Discrete Homework 1 (Attempt) Solutions

Question 1. (Exercise 1.2) Explain the error in the following "proof" that $2 = 1$:
Let $x = y$. Then:

$$\begin{aligned} x^2 &= xy \\ x^2 - y^2 &= xy - y^2 \\ (x+y)(x-y) &= y(x-y) \\ x + y &= y \\ 2y &= y \\ 2 &= 1. \end{aligned}$$

Solution:
The critical flaw in this argument occurs when moving from $(x+y)(x-y) = y(x-y)$ to $x + y = y$. Since we assumed $x = y$, it follows that $x - y = 0$. This implies that we are dividing both sides of the equation by 0, which is undefined in mathematics. Division by zero is not a permissible algebraic operation because it leads to contradictions and loss of information about the original equation. Therefore, the step from line 3 to line 4 is invalid. Additionally, in the final step, $2y = y$ does not imply $2 = 1$ unless $y \neq 0$, but in our assumption, $x = y$ and $x - y = 0$, which forces $y = 0$, making all the later manipulations undefined.

Question 2. (Exercise 1.5) If I remove two squares of different colors from an 8×8 chessboard, must the result have a perfect cover?
Solution:
Yes, the result will always have a perfect cover if the two removed squares are of opposite colors. Here is why: a domino covers two adjacent squares on the board, which are always of opposite color (since adjacent squares alternate in color). A standard 8×8 chessboard contains 32 white and 32 black squares. Removing two squares of opposite color results in a board with 31 squares of each color. Since each domino covers one white and one black square, we need an equal number of each color to perfectly tile the board. Hence, the condition is satisfied. In fact, the board can always be covered by dominoes in such cases, and one can construct such a covering by using a Hamiltonian path that visits each square exactly once and placing dominoes on adjacent pairs along the path.

Question 3. (Exercise 1.6) If I remove four squares—two black, two white—from an 8×8 chessboard, must the result have a perfect cover?
Solution:
No, a perfect cover is not guaranteed. A counterexample demonstrates the issue. Consider removing the four squares $a1$, $a2$, $b1$, and $b2$. These form the top-left 2×2 corner of the board. Once these four squares are removed, the remaining portion of the board cannot be completely tiled with dominoes. Specifically, the square $a3$ now only has one adjacent square ($b3$), and this pattern causes a mismatch in connectivity, where the board becomes locally untileable despite the color count being balanced. Therefore, color balancing is a necessary but not sufficient condition for tiling with dominoes.

Question 4. (Exercise 1.7)
(a) Suppose there is a knight on every square of a 7×7 chessboard. Is it possible for every one of these knights to simultaneously make a legal move?
Solution:
No, it is not possible. First, a 7×7 board has 49 squares. A knight always moves from a square of one color to a square of the opposite color. Since the board has an odd number of squares, the number of black squares is 25 and the number of white squares is 24. If a knight occupies each square and all knights move simultaneously, each must move to a square of the opposite color. But that would mean 25 knights are trying to land on 24 squares of the opposite color, which is impossible due to the pigeonhole principle. Hence, such a simultaneous legal move for all knights cannot occur.
(b) Suppose there is a knight on every square of an 8×8 chessboard. Is it possible for every one of these knights to simultaneously make a legal move?
Solution:
Yes, it is possible. An 8×8 chessboard has an even number of both white and black squares (32 each). There exists a well-known construct called a knight's tour, which is a sequence of knight moves that visits every square exactly once. If every knight moves along this tour to the next square, each knight lands on a unique square and every move is legal. Thus, such a configuration allows all knights to move simultaneously without conflicts.

Question 5. (Exercise 1.9) Let n be a positive integer. Prove that if one selects any $n + 1$ numbers from the set $\{1, 2, 3, ..., 2n\}$, then two of the selected numbers must sum to $2n + 1$.
Examples: From $\{1, 2, 3, 4, 5, 6\}$, choose $\{1, 3, 4, 5\}$. We have $3 + 4 = 7$.
From $\{1, 2, ..., 8\}$, choose $\{1, 3, 4, 5, 6\}$. We have $4 + 5 = 9$. From $\{1, ..., 10\}$, choose $\{1, 2, 5, 7, 8, 9\}$. We have $2 + 9 = 11$.
Proof:
We partition the set $\{1, 2, ..., 2n\}$ into n disjoint pairs: $\{1, 2n\}, \{2, 2n - 1\}, ..., \{n, n + 1\}$. Each pair sums to $2n+1$. These are our n pigeonholes. Now, we are selecting $n+1$ elements from the full set, so by the pigeonhole principle, at least one pair must be completely chosen. Hence, some two numbers among the selected ones must sum to $2n + 1$, as desired.

Question 6. (Exercise 1.16) Give an example of 100 numbers from $\{1, 2, ..., 200\}$ such that none divides any other.

Solution:

Consider the set $\{101, 102, ..., 200\}$, which contains exactly 100 elements. For any two distinct numbers $a < b$ in this set, $b/a < 2$, which means b is not at least twice as large as a, and hence a cannot divide b. Therefore, no element in this set divides any other. This construction proves that Proposition 1.11, which states that any set of more than 100 elements from $\{1, 2, ..., 200\}$ must contain a pair where one divides the other, is optimal.

Question 7. (Exercise 1.17) Prove that any set of seven integers contains a pair whose sum or difference is divisible by 10. Also, before your proof, write down three different sets of seven integers, and for each set locate a pair whose sum or difference is divisible by 10.

Examples:

Set 1: $\{-3, 7, 15, 2, 11, -8, 0\} \to 7 + 3 = 10$
Set 2: $\{-14, 16, 1, 9, 23, -21, 0\} \to 1 + 9 = 10$
Set 3: $\{4, -6, 13, -3, 5, 12, 8\} \to -6 + 16 = 10$

Proof:

Let the objects be the last digits (modulo 10) of the integers. There are 10 residue classes modulo 10: $\{0, 1, ..., 9\}$. We group them into 6 classes that can potentially result in sums or differences divisible by 10: $\{0\}, \{5\}, \{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}$. Each group contains residues that sum to 10 or differ by 10. With 7 integers and only 6 groups, by the pigeonhole principle at least one group must contain at least two numbers. There are two cases:

- If two numbers have the same last digit (e.g., both end in 0), then their difference is divisible by 10.
- If two numbers belong to a group like $\{1, 9\}$ or $\{3, 7\}$, then their sum is divisible by 10.

Thus, any set of 7 integers must contain such a pair.

Question 8. (Exercise 1.18) Prove that if one chooses any 19 points from the interior of a 6×4 rectangle and no three of them form a straight line, then there must exist four of these points which form a quadrilateral of area at most 4.

Solution:

We partition the 6×4 rectangle into six 2×2 subrectangles. Each of these has an area of 4. Consider each such square to be a box. We now place each of the 19 chosen points into the square (box) that contains it. By the pigeonhole principle, since there are 6 boxes and 19 points, at least one box must contain at least $\lceil 19/6 \rceil = 4$ points.

Within a 2×2 square, any four points, assuming no three are collinear, must form a quadrilateral. Since this quadrilateral is entirely contained within the 2×2 square, its area must be at most 4. Therefore, the condition is satisfied.

Question 9. (Exercise 1.22) The following conjectures are all false. Prove that they are false by finding a counterexample to each.

(a) Conjecture: If x and y are real numbers, then $|x+y| = |x| + |y|$.

Counterexample: Let $x = -2$, $y = 1$. Then:

$$|x+y| = |-1| = 1 \quad \text{but} \quad |x| + |y| = 2 + 1 = 3.$$

Hence, $|x+y| \neq |x| + |y|$.

(b) Conjecture: If x is a real number, then $x^2 < x^4$.

Counterexample: Let $x = 0.5$. Then:

$$x^2 = 0.25 \quad \text{and} \quad x^4 = 0.0625,$$

so $x^2 > x^4$. Therefore, the inequality fails for $0 < x < 1$.

(c) Conjecture: Suppose x and y are real numbers. If $|x+y| = |x-y|$, then $y = 0$.

Counterexample: Let $x = 0$, $y = 2$. Then:

$$|x+y| = |2| = 2, \quad |x-y| = |-2| = 2,$$

but $y \neq 0$. Thus, the conjecture is false.

Question 10. (Exercise 1.27) Prove that, for every $n \geq 2$, there does not exist an $n \times n$ antimagic square where each entry is -1, 0, or 1.

Solution:

In an $n \times n$ matrix, there are n rows, n columns, and 2 diagonals, totaling $2n+2$ line sums. Each entry is from the set $\{-1, 0, 1\}$, so the sum of any row/column/diagonal lies in the interval $[-n, n]$, which gives at most $2n+1$ possible distinct values.

Since there are $2n+2$ sums and only $2n+1$ possible values, the pigeonhole principle implies that at least two of the sums must be equal. Thus, it is impossible for all $2n+2$ sums to be distinct, so the square cannot be antimagic.