Example 1 Convert $40° 20'$ into radian measure.

Solution We know that $180° = \pi$ radian.

Hence $40° 20' = 40\frac{1}{3}$ degree $= \frac{\pi}{180} \times \frac{121}{3}$ radian $= \frac{121\pi}{540}$ radian.

Therefore $40° 20' = \frac{121\pi}{540}$ radian.

Example 2 Convert 6 radians into degree measure.

Solution We know that π radian $= 180°$.

Hence

$$6 \text{ radians} = \frac{180}{\pi} \times 6 \text{ degree} = \frac{1080 \times 7}{22} \text{degree}$$

$$= 343\frac{7}{11} \text{degree} \quad = 343° + \frac{7 \times 60}{11} \text{ minute} \quad [\text{as } 1° = 60']$$

$$= 343° + 38' + \frac{2}{11} \text{ minute} \quad [\text{as } 1' = 60'']$$

$$= 343° + 38' + 10.9'' \quad = 343°38' 11'' \text{ approximately.}$$

Hence 6 radians = $343° 38' 11''$ approximately.

Example 3 Find the radius of the circle in which a central angle of $60°$ intercepts an arc of length 37.4 cm (use $\pi = \frac{22}{7}$).

Solution Here l = 37.4 cm and $\theta = 60° = \frac{60\pi}{180}$ radian $= \frac{\pi}{3}$

Hence, by $r = \frac{l}{\theta}$, we have

$$r = \frac{37.4\times3}{\pi} = \frac{37.4\times3\times7}{22} = 35.7 \text{ cm}$$

Example 4 The minute hand of a watch is 1.5 cm long. How far does its tip move in 40 minutes? (Use π = 3.14).

Solution In 60 minutes, the minute hand of a watch completes one revolution. Therefore, in 40 minutes, the minute hand turns through $\frac{2}{3}$ of a revolution. Therefore, $\theta = \frac{2}{3} \times 360°$ or $\frac{4\pi}{3}$ radian. Hence, the required distance travelled is given by

$$l = r\,\theta = 1.5 \times \frac{4\pi}{3} \text{ cm} = 2\pi \text{ cm} = 2 \times 3.14 \text{ cm} = 6.28 \text{ cm}.$$

Example 5 If the arcs of the same lengths in two circles subtend angles 65°and 110° at the centre, find the ratio of their radii.

Solution Let r_1 and r_2 be the radii of the two circles. Given that

$$\theta_1 = 65° = \frac{\pi}{180}\times65 = \frac{13\pi}{36} \text{ radian}$$

and $$\theta_2 = 110° = \frac{\pi}{180}\times110 = \frac{22\pi}{36} \text{ radian}$$

Let l be the length of each of the arc. Then $l = r_1\theta_1 = r_2\theta_2$, which gives

$$\frac{13\pi}{36}\times r_1 = \frac{22\pi}{36}\times r_2, \text{ i.e., } \frac{r_1}{r_2} = \frac{22}{13}$$

Hence $r_1 : r_2 = 22 : 13$.

Example 6 If $\cos x = -\frac{3}{5}$, x lies in the third quadrant, find the values of other five trigonometric functions.

Solution Since $\cos x = -\frac{3}{5}$, we have $\sec x = -\frac{5}{3}$

Now $\sin^2 x + \cos^2 x = 1$, i.e., $\sin^2 x = 1 - \cos^2 x$

or $\sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Hence $\sin x = \pm \frac{4}{5}$

Since x lies in third quadrant, $\sin x$ is negative. Therefore

$$\sin x = -\frac{4}{5}$$

which also gives

$$\operatorname{cosec} x = -\frac{5}{4}$$

Further, we have

$$\tan x = \frac{\sin x}{\cos x} = \frac{4}{3} \quad \text{and} \quad \cot x = \frac{\cos x}{\sin x} = \frac{3}{4}.$$

Example 7 If $\cot x = -\frac{5}{12}$, x lies in second quadrant, find the values of other five trigonometric functions.

Solution Since $\cot x = -\frac{5}{12}$, we have $\tan x = -\frac{12}{5}$

Now $$\sec^2 x = 1 + \tan^2 x = 1 + \frac{144}{25} = \frac{169}{25}$$

Hence $$\sec x = \pm \frac{13}{5}$$

Since x lies in second quadrant, $\sec x$ will be negative. Therefore

$$\sec x = -\frac{13}{5},$$

which also gives

$$\cos x = -\frac{5}{13}$$

Further, we have

$$\sin x = \tan x \cos x = (-\frac{12}{5}) \times (-\frac{5}{13}) = \frac{12}{13}$$

and $$\operatorname{cosec} x = \frac{1}{\sin x} = \frac{13}{12}.$$

Example 8 Find the value of $\sin \frac{31\pi}{3}$.

Solution We know that values of $\sin x$ repeats after an interval of 2π. Therefore

$$\sin \frac{31\pi}{3} = \sin (10\pi + \frac{\pi}{3}) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Example 9 Find the value of cos (–1710°).

Solution We know that values of cos x repeats after an interval of 2π or 360°.

Therefore, $\cos(-1710°) = \cos(-1710° + 5 \times 360°)$

$$= \cos(-1710° + 1800°) = \cos 90° = 0.$$

Example 10 Prove that

$$3\sin\frac{\pi}{6}\sec\frac{\pi}{3}-4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}=1$$

Solution We have

$$\text{L.H.S.} = 3\sin\frac{\pi}{6}\sec\frac{\pi}{3}-4\sin\frac{5\pi}{6}\cot\frac{\pi}{4}$$

$$= 3 \times \frac{1}{2} \times 2 - 4 \sin\left(\pi-\frac{\pi}{6}\right) \times 1 = 3 - 4 \sin\frac{\pi}{6}$$

$$= 3 - 4 \times \frac{1}{2} = 1 = \text{R.H.S.}$$

Example 11 Find the value of sin 15°.

Solution We have

$$\sin 15° = \sin (45° - 30°)$$

$$= \sin 45° \cos 30° - \cos 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\times\frac{1}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 12 Find the value of $\tan\frac{13\pi}{12}$.

Solution We have

$$\tan\frac{13\pi}{12} = \tan\left(\pi+\frac{\pi}{12}\right) = \tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4}-\frac{\pi}{6}\right)$$

$$= \frac{\tan\frac{\pi}{4}-\tan\frac{\pi}{6}}{1+\tan\frac{\pi}{4}\tan\frac{\pi}{6}} = \frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$$

Example 13 Prove that

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x+\tan y}{\tan x-\tan y}.$$

Solution We have

$$\text{L.H.S.} \quad = \frac{\sin(x+y)}{\sin(x-y)} = \frac{\sin x\cos y+\cos x\sin y}{\sin x\cos y-\cos x\sin y}$$

Dividing the numerator and denominator by $\cos x \cos y$, we get

$$\frac{\sin(x+y)}{\sin(x-y)} = \frac{\tan x+\tan y}{\tan x-\tan y}.$$

Example 14 Show that

$$\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$$

Solution We know that $3x = 2x + x$

Therefore, $\tan 3x = \tan(2x + x)$

or $\tan 3x = \dfrac{\tan 2x+\tan x}{1-\tan 2x\tan x}$

or $\tan 3x - \tan 3x \tan 2x \tan x = \tan 2x + \tan x$

or $\tan 3x - \tan 2x - \tan x = \tan 3x \tan 2x \tan x$

or $\tan 3x \tan 2x \tan x = \tan 3x - \tan 2x - \tan x$.

Example 15 Prove that

$$\cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)=\sqrt{2}\cos x$$

Solution Using the Identity 20(i), we have

L.H.S. $= \cos\left(\frac{\pi}{4}+x\right)+\cos\left(\frac{\pi}{4}-x\right)$

$$= 2\cos\left(\frac{\frac{\pi}{4}+x+\frac{\pi}{4}-x}{2}\right)\cos\left(\frac{\frac{\pi}{4}+x-(\frac{\pi}{4}-x)}{2}\right)$$

$$= 2\cos\frac{\pi}{4}\cos x = 2\times\frac{1}{\sqrt{2}}\cos x = \sqrt{2}\cos x = \text{R.H.S.}$$

Example 16 Prove that $\frac{\cos 7x+\cos 5x}{\sin 7x-\sin 5x} = \cot x$

Solution Using the Identities 20 (i) and 20 (iv), we get

L.H.S. $= \frac{2\cos\frac{7x+5x}{2}\cos\frac{7x-5x}{2}}{2\cos\frac{7x+5x}{2}\sin\frac{7x-5x}{2}} = \frac{\cos x}{\sin x} = \cot x = \text{R.H.S.}$

Example 17 Prove that $= \frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \tan x$

Solution We have

L.H.S. $= \frac{\sin 5x-2\sin 3x+\sin x}{\cos 5x-\cos x} = \frac{\sin 5x+\sin x-2\sin 3x}{\cos 5x-\cos x}$

$$= \frac{2\sin 3x\cos 2x-2\sin 3x}{-2\sin 3x\sin 2x} = -\frac{\sin 3x(\cos 2x-1)}{\sin 3x\sin 2x}$$

$$= \frac{1-\cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \tan x = \text{R.H.S.}$$

Example 18 If $\sin x = \frac{3}{5}$, $\cos y = -\frac{12}{13}$, where x and y both lie in second quadrant, find the value of $\sin (x + y)$.

Solution We know that

$$\sin (x + y) = \sin x \cos y + \cos x \sin y \quad \ldots (1)$$

Now $\quad \cos^2 x = 1 - \sin^2 x = 1 - \frac{9}{25} = \frac{16}{25}$

Therefore $\quad \cos x = \pm\frac{4}{5}$.

Since x lies in second quadrant, $\cos x$ is negative.

Hence $\quad \cos x = -\frac{4}{5}$

Now $\quad \sin^2 y = 1 - \cos^2 y = 1 - \frac{144}{169} = \frac{25}{169}$

i.e. $\quad \sin y = \pm\frac{5}{13}$.

Since y lies in second quadrant, hence $\sin y$ is positive. Therefore, $\sin y = \frac{5}{13}$. Substituting the values of $\sin x$, $\sin y$, $\cos x$ and $\cos y$ in (1), we get

$$\sin(x + y) = \frac{3}{5}\times\left(-\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\times\frac{5}{13} = -\frac{36}{65}-\frac{20}{65}=-\frac{56}{65}.$$

Example 19 Prove that

$$\cos 2x \cos\frac{x}{2}-\cos 3x \cos\frac{9x}{2} = \sin 5x \sin \frac{5x}{2}.$$

Solution We have

$$\text{L.H.S.} \quad = \frac{1}{2}\left[2\cos 2x \cos\frac{x}{2} - 2\cos\frac{9x}{2}\cos 3x\right]$$

$$= \frac{1}{2}\left[\cos\left(2x+\frac{x}{2}\right)+\cos\left(2x-\frac{x}{2}\right)-\cos\left(\frac{9x}{2}+3x\right)-\cos\left(\frac{9x}{2}-3x\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5x}{2}+\cos\frac{3x}{2}-\cos\frac{15x}{2}-\cos\frac{3x}{2}\right] = \frac{1}{2}\left[\cos\frac{5x}{2}-\cos\frac{15x}{2}\right]$$

$$= \frac{1}{2}\left[-2\sin\left\{\frac{\frac{5x}{2}+\frac{15x}{2}}{2}\right\}\sin\left\{\frac{\frac{5x}{2}-\frac{15x}{2}}{2}\right\}\right]$$

$$= -\sin 5x \sin\left(-\frac{5x}{2}\right) = \sin 5x \sin\frac{5x}{2} = \text{R.H.S.}$$

Example 20 Find the value of $\tan\frac{\pi}{8}$.

Solution Let $x = \frac{\pi}{8}$. Then $2x = \frac{\pi}{4}$.

Now $\qquad \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

or $\qquad \tan\frac{\pi}{4} = \dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$

Let $y = \tan\frac{\pi}{8}$. Then $1 = \dfrac{2y}{1-y^2}$

or $\qquad y^2 + 2y - 1 = 0$

Therefore $\qquad y = \dfrac{-2\pm 2\sqrt{2}}{2} = -1\pm\sqrt{2}$

Since $\frac{\pi}{8}$ lies in the first quadrant, $y = \tan\frac{\pi}{8}$ is positve. Hence

$$\tan\frac{\pi}{8} = \sqrt{2} - 1.$$

Example 21 If $\tan x = \frac{3}{4}, \pi < x < \frac{3\pi}{2}$, find the value of $\sin\frac{x}{2}$, $\cos\frac{x}{2}$ and $\tan\frac{x}{2}$.

Solution Since $\pi < x < \frac{3\pi}{2}$, $\cos x$ is negative.

Also $\frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4}$.

Therefore, $\sin\frac{x}{2}$ is positive and $\cos\frac{x}{2}$ is negative.

Now $\sec^2 x = 1 + \tan^2 x = 1 + \frac{9}{16} = \frac{25}{16}$

Therefore $\cos^2 x = \frac{16}{25}$ or $\cos x = -\frac{4}{5}$ (Why?)

Now $2\sin^2\frac{x}{2} = 1 - \cos x = 1 + \frac{4}{5} = \frac{9}{5}$.

Therefore $\sin^2\frac{x}{2} = \frac{9}{10}$

or $\sin\frac{x}{2} = \frac{3}{\sqrt{10}}$ (Why?)

Again $2\cos^2\frac{x}{2} = 1 + \cos x = 1 - \frac{4}{5} = \frac{1}{5}$

Therefore $\cos^2\frac{x}{2} = \frac{1}{10}$

or $\cos\frac{x}{2} = -\frac{1}{\sqrt{10}}$ (Why?)

Hence $\tan\frac{x}{2} = \dfrac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \dfrac{3}{\sqrt{10}} \times \left(\dfrac{-\sqrt{10}}{1}\right) = -3.$

Example 22 Prove that $\cos^2 x + \cos^2\left(x+\frac{\pi}{3}\right) + \cos^2\left(x-\frac{\pi}{3}\right) = \frac{3}{2}$.

Solution We have

$$\text{L.H.S.} = \frac{1+\cos 2x}{2} + \frac{1+\cos\left(2x+\frac{2\pi}{3}\right)}{2} + \frac{1+\cos\left(2x-\frac{2\pi}{3}\right)}{2}.$$

$$= \frac{1}{2}\left[3+\cos 2x+\cos\left(2x+\frac{2\pi}{3}\right)+\cos\left(2x-\frac{2\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\frac{2\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x+2\cos 2x\cos\left(\pi-\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-2\cos 2x\cos\frac{\pi}{3}\right]$$

$$= \frac{1}{2}\left[3+\cos 2x-\cos 2x\right] = \frac{3}{2} = \text{R.H.S.}$$