Discrete Math Chapter 6 (Attempt) Solutions

Question 1. (Exercise 6.1) Explain in your own words the difference between the contrapositive, the converse, and a counterexample.
Solution:
- The
textbfcontrapositive of "If P, then Q" is "If not Q, then not P." It is logically equivalent to the original implication.
- The
textbfconverse is "If Q, then P." It may not be logically equivalent.
- A
textbfcounterexample disproves a general statement by showing one case where the premise is true but the conclusion is false.

Question 2. (Exercise 6.2) Give 4 examples of implications, and for each write the contrapositive.
Solution:
Real-world:
(1) If it rains, then the ground gets wet.
Contrapositive: If the ground is not wet, then it did not rain.
(2) If I study hard, then I will pass the test.
Contrapositive: If I do not pass the test, then I did not study hard.
Mathematical:
(3) If n is even, then n^2 is even.
Contrapositive: If n^2 is not even, then n is not even.
(4) If $x > 3$, then $x^2 > 9$.
Contrapositive: If $x^2 \leq 9$, then $x \leq 3$.

Question 3. (Nearly Exercise 6.3)
(a) What is the contrapositive of: "If $2n^2 5n + 3$ is odd, then n is even"?
Solution:
If n is not even, then $2n^2 5n + 3$ is not odd.

(b) Suppose $n \in \mathbb{Z}$. Prove that if $2n^2 5n + 3$ is odd, then n is even.
Proof:
We prove the contrapositive: If n is not even (i.e., n is odd), then $2n^2 5n + 3$ is even.

Let $n = 2k + 1$ for $k \in \mathbb{Z}$. Then:

$$\begin{aligned} 2n^2 5n + 3 &= 2(2k+1)^2 5(2k+1) + 3 \\ &= 2(4k^2 + 4k + 1)10k5 + 3 \\ &= 8k^2 + 8k + 210k2 \\ &= 8k^2 2k = 2(4k^2 k). \end{aligned}$$

Which is even. Thus, the original implication is true.

Question 4. (Nearly Exercise 6.5(c)) Prove: If $n^2 + 2n + 3$ is even, then n is odd.
Proof:
We use the contrapositive. Suppose n is not odd, i.e., n is even. Let $n = 2k$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} n^2 + 2n + 3 &= (2k)^2 + 2(2k) + 3 \\ &= 4k^2 + 4k + 3 = 2(2k^2 + 2k + 1) + 1. \end{aligned}$$

Which is odd, i.e., not even. Hence, by contrapositive, if $n^2 + 2n + 3$ is even, then n is odd.

Question 5. (Exercise 6.6(d)) Suppose $n \in \mathbb{Z}$. Prove that if $3 \mid (n^2 1)$, then $3 \mid n$.
Proof:
We prove the contrapositive: If $3 \nmid n$, then $3 \mid (n^2 1)$.
By the division algorithm, if $3 \nmid n$, then $n = 3q + 1$ or $3q + 2$.
Case 1: $n = 3q + 1$:

$$n^2 1 = (3q+1)^2 1 = 9q^2 + 6q = 3(3q^2 + 2q).$$

Case 2: $n = 3q + 2$:

$$n^2 1 = (3q+2)^2 1 = 9q^2 + 12q + 3 = 3(3q^2 + 4q + 1).$$

In both cases, $3 \mid (n^2 1)$. Thus, the implication holds.

Question 6. (Exercise 6.8(e)) Suppose $x \in \mathbb{R}$. Prove: If $x^3 + x > 0$, then $x > 0$.
Proof:
We prove the contrapositive: If $x \leq 0$, then $x^3 + x \leq 0$.
Since $x^2 \geq 0$, multiplying both sides of $x \leq 0$ by x^2 (which is non-negative) gives $x^3 \leq 0$.
Also, since $x \leq 0$, adding the two inequalities: $x^3 + x \leq 0$.

Question 7. (Exercise 6.10) Let $F_1 = F_2 = 1$, and $F_n = F_{n1} + F_{n2}$ for $n \geq 3$. Prove: If F_n is not a perfect cube, then $n \notin \{1, 2, 6\}$.
Proof:
We prove the contrapositive: If $n \in \{1, 2, 6\}$, then F_n is a perfect cube.

- $F_1 = 1 = 1^3$
- $F_2 = 1 = 1^3$
- $F_6 = 8 = 2^3$

So in each case, F_n is a perfect cube. Hence, the original statement is true.

Question 8. (Exercise 6.12(c)) Prove that $(n+1)^2 1$ is even if and only if n is even.
Proof:
(⇒) Assume n is even. Let $n = 2k$. Then:

$$(n+1)^2 1 = (2k+1)^2 1 = 4k^2 + 4k = 2(2k^2 + 2k).$$

So it is even.

(⇐) Assume $(n+1)^2 1$ is even. We prove the contrapositive: if n is odd, then $(n+1)^2 1$ is odd.
Let $n = 2k + 1$. Then:

$$(n+1)^2 1 = (2k+2)^2 1 = 4k^2 + 8k + 41 = 2(2k^2 + 4k + 1) + 1.$$

Which is odd. Hence, original claim holds.

Question 9. (Exercise 6.15) Come up with a false real-world claim and disprove it by counterexample.
Claim: No math textbook has ever included a meme.
Counterexample: Jay Cummings' "Proofs: A Long-Form Mathematics Textbook" includes memes.

Question 10. (Exercise 6.16 parts (d), (i), (o)) The following statements are all false. Give a counterexample for each.
(d) Claim: If $|x + y| = |xy|$, then $y = 0$.
Counterexample: Let $x = 0$, $y = 4$. Then $|x + y| = |4| = 4 = |xy| = |-4|$ but $y \neq 0$.
(i) Claim: If $n \in \mathbb{N}$, then $2n^2 4n + 31$ is prime.
Counterexample: Let $n = 31$. Then:

$$2n^2 4n + 31 = 2(31)^2 4(31) + 31 = 1922 = 31 \cdot 62.$$

Not prime.
(o) Claim: $A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$.
Counterexample: Let $A = \{1, 2, 3\}$, $B = \{2\}$, $C = \{3\}$. Then:

$$A \setminus (B \cap C) = A \setminus \emptyset = A = \{1, 2, 3\},$$

$$(A \setminus B) \cap (A \setminus C) = \{1, 3\} \cap \{1, 2\} = \{1\}.$$

Sets are not equal. Hence, the statement is false.