Note In this chapter

1. We shall follow the notation, namely $A = [a_{ij}]_{m \times n}$ to indicate that A is a matrix of order $m \times n$.
2. We shall consider only those matrices whose elements are real numbers or functions taking real values.

We can also represent any point (x, y) in a plane by a matrix (column or row) as $\begin{bmatrix} x \\ y \end{bmatrix}$ (or $[x, y]$). For example point P(0, 1) as a matrix representation may be given as

$$P = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ or } [0 \ 1].$$

Observe that in this way we can also express the vertices of a closed rectilinear figure in the form of a matrix. For example, consider a quadrilateral ABCD with vertices A (1, 0), B (3, 2), C (1, 3), D (–1, 2).

Now, quadrilateral ABCD in the matrix form, can be represented as

$$X = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 2 & 3 & 2 \end{bmatrix}_{2 \times 4} \end{array} \quad \text{or} \quad Y = \begin{array}{c} A \\ B \\ C \\ D \end{array} \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 1 & 3 \\ -1 & 2 \end{bmatrix}_{4 \times 2}$$

Thus, matrices can be used as representation of vertices of geometrical figures in a plane.

Now, let us consider some examples.

Example 1 Consider the following information regarding the number of men and women workers in three factories I, II and III

	Men workers	Women workers
I	30	25
II	25	31
III	27	26

Represent the above information in the form of a 3 × 2 matrix. What does the entry in the third row and second column represent?

Solution The information is represented in the form of a 3 × 2 matrix as follows:

$$A = \begin{bmatrix} 30 & 25 \\ 25 & 31 \\ 27 & 26 \end{bmatrix}$$

The entry in the third row and second column represents the number of women workers in factory III.

Example 2 If a matrix has 8 elements, what are the possible orders it can have?

Solution We know that if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 8 elements, we will find all ordered pairs of natural numbers, whose product is 8.
Thus, all possible ordered pairs are (1, 8), (8, 1), (4, 2), (2, 4)

Hence, possible orders are 1 × 8, 8 ×1, 4 × 2, 2 × 4

Example 3 Construct a 3 × 2 matrix whose elements are given by $a_{ij} = \frac{1}{2}|i - 3j|$.

Solution In general a 3 × 2 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$.

Now $\quad a_{ij} = \frac{1}{2}|i - 3j|$, $i = 1, 2, 3$ and $j = 1, 2$.

Therefore

$$a_{11} = \frac{1}{2}|1 - 3 \times 1| = 1 \qquad a_{12} = \frac{1}{2}|1 - 3 \times 2| = \frac{5}{2}$$

$$a_{21} = \frac{1}{2}|2 - 3 \times 1| = \frac{1}{2} \qquad a_{22} = \frac{1}{2}|2 - 3 \times 2| = 2$$

$$a_{31} = \frac{1}{2}|3 - 3 \times 1| = 0 \qquad a_{32} = \frac{1}{2}|3 - 3 \times 2| = \frac{3}{2}$$

Hence the required matrix is given by $A = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}$.

(vii) **Zero matrix**

A matrix is said to be *zero matrix* or *null matrix* if all its elements are zero.

For example, [0], $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, [0, 0] are all zero matrices. We denote zero matrix by O. Its order will be clear from the context.

3.3.1 *Equality of matrices*

Definition 2 Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ are said to be equal if

(i) they are of the same order

(ii) each element of A is equal to the corresponding element of B, that is $a_{ij} = b_{ij}$ for all i and j.

For example, $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are equal matrices but $\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$ are not equal matrices. Symbolically, if two matrices A and B are equal, we write A = B.

If $\begin{bmatrix} x & y \\ z & a \\ b & c \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 2 & \sqrt{6} \\ 3 & 2 \end{bmatrix}$, then $x = -1.5$, $y = 0$, $z = 2$, $a = \sqrt{6}$, $b = 3$, $c = 2$

Example 4 If $\begin{bmatrix} x+3 & z+4 & 2y-7 \\ -6 & a-1 & 0 \\ b-3 & -21 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 3y-2 \\ -6 & -3 & 2c+2 \\ 2b+4 & -21 & 0 \end{bmatrix}$

Find the values of a, b, c, x, y and z.

Solution As the given matrices are equal, therefore, their corresponding elements must be equal. Comparing the corresponding elements, we get

$$x + 3 = 0, \qquad z + 4 = 6, \qquad 2y - 7 = 3y - 2$$
$$a - 1 = -3, \qquad 0 = 2c + 2 \qquad b - 3 = 2b + 4,$$

Simplifying, we get

$$a = -2,\ b = -7,\ c = -1,\ x = -3,\ y = -5,\ z = 2$$

Example 5 Find the values of a, b, c, and d from the following equation:

$$\begin{bmatrix} 2a+b & a-2b \\ 5c-d & 4c+3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix}$$

Solution By equality of two matrices, equating the corresponding elements, we get

$$2a + b = 4 \qquad 5c - d = 11$$

$$a - 2b = -3 \qquad 4c + 3d = 24$$

Solving these equations, we get

$$a = 1,\ b = 2,\ c = 3 \text{ and } d = 4$$

EXERCISE 3.1

1. In the matrix $A = \begin{bmatrix} 2 & 5 & 19 & -7 \\ 35 & -2 & \frac{5}{2} & 12 \\ \sqrt{3} & 1 & -5 & 17 \end{bmatrix}$, write:

(i) The order of the matrix, (ii) The number of elements,

(iii) Write the elements $a_{13}, a_{21}, a_{33}, a_{24}, a_{23}$.

2. If a matrix has 24 elements, what are the possible orders it can have? What, if it has 13 elements?

3. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

4. Construct a 2 × 2 matrix, $A = [a_{ij}]$, whose elements are given by:

(i) $a_{ij} = \frac{(i+j)^2}{2}$ (ii) $a_{ij} = \frac{i}{j}$ (iii) $a_{ij} = \frac{(i+2j)^2}{2}$

5. Construct a 3 × 4 matrix, whose elements are given by:

(i) $a_{ij} = \frac{1}{2}|-3i + j|$ (ii) $a_{ij} = 2i - j$

6. Find the values of x, y and z from the following equations:

(i) $\begin{bmatrix} 4 & 3 \\ x & 5 \end{bmatrix} = \begin{bmatrix} y & z \\ 1 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$ (iii) $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

7. Find the value of a, b, c and d from the equation:

$$\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$$

This new matrix is the **sum** of the above two matrices. We observe that the sum of two matrices is a matrix obtained by adding the corresponding elements of the given matrices. Furthermore, the two matrices have to be of the same order.

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is a 2×3 matrix and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$ is another 2×3 matrix. Then, we define $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$.

In general, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$. Then, the sum of the two matrices A and B is *defined* as a matrix $C = [c_{ij}]_{m \times n}$, where $c_{ij} = a_{ij} + b_{ij}$, for all possible values of i and j.

Example 6 Given $A = \begin{bmatrix} \sqrt{3} & 1 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & \sqrt{5} & 1 \\ -2 & 3 & \frac{1}{2} \end{bmatrix}$, find A + B

Since A, B are of the same order 2×3. Therefore, addition of A and B is defined and is given by

$$A + B = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 1-1 \\ 2-2 & 3+3 & 0+\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2+\sqrt{3} & 1+\sqrt{5} & 0 \\ 0 & 6 & \frac{1}{2} \end{bmatrix}$$

Note

1. We emphasise that if A and B are not of the same order, then A + B is not defined. For example if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then A + B is not defined.
2. We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.

3.4.2 *Multiplication of a matrix by a scalar*

Now suppose that Fatima has doubled the production at a factory A in all categories (refer to 3.4.1).

Previously quantities (in standard units) produced by factory A were

$$\begin{array}{c c} & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 80 & 60 \\ 75 & 65 \\ 90 & 85 \end{bmatrix} \end{array}$$

Revised quantities produced by factory A are as given below:

$$\begin{array}{c c} & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2\times 80 & 2\times 60 \\ 2\times 75 & 2\times 65 \\ 2\times 90 & 2\times 85 \end{bmatrix} \end{array}$$

This can be represented in the matrix form as $\begin{bmatrix} 160 & 120 \\ 150 & 130 \\ 180 & 170 \end{bmatrix}$. We observe that the new matrix is obtained by multiplying each element of the previous matrix by 2.

In general, we may define *multiplication of a matrix* by a scalar as follows: if $A = [a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix which is obtained by multiplying each element of A by the scalar k.

In other words, $kA = k[a_{ij}]_{m \times n} = [k(a_{ij})]_{m \times n}$, that is, $(i, j)^{\text{th}}$ element of kA is ka_{ij} for all possible values of i and j.

For example, if $A = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then

$$3A = 3\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

Negative of a matrix The negative of a matrix is denoted by $-A$. We define $-A = (-1) A$.

For example, let $A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}$, then – A is given by

$$-A = (-1)A = (-1)\begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

Difference of matrices If $A = [a_{ij}]$, $B = [b_{ij}]$ are two matrices of the same order, say $m \times n$, then difference A – B is defined as a matrix $D = [d_{ij}]$, where $d_{ij} = a_{ij} - b_{ij}$, for all value of i and j. In other words, D = A – B = A + (–1) B, that is sum of the matrix A and the matrix – B.

Example 7 If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find 2A – B.

Solution We have

$$2A - B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

3.4.3 *Properties of matrix addition*

The addition of matrices satisfy the following properties:

(i) **Commutative Law** If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, then A + B = B + A.

Now $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ (addition of numbers is commutative)

$= ([b_{ij}] + [a_{ij}]) = B + A$

(ii) **Associative Law** For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$, (A + B) + C = A + (B + C).

Now $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}]$

$= [a_{ij} + (b_{ij} + c_{ij})]$ (Why?)

$= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C)$

(iii) **Existence of additive identity** Let A = $[a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then A + O = O + A = A. In other words, O is the additive identity for matrix addition.

(iv) **The existence of additive inverse** Let A = $[a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as – A = $[-a_{ij}]_{m \times n}$ such that A + (– A) = (– A) + A= O. So – A is the additive inverse of A or negative of A.

3.4.4 *Properties of scalar multiplication of a matrix*

If A = $[a_{ij}]$ and B = $[b_{ij}]$ be two matrices of the same order, say $m \times n$, and k and l are scalars, then

(i) $k(\text{A} + \text{B}) = k\,\text{A} + k\text{B}$, (ii) $(k + l)\text{A} = k\,\text{A} + l\,\text{A}$

(ii) $$k\,(\text{A} + \text{B}) = k\,([a_{ij}] + [b_{ij}])$$
$$= k\,[a_{ij} + b_{ij}] = [k\,(a_{ij} + b_{ij})] = [(k\,a_{ij}) + (k\,b_{ij})]$$
$$= [k\,a_{ij}] + [k\,b_{ij}] = k\,[a_{ij}] + k\,[b_{ij}] = k\text{A} + k\text{B}$$

(iii) $$(k + l)\,\text{A} = (k + l)\,[a_{ij}]$$
$$= [(k + l)\,a_{ij}] + [k\,a_{ij}] + [l\,a_{ij}] = k\,[a_{ij}] + l\,[a_{ij}] = k\,\text{A} + l\,\text{A}$$

Example 8 If $\text{A} = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $\text{B} = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find the matrix X, such that 2A + 3X = 5B.

Solution We have 2A + 3X = 5B

or $2\text{A} + 3\text{X} - 2\text{A} = 5\text{B} - 2\text{A}$

or $2\text{A} - 2\text{A} + 3\text{X} = 5\text{B} - 2\text{A}$ (Matrix addition is commutative)

or $\text{O} + 3\text{X} = 5\text{B} - 2\text{A}$ (– 2A is the additive inverse of 2A)

or $3\text{X} = 5\text{B} - 2\text{A}$ (O is the additive identity)

or $$\text{X} = \frac{1}{3}(5\text{B} - 2\text{A})$$

or $$\text{X} = \frac{1}{3}\left(5\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix} - 2\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}\right) = \frac{1}{3}\left(\begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix} + \begin{bmatrix} -16 & 0 \\ -8 & 4 \\ -6 & -12 \end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix} 10-16 & -10+0 \\ 20-8 & 10+4 \\ -25-6 & 5-12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix} = \begin{bmatrix} -2 & \frac{-10}{3} \\ 4 & \frac{14}{3} \\ \frac{-31}{3} & \frac{-7}{3} \end{bmatrix}$$

Example 9 Find X and Y, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

Solution We have $(X+Y)+(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

or $$(X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$$

or $$X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

Also $$(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

or $$(X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

or $$Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

Example 10 Find the values of x and y from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Solution We have

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $\begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

or $2x + 3 = 7$ and $2y - 4 = 14$ (Why?)

or $2x = 7 - 3$ and $2y = 18$

or $x = \frac{4}{2}$ and $y = \frac{18}{2}$

i.e. $x = 2$ and $y = 9$.

Example 11 Two farmers Ramkishan and Gurcharan Singh cultivates only three varieties of rice namely Basmati, Permal and Naura. The sale (in Rupees) of these varieties of rice by both the farmers in the month of September and October are given by the following matrices A and B.

September Sales (in Rupees)

$$A = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 10{,}000 & 20{,}000 & 30{,}000 \\ 50{,}000 & 30{,}000 & 10{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

October Sales (in Rupees)

$$B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 5000 & 10{,}000 & 6000 \\ 20{,}000 & 10{,}000 & 10{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(i) Find the combined sales in September and October for each farmer in each variety.

(ii) Find the decrease in sales from September to October.

(iii) If both farmers receive 2% profit on gross sales, compute the profit for each farmer and for each variety sold in October.

Solution

(i) Combined sales in September and October for each farmer in each variety is given by

$$A + B = \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 15{,}000 & 30{,}000 & 36{,}000 \\ 70{,}000 & 40{,}000 & 20{,}000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(ii) Change in sales from September to October is given by

$$A - B = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 5000 & 10,000 & 24,000 \\ 30,000 & 20,000 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

(iii) 2% of B $= \frac{2}{100} \times B = 0.02 \times B$

$$= 0.02 \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 5000 & 10,000 & 6000 \\ 20,000 & 10,000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} \text{Basmati} & \text{Permal} & \text{Naura} \end{array} \\ \begin{bmatrix} 100 & 200 & 120 \\ 400 & 200 & 200 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Ramkishan} \\ \text{Gurcharan Singh} \end{array}$$

Thus, in October Ramkishan receives ₹ 100, ₹ 200 and ₹ 120 as profit in the sale of each variety of rice, respectively, and Grucharan Singh receives profit of ₹ 400, ₹ 200 and ₹ 200 in the sale of each variety of rice, respectively.

3.4.5 *Multiplication of matrices*

Suppose Meera and Nadeem are two friends. Meera wants to buy 2 pens and 5 story books, while Nadeem needs 8 pens and 10 story books. They both go to a shop to enquire about the rates which are quoted as follows:

Pen – ₹ 5 each, story book – ₹ 50 each.

How much money does each need to spend? Clearly, Meera needs ₹ $(5 \times 2 + 50 \times 5)$ that is ₹ 260, while Nadeem needs $(8 \times 5 + 50 \times 10)$ ₹, that is ₹ 540. In terms of matrix representation, we can write the above information as follows:

Requirements **Prices per piece (in Rupees)** **Money needed (in Rupees)**

$$\begin{bmatrix} 2 & 5 \\ 8 & 10 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 50 \end{bmatrix} \qquad \begin{bmatrix} 5 \times 2 + 5 \times 50 \\ 8 \times 5 + 10 \times 50 \end{bmatrix} = \begin{bmatrix} 260 \\ 540 \end{bmatrix}$$

Suppose that they enquire about the rates from another shop, quoted as follows:

pen – ₹ 4 each, story book – ₹ 40 each.

Now, the money required by Meera and Nadeem to make purchases will be respectively ₹ $(4 \times 2 + 40 \times 5)$ = ₹ 208 and ₹ $(8 \times 4 + 10 \times 40)$ = ₹ 432 and is given by $\mathrm{CD} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$. This is a 2×2 matrix in which each entry is the sum of the products across some row of C with the corresponding entries down some column of D. These four computations are

Entry in first row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} (1)(2) + (-1)(-1) + (2)(5) & ? \\ ? & ? \end{bmatrix}$

Entry in first row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & (1)(7) + (-1)(1) + 2(-4) \\ ? & ? \end{bmatrix}$

Entry in second row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 0(2) + 3(-1) + 4(5) & ? \end{bmatrix}$

Entry in second row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 17 & 0(7) + 3(1) + 4(-4) \end{bmatrix}$

Thus $\mathrm{CD} = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}$

Example 12 Find AB, if $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$.

Solution The matrix A has 2 columns which is equal to the number of rows of B. Hence AB is defined. Now

$$\mathrm{AB} = \begin{bmatrix} 6(2) + 9(7) & 6(6) + 9(9) & 6(0) + 9(8) \\ 2(2) + 3(7) & 2(6) + 3(9) & 2(0) + 3(8) \end{bmatrix}$$

$$= \begin{bmatrix} 12 + 63 & 36 + 81 & 0 + 72 \\ 4 + 21 & 12 + 27 & 0 + 24 \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}$$

Remark If AB is defined, then BA need not be defined. In the above example, AB is defined but BA is not defined because B has 3 column while A has only 2 (and not 3) rows. If A, B are, respectively $m \times n$, $k \times l$ matrices, then both AB and BA are defined **if and only if** $n = k$ and $l = m$. In particular, if both A and B are square matrices of the same order, then both AB and BA are defined.

Non-commutativity of multiplication of matrices

Now, we shall see by an example that even if AB and BA are both defined, it is not necessary that AB = BA.

Example 13 If $A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, then find AB, BA. Show that AB ≠ BA.

Solution Since A is a 2×3 matrix and B is 3×2 matrix. Hence AB and BA are both defined and are matrices of order 2×2 and 3×3, respectively. Note that

$$AB = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2-8+6 & 3-10+3 \\ -8+8+10 & -12+10+5 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$$

and $$BA = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 2-12 & -4+6 & 6+15 \\ 4-20 & -8+10 & 12+25 \\ 2-4 & -4+2 & 6+5 \end{bmatrix} = \begin{bmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{bmatrix}$$

Clearly AB ≠ BA

In the above example both AB and BA are of different order and so AB ≠ BA. But one may think that perhaps AB and BA could be the same if they were of the same order. But it is not so, here we give an example to show that even if AB and BA are of same order they may not be same.

Example 14 If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

and $$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ Clearly AB ≠ BA.

Thus matrix multiplication is not commutative.

Note This does not mean that AB ≠ BA for every pair of matrices A, B for which AB and BA, are defined. For instance,

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, then $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$

Observe that multiplication of diagonal matrices of same order will be commutative.

Zero matrix as the product of two non zero matrices

We know that, for real numbers a, b if $ab = 0$, then either $a = 0$ or $b = 0$. This need not be true for matrices, we will observe this through an example.

Example 15 Find AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

Solution We have $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

3.4.6 *Properties of multiplication of matrices*

The multiplication of matrices possesses the following properties, which we state without proof.

1. **The associative law** For any three matrices A, B and C. We have (AB) C = A (BC), whenever both sides of the equality are defined.
2. **The distributive law** For three matrices A, B and C.
   (i) A (B+C) = AB + AC
   (ii) (A+B) C = AC + BC, whenever both sides of equality are defined.
3. **The existence of multiplicative identity** For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

Now, we shall verify these properties by examples.

Example 16 If $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$, find A(BC), (AB)C and show that (AB)C = A(BC).

Solution We have $AB = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1+0+1 & 3+2-4 \\ 2+0-3 & 6+0+12 \\ 3+0-2 & 9-2+8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix}$

$$(AB)\ (C) = \begin{bmatrix} 2 & 1 \\ -1 & 18 \\ 1 & 15 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2+2 & 4+0 & 6-2 & -8+1 \\ -1+36 & -2+0 & -3-36 & 4+18 \\ 1+30 & 2+0 & 3-30 & -4+15 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$$

Now $\quad BC = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+6 & 2+0 & 3-6 & -4+3 \\ 0+4 & 0+0 & 0-4 & 0+2 \\ -1+8 & -2+0 & -3-8 & 4+4 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$$

Therefore $\quad A(BC) = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 2 & -3 & -1 \\ 4 & 0 & -4 & 2 \\ 7 & -2 & -11 & 8 \end{bmatrix}$

$$= \begin{bmatrix} 7+4-7 & 2+0+2 & -3-4+11 & -1+2-8 \\ 14+0+21 & 4+0-6 & -6+0-33 & -2+0+24 \\ 21-4+14 & 6+0-4 & -9+4-22 & -3-2+16 \end{bmatrix}$$

$= \begin{bmatrix} 4 & 4 & 4 & -7 \\ 35 & -2 & -39 & 22 \\ 31 & 2 & -27 & 11 \end{bmatrix}$. Clearly, (AB) C = A (BC)

Example 17 If $A=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}, B=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}, C=\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}$

Calculate AC, BC and (A + B)C. Also, verify that (A + B)C = AC + BC

Solution Now, $A+B=\begin{bmatrix}0 & 7 & 8\\ -5 & 0 & 10\\ 8 & -6 & 0\end{bmatrix}$

So $$(A+B)\,C=\begin{bmatrix}0 & 7 & 8\\ -5 & 0 & 10\\ 8 & -6 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}=\begin{bmatrix}0-14+24\\ -10+0+30\\ 16+12+0\end{bmatrix}=\begin{bmatrix}10\\ 20\\ 28\end{bmatrix}$$

Further $$AC=\begin{bmatrix}0 & 6 & 7\\ -6 & 0 & 8\\ 7 & -8 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}=\begin{bmatrix}0-12+21\\ -12+0+24\\ 14+16+0\end{bmatrix}=\begin{bmatrix}9\\ 12\\ 30\end{bmatrix}$$

and $$BC=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}\begin{bmatrix}2\\ -2\\ 3\end{bmatrix}=\begin{bmatrix}0-2+3\\ 2+0+6\\ 2-4+0\end{bmatrix}=\begin{bmatrix}1\\ 8\\ -2\end{bmatrix}$$

So $$AC+BC=\begin{bmatrix}9\\ 12\\ 30\end{bmatrix}+\begin{bmatrix}1\\ 8\\ -2\end{bmatrix}=\begin{bmatrix}10\\ 20\\ 28\end{bmatrix}$$

Clearly, $$(A+B)\,C=AC+BC$$

Example 18 If $A=\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}$, then show that $A^3-23A-40\,I=O$

Solution We have $A^2=A.A=\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 3 & -2 & 1\\ 4 & 2 & 1\end{bmatrix}=\begin{bmatrix}19 & 4 & 8\\ 1 & 12 & 8\\ 14 & 6 & 15\end{bmatrix}$

So $$A^3 = A\,A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

Now

$$A^3 - 23A - 40I = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Example 19 In a legislative assembly election, a political group hired a public relations firm to promote its candidate in three ways: telephone, house calls, and letters. The cost per contact (in paise) is given in matrix A as

$$A = \begin{array}{c} \textit{Cost per contact} \\ \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Telephone} \\ \text{Housecall} \\ \text{Letter} \end{array}$$

The number of contacts of each type made in two cities X and Y is given by

$$B = \begin{array}{c} \begin{array}{ccc} \text{Telephone} & \text{Housecall} & \text{Letter} \end{array} \\ \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10,000 \end{bmatrix} \end{array} \begin{array}{l} \\ \to X \\ \to Y \end{array}$$. Find the total amount spent by the group in the two cities X and Y.

Solution We have

$$BA = \begin{bmatrix} 40,000 + 50,000 + 250,000 \\ 120,000 + 100,000 + 500,000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$

$$= \begin{bmatrix} 340,000 \\ 720,000 \end{bmatrix} \begin{matrix} \to X \\ \to Y \end{matrix}$$

So the total amount spent by the group in the two cities is 340,000 paise and 720,000 paise, i.e., ₹3400 and ₹7200, respectively.

EXERCISE 3.2

1. Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}, C = \begin{bmatrix} -2 & 5 \\ 3 & 4 \end{bmatrix}$

Find each of the following:

(i) A + B (ii) A – B (iii) 3A – C

(iv) AB (v) BA

2. Compute the following:

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} a^2+b^2 & b^2+c^2 \\ a^2+c^2 & a^2+b^2 \end{bmatrix} + \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

(iii) $\begin{bmatrix} -1 & 4 & -6 \\ 8 & 5 & 16 \\ 2 & 8 & 5 \end{bmatrix} + \begin{bmatrix} 12 & 7 & 6 \\ 8 & 0 & 5 \\ 3 & 2 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} \cos^2 x & \sin^2 x \\ \sin^2 x & \cos^2 x \end{bmatrix} + \begin{bmatrix} \sin^2 x & \cos^2 x \\ \cos^2 x & \sin^2 x \end{bmatrix}$

3. Compute the indicated products.

(i) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ (ii) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & -3 & 5 \\ 0 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}$ (v) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 3 & 1 \end{bmatrix}$

20. The bookshop of a particular school has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are ₹ 80, ₹ 60 and ₹ 40 each respectively. Find the total amount the bookshop will receive from selling all the books using matrix algebra.

Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times k$, $2 \times p$, $n \times 3$ and $p \times k$, respectively. Choose the correct answer in Exercises 21 and 22.

21. The restriction on n, k and p so that PY + WY will be defined are:

(A) $k = 3, p = n$ (B) k is arbitrary, $p = 2$

(C) p is arbitrary, $k = 3$ (D) $k = 2, p = 3$

22. If $n = p$, then the order of the matrix $7X – 5Z$ is:

(A) $p \times 2$ (B) $2 \times n$ (C) $n \times 3$ (D) $p \times n$

3.5. Transpose of a Matrix

In this section, we shall learn about transpose of a matrix and special types of matrices such as symmetric and skew symmetric matrices.

Definition 3 If $A = [a_{ij}]$ be an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the *transpose* of A. Transpose of the matrix A is denoted by A' or (A^T). In other words, if $A = [a_{ij}]_{m \times n}$, then $A' = [a_{ji}]_{n \times m}$. For example,

$$\text{if } A = \begin{bmatrix} 3 & 5 \\ \sqrt{3} & 1 \\ 0 & \frac{-1}{5} \end{bmatrix}_{3 \times 2}, \text{ then } A' = \begin{bmatrix} 3 & \sqrt{3} & 0 \\ 5 & 1 & \frac{-1}{5} \end{bmatrix}_{2 \times 3}$$

3.5.1 Properties of transpose of the matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any matrices A and B of suitable orders, we have

(i) $(A')' = A$, (ii) $(kA)' = kA'$ (where k is any constant)

(iii) $(A + B)' = A' + B'$ (iv) $(A B)' = B' A'$

Example 20 If $A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, verify that

(i) $(A')' = A$, (ii) $(A + B)' = A' + B'$,

(iii) $(kB)' = kB'$, where k is any constant.

Solution

(i) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix} \Rightarrow (A')' = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix} = A$$

Thus $(A')' = A$

(ii) We have

$$A = \begin{bmatrix} 3 & \sqrt{3} & 2 \\ 4 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} \Rightarrow A + B = \begin{bmatrix} 5 & \sqrt{3}-1 & 4 \\ 5 & 4 & 4 \end{bmatrix}$$

Therefore $$(A + B)' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Now $$A' = \begin{bmatrix} 3 & 4 \\ \sqrt{3} & 2 \\ 2 & 0 \end{bmatrix}, B' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix},$$

So $$A' + B' = \begin{bmatrix} 5 & 5 \\ \sqrt{3}-1 & 4 \\ 4 & 4 \end{bmatrix}$$

Thus $(A + B)' = A' + B'$

(iii) We have

$$kB = k\begin{bmatrix} 2 & -1 & 2 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2k & -k & 2k \\ k & 2k & 4k \end{bmatrix}$$

Then $$(kB)' = \begin{bmatrix} 2k & k \\ -k & 2k \\ 2k & 4k \end{bmatrix} = k\begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & 4 \end{bmatrix} = kB'$$

Thus $(kB)' = kB'$

Example 21 If $A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$, verify that $(AB)' = B'A'$.

Solution We have

$$A = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & -6 \end{bmatrix}$$

then $$AB = \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & -6 \end{bmatrix} = \begin{bmatrix} -2 & -6 & 12 \\ 4 & 12 & -24 \\ 5 & 15 & -30 \end{bmatrix}$$

Now $$A' = [-2\ 4\ 5]\ ,\ B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} \begin{bmatrix} -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix} = (AB)'$$

Clearly $(AB)' = B'A'$

3.6 Symmetric and Skew Symmetric Matrices

Definition 4 A square matrix $A = [a_{ij}]$ is said to be *symmetric* if $A' = A$, that is, $[a_{ij}] = [a_{ji}]$ for all possible values of i and j.

For example $A = \begin{bmatrix} \sqrt{3} & 2 & 3 \\ 2 & -1.5 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ is a symmetric matrix as $A' = A$

Definition 5 A square matrix $A = [a_{ij}]$ is said to be *skew symmetric* matrix if $A' = -A$, that is $a_{ji} = -a_{ij}$ for all possible values of i and j. Now, if we put $i = j$, we have $a_{ii} = -a_{ii}$. Therefore $2a_{ii} = 0$ or $a_{ii} = 0$ for all i's.

This means that all the diagonal elements of a skew symmetric matrix are zero.

is symmetric matrix and $\frac{1}{2}(A-A')$ is skew symmetric matrix. Thus, any square matrix can be expressed as the sum of a symmetric and a skew symmetric matrix.

Example 22 Express the matrix $B=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution Here

$$B' = \begin{bmatrix}2 & -1 & 1\\ -2 & 3 & -2\\ -4 & 4 & -3\end{bmatrix}$$

Let $$P = \frac{1}{2}(B+B') = \frac{1}{2}\begin{bmatrix}4 & -3 & -3\\ -3 & 6 & 2\\ -3 & 2 & -6\end{bmatrix} = \begin{bmatrix}2 & \frac{-3}{2} & \frac{-3}{2}\\ \frac{-3}{2} & 3 & 1\\ \frac{-3}{2} & 1 & -3\end{bmatrix},$$

Now $$P' = \begin{bmatrix}2 & \frac{-3}{2} & \frac{-3}{2}\\ \frac{-3}{2} & 3 & 1\\ \frac{-3}{2} & 1 & -3\end{bmatrix} = P$$

Thus $P = \frac{1}{2}(B+B')$ is a symmetric matrix.

Also, let $$Q = \frac{1}{2}(B-B') = \frac{1}{2}\begin{bmatrix}0 & -1 & -5\\ 1 & 0 & 6\\ 5 & -6 & 0\end{bmatrix} = \begin{bmatrix}0 & \frac{-1}{2} & \frac{-5}{2}\\ \frac{1}{2} & 0 & 3\\ \frac{5}{2} & -3 & 0\end{bmatrix}$$

Then $$Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q$$

Thus $$Q = \frac{1}{2}(B - B') \text{ is a skew symmetric matrix.}$$

Now $$P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

EXERCISE 3.3

1. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

2. If $A = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$, (ii) $(A - B)' = A' - B'$

3. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then verify that

(i) $(A + B)' = A' + B'$ (ii) $(A - B)' = A' - B'$

Miscellaneous Examples

Example 23 If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$.

Solution We shall prove the result by using principle of mathematical induction.

We have $\quad P(n)$: If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, $n \in \mathbf{N}$

$$P(1): A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ so } A^1 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Therefore, $\quad$ the result is true for $n = 1$.

Let the result be true for $n = k$. So

$$P(k): A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \text{ then } A^k = \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$$

Now, we prove that the result holds for $n = k + 1$

Now $\quad A^{k+1} = A \cdot A^k = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos\theta\cos k\theta - \sin\theta\sin k\theta & \cos\theta\sin k\theta + \sin\theta\cos k\theta \\ -\sin\theta\cos k\theta + \cos\theta\sin k\theta & -\sin\theta\sin k\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta + k\theta) & \sin(\theta + k\theta) \\ -\sin(\theta + k\theta) & \cos(\theta + k\theta) \end{bmatrix} = \begin{bmatrix} \cos(k+1)\theta & \sin(k+1)\theta \\ -\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$. Thus by principle of mathematical induction, we have $A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$, holds for all natural numbers.

Example 24 If A and B are symmetric matrices of the same order, then show that AB is symmetric if and only if A and B commute, that is AB = BA.

Solution Since A and B are both symmetric matrices, therefore $A' = A$ and $B' = B$.

Let AB be symmetric, then $(AB)' = AB$

But $(AB)' = B'A' = BA$ (Why?)

Therefore $BA = AB$

Conversely, if AB = BA, then we shall show that AB is symmetric.

Now $(AB)' = B'A'$

$= B\,A$ (as A and B are symmetric)

$= AB$

Hence AB is symmetric.

Example 25 Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}, C = \begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}$. Find a matrix D such that CD – AB = O.

Solution Since A, B, C are all square matrices of order 2, and CD – AB is well defined, D must be a square matrix of order 2.

Let $D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then CD – AB = 0 gives

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} = O$$

or $$\begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or $$\begin{bmatrix} 2a+5c-3 & 2b+5d \\ 3a+8c-43 & 3b+8d-22 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

By equality of matrices, we get

$$2a + 5c - 3 = 0 \qquad \ldots (1)$$

$$3a + 8c - 43 = 0 \qquad \ldots (2)$$

$$2b + 5d = 0 \qquad \ldots (3)$$

and $$3b + 8d - 22 = 0 \qquad \ldots (4)$$

Solving (1) and (2), we get $a = -191$, $c = 77$. Solving (3) and (4), we get $b = -110$, $d = 44$.

Therefore $$\mathrm{D} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$

Miscellaneous Exercise on Chapter 3

1. If A and B are symmetric matrices, prove that AB – BA is a skew symmetric matrix.

2. Show that the matrix B′AB is symmetric or skew symmetric according as A is symmetric or skew symmetric.

3. Find the values of x, y, z if the matrix $\mathrm{A} = \begin{bmatrix} 0 & 2y & z \\ x & y & -z \\ x & -y & z \end{bmatrix}$ satisfy the equation $\mathrm{A}'\mathrm{A} = \mathrm{I}$.

4. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = \mathrm{O}$?

5. If $\mathrm{A} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $\mathrm{A}^2 - 5\mathrm{A} + 7\mathrm{I} = 0$.

6. Find x, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix} = \mathrm{O}$

7. A manufacturer produces three products x, y, z which he sells in two markets. Annual sales are indicated below:

Market	**Products**		
I	10,000	2,000	18,000
II	6,000	20,000	8,000