Example 1 If $(x + 1, y - 2) = (3,1)$, find the values of x and y.

Solution Since the ordered pairs are equal, the corresponding elements are equal.

Therefore $\quad x + 1 = 3$ and $y - 2 = 1$.

Solving we get $\quad x = 2$ and $y = 3$.

Example 2 If P = $\{a, b, c\}$ and Q = $\{r\}$, form the sets P × Q and Q × P. Are these two products equal?

Solution By the definition of the cartesian product,

$$P \times Q = \{(a, r), (b, r), (c, r)\} \text{ and } Q \times P = \{(r, a), (r, b), (r, c)\}$$

Since, by the definition of equality of ordered pairs, the pair (a, r) is not equal to the pair (r, a), we conclude that P × Q ≠ Q × P.

However, the number of elements in each set will be the same.

Example 3 Let A = {1,2,3}, B = {3,4} and C = {4,5,6}. Find

(i) $A \times (B \cap C)$ (ii) $(A \times B) \cap (A \times C)$

(iii) $A \times (B \cup C)$ (iv) $(A \times B) \cup (A \times C)$

Solution (i) By the definition of the intersection of two sets, $(B \cap C) = \{4\}$.

Therefore, $A \times (B \cap C) = \{(1,4), (2,4), (3,4)\}$.

(ii) Now $(A \times B) = \{(1,3), (1,4), (2,3), (2,4), (3,3), (3,4)\}$

and $(A \times C) = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

Therefore, $(A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$.

(iii) Since, $(B \cup C) = \{3, 4, 5, 6\}$, we have

$A \times (B \cup C) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

(iv) Using the sets A × B and A × C from part (ii) above, we obtain

$(A \times B) \cup (A \times C) = \{(1,3), (1,4), (1,5), (1,6), (2,3), (2,4), (2,5), (2,6), (3,3), (3,4), (3,5), (3,6)\}$.

Example 4 If P = {1, 2}, form the set P × P × P.

Solution We have, P × P × P = {(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)}.

Example 5 If **R** is the set of all real numbers, what do the cartesian products **R** × **R** and **R** × **R** × **R** represent?

Solution The Cartesian product **R** × **R** represents the set **R** × **R**={(x, y) : x, y ∈ **R**} which represents the *coordinates of all the points in two dimensional space* and the cartesian product **R** × **R** × **R** represents the set **R** × **R** × **R** ={(x, y, z) : x, y, z ∈ **R**} which represents the *coordinates of all the points in three-dimensional space*.

Example 6 If A × B ={(p, q),(p, r), (m, q), (m, r)}, find A and B.

Solution

A = set of first elements = {p, m}

B = set of second elements = {q, r}.

Example 7 Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by

$R = \{(x, y) : y = x + 1\}$

(i) Depict this relation using an arrow diagram.

(ii) Write down the domain, codomain and range of R.

Solution (i) By the definition of the relation,

$R = \{(1,2), (2,3), (3,4), (4,5), (5,6)\}$.

Example 8 The Fig 2.6 shows a relation between the sets P and Q. Write this relation (i) in set-builder form, (ii) in roster form. What is its domain and range?

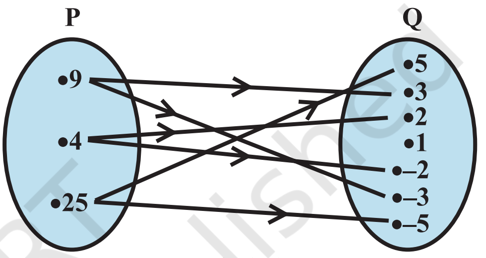


Fig 2.6

Solution It is obvious that the relation R is "x is the square of y".

(i) In set-builder form, R = $\{(x, y): x$ is the square of $y, x \in$ P, $y \in$ **Q**$\}$

(ii) In roster form, R = {(9, 3), (9, –3), (4, 2), (4, –2), (25, 5), (25, –5)}

The domain of this relation is {4, 9, 25}.

The range of this relation is {– 2, 2, –3, 3, –5, 5}.

Note that the element 1 is not related to any element in set P.

The set Q is the codomain of this relation.

Note The total number of relations that can be defined from a set A to a set B is the number of possible subsets of A × B. If n(A) = p and n(B) = q, then n (A × B) = pq and the total number of relations is 2^{pq}.

Example 9 Let A = {1, 2} and B = {3, 4}. Find the number of relations from A to B.

Solution We have,

A × B = {(1, 3), (1, 4), (2, 3), (2, 4)}.

Since n (A×B) = 4, the number of subsets of A×B is 2^4. Therefore, the number of relations from A into B will be 2^4.

Remark A relation R from A to A is also stated as a relation on A.

The function f from A to B is denoted by f: A → B.

Looking at the previous examples, we can easily see that the relation in Example 7 is not a function because the element 6 has no image.

Again, the relation in Example 8 is not a function because the elements in the domain are connected to more than one images. Similarly, the relation in Example 9 is also not a function. (*Why*?) In the examples given below, we will see many more relations some of which are functions and others are not.

Example 10 Let **N** be the set of natural numbers and the relation R be defined on N such that $R = \{(x, y) : y = 2x,\ x, y \in \mathbf{N}\}$.

What is the domain, codomain and range of R? Is this relation a function?

Solution The domain of R is the set of natural numbers **N**. The codomain is also **N**. The range is the set of even natural numbers.

Since every natural number n has one and only one image, this relation is a function.

Example 11 Examine each of the following relations given below and state in each case, giving reasons whether it is a function or not?

(i) $R = \{(2,1),(3,1),(4,2)\}$, (ii) $R = \{(2,2),(2,4),(3,3),(4,4)\}$

(iii) $R = \{(1,2),(2,3),(3,4),(4,5),(5,6),(6,7)\}$

Solution (i) Since 2, 3, 4 are the elements of domain of R having their unique images, this relation R is a function.

(ii) Since the same first element 2 corresponds to two different images 2 and 4, this relation is not a function.

(iii) Since every element has one and only one image, this relation is a function.

Example 12 Let **N** be the set of natural numbers. Define a real valued function $f : \mathbf{N} \to \mathbf{N}$ by $f(x) = 2x + 1$. Using this definition, complete the table given below.

x	1	2	3	4	5	6	7
y	$f(1) = ...$	$f(2) = ...$	$f(3) = ...$	$f(4) = ...$	$f(5) = ...$	$f(6) = ...$	$f(7) = ...$

Solution The completed table is given by

x	1	2	3	4	5	6	7
y	$f(1) = 3$	$f(2) = 5$	$f(3) = 7$	$f(4) = 9$	$f(5) = 11$	$f(6) = 13$	$f(7) = 15$

The graph is a line parallel to x-axis. For example, if $f(x)=3$ for each $x\in\mathbf{R}$, then its graph will be a line as shown in the Fig 2.9.

(iii) **Polynomial function** A function $f : \mathbf{R} \to \mathbf{R}$ is said to be *polynomial function* if for each x in $\mathbf{R}$, $y = f(x) = a_0 + a_1x + a_2x^2 + ...+ a_n x^n$, where n is a non-negative integer and $a_0, a_1, a_2,...,a_n\in\mathbf{R}$.

The functions defined by $f(x) = x^3 - x^2 + 2$, and $g(x) = x^4 + \sqrt{2}\,x$ are some examples of polynomial functions, whereas the function h defined by $h(x) = x^{\frac{2}{3}} + 2x$ is not a polynomial function.(*Why*?)

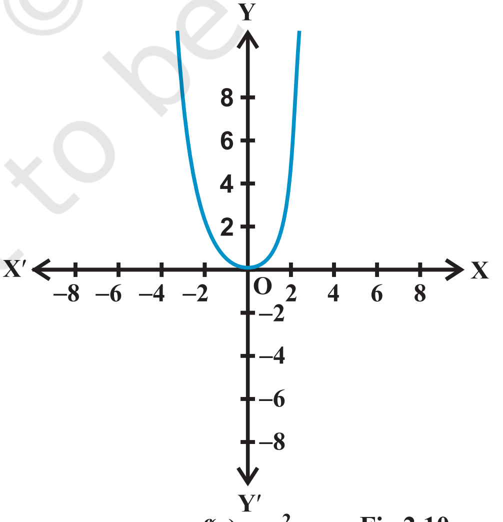
Example 13 Define the function f: $\mathbf{R} \to \mathbf{R}$ by $y = f(x) = x^2$, $x \in \mathbf{R}$. Complete the Table given below by using this definition. What is the domain and range of this function? Draw the graph of f.

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$									

Solution The completed Table is given below:

x	-4	-3	-2	-1	0	1	2	3	4
$y = f(x) = x^2$	16	9	4	1	0	1	4	9	16

Domain of $f = \{x : x\in\mathbf{R}\}$. Range of $f = \{x^2 : x \in \mathbf{R}\}$. The graph of f is given by Fig 2.10



$f(x) = x^2$ **Fig 2.10**

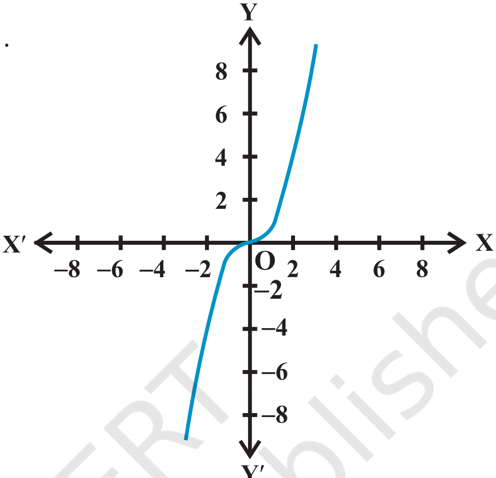
Example 14 Draw the graph of the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$, $x \in \mathbf{R}$.

Solution We have

$f(0) = 0$, $f(1) = 1$, $f(-1) = -1$, $f(2) = 8$, $f(-2) = -8$, $f(3) = 27$; $f(-3) = -27$, etc.

Therefore, $f = \{(x, x^3) : x \in \mathbf{R}\}$.

The graph of f is given in Fig 2.11.



$f(x) = x^3$

Fig 2.11

(iv) **Rational functions** are functions of the type $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$.

Example 15 Define the real valued function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by $f(x) = \dfrac{1}{x}$, $x \in \mathbf{R} - \{0\}$. Complete the Table given below using this definition. What is the domain and range of this function?

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \dfrac{1}{x}$	...	...	...	...	...	...	...	...	...

Solution The completed Table is given by

x	–2	–1.5	–1	–0.5	0.25	0.5	1	1.5	2
$y = \dfrac{1}{x}$	– 0.5	– 0.67	–1	– 2	4	2	1	0.67	0.5

Example 16 Let $f(x) = x^2$ and $g(x) = 2x + 1$ be two real functions. Find

$$(f + g)\ (x),\ (f - g)\ (x),\ (fg)\ (x), \left(\frac{f}{g}\right)(x).$$

Solution We have,

$$(f + g)\ (x) = x^2 + 2x + 1,\ (f - g)\ (x) = x^2 - 2x - 1,$$

$$(fg)\ (x) = x^2\ (2x + 1) = 2x^3 + x^2,\ \left(\frac{f}{g}\right)(x) = \frac{x^2}{2x+1},\ x \neq -\frac{1}{2}$$

Example 17 Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f + g)\ (x)$, $(f - g)\ (x)$, $(fg)\ (x)$ and $\left(\frac{f}{g}\right)(x)$.

Solution We have

$$(f + g)\ (x) = \sqrt{x} + x,\ (f - g)\ (x) = \sqrt{x} - x,$$

$$(fg)\ x = \sqrt{x}(x) = x^{\frac{3}{2}} \text{ and } \left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{x} = x^{-\frac{1}{2}},\ x \neq 0$$

Miscellaneous Examples

Example 18 Let **R** be the set of real numbers. Define the real function

$$f: \mathbf{R} \rightarrow \mathbf{R} \text{ by } f(x) = x + 10$$

and sketch the graph of this function.

Solution Here $f(0) = 10, f(1) = 11, f(2) = 12, ..., f(10) = 20$, etc., and

$f(-1) = 9, f(-2) = 8, ..., f(-10) = 0$ and so on.

Therefore, shape of the graph of the given function assumes the form as shown in Fig 2.16.

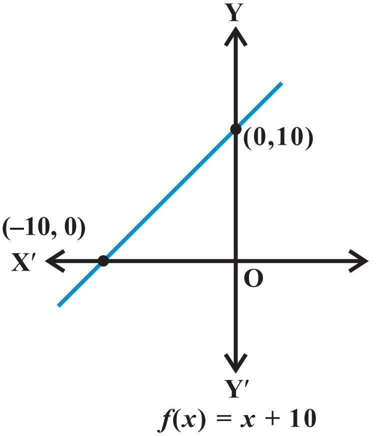


Fig 2.16

Remark The function f defined by $f(x) = mx + c$, $x \in \mathbf{R}$, is called *linear function*, where m and c are constants. Above function is an example of a *linear function*.

Example 19 Let R be a relation from **Q** to **Q** defined by R = $\{(a,b): a,b \in \mathbf{Q}$ and $a - b \in \mathbf{Z}\}$. Show that

(i) $(a,a) \in$ R for all $a \in \mathbf{Q}$

(ii) $(a,b) \in$ R implies that $(b, a) \in$ R

(iii) $(a,b) \in$ R and $(b,c) \in$ R implies that $(a,c) \in$R

Solution (i) Since, $a - a = 0 \in \mathbf{Z}$, if follows that $(a, a) \in$ R.

(ii) $(a,b) \in$ R implies that $a - b \in \mathbf{Z}$. So, $b - a \in \mathbf{Z}$. Therefore, $(b, a) \in$ R

(iii) (a, b) and $(b, c) \in$ R implies that $a - b \in \mathbf{Z}$. $b - c \in \mathbf{Z}$. So, $a - c = (a - b) + (b - c) \in \mathbf{Z}$. Therefore, $(a,c) \in$ R

Example 20 Let $f = \{(1,1), (2,3), (0,-1), (-1,-3)\}$ be a linear function from **Z** into **Z**. Find $f(x)$.

Solution Since f is a linear function, $f(x) = mx + c$. Also, since $(1, 1), (0, -1) \in$ R, $f(1) = m + c = 1$ and $f(0) = c = -1$. This gives $m = 2$ and $f(x) = 2x - 1$.

Example 21 Find the domain of the function $f(x) = \dfrac{x^2 + 3x + 5}{x^2 - 5x + 4}$

Solution Since $x^2 - 5x + 4 = (x - 4)(x - 1)$, the function f is defined for all real numbers except at $x = 4$ and $x = 1$. Hence the domain of f is $\mathbf{R} - \{1, 4\}$.

Example 22 The function f is defined by

$$f(x) = \begin{cases} 1 - x, & x < 0 \\ 1 & , x = 0 \\ x + 1, & x > 0 \end{cases}$$

Draw the graph of $f(x)$.

Solution Here, $f(x) = 1 - x$, $x < 0$, this gives

$f(-4) = 1 - (-4) = 5;$

$f(-3) = 1 - (-3) = 4,$

$f(-2) = 1 - (-2) = 3$

$f(-1) = 1 - (-1) = 2;$ etc,

and $f(1) = 2, f(2) = 3, f(3) = 4$

$f(4) = 5$ and so on for $f(x) = x + 1$, $x > 0$.

Thus, the graph of f is as shown in Fig 2.17

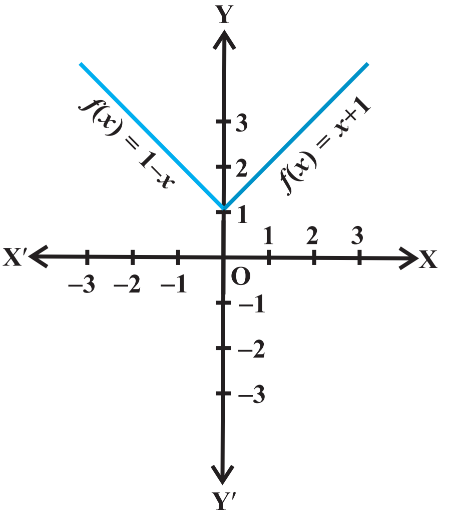


Fig 2.17