Linear Algebra Refresher Part 2C

Singular Value Decomposition

Instructors

Mousum Dutta
Chief Data Scientist, Spotle.ai
IIT Kharagpur

Dr Soumen Nandi
Research Scientist
Indian Statistical Institute

What is Singular Value Decomposition?

Singular Value Decompositions or SVD is the decomposition of, any real matrix A_{mxn}, where ($n \leq m$) as follows:

$$\boldsymbol{A = UDV^T}$$

where:

U is an ($m \times m$) orthogonal matrix containing the eigenvectors of the symmetric matrix AA^T.

D is a ($m \times n$) diagonal matrix, containing the singular values of matrix A arranged in the diagonal in descending order. The number of non zero diagonal elements of D corresponds to the rank r of A.

V^T is an ($n \times n$) orthogonal matrix, containing the Eigenvectors of the symmetric matrix A^TA.

Explaining the terms used in the definition of SVD

Orthogonal Matrix is a square matrix whose columns and rows are orthogonal unit vectors or orthonormal vectors. Consider a 2x2 matrix given by:

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

then for an orthogonal matrix, $p^2 + q^2 = 1$,
$r^2 + s^2 = 1$ and
$pr + qs = 0$. (That is the inner dot product of 2 vectors = 0).

For any orthogonal matrix U, $U^T U = U U^T = I$

Diagonal Matrix is a matrix where non-diagonal elements are all zero. e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Singular Values are the square roots of the eigenvalues of the corresponding square gram matrix of a given matrix.

Rank is the maximum number of linearly independent vectors in a matrix also given by the number of non-zero rows in its row-echeleon matrix.

SVD - Properties

The following properties of SVD are useful in real-life use-cases, e.g. image compression, reduction of training data in machine learning and many more:

1. $AA^T = UDV^T \times VDU^T$

   $= UD^2U^T$

Similarly $A^TA = VD^2V^T$

U contains the eigenvectors of AA^T along its columns. These are also called left singular vectors.

V contains the eigenvectors of A^TA along its columns. These are also called the right singular vectors.

2. The singular values of D: d1, d2, d3,..are unique but U and V are not unique.

3. The singular values of D are arranged in descending order along the diagonal.

4. The rank of A = number of non-zero singular values in D.

SVD Use Case – Image Compression

An image can be compressed using SVD as follows:

1. An image of size mxn pixels can be represented by a mxn matrix say A. Using SVD A can be represented as **$A = UDV^T$**

2. To reduce the matrix using SVD, recall that D contains the singular values of A arranged in descending order along its diagonal. The first singular values consist of maximum information and the amount of information declines as you go down the diagonal. These lower order singular values can be discarded without causing image distortion.

   Assuming only k singular values are retained,

   and writing $A = UDV^T$

   $A = \sum u_i d_i v_i^T$, where i iterates till k and k < r, r is the rank of image matrix A.

The storage space required to store the mxn matrix of rank r is therefore reduced.

SVD Use Case – Image Compression

As an example consider the coins image, available in scikit-learn package. The 303 x 384 grayscale image can be represented by a 303 x 384 matrix where each pixel can take a value between 0 and 255.

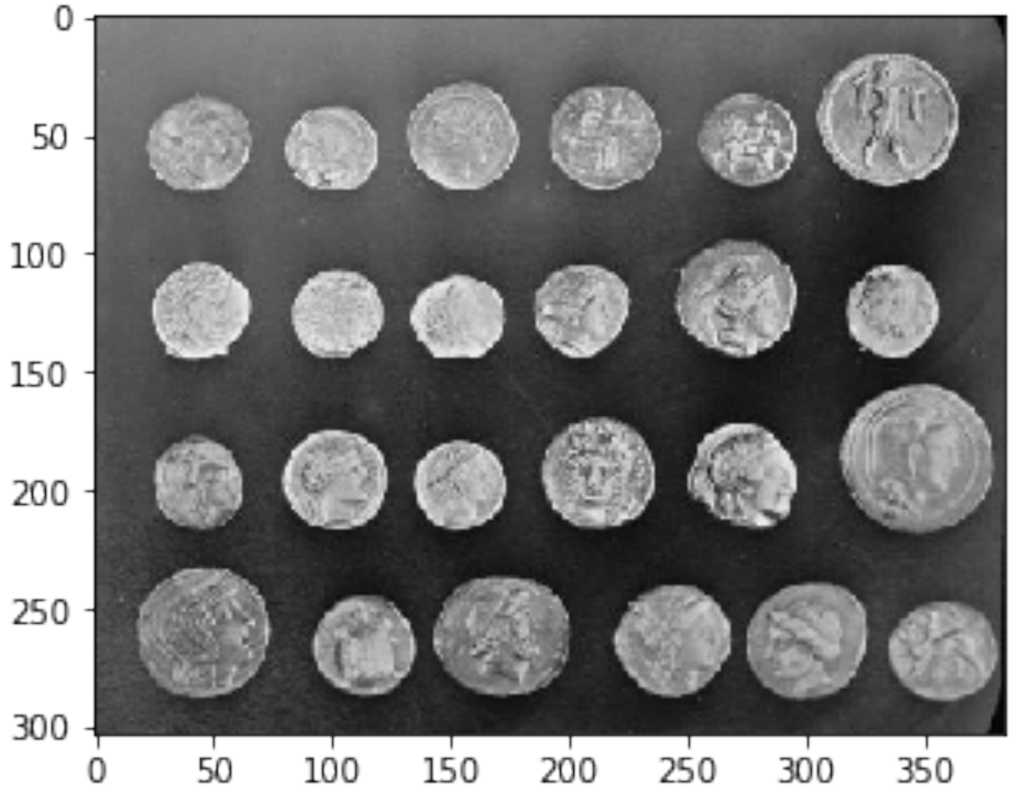
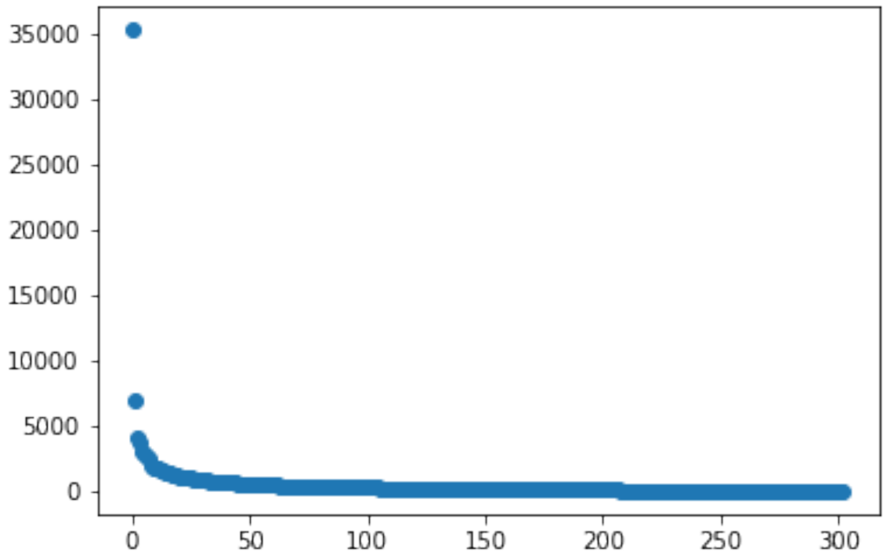


Fig 1 – Original Image Before Compression

SVD Use Case – Image Compression



When we plot singular values of this matrix, we get a graph as above. The image mostly is explained by its first 50 mods.

SVD Use Case – Image Compression

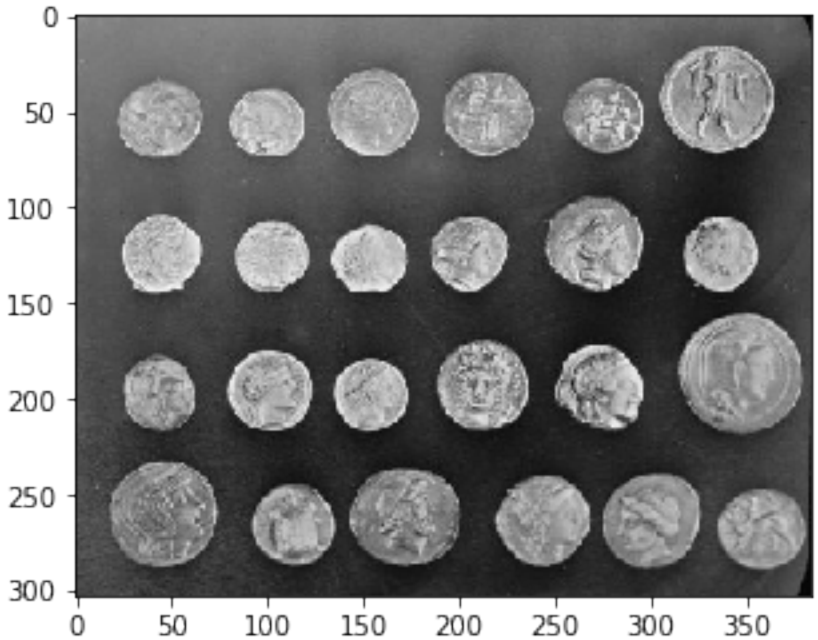


Fig 1 – Before Compression

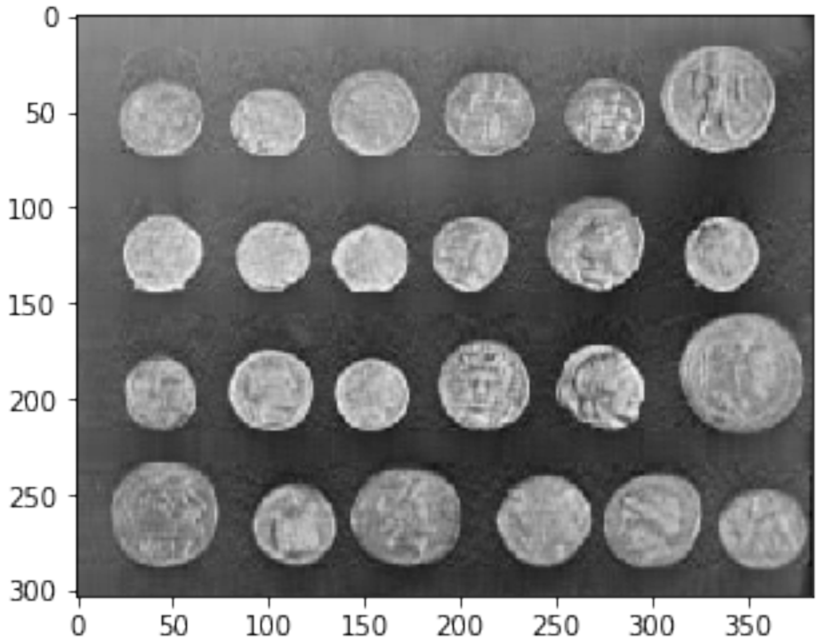


Fig 2 – After Compression