# Discrete Math Chapter 2 (Attempt) Solutions

**Question 1.** (Exercise 2.3 part (c)) Prove that the product of two odd integers is odd.
**Solution:**
Let $m$ and $n$ be odd integers. By definition of oddness, there exist integers $a$ and $b$ such that $m = 2a + 1$ and $n = 2b + 1$. Now compute:

$$\begin{aligned} mn &= (2a+1)(2b+1) \\ &= 4ab + 2a + 2b + 1 \\ &= 2(2ab + a + b) + 1. \end{aligned}$$

Since $2ab + a + b$ is an integer, it follows that $mn$ is of the form $2k + 1$, which by definition is odd. Hence, the product of two odd integers is odd.

**Question 2.** (Nearly Exercise 2.5(a)) Suppose that $n$ is an integer. Prove that if $n$ is odd, then $n^2 + 6n + 5$ is even.
**Solution:**
Assume $n$ is odd. Then there exists an integer $a$ such that $n = 2a + 1$. Compute:

$$\begin{aligned} n^2 + 6n + 5 &= (2a+1)^2 + 6(2a+1) + 5 \\ &= 4a^2 + 4a + 1 + 12a + 6 + 5 \\ &= 4a^2 + 16a + 12 \\ &= 2(2a^2 + 8a + 6). \end{aligned}$$

Since $2a^2 + 8a + 6$ is an integer, this expression is even by definition. Therefore, if $n$ is odd, then $n^2 + 6n + 5$ is even.

**Question 3.** (Nearly Exercise 2.8(b)) If $n$ is an integer, then $5n^2 + n + 3$ is odd.
**Example:** Let $n = 3$. Then:

$$5n^2 + n + 3 = 5(9) + 3 + 3 = 51,$$

which is odd.
**Proof:**
We proceed by cases based on the parity of $n$.
**Case 1:** $n$ is even. Then $n = 2a$ for some integer $a$. Compute:

$$\begin{aligned} 5n^2 + n + 3 &= 5(2a)^2 + 2a + 3 \\ &= 20a^2 + 2a + 3 \\ &= 2(10a^2 + a + 1) + 1. \end{aligned}$$

This is of the form $2k+1$, hence odd.

**Case 2:** $n$ is odd. Then $n = 2a+1$ for some integer $a$. Compute:

$$\begin{aligned} 5n^2 + n + 3 &= 5(2a+1)^2 + (2a+1) + 3 \\ &= 5(4a^2 + 4a + 1) + 2a + 1 + 3 \\ &= 20a^2 + 20a + 5 + 2a + 1 + 3 \\ &= 20a^2 + 22a + 9 \\ &= 2(10a^2 + 11a + 4) + 1. \end{aligned}$$

Again, this is of the form $2k+1$, and is odd. Thus, $5n^2 + n + 3$ is odd for all integers $n$.

**Question 4.** (Exercise 2.10 (a) and (c))
**(a)** Prove that if $m \mid n$, then $m^2 \mid n^2$.
**Solution:**
If $m \mid n$, then by definition of divisibility, $n = md$ for some integer $d$. Then:

$$n^2 = (md)^2 = m^2 d^2.$$

Since $d^2$ is an integer, it follows that $m^2 \mid n^2$.
**(c)** Prove that if $m \mid n$ and $m \mid t$, then $m \mid (n+t)$.
**Solution:**
If $m \mid n$ and $m \mid t$, then $n = md$ and $t = m\ell$ for some integers $d$ and $\ell$. Then:

$$n + t = md + m\ell = m(d + \ell),$$

which shows $m \mid (n+t)$.

**Question 5.** (Exercise 2.15)
**(a)** Prove that 4 divides $1 + (-1)^n(2n-1)$ for all integers $n$.
**Solution:**
We proceed by cases.

**Case 1:** $n$ is even, $n = 2a$. Then:

$$1 + (-1)^n(2n-1) = 1 + (2n-1) = 1 + (4a-1) = 4a.$$

Hence, divisible by 4.

**Case 2:** $n$ is odd, $n = 2a+1$. Then:

$$1 + (-1)^n(2n-1) = 1 - (4a+1) = -4a.$$

Again, divisible by 4. So in all cases, $4 \mid 1 + (-1)^n(2n-1)$.
**(b)** Prove that every multiple of 4 can be written as $1 + (-1)^n(2n-1)$ for some positive integer $n$.
**Solution:**
Let $4k$ be any multiple of 4.

If $k > 0$, let $n = 2k$. Then:

$$1 + (-1)^n(2n-1) = 1 + (4k-1) = 4k.$$

If $k \leq 0$, let $n = -2k + 1$, which is positive. Then:

$$1 + (-1)^n(2n - 1) = 1 - (-4k + 1) = 4k.$$

Thus, for any multiple of 4, such an $n$ exists.

**Question 6.** (Nearly Exercise 2.16)
**Solution:**
(a) $17 \div 5$: quotient $q = 3$, remainder $r = 2$. (b) $5 \div 17$: $q = 0$, $r = 5$. (c) $-10 \div 3$: $q = -4$, $r = 2$ (since $-10 = -4 \cdot 3 + 2$).

**Question 7.** (Nearly Exercise 2.20) Determine the remainder when $4^{301}$ is divided by 17.
**Solution:**
First, observe:

$$4^2 = 16 \equiv -1 \pmod{17}.$$

Then:

$$(4^2)^{150} = 16^{150} \equiv (-1)^{150} = 1 \pmod{17}.$$

Thus:

$$4^{300} \equiv 1 \pmod{17}, \quad \text{and } 4^{301} = 4^{300} \cdot 4 \equiv 1 \cdot 4 = 4 \pmod{17}.$$

So the remainder is 4.

**Question 8.** (Exercise 2.21) Suppose $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$. Prove:
**(a)** $a - c \equiv b - d \pmod{m}$
**Solution:**
Given $m \mid (a - b)$ and $m \mid (c - d)$. Subtract:

$$(a - c) - (b - d) = (a - b) - (c - d) \Rightarrow m \mid [(a - c) - (b - d)],$$

which implies $a - c \equiv b - d \pmod{m}$.
**(b)** $ac \equiv bd \pmod{m}$
**Solution:**
From above, write $a = b + mk$, $c = d + m\ell$ for some integers $k, \ell$. Then:

$$\begin{aligned} ac &= (b + mk)(d + m\ell) = bd + bm\ell + dmk + m^2k\ell \\ &= bd + m(b\ell + dk + mk\ell). \end{aligned}$$

So $ac - bd$ is divisible by $m$, and thus $ac \equiv bd \pmod{m}$.

**Question 9.** (Exercise 2.22) Prove that if $p \mid a$ and $q \mid a$ with $p, q$ distinct primes, then $pq \mid a$.
**Examples:**
$6 \mid 60$ since $2 \mid 60$ and $3 \mid 60$
$15 \mid 90$ since $3 \mid 90$ and $5 \mid 90$
$35 \mid 105$ since $5 \mid 105$ and $7 \mid 105$

**Proof:**
Let $a$ be an integer such that $p \mid a$ and $q \mid a$, with $\gcd(p, q) = 1$ (since both are primes). Then:

$$a = pk = q\ell \Rightarrow pk = q\ell.$$

Since $\gcd(p, q) = 1$, it follows from the property of divisibility that $q \mid k$, i.e., $k = qt$. Thus:

$$a = pqt \Rightarrow pq \mid a.$$

**Question 10.** (Exercise 2.25) Prove that for every integer $n$, either $n^2 \equiv 0 \pmod 4$ or $n^2 \equiv 1 \pmod 4$.
**Solution:**
**Case 1:** $n$ is even. Then $n = 2a$, so:

$$n^2 = 4a^2 \Rightarrow n^2 \equiv 0 \pmod 4.$$

**Case 2:** $n$ is odd. Then $n = 2a + 1$, so:

$$n^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1 \Rightarrow n^2 \equiv 1 \pmod 4.$$

Thus, the result holds for all integers $n$.