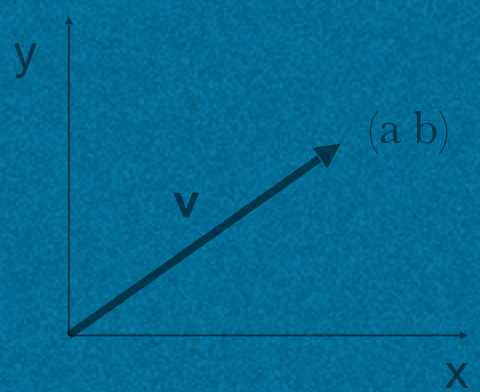
Linear Algebra Refresher - Part 1

$$\boldsymbol{x} = \begin{pmatrix} height \\ weight \\ age \end{pmatrix}$$

$$\boldsymbol{x}_n = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_n \end{pmatrix}$$

Scalars And Vectors



$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$

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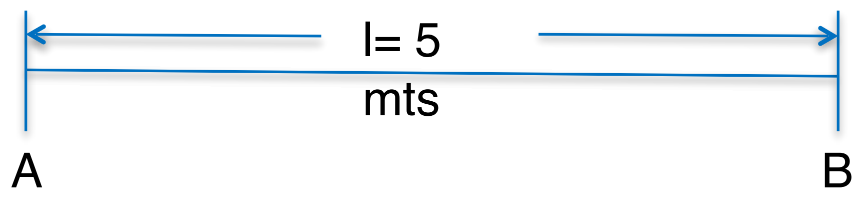
Outline

- Scalars and Vectors
- Vector Norm
- Operations on Vectors
- Linearly independent vectors

Scalar

A scalar represents a real number, or any quantity that can be measured using a single real number. Temperature, length, and mass are all scalars. A scalar has magnitude but no direction.



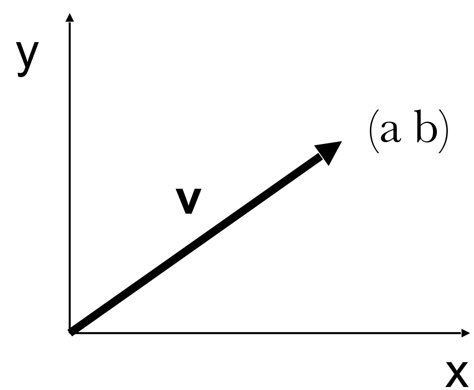
The length of line AB is a scalar with magnitude 5 metres

What is a Vector?

- Vectors, in linear algebra, are ordered arrays of numbers: e.g. height, weight and age of a person or the length, width and height of a cube.

- Think of a vector as a directed line segment in N-dimensions! (has “length” and “direction”)

$$\boldsymbol{x} = \begin{pmatrix} height \\ weight \\ age \end{pmatrix}$$



$$\boldsymbol{x}_n = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \dots \\ x_n \end{pmatrix}$$

Vector Addition

$$\mathbf{x} + \mathbf{y} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

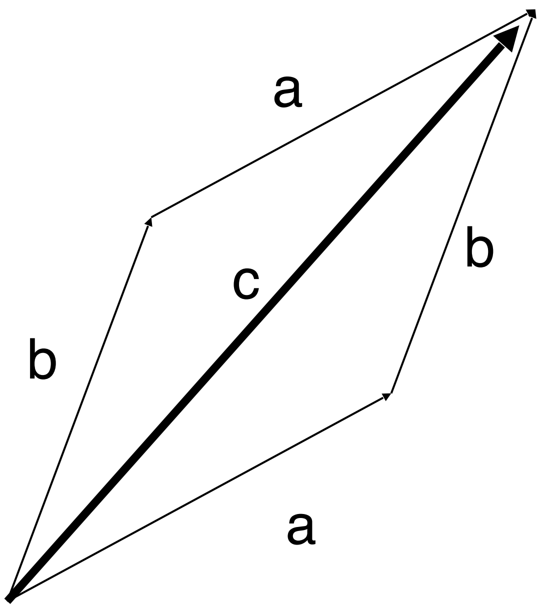
Two or more vectors of the same dimension can be added together to produce a resultant vector. The resultant vector will also be of the same dimension as that of the vectors being added.

Algebraically, summation of two vectors is done by adding each element of the vectors.

Vector Addition - Properties

1. Vector addition is commutative. Therefore for 2 vectors x and y, $x + y = y + x$
2. Vector addition is associative. Therefore for 3 vectors: a, b and c: $a + (b+c) = (a + b) +c$
3. Vector addition is distributive with respect to scalar multiplication. So for scalar π, $\pi(a + b) = \pi a + \pi b$

The Parallelogram Law Of Vector Addition



The Parallelogram law of vector addition states: If two vectors are considered to be the adjacent sides of a parallelogram, then the resultant of two vectors is given by the vector that is a diagonal passing through the point of contact of two vectors.

In the adjacent diagram:

$$\vec{c} = \vec{a} + \vec{b}$$

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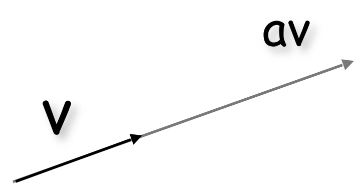
Scalar Product

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$

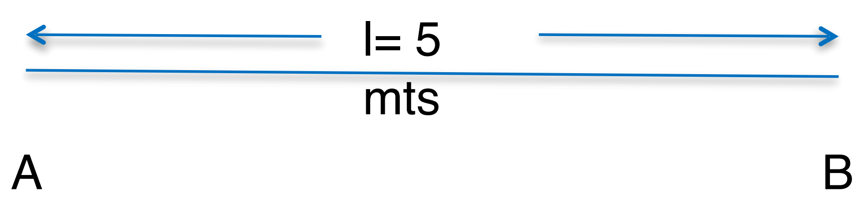


- The scalar product of a vector is simply the multiplication of a vector by a scalar quantity. This changes the magnitude of the vector, keeping the direction same.

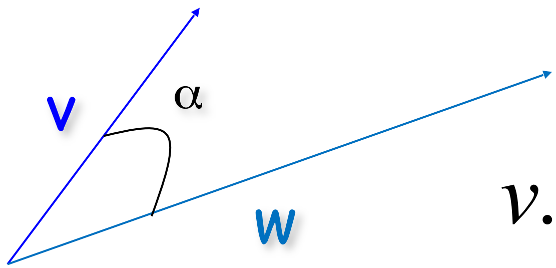
- The process is called as scaling

Norm Of A Vector

- The norm of a vector is simply its magnitude denoted by $||\vec{x}||$
- Mathematically $||\vec{x}|| = \text{sqrt}\,(x_1^2 + x_2^2 + \cdot\cdot + ..x_n^2)$
- Consider a vector $\vec{x}$ = [3 4] then $||\vec{x}||$ = sqrt(3^2 + 4^2) = 5
- In example below $||\ \overrightarrow{AB}\ ||$ = length of AB =5



Inner (dot) Product



$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2 . y_2$$

$$v.w = (x_1, x_2).(y_1, y_2) = \| v \| \cdot \| w \| \cos\alpha$$

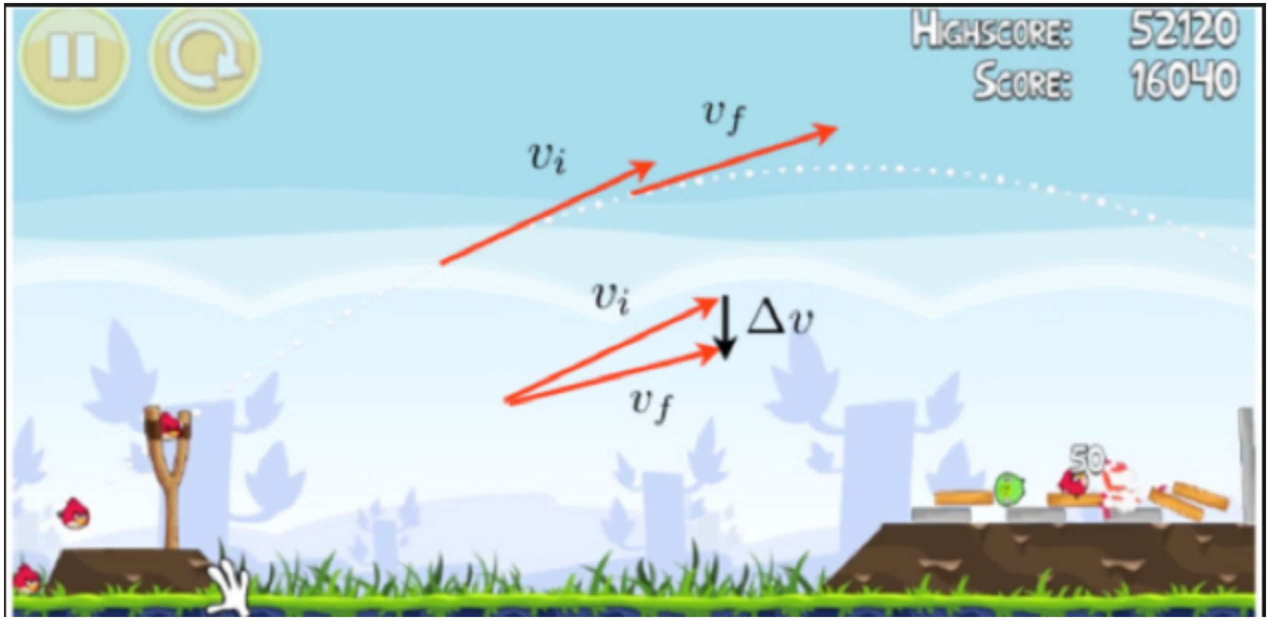
Dot product of two vectors v and w is a scalar quantity that is equal to the sum of pair-wise products of vectors v and w.

Geometrically, Dot product of two vectors v and w can be defined as a scalar quantity that is equal to the product of magnitudes of vectors multiplied by the cosine of the angle between vector.

For orthogonal vectors a and b, or vectors perpendicular to each other, their dot product = ||a||.||b|| * cos (90) = 0

Dot Product - Applications

For 2 vectors, v and w
v.w = llvll llwll cos å
where å is the angle between v and w



Dot product of two vectors can be used to calculate the cosine of the angle between them and hence understand their relative direction. If the dot product for 2 vectors is negative then it follows the cosine of the angle between them is negative, therefore the vectors are at an obtuse angle or moving away from each other. Similarly if the dot product is zero, the vectors are perpendicular to each other.

This is used in gaming applications and electromagnetic theory to understand relative motions of characters and particles.

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Inner (dot) Product - Properties

The dot product is always more than zero or equal to zero when it comes to a vector with itself: $a \cdot a \geq 0$

The dot product of a vector with itself is zero only if the vector is the zero vector: $a \cdot a = 0 \iff a = 0$

The dot product of a vector with itself is equal to the square of its magnitude: $a \cdot a = |a|^2$

The dot product operation is communicative: $a \cdot b = b \cdot a$

If the dot product of two not zero vectors is zero, then note that these vectors are orthogonal: $a \neq 0,\ b \neq 0,\ a \cdot b = 0 \iff a \perp b$

$(\alpha a) \cdot b = \alpha (a \cdot b)$

The dot product operation is distributive: $(a + b) \cdot c = a \cdot c + b \cdot c$

Inner (dot) Product – Solved Examples

Two vectors a and b of magnitude 3 and 4 respectively are parallel to each other. What is their dot product?

a.b = ||a|| * ||b|| * cos0
= 3*4*1
= 12

In the example above, b is rotated such that a and b are now at 45 degree to each other. What is their dot product?

a.b = ||a|| * ||b|| * cos45
= 3.4.1/sqrt(2)
= 6.sqrt(2)

Linear Independence Of Vectors

A set of vectors (v_1, v_2, .., v_n) are linearly independent if no vector of the set can be represented as a linear combination (only using scalar multiplication and vector additions) of other vectors.

v_1, v_2, .., v_n are said to be dependent if any of them say v1 can be expressed as: $v_1 = a2 * v_2 + a3 * v_3 + \ldots + an * v_n$, where a1, a2 … an are scalars.

Solved Example:

u =[2 2], v=[1 1], w=[3 3]

Prove that u, v and w are linearly dependant.
From the given values, we can derive:
v = 0.5* u +0*w
w = u +v
Therefore by definition, u, v and w are linearly dependant