#BeCareerReady

Linear Algebra Refresher
Part 2A

An introduction to Lower Upper Decomposition of a Matrix

Lower Upper (LU) Decompositions

For any nonsingular square matrix $A_{n \times n}$, there exists two matrices L and U such that A = LU, where L is a lower triangular matrix and U is an upper triangular matrix.

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \qquad L = \begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}$$

Note that LU decompositions are not unique. Given a matrix A that can be expressed as a product of L and U, the following holds.

$$A = LU \Rightarrow A = (\lambda L)\left(\frac{1}{\lambda}U\right) \forall \lambda \neq 0$$

Some Terms And Definitions

Lower Triangular Matrix is a square matrix with all the elements above the principal diagonal 0.

For example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

Upper Triangular Matrix is a square matrix where all elements below the principal elements are 0.

For example:

$$\begin{bmatrix} 1 & 6 & 7 \\ 0 & 2 & 4 \\ 0 & 0 & 8 \end{bmatrix}$$

How Are LU Decompositions Used?

The LU decomposition method is used by systems to solve simultaneous equations and applied to real world problems like analysing call data records, calculating current in a circuit etc.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

For example, the above system of equations can solved by expressing the problems as AX = B where A is the co-efficient matrix. A can then be substituted by LU to give the value of X.

LU Decompositions and Solving Simultaneous Equations

System of Linear Equations:

Suppose LU decompositions of A is

$$LU\boldsymbol{x} = \boldsymbol{b} \text{ or } L\boldsymbol{y} = \boldsymbol{b} \text{ where } \boldsymbol{y} = U\boldsymbol{x}$$

We can solve $Ly = b$ by forward substitution and get y

Then we can solve $y = Ux$ by backward substitution and get x

LU Decompositions and Solving Simultaneous Equations

Let us illustrate with an example:

$$\begin{pmatrix} 2 & 1 \\ -4 & -6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

1. Express equation in matrix form, AX = B

$$\begin{pmatrix} 2 & 1 \\ -4 & -6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & -4 \end{pmatrix}$$

2. The coefficient matrix A can be represented by a LU product as indicated. Therefore Ly = B

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Longrightarrow y_1 = 1,\ y_2 = 4$$

3. Solve for y by forward substitution

$$\begin{pmatrix} 2 & 1 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Longrightarrow x_2 = -1,\ x_1 = 1$$

4. Solve for x by backward substitution