8+\ldots+2^k+2^{k+1} = 2^{k+2} - 1$
Now we just have to simplify the LHS and RHS respectively. So,
LHS:

$$\underbrace{1+2+4+8+\ldots+2^k+\ldots}_{\text{inductive hypothesis}}+2^{k+1} = (2^{k+1}-1)+2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1 \quad \text{as desired.}$$

4. Conclusion: We have shown that LHS = RHS and therefore by induction, $P(n)$ holds for all $n \in \mathbb{N}_0$. □

One final remark: There are a lot more intricacies to mathematical induction, including the principle of strong induction (not covered in the text). Feel free to explore this topic on your own!