Linear Algebra Refresher Part 2B

An introduction to
Eigenvalues and Eigenvectors

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Eigenvalues and Eigenvectors - Definition

An eigenvector (also known as a characteristic vector, proper vector or latent vector) of a linear transformation is a non-zero vector that gets multiplied by a scalar factor when the linear transformation is applied to it.

For example consider the vector given by **x** = Ci + 0j where C is a constant. This is a vector parallel to the x axis. Now if the x axis is stretched horizontally (to the right) to twice the size. The vector by **x** = Ci + 0j will shift to the right and becomes x1 = 2Ci + 0j. The direction remains unchanged and x1 is parallel to x. The scalar value by which the vector **x** is multiplied to get the new vector **x1** (2 in this case) is called the eigenvalue.

Eigenvalues and Eigenvectors – Formal Notation

Expressed formally, an eigenvector and an eigenvalue can be represented by:

$T(x) = \lambda x$

Where

1. T is a linear transformation applied to Vector space V over a field F that maps V on to itself.
2. x, a non-zero vector in V, is an eigenvector in V.
3. Lambda is a scalar in field F and is an eigenvalue of the above linear transformation.

Eigenvalues and Eigenvectors – Matrix Notation

Any non-zero vector x is called an eigenvector of a square matrix A if there exists a lambda such that

$$\boldsymbol{Ax = \lambda x}$$

where λ is a scalar value

The set of Eigenvalues for a given linear transformation is unique but Eigenvector is not unique.

λ is computed from the characteristic equation $| A - \lambda I | = 0$
For $\lambda = \lambda_i$ the eigenvector is the solution of $\boldsymbol{x}$ from system of linear equation $(A - \lambda_i)x = 0$.

Eigenvalues and Eigenvectors – Symmetric Matrix

If A is a real symmetric matrix (where A = Transpose o A), for example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the distinct eigenvectors of A, which in this case are (1,0) and (0,1) are orthogonal to each other.

Eigenvalues and Eigenvectors - Example

Consider the following matrix A given by:

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(1-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 2, \lambda = 1$$

Case 1: $\lambda = 2$ (largest)

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longrightarrow x_1 = 2x_1,\ x_1 + 2x_2 = 2x_2$$

$x_1 = 0$, x_2 can take any value except zero for non-obvious solution.

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the eigenvector with unit length.

Eigenvalues and Eigenvectors - Example

Case 2: $\lambda = 1$ (smallest)

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longrightarrow x_1 = x_1, \; x_1 + 2x_2 = x_2$$

$x_1 = -x_2$ for any value except zero for non-obvious solution.

$\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is the corresponding eigenvector with unit length.

Use of Eigenvalues and Eigenvectors

1. Facial Recognition
Eigenvectors have several applications in image transformation, **they** are used in principal component analysis for dimensionality reduction in face recognition.

2. Page Ranking
Eigenvectors are used to compute pageranks
PageRank is a “vote”, by all the other pages on the Web, about how important a page is. A page B votes for a page A by linking to it. If there’s no link there’s no vote. Given that this has to be solved by an iterating over the link or transition matrix, the solution looks like an Eigen Vector.

Img src: brown.edu

Use of Eigenvalues and Eigenvectors – Page Rank Algorithm Example

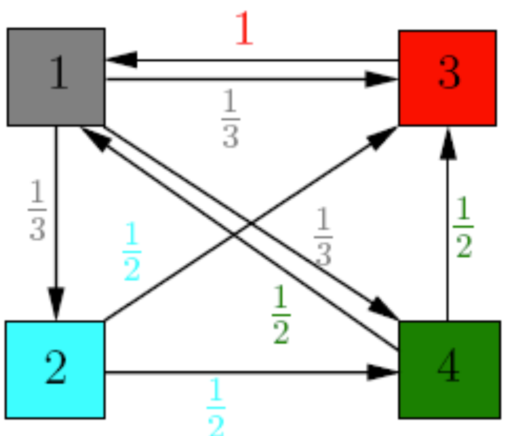
Suppose we have 4 pages. Page 1 has links to other 3 pages. Page 2 has links to Page 3 and Page 4. Page 3 has link to Page 1 only. And Page 4 has links to Page 1 and Page 3.

Each page should transfer evenly its importance to the pages that it links to. Page1 has 3 outgoing edges, so it will pass on 1/3 of its importance to each of the other 3 pages. Page3 has only one outgoing edge, so it will pass on all of its importance to node 1. In general, if a node has *k* outgoing edges, it will pass on *1/k* of its importance to each of the pages that it links to. Let us better visualize the process in a graph by assigning weights to each edge.

Use of Eigenvalues and Eigenvectors

Worked Example:

Transition Graph



The Transition/ Link Matrix:

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

The Page Rank of the 4 pages is given by:

Eigenvector corresponding to largest eigenvalue $\begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}$

src: pi.math.cornell.edu