# Chapter 6: Sequences, Recurrence Relations, and Solving LHRR

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## 6.1: Sequences:

Before we talk about sequences and recurrence relations, lets establish some notation:

- The set of natural numbers is denoted $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- The set of whole numbers is denoted $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$
- The set of integers is denoted $\mathbb{Z} = \{0, 1, -1, 2, -2, \ldots\}$.
  - The set of positive integers (whole numbers) has the notation $\mathbb{Z}_{>0}$.
  - We can use the same logic for other such sets.
- The set of rational numbers is denoted $\mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z}, b \neq 0 \right\}$.
- The set of real numbers is denoted as $\mathbb{R} = \mathbb{Q} + \mathbb{Q}'$ (rationals + irrationals). Note: This is **not** is a complete description of the real numbers. A class such as real analysis will explore that in detail.

**We will talk more about basic set theory later on. These are just common symbols you will encounter when working with proofs and/or other topics.**

Definition 6.1: A sequence is an ordered list of of elements (not just numbers; they can also be functions). Furthermore, a sequence can be thought of as a function that maps a subset (the domain) $\mathbb{N}$ to a set $S$ (the range: the output space).

- We denote a sequence of terms as $\{a_n\}$.
- A sequence can start at any index, but by convention it usually starts at 0 or 1.
  - There are two types of sequences: finite sequences and infinite sequences.
    - A finite sequence is denoted $\{a_n\}_{n=m}^{n=k}$ with starting index $m$ and final index $k$, where $m \leq k$. In other words: $a_m, a_{m+1}, \ldots, a_k$.
    - An infinite sequence is denoted $\{a_n\}_{n=m}^{\infty}$ with starting index $m$ and defined for all indices $k$ such that $k \geq m$. In other words: $a_m, a_{m+1}, \ldots$
- Sometimes, a sequence can have a closed form (an explicit formula) for the $n^{\text{th}}$ term of the sequence.

Here's some examples of sequences:

- $\{2, 4, 6, 8, \ldots\}$
- $a_n = 2^n$, where $n \geq 1$
- $\{3k^2+1\}_{k=4}^{\infty}$
- $f(n) = 2n - 1$, where $n \geq 1$

Behavior of Sequences: Consider an infinite sequence $\{a_n\}_{j=1}^{\infty}$.

- $a_n$ is said to be increasing if: $a_1 \leq a_2 \leq a_3 \ldots \leq a_n \leq \ldots$
  - Ex: 1, 2, 2, 4, 5, 5, 8, …
- $a_n$ is said to be monotonic (strictly) increasing if: $a_1 < a_2 < a_3 \ldots < a_n < \ldots$
  - Ex: 1, 2, 3, 4, …
- $a_n$ is said to be decreasing if: $a_1 \geq a_2 \geq a_3 \ldots \geq a_n \geq \ldots$
  - Ex: 5, 4, 4, 3, -1, 0, 7, …
- $a_n$ is said to be monotonic (strictly) decreasing if: $a_1 > a_2 > a_3 \ldots > a_n > \ldots$
  - Ex: 5, 4, 3, 2, 1, …

There are two special types of sequences:

- <u>Arithmetic sequences</u>, where each term $a_n$ is found by adding the common ratio, $d$. As it turns out, there is a closed form for this: $a_n = a_1 + d(n - 1)$ where $n \geq 1$. Note: $a_1$ denotes the first term of the sequence.
- <u>Geometric sequences</u>, where each term $a_n$ is found by multiplying the common ratio, $r$. There is also a closed form for this: $a_n = a_1 \cdot r^{n-1}$ where $n \geq 1$.

Ex: Define $f(k) = k^2$ where $k \in \mathbb{N}_{\geq 2}$. Find the first three terms of the sequence.

Solution:

Observe that the starting index is $k = 2$. So, $f(2) = 2^2 = 4$, $f(3) = 3^2 = 9$, $f(4) = 4^2 = 16$. Therefore, the first three terms of the sequence $f(k)$ are $\{4, 9, 16\}$.

Ex: Consider the sequence $a_n = \{4, 9, 14, 19, 24, 29, \ldots\}$.

i. Find a close form for $a_n$.

ii. Find $a_{50}$.

Solution:

i. Notice that $a_n$ is in arithmetic progression i.e. $d = 5$ is being added to each term. Therefore, we can use the explicit formula for an arithmetic sequence.

Also, note that $a_1 = 4$. Therefore, $a_n = 4 + 5(n - 1)$.

ii. $a_{50} = 4 + 5(50 - 1) = 249$.

Ex: The 11th term of an arithmetic sequence is 52, and the 19th term is 92. Find an explicit formula for the sequence, $a_n$.

Solution: Since $a_n$ is an arithmetic sequence, we can use the formula $a_n = a_1 + d(n - 1)$. However, we don't know how to the first term nor the common difference. As a result, we can substitute the given information into the general formula and set-up/solve the system of equations. In other words,

$$\begin{cases} a_{11} = a_1 + d(11-1) = 52 \\ a_{19} = a_1 + d(19-1) = 92 \end{cases} \Rightarrow \begin{cases} a_1 + 10d = 52 \\ a_1 + 18d = 92 \end{cases} \Rightarrow (a_1, d) = (2, 5)$$

Therefore, the explicit formula is given by $a_n = 2 + 5(n - 1)$.

Ex: Consider the geometric sequence, $a_n$, such that $a_1 = 3$ and $r = 2$ and $n \in \mathbb{N}$.

i. Find a closed form for $a_n$.

ii. Find $a_{10}$.

Solution:

i. Since $a_n$ is a geometric sequence, we can use the explicit formula and substitute the given information. We already know the first term and the common ratio, so: $a_n = 3(2)^{n-1}$.

ii. $a_{10} = 3(2)^{10-1} = 1536$.

Ex: The third term of a geometric sequence is 63/4 and the sixth term is 1701/32. Find the fifth term.

Solution: Since $a_n$ is a geometric sequence, we can use the formula $a_n = a_1 \cdot r^{n-1}$ .

However, we don't know how to the first term nor the common difference. As a result, we can substitute the given information into the general formula and set-up/solve the system of equations.

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Ex Cont'd: After the substitution, we end up with the following system of equations:

$$\begin{cases} a_3 = a_1 \cdot r^{3-1} = \dfrac{63}{4} \\ a_6 = a_1 \cdot r^{6-1} = \dfrac{1701}{32} \end{cases}$$

We can do divide $a_6$ by $a_3$ to eliminate the $a_1$ term, thus making things slightly easier. In other words, $\frac{a_6}{a_3} = \frac{a_1 \cdot r^5}{a_1 \cdot r^2} = \frac{1701/32}{63/4} \Rightarrow r^3 = \frac{27}{8} \Rightarrow r = \sqrt[3]{\frac{27}{8}} = \frac{3}{2}$ . Now that we found $r$, we can substitute this into either of the two equations from above.

Let's substitute it into equation (1) to make life a little easier: $\frac{63}{4} = a_1 \left(\frac{3}{2}\right)^2 \Rightarrow \frac{9}{4} a_1 = \frac{64}{4} \Rightarrow a_1 = 7$ .

Thus, we found $a_1$ and $r$ i.e. $(a_1, r) = (7, 3/2)$. And after all that algebra we obtain $a_n = 7\left(\frac{3}{2}\right)^{n-1}$ . Using the explicit solution, it's clear $a_5 = 7\left(\frac{3}{2}\right)^{5-1} = 567/16$ .

## 6.2: Recurrence Relations

Definition 6.2: A recurrence relation is a rule that defines a term, $a_n$, as a function of previous terms in the sequence.

Some Special Recurrence Relations:

- For arithmetic sequences: $\begin{cases} a_0 = a \quad (\text{initial value}) \\ a_n = d + a_{n-1} \text{ for } n \geq 1 \quad (\text{recurrance relation}) \end{cases}$
- For geometric sequences: $\begin{cases} a_0 = a \quad (\text{initial value}) \\ a_n = r \cdot a_{n-1} \text{ for } n \geq 1 \quad (\text{recurrance relation}) \end{cases}$

Ex: Write a recursive rule for $a_n = -6 + 8n$.

Solution: Clearly, $a_0 = -6$ and $d = 8$. Therefore,

$$\begin{cases} a_0 = -6 \\ a_n = 8 + a_{n-1} \quad \text{where } n \geq 1 \end{cases}$$

Ex: Write an explicit formula for each recurrence relation:

i. $a_1 = 5$, $a_n = a_{n-1} - 2$.

ii. $a_1 = 10$, $a_n = 2a_{n-1}$.

Solution:

i. This is an arithmetic sequence since it's in the form (given above). Furthermore, $a_1 = 5$, is given and $d = -2$ from the recurrence equation. Therefore, $a_n = 5 - 2(n - 1)$.

ii. This is a geometric sequence since it's in the form (given above). Furthermore, $a_1 = 10$, is given and $r = 2$. Therefore, $a_n = 10(2)^{n-1}$.

Ex: A lake initially contains 500 fish. Each year, the population declines 30% due fishing and other causes, so the lake is restocked with 400 fish.

i. Write a recurrence relation, $a_n$, of fish at the start of the $n^{\text{th}}$ year.

ii. Using your answer from part (i), find the fish population after 5 years.

Ex Cont'd (Solution):

i. To solve this problem, it is best to make a table where $n$ is the # of years and $a_n$ is the fish population at year $n$. Also notice that 400 fish are added each year with a decline in population of 30%/year.

| $n$ | $a_n$ |
|---|---|
| 0 | 5200 |
| 1 | $(a_0 - 0.3a_0) + 400 = 0.7a_0 + 400$ # |
| 2 | $(a_1 - 0.3a_1) + 400 = 0.7a_1 + 400$ # |
| ⋮ | ⋮ |

Hopefully now, the pattern is obvious. Therefore we have: $\begin{cases} a_0 = 5200 \\ a_n = 0.7a_{n-1} + 400 \quad \text{where} \quad n \geq 1 \end{cases}$

ii. Using a graphing utility, the fish population after 5 years is given by $a_5 \cong 1984$ (fish).

For completeness, let's define the Fibonacci Sequence (which we will find a closed form for in section 6.3).

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \quad \text{where} \quad n \geq 2 \end{cases}$$

Notice that the next term is the sum of the previous two terms (you can check this for yourself).

Ex: Let $T_n$ be the sum of the first $n$ natural numbers.

i. Make a table of values for $n$, $T_n$, and $T_{n+1}$.

ii. Based on your table, write an expression (in terms of $n$) for $T_n + T_{n+1}$.

iii. Prove your result by mathematical induction on $n$.

Solution: Refer to Chapter 5 Notes.

## 6.3: Solving LHRR

We have seen examples of recurrence relations in section 6.2 We will expand on this a bit, in this section. Here's a table from zybooks on the different classifications for recurrence relations.

Table 6.3.1: Examples illustrating linear and non-linear recurrence relations.

| Recurrence relation | Type |
|---|---|
| $b_n = b_{n-1} + (b_{n-2} \cdot b_{n-3})$ | Non-linear |
| $c_n = 3n \cdot c_{n-1}$ | Non-linear |
| $d_n = d_{n-1} + (d_{n-2})^2$ | Non-linear |
| $f_n = 3f_{n-1} - 2f_{n-2} + f_{n-4} + n^2$ | Linear, degree 4. Non-homogeneous. |
| $g_n = g_{n-1} + g_{n-3} + 1$ | Linear, degree 3. Non-homogeneous. |
| $h_n = 2h_{n-1} - h_{n-2}$ | Linear, degree 2. Homogeneous. |
| $s_n = 2s_{n-1} - \sqrt{3}s_{n-5}$ | Linear, degree 5. Homogeneous. |

Definition 6.3: A <u>linear homogeneous recurrence relation (LHRR)</u> with constant coefficients and degree $k$ has the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \text{ where } c_1, \dots, c_k \in \mathbb{R} \text{ and } c_i \neq 0.$$

Motivation: Most of the time, recurrence relations can be hard to solve. As such, we assume the solution to the recurrence relation is of the form $a_n = r^n$, which allows us to convert it into a polynomial equation that is easier to solve. You will see why this assumption makes sense in the next couple of examples. It is assumed you have knowledge of: factoring, synthetic or polynomial division, the quadratic formula (or other such formulas to solve quadratic equations), and the rational roots theorem (although we probably won't need it). Finally, we will focus on recurrence relations with distinct and repeated roots of the characteristic equation. Let's get started!

Steps for Solving LHRR:

1. Assume $a_n = r^n$ is a solution to the recurrence relation.
2. Divide by $r^{n-k}$ to obtain the characteristic equation.
3. Solve the characteristic equation.
4. Assume the solution is of the form $a_n = \alpha_1 r^n + \alpha_2 r^n + \dots + \alpha_k r^n$ with coefficients $\alpha_1, \dots, \alpha_k$.

Lets apply these steps to the general case.

1. Assume $a_n = r^n$ is a solution to the recurrence relation. Therefore: $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$.
2. Divide by $r^{n-k}$: $\dfrac{r^n}{r^{n-k}} = \dfrac{c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}}{r^{n-k}} \Rightarrow r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k$
3. For simplicity, take the degree 2 characteristic equation. We have $r^2 - c_1 r - c_2 = 0$ and solve.
4. The basic solution is therefore $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ where $r_1$ and $r_2$ are roots of the equation.

This may seem confusing at first, so lets do an example!

Ex: Find a basic solution to $a_n = 7a_{n-1} - 10a_{n-2}$ where $n \in \mathbb{Z}_{\geq 2}$ with initial conditions $a_0 = 2$ and $a_1 = 1$.

Solution: We will go to through this 4-step process.

1. Assume $a_n = r^n$ is a solution to $a_n = 7a_{n-1} - 10a_{n-2}$. Therefore $r^n = 7r^{n-1} - 10r^{n-2}$ .
2. Divide by $r^{n-k}$ (with $k = 2$): $\dfrac{r^n}{r^{n-2}} = \dfrac{7r^{n-1} - 10r^{n-2}}{r^{n-2}} \Rightarrow r^2 = 7r - 10 \Rightarrow r^2 - 7r - 10 = 0$ .
3. Solving the characteristic equation: $r^2 - 7r - 10 = 0 \Rightarrow (r-2)(r-5) = 0 \Rightarrow r_1 = 2, r_2 = 5$ .
4. Substitute this into the basic solution and use initial conditions: $a_n = \alpha_1 \cdot 2^n + \alpha_2 \cdot 5^n$ .
   This leads to a system of equations:
   $$\begin{cases} a_0 = \alpha_1 \cdot 2^0 + \alpha_2 \cdot 5^0 = 2 \\ a_1 = \alpha_1 \cdot 2^1 + \alpha_2 \cdot 5^1 = 1 \end{cases} \Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 2 \\ 2\alpha_1 + 5\alpha_2 = 1 \end{cases} \Rightarrow (\alpha_1, \alpha_2) = (3, -1)$$

So, the basic solution with the initial conditions is given by: $a_n = 3 \cdot 2^n - 5^n$

Theorem 6.4: Taking the case of second degree recurrence relations (although this can be generalized to degree $k$), $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ is a solution to $a_n = c_1 a_{n-1} + c_2 + a_{n-2}$ .

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Proof of Theorem 6.4: We want to show that $a_n=\alpha_1 r_1^n+\alpha_2 r_2^n$ is a solution to $a_n=c_1 a_{n-1}+c_2+a_{n-2}$ . By our assumption, we can substitute this into the recurrence relation. In other words,

$$a_n=c_1(\alpha_1 r_1^{n-1}+\alpha_2 r_2^{n-1})+c_2(\alpha_1 r_1^{n-2}+\alpha_2 r_2^{n-2})=c_1\alpha_1 r_1^{n-1}+c_2\alpha_1 r_1^{n-2}+c_1\alpha_2 r_2^{n-1}+c_2\alpha_2 r_2^{n-2}$$

Grouping and factoring terms: $a_n=\alpha_1 r_1^{n-2}(c_1 r_1+c_2)+\alpha_2 r_2^{n-2}(c_1 r_2+c_2)$

Recall from our assumption that $r_1$ and $r_2$ are roots of the second-order characteristic equation i.e. $r^2-c_1 r-c_2=0$ .

Substituting roots $\{r_1, r_2\}$ we obtain: $\begin{cases} r_1^2-c_1 r_1-c_2=0 \\ r_2^2-c_1 r_2+c_2=0 \end{cases} \Rightarrow \begin{cases} r_1^2=c_1 r_1+c_2 \\ r_2^2=c_1 r_2+c_2 \end{cases}$

So, we have $a_n=\alpha_1 r_1^{n-2} r_1^2+\alpha_2 r_2^{n-2} r_2^2=\alpha_1 r_1^n+\alpha_2 r_2^n$ , which is exactly we assumed. Therefore, our assumption is correct.

Now we want to show ∃ $\alpha_1$, $\alpha_2$ such that $a_n=\alpha_1 r_1^n+\alpha_2 r_2^n$ satisfies initial conditions $a_0 = k_1$ and $a_1 = k_2$.

Solving the 2 by 2 system of equations, we obtain $\begin{cases} \alpha_1=\dfrac{k_1-k_0 r_2}{r_1-r_2} \\ \alpha_2=\dfrac{k_0 r_1-k_1}{r_1-r_2} \end{cases}$ .

Hence, Theorem 6.4 is proven true. □

In general, for the linear characteristic equation $p(x) = 0$ where $r$ is a root with multiplicity $m$ (a repeated root), then the following satisfy the recurrence relation:

$$a_n=r^n, a_n=n\cdot r^n, a_n=n^2\cdot r^n, \dots, a_n=n^{m-1}\cdot r^n .$$

As promised, we will now find a closed form solution for the $n^{\text{th}}$ term of the Fibonacci Sequence. To do so, lets use the 4-step process:

1. Assume $F_n = r^n$ is a solution to $F_n = F_{n-1} + F_{n-2}$ where $n \in \mathbb{Z}_{\geq 2}$ with initial conditions $F_0 = 0$ and $F_1 = 1$. Therefore, $r^n=r^{n-1}+r^{n-2}$ .
2. Divide by $r^{n-k}$ (with $k = 2$): $\dfrac{r^n}{r^{n-2}}=\dfrac{r^{n-1}+r^{n-2}}{r^{n-2}} \Rightarrow r^2=r+1 \Rightarrow r^2-r-1=0$ .
3. Solving the characteristic equation:
   $r^2-7r-10=0 \Rightarrow r=\dfrac{1}{2}\pm\sqrt{\left(-\dfrac{1}{2}\right)^2+1} \Rightarrow r_1=\dfrac{1+\sqrt{5}}{2}, r_2=\dfrac{1-\sqrt{5}}{2}$ .
4. Substitute this into the basic solution and use initial conditions:
   $F_n=\alpha_1\cdot\left(\dfrac{1+\sqrt{5}}{2}\right)^n+\alpha_2\cdot\left(\dfrac{1-\sqrt{5}}{2}\right)^n$ .This leads to a system of equations:

$$\begin{cases} F_0=\alpha_1\cdot\left(\dfrac{1+\sqrt{5}}{2}\right)^0+\alpha_2\cdot\left(\dfrac{1-\sqrt{5}}{2}\right)^0=0 \\ F_1=\alpha_1\cdot\left(\dfrac{1+\sqrt{5}}{2}\right)^1+\alpha_2\cdot\left(\dfrac{1-\sqrt{5}}{2}\right)^1=1 \end{cases} \Rightarrow \begin{cases} \alpha_1+\alpha_2=0 \\ \alpha_1\cdot\left(\dfrac{1+\sqrt{5}}{2}\right)+\alpha_2\cdot\left(\dfrac{1-\sqrt{5}}{2}\right)=1 \end{cases} \Rightarrow (\alpha_1,\alpha_2)=\left(\frac{1}{\sqrt{5}},-\frac{1}{\sqrt{5}}\right)$$

So, the basic solution with the initial conditions is given by: $F_n=\dfrac{1}{\sqrt{5}}\cdot\left(\dfrac{1+\sqrt{5}}{2}\right)^n-\dfrac{1}{\sqrt{5}}\cdot\left(\dfrac{1-\sqrt{5}}{2}\right)^n$