Example 1 Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Solution Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$. Then, $\sin y = \frac{1}{\sqrt{2}}$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. Therefore, principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is $\frac{\pi}{4}$

Example 2 Find the principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

Solution Let $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right) = y$. Then,

$$\cot y = \frac{-1}{\sqrt{3}} = -\cot\left(\frac{\pi}{3}\right) = \cot\left(\pi - \frac{\pi}{3}\right) = \cot\left(\frac{2\pi}{3}\right)$$

We know that the range of principal value branch of $\cot^{-1}$ is $(0, \pi)$ and $\cot\left(\frac{2\pi}{3}\right) = \frac{-1}{\sqrt{3}}$. Hence, principal value of $\cot^{-1}\left(\frac{-1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$

Example 3 Show that

(i) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x,\ -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

(ii) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x,\ \frac{1}{\sqrt{2}} \le x \le 1$

Solution

(i) Let $x = \sin\theta$. Then $\sin^{-1}x = \theta$. We have

$$\begin{aligned}\sin^{-1}\left(2x\sqrt{1-x^2}\right) &= \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)\\ &= \sin^{-1}(2\sin\theta\cos\theta) = \sin^{-1}(\sin 2\theta) = 2\theta\\ &= 2\sin^{-1}x\end{aligned}$$

(ii) Take $x = \cos\theta$, then proceeding as above, we get, $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\cos^{-1}x$

Example 4 Express $\tan^{-1}\frac{\cos x}{1-\sin x},\ \frac{-3\pi}{2} < x < \frac{\pi}{2}$ in the simplest form.

Solution We write

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left[\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left[\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right] = \tan^{-1}\left[\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = \frac{\pi}{4}+\frac{x}{2}$$

Example 5 Write $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$, $x > 1$ in the simplest form.

Solution Let $x = \sec \theta$, then $\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta$

Therefore, $\cot^{-1}\frac{1}{\sqrt{x^2-1}} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1} x$, which is the simplest form.

Example 6 Find the value of $\sin^{-1}(\sin\frac{3\pi}{5})$

Solution We know that $\sin^{-1}(\sin x) = x$. Therefore, $\sin^{-1}(\sin\frac{3\pi}{5}) = \frac{3\pi}{5}$

But $\frac{3\pi}{5} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal branch of $\sin^{-1} x$

However $\sin(\frac{3\pi}{5}) = \sin(\pi - \frac{3\pi}{5}) = \sin\frac{2\pi}{5}$ and $\frac{2\pi}{5} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Therefore $\sin^{-1}(\sin\frac{3\pi}{5}) = \sin^{-1}(\sin\frac{2\pi}{5}) = \frac{2\pi}{5}$