Discrete Math Chapter 4 (Attempt) Solutions

Question 4. (Exercise 4.5(e)) Prove that $1 + 2^n \leq 3^n$ for all $n \in \mathbb{N}$.
Solution:
We prove this by induction on n.

Base Case: Try $n = 1, 2, 3$.

$$1 + 2^1 = 3 \leq 3^1 = 3, \quad 1 + 2^2 = 5 \leq 9 = 3^2, \quad 1 + 2^3 = 9 \leq 27 = 3^3.$$

All hold.

Inductive Step: Assume $1 + 2^k \leq 3^k$ for some $k \in \mathbb{N}$. Then:

$$\begin{aligned} 1 + 2^{k+1} &= 1 + 2 \cdot 2^k = 2(1 + 2^k) - 1 \\ &\leq 2 \cdot 3^k - 1 \leq 3^{k+1}. \end{aligned}$$

Since $3^k > 0$, and $2 \cdot 3^k - 1 < 3 \cdot 3^k$, this confirms the step.

Conclusion: The inequality holds for all $n \in \mathbb{N}$.

Question 5. (Exercise 4.8) Prove that $\tilde{F}_0 \cdot \tilde{F}_1 \cdot \cdots \cdot \tilde{F}_n = \tilde{F}_{n+1} - 2$ where $\tilde{F}_n = 2^{2^n} + 1$.
Solution:
We prove by induction on n.

Base Case: $n = 0$. Then:

$$\tilde{F}_0 = 2^{2^0} + 1 = 2^1 + 1 = 3, \quad \tilde{F}_1 = 2^{2^1} + 1 = 5.$$

And:

$$\tilde{F}_0 = \tilde{F}_1 - 2.$$

True.

Inductive Step: Assume $\tilde{F}_0 \cdot \tilde{F}_1 \cdot \cdots \cdot \tilde{F}_k = \tilde{F}_{k+1} - 2$. Multiply both sides by $\tilde{F}_{k+1}$:

$$\begin{aligned} (\tilde{F}_{k+1} - 2) \cdot \tilde{F}_{k+1} &= \tilde{F}_{k+1}^2 - 2\tilde{F}_{k+1} \\ &= (2^{2^{k+1}} + 1)^2 - 2(2^{2^{k+1}} + 1) \\ &= 2^{2^{k+2}} - 1 = \tilde{F}_{k+2} - 2. \end{aligned}$$

Thus, the result holds for $k + 1$.

Conclusion: By induction, the identity holds for all $n \in \mathbb{N}_0$.

Question 6. (Exercise 4.10) Explain the error in the "proof" of the Fake Proposition 4.11 that claims all people have the same name.

Solution:
The flaw is in the induction step. The argument assumes that in a group of $k+1$ people, the first k and the last k overlap, ensuring a common name. However, for $k = 1$, the two groups are disjoint (each has just one person). Without overlap, there is no link to establish identical names. Thus, the logic fails at $k = 1$.

Question 7. (Exercise 4.15) Prove that if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.
Solution:
We prove by induction on n.
Base Case: $n = 0$. Then $A = \emptyset$. $\mathcal{P}(A) = \{\emptyset\}$, so $|\mathcal{P}(A)| = 1 = 2^0$.
Inductive Step: Assume $|\mathcal{P}(A)| = 2^k$ for a set A with $|A| = k$.
Let A' be a set of size $k+1$. Choose an element $a \in A'$. Then:

- Subsets that contain a: One for each subset of $A' \setminus \{a\}$.
- Subsets that do not contain a: All subsets of $A' \setminus \{a\}$.

So:
$$|\mathcal{P}(A')| = 2^k + 2^k = 2^{k+1}.$$
Conclusion: The number of subsets of a set of size n is 2^n.

Question 8. (Exercise 4.24) Disprove the conjecture: $1 + \frac{1}{2} + \cdots + \frac{1}{n} < 3$ for all $n \in \mathbb{N}$.
Solution:
Try $n = 11$:
$$H_{11} = 1 + \frac{1}{2} + \cdots + \frac{1}{11} \approx 3.019.$$
This is greater than 3, so the conjecture is false. A correct bound would involve $\ln(n) + \gamma$ where γ is Euler-Mascheroni constant.

Question 9. (Exercise 4.28) Prove that for all $n \geq 4$, we can place n non-attacking rooks on an $n \times n$ board with none on either diagonal.
Solution:
We proceed by strong induction.
Base Cases: For $n = 4, 5, 6, 7$, one can explicitly construct such placements (omitted here for brevity).
Inductive Step: Assume we can place $k, k+1, k+2, k+3$ rooks as required. Consider a $(k+4) \times (k+4)$ board.
Place 4 rooks in the corners of the board so that they avoid diagonals (e.g., top two rows and bottom two columns). Then, place k non-attacking rooks in the center $k \times k$ subgrid, which also avoids diagonals by hypothesis.
Thus, all $k+4$ rooks are non-attacking and avoid both diagonals.
Conclusion: By induction, such placements exist for all $n \geq 4$.

Question 10. (Exercise 4.31(a)) Prove that:
$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

Solution:
We use induction on n.

Base Case: $n = 1$. Left: $F_1 = 1$, Right: $F_3 - 1 = 2 - 1 = 1$. True.

Inductive Step: Assume:

$$F_1 + \cdots + F_k = F_{k+2} - 1.$$

Then:

$$\begin{aligned} F_1 + \cdots + F_{k+1} &= F_{k+2} - 1 + F_{k+1} \\ &= F_{k+1} + F_{k+2} - 1 = F_{k+3} - 1. \end{aligned}$$

By Fibonacci definition, this holds.

Conclusion: The identity is true for all $n \in \mathbb{N}$.