Credit Risk Models

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Abstract

This paper reviews the literature on credit risk models. Topics included are structural and reduced form models, incomplete information, credit derivatives, and default contagion. It is argued that reduced form models and not structural models are appropriate for the pricing and hedging of credit-risky securities. Directions for future research are discussed.

1. INTRODUCTION

From a historical perspective, credit risk is the final frontier of option pricing theory. Valuing credit-risky securities necessitates an understanding of standard option pricing theory, but under a stochastic term structure of interest rates. Standard option pricing theory came first with the development of the Black-Scholes-Merton model (Black & Scholes 1973, Merton 1973), followed over a decade later by the term structure of interest rates option pricing technology of Heath et al. (1992). Extending the Heath-Jarrow-Morton (HJM) model to include default—credit risk—yields a fresh and currently active research area within financial economics.

Credit risk arises whenever two counterparties engage in borrowing and lending. Borrowing can be in cash, which is the standard case, or it can be through the "shorting" of securities. Shorting a security is selling a security one does not own. To do this, the security must first be borrowed from an intermediate counterparty, with an obligation to return the borrowed security at a later date. The borrowing part of this shorting transaction involves credit risk. Because the majority of transactions in financial and commodity markets involve some sort of borrowing, understanding the economics of credit risk is fundamental to the broader understanding of economics itself.[1]

Another reason for studying credit risk is the recent expansion in the trading of credit derivatives. Trading in credit derivatives began in the early 1990s. Credit derivatives are financial contracts whose payoffs depend on whether some credit entity (e.g., a person, corporation, bank, government, or trust[2]) defaults on its debt. Traded are many different varieties of credit default swaps (CDS), credit derivatives on baskets, and collateralized default obligations (CDO). In 2007, the International Swaps and Derivatives Association estimated the outstanding notional in credit derivatives to be over 62 trillion dollars. In contrast, the more mature equity derivatives market was estimated to be just under 10 trillion dollars.[3]

This paper reviews the financial economics literature relating to the modeling of credit risk. The empirical literature is only discussed as needed to motivate the development of various models. An outline for this paper is as follows. Section 2 provides a brief historical overview of the literature. Sections 3 and 4 discuss structural models and reduced form models, respectively. Section 5 discusses how incomplete information affects credit risk modeling. Section 6 discusses credit derivatives in the context of a reduced form model, and Section 7 concludes.

2. A HISTORICAL OVERVIEW

The first model for studying credit risk, termed the structural approach, was introduced by Merton (1970, 1974). As the first contribution to credit risk, Merton's original model was purposely simple. Merton considered credit risk in the context of a firm issuing only a single zero-coupon bond with a fixed maturity date. In this structure, default can only

[1]For example, credit risk is essential to understanding business cycles; see Bernanke & Gertler (1989) and Kiyotaki & Moore (1997).

[2]A trust is a legal entity that holds a collection of assets against which debt is issued. The debt are often termed asset-backed securities. Examples of asset-backed securities would be bonds issued against a pool of subprime residential mortgages.

[3]See http://www.isda.org and market surveys.

occur on the debt's maturity date, and the firm's default probability is the likelihood that the firm's asset value at maturity will exceed the face value of the debt. Shortly thereafter, extensions to address this simple liability structure were explored by Black & Cox (1976), Geske (1977), Mason & Bhattacharya (1981), Longstaff & Schwartz (1995), and Zhou (2001), among others. These extensions characterize default as the first hitting time of the firm's asset value to a given barrier where the barrier is determined by the firm's liability structure. The recovery rate of the debt in default is determined by the liabilities' priorities. Usually, the firm's asset value is assumed to follow a diffusion process, as in Merton's (1974) original paper. Extensions of Merton's model that endogenize the firm's capital structure (see Leland 1994, Leland & Toft 1996, and Anderson & Sundaresan 1996) were also studied.

The structural approach to credit risk modeling has a well-known empirical shortcoming: The firm's assets are neither traded nor observable (see Lando 2004 and Duffie & Singleton 2003). To address this shortcoming, an alternative approach to modeling credit risk was developed by Jarrow & Turnbull (1992, 1995). This alternative approach has become known as the reduced form model. Reduced form models take as exogenous both the firm's default time and its recovery rate process. Usually, the default time is modeled as the first jump time of a point process, and the recovery rate process a constant proportion of an observed market price for a traded debt instrument. Reduced form models have a nice characteristic. They enable one to value credit derivatives using only the HJM default-free interest rate pricing technology by adjusting the default-free spot rate of interest process to reflect the default intensity. Of course, this results in simplified computational procedures.

As the credit derivative markets expanded in the last two decades, so have extensions to the reduced form model. To consider credit rating migration, Jarrow et al. (1997) introduced a Markov chain model where the states correspond to credit ratings. Next, there was the issue of default correlation for pricing credit derivatives on baskets (e.g., CDO). This correlation was first handled with Cox processes (Lando 1998). The use of Cox processes induces default correlations across firms through common state variables that drive the default intensities. But when conditioning on the state variables, defaults are assumed to be independent across firms. If this structure is true, then after conditioning defaults may be diversifiable in a large portfolio and require no additional risk premia (see Jarrow et al. 2005). However, this is not the only mechanism through which default correlations can be generated. Default correlations are also possible through competitive industry considerations. This type of default contagion is a form of so-called counterparty risk, and it was first studied in the context of a reduced form model by Jarrow & Yu (2005). Counterparty risk in a reduced form model, an issue in and of itself, was previously studied by Jarrow & Turnbull (1995, 1997) and Duffie & Huang (1996).

An important contribution to the credit risk literature was clarifying the relation between structural and reduced form models using the information sets employed in their construction. Structural models use the management's information set, whereas reduced form models use the market's. Indeed, the manager has access to the firm's asset values, but the market does not. The first paper making this connection was by Duffie & Lando (2001), who viewed the market as having the management's information set plus noise, due to the accounting process. Noise in the market's information set transforms the first hitting time of the firm's asset value to a barrier into the first jump time of a point process. An alternative view is that the market has a coarser partitioning of management's information, that is, less of it. Both views are reasonable, but their mathematics are quite different. The second approach, first explored by Cetin et al. (2004), also transforms the first hitting time of the firm's asset value to a barrier into the first jump time of a point process. Related references include Kusuoka (1999), Blanchet-Scalliet & Jeanblanc (2004), and Jarrow et al. (2007).

Incomplete information also can affect correlations in credit risk spreads. A default by one firm may cause another firm's default intensity to increase as the market learns about the reasons for the realized default. This, in turn, will lead to an increase in credit spreads. This information effect corresponds to the concept of frailty in hazard rate estimation; see Schonbucher (2004), Duffie et al. (2006), and Chava et al. (2006).

As our understanding of the default process has increased, modeling the recovery rate has taken on new importance. Early credit risk models had the recovery rate take one of two simple forms. Jarrow & Turnbull (1995) had the recovery rate be a fixed fraction of an otherwise equivalent Treasury security (termed recovery of Treasury), whereas Madan & Unal (1998) and Duffie & Singleton (1999) had the recovery rate be a fixed fraction of the debt's value an instant before default (termed recovery of market value). Recent extensions for stochastic recovery rates include Bakshi et al. (2001), Guo et al. (2007), and Guo et al. (2008). These extensions have important implications for the pricing of defaulted debt that trade in distressed debt markets (see Altman 1998 for a discussion of distressed debt markets).

3. STRUCTURAL MODELS

The structural model for credit risk, sometimes referred to as contingent claims modeling, was introduced by Merton (1970, 1974). This section discusses only the pricing of zero-coupon bonds. More complex credit-risky securities are discussed in a subsequent section.

3.1. Merton's Model

To understand the intuition behind the structural model, it is best to start with Merton's (1974) original model. We consider a continuous-time, continuous-trading model with time horizon $[0, \tau^*]$. We are given a filtered probability space $[\Omega, G, (G_t)_{[t \in 0, \tau^*]}, P]$ satisfying the usual conditions (see Protter 2004), where P is the statistical probability measure.

3.1.1. The Firm's Liability Structure. We consider a firm with the liability structure given in **Table 1**. The firm has assets with time t value V_t, against which it has issued debt. The debt consists of a single zero-coupon bond with face value K and maturity date $T \leq \tau^*$. We let D_t be the time t value of the firm's debt, and we let E_t be the time t value of the firm's equity. Of course, the accounting identity

$$V_t = D_t + E_t \tag{1}$$

holds for all times t.

Table 1 A firm's balance sheet

Assets	Liabilities
V_t	D_t (zero-coupon bond: face value K, maturity T)
	E_t

In this firm, the time T value of the firm's debt and equity is completely determined by the debt's covenants. At the debt's maturity, the equity holders decide whether to pay off the debt. They will do so if and only if the value of the firm's assets exceeds the debt's face value. Otherwise, default occurs. Hence, the time T value of the equity can be written as

$$E_T = \max\{V_T - K, 0\}. \tag{2}$$

The equity's value is equivalent to the payoff from a European call option on the firm's value, with maturity date T and strike price K. In essence, the debt holders own the firm. However, the equity holders have the option to buy the firm back from the debt holders at time T for K dollars.

As a consequence, the debt's value at maturity is equal to the face value of the debt or the firm's asset value, whichever is smaller:

$$\begin{aligned} D_T &= \min\{K, V_T\} \\ &= V_T - \max\{V_T - K, 0\} \\ &= K - \max\{K - V_T, 0\}. \end{aligned} \tag{3}$$

As indicated, this can alternatively be written in two equivalent ways. The first is that the debt is equal to the firm's value less the equity's value, which is equal to the payoff from a European call option on the firm's asset value (as previously discussed). The second is that the debt's time T payoff is equal to the face value of the debt less the payoff to a European put option on the firm's value, with maturity date T and strike price K. From this perspective the debt holder's original principal will be returned if and only if the put option expires worthless. In essence, the debt holders are writing an insurance policy on the firm's assets to the equity holders. The insurance policy's notional is the face value of the debt, and the interest on the debt is the compensation earned for writing this insurance policy. This last interpretation makes it clear that the debt is worth less than its face value and that the interest earned on the debt should exceed the so-called riskless rate.

3.1.2. The Firm's Default Probability and Recovery Rate. The firm's default probability and recovery rate are determined by the previous structure. Indeed, note that for this firm default can only happen at time T. Assuming that the firm's asset value follows a stochastic process adapted to the filtration $(G_t)_{t\in[0,\tau^*]}$, the likelihood of default is given by

$$P\{V_T \le K\}. \tag{4}$$

Also, the recovery rate is determined by the firm's liability structure. As a percentage of the face value, the recovery rate given default is

$$\delta_T \equiv \frac{V_T}{K} \in [0,1] \text{ if } V_T \le K. \tag{5}$$

These quantities are used in subsequent sections for risk management computations.

3.1.3. Valuing the Firm's Debt. To value the firm's debt, we need to add more structure. As in Merton (1974), we use the standard arbitrage-free pricing methodology. First, we assume that the default-free spot rate of interest r is a constant. We let a money market account trade, initialized with a dollar at time 0, where the money market account earns interest at the default-free spot rate. Its time t value is

$$B_t \equiv e^{rt}. \tag{6}$$

Next, we assume that markets are frictionless and competitive. A frictionless market is one where there are no transaction costs, no restrictions on trades (e.g., short-sales restrictions), and no taxes. A competitive market is one where all traders act as price takers. A price taker is a trader who believes that his trades have no impact on market prices.

We also assume that the firm's assets trade in this market and follow a geometric Brownian motion,

$$dV_t = \mu V_t dt + \sigma V_t dW_t, \tag{7}$$

where μ and $\sigma > 0$ are constants, and W_t is a Brownian motion on the filtered probability space $[\Omega, G, (Gt)_{t\in[0,\tau^*]}, P]$. This assumption implies that the firm's asset value over $[0,T]$ is lognormally distributed with mean $\left(\mu - \frac{1}{2}\sigma^2\right)T$ and volatility $\sigma\sqrt{T}$.

Lastly, we assume that there exists an equivalent[4] probability measure Q, such that

$$\frac{V_t}{B_t} \tag{8}$$

is a Q-martingale.

This probability Q is known as an equivalent martingale probability. Using the first fundamental theorem of asset pricing (see Harrison & Pliska 1981), this implies that the market is arbitrage free.

Because the volatility of the firm's value process is a constant, it can be shown that the market is complete. In a complete market, by the second fundamental theorem of asset pricing (see Harrison & Pliska 1981, 1983), any derivative (contingent claim) written on the firm's asset value can be valued using martingale pricing. In particular, letting a derivative's time T cash flow be denoted by $h(V_T)$, where $h : \mathbb{R} \to \mathbb{R}$, is suitably measurable and bounded such that $\tilde{E}(|h(V_T)|) < \infty$; then the derivative's time t value is

$$\tilde{E}\left(h(V_T)|G_t\right)e^{-r(T-t)},$$

where $\tilde{E}(\cdot)$ corresponds to expectation under the martingale probability Q.

Using this expression, the value of the firm's debt is easily seen to be

$$\begin{aligned} D_t &= \tilde{E}(\min\{K, V_T\}|G_t)e^{-r(T-t)} \\ &= V_t - \tilde{E}\left(\max\{V_T - K, 0\}|G_t\right)e^{-r(T-t)}. \end{aligned} \tag{9}$$

The second term in expression 9 is just the value of a European call option under geometric Brownian motion—the well-known Black-Scholes formula. Using this formula, we can rewrite the debt issue's value as

$$D_t = V_t N(-d_1) + Ke^{-r(T-t)}N(d_2), \tag{10}$$

where $N(\cdot) \equiv N(\cdot:0,1)$ is the standard normal distribution function (with mean 0, variance 1):

[4]This means that the probability measures P and Q agree on zero-probability events.

$$d_1 \equiv \frac{\log\left(\frac{V_t}{K}\right) + r(T-t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 \equiv d_1 - \sigma\sqrt{T-t}.$$

This expression has a nice economic interpretation. The debt's value is equal to the risk-adjusted probability of default times the firm's value plus the risk-adjusted probability of no default times the (discounted) face value of the debt. Comparative statics on this debt's price reveals that it is increasing in V_t and K and decreasing in r, σ, and $(T-t)$. Because the firm's volatility σ is under the control of management, the decreasing value of the debt as σ increases leads to the shareholder/debt holder conflict (see Ross et al. 1993, p. 640).

3.1.4. Credit Risk Spreads. Important in the practical application of credit risk models is the determination of the credit risk spread. To define the credit risk spread, we must first define a bond's yield. The yield on a zero-coupon bond is implicitly defined as the solution $y(t,T)$ to

$$D_t = Ke^{-y(t,T)(T-t)}. \tag{11}$$

That is, the yield is the constant discount rate (per unit time) that equates the present value of the face value of the debt (as if paid for sure) to its market price. The credit spread $s(t,T)$ is the magnitude by which the bond's yield exceeds the default-free spot rate of interest, specifically

$$\begin{aligned} s(t,T) &= y(t,T) - r \\ &= \frac{\ln\left(\frac{K}{D_t}\right)}{(T-t)} - r. \end{aligned} \tag{12}$$

Under Merton's (1974) model, this has an explicit representation (substitute expression 10 into expression 12). It can be shown that

$$\lim_{(T-t)\to 0} s(t,T) = 0 \text{ if } V_t > K. \tag{13}$$

That is, the credit spread goes to zero as the maturity of the bond vanishes. This follows directly from the geometric Brownian motion assumption. As the debt matures, if the firm's asset value exceeds the face value of the debt, due to the Brownian motion–based evolution it eventually becomes certain that the firm's asset value will not default (see Lando 2004, p. 15).

3.1.5. Default Contagion. An important issue in the risk management of fixed income portfolios or the pricing and hedging of credit derivatives on baskets is default correlation or default contagion across firms, sometimes known as systemic risk. This section considers default contagion within the context of Merton's (1974) model. By assumption, we consider a collection of n firms, each with the liability structure contained in **Table 2**. All firms issue zero-coupon bonds with the same maturity date T, but with different face values K_i. Also, the firms can have correspondingly different asset values, debt values, and equity values.

We assume that all of these firms' assets trade continuously in frictionless and competitive markets following geometric Brownian motions,

$$dV_i(t) = \mu_i V_i(t)dt + \sigma_i V_i(t)dW_i(t), \tag{14}$$

where μ_i and $\sigma_i > 0$ are constants for $i = 1,\ldots,n$, and $[W_1(t),\ldots,W_n(t)]$ is an n-dimensional Brownian motion on the filtered probability space $[\Omega, G, (G_t)_{t\in[0,\tau]}, P]$ with correlation structure $dW_i(t)dW_j(t) = \rho_{ij}dt$ for $i, j = 1,\ldots, n$.[5]

Table 2 Firm i's balance sheet for $i = 1,\ldots, n$

Assets	Liabilities
$V_i(t)$	$D_i(t)$ (zero-coupon bond: face value K_i, maturity T)
	$E_i(t)$

Consider the time horizon $[0,T]$. Default contagion is characterized by determining the joint default probability of the firms at time T. Under the geometric Brownian motion assumption (expression 14), this joint default distribution function is given by

$$P\{[V_1(T) \le K_1],\ldots,[V_n(T) \le K_n]\} = N(\log K_1,\ldots,\log K_n : \boldsymbol{\mu}, \boldsymbol{\rho}), \tag{15}$$

where $N(\cdot,\ldots,\cdot : \boldsymbol{\mu}, \boldsymbol{\rho})$ is an n-dimensional cumulative normal distribution function with mean vector $\boldsymbol{\mu} = \left(\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)T,\ldots,\left(\mu_n - \frac{1}{2}\sigma_n^2\right)T\right)$ and correlation matrix $\boldsymbol{\rho} = \left[\rho_{ij}T\right]_{i,j=1,\ldots,n}$. Using this joint default distribution function, one can easily compute the probability that any subcollection of firms will default at time T.

3.1.6. Risk Management. Next, we consider a portfolio consisting of these n firms' debt issues with portfolio weights $w_i \ge 0$ where $\sum_{i=1}^n w_i = 1$. The dollar loss, given default for this portfolio over $[0,T]$, is defined by

$$L \equiv \sum_{i=1}^n w_i 1_{\{V_i(T)\le K_i\}} L_i K_i \ge 0,$$

where

$$L_i \equiv 1 - \delta_i(T) = 1 - \frac{V_i(T)}{K_i} \ge 0 \text{ if } V_i(T) \le K_i.$$

Risk management statistics are computed using the distribution function for these losses given by

$$P\{L \le u\} \text{ for all } u \in [0,\infty) \tag{16}$$

under the statistical probability. Valuation of credit derivatives on baskets uses this distribution function under the martingale probability. For simplicity (and without loss of generality), we discuss the computation of this loss distribution only under the statistical probability.

Using standard probability methods, this distribution can be computed given two inputs:[6] (a) the joint default distribution function in expression 15 and (b) the joint distribution function for the percentage losses given default. Under the geometric Brownian motion assumption (expression 14), this joint distribution function for losses given default is

[5]Note the notation changes the subscript referencing time to the subscript referencing a specific firm. Time is reintroduced as an explicit functional argument of the price process.

[6]The proof of this assertion is contained in the appendix (Section 8).

$$P\{L_1 \leq u_1, ..., L_n \leq u_n | V_1(T) \leq K_1, ..., V_n(T) \leq K_n\}$$
$$= N(-\log(1-u_1)K_1, ..., -\log(1-u_n)K_n | V_1(T) \leq K_1, ..., V_n(T) \leq K_n : \boldsymbol{\mu}, \boldsymbol{\rho}) \quad (17)$$

for all $(u_1, ..., u_n) \in [0, \infty)^n$, where $N(\cdot, ..., \cdot | V_1(T) \leq K_1, ..., V_n(T) \leq K_n : \boldsymbol{\mu}, \boldsymbol{\rho})$ is an n-dimensional conditional normal distribution function with mean vector $\boldsymbol{\mu} = \left(\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)T, ..., \left(\mu_n - \frac{1}{2}\sigma_n^2\right)T\right)$ and correlation matrix $\boldsymbol{\rho} = \left[\rho_{ij}T\right]_{i,j=1,...,n}$.[7]
Using this joint distribution function for losses given default, one can easily compute the probability of any subcollection of firm losses given default at time T.

3.1.7. Loss Distributions Using Copulas.

Merton's (1974) model for the loss distribution has been extended for industry usage via the application of copulas (see Bluhm et al. 2003). To illustrate this industry modification, we first need to define a default indicator for firm i:

$$\chi_i \equiv \begin{cases} 1 \text{ if firm } i \text{ defaults at } T \\ 0 \quad \text{otherwise} \end{cases}.$$

In Merton's model, $\chi_i = 1_{\{V_i(T) \leq K_i\}}$, but other formulations are also possible.

The motivation for this modification is that we cannot directly observe (or easily estimate) the joint default distribution

$$F(u_1, ..., u_n) \equiv P\{\chi_1 \leq u_1, ..., \chi_n \leq u_n\}$$

for all $(u_1, ..., u_n) \in [0,1]^n$, but we can observe and estimate the marginal default distributions. The marginal default distribution for firm i is given by

$$P\{\chi_i \leq u_i\} = \begin{cases} 1 - \pi_i \text{ if } u_i \in [0,1) \\ 1 \text{ if } u_i = 1 \end{cases} \equiv b_i(u_i\,;\,\pi_i),$$

where

$$\pi_i \equiv P\{\chi_i = 1\}.$$

We see that the marginal default distribution is the binomial distribution function with parameter π_i. In Merton's model, for example, the parameter π_i is determined by the firm's asset value falling below a barrier:

$$\pi_i = N\left(\log K_i : \left(\mu_i - \frac{1}{2}\sigma_i^2\right)T, \sigma_i^2 T\right).$$

To overcome this difficulty in directly estimating the joint default distribution function, one can employ a copula function. A copula function is used to generate a joint distribution function consistent with a given set of marginals. More formally, a copula function is any n-dimensional distribution function on $[0,1]^n$ such that all of its marginal distributions are uniformly distributed on [0,1]. Examples include the Gaussian copula or the t-copula (see Bluhm et al. 2003, p. 105).

[7]The proof of this expression follows.
$P\{L_1 \leq u_1, ..., L_n \leq u_n | V_1(T) < K_1, ..., V_n(T) < K_n)$
$= P\{\log V_1 \geq \log(1-u_1)K_1, ..., \log V_n \geq \log(1-u_n)K_1 | V_1(T) < K_1, ..., V_n(T) < K_n)$.
Noting that $(\log V_1, ..., \log V_n)$ is normally distributed, and using the symmetry of the normal distribution function, gives the result.

The industry extension replaces two of Merton's original assumptions with the following.

> Assumption 1. First, we take as exogenous a given copula,
> $C(u_1,\dots,u_n : \theta)$,
> which depends on a parameter vector θ.

A copula is usually selected that is simple to compute. Then, using this copula and the (observed) marginal default distributions, the joint default distribution function is defined by

$$F(u_1,\dots,u_n) \equiv C[b_1(u_1;\pi_1),\dots,b_n(u_n;\pi_n):\theta].$$

This joint distribution function, by construction, will have the marginal distributions given by $b_i\ (u_i\ ;\ \pi_i)$ for all i (for a proof see Bluhm et al. 2003). This assumption replaces expression 15 in Merton's (1974) model. Second, we add the following assumption.

> Assumption 2. Losses given default for firm i are independent of the losses given default for firm j, for $i \neq j$.

Then,

$$P\{L_1 \le u_1,\dots,L_n \le u_n|\chi_1 = 1,\dots,\chi_n = 1\}$$

$$= P\{L_1 \le u_1|\chi_1 = 1\}\cdots P\{L_n \le u_n|\chi_n = 1\}.$$

Under assumption 2, knowledge of just the marginal distributions for losses given default are sufficient to compute the joint loss distribution given default. For example, one might use a lognormal distribution with default barrier K_i:

$$\begin{aligned}&P\{L_i \le u_i|\chi_i = 1\}\\&\quad = N\left(-\log(1-u_i)K_i|V_i(T) \le K_i : \left(\mu_i - \frac{1}{2}\sigma_i^2\right)T, \sigma_i^2 T\right).\end{aligned}$$

Other choices are, of course, possible. This assumption replaces expression 17 in Merton's model.

Combined, these two assumptions yield the standard industry modification of Merton's model. As seen, this modification enables one to compute expression 16 given only knowledge of the marginals plus the copula's parameter vector θ. Various implementations of the copula approach are given in Bluhm et al. (2003).

3.2. Extensions

Merton's (1974) model is valuable for generating an economic understanding of credit risk. However, from a practical perspective, it is too simple a credit risk model to provide a useful characterization of actual fixed income markets. This is because its underlying assumptions do not reflect critical aspects of credit risk markets. Recall that Merton's model is based on five assumptions: (*a*) constant interest rates, (*b*) geometric Brownian motion for the firm's asset value process, (*c*) that the firm's liabilities consist of a single zero-coupon bond, (*d*) that recovery rates in default follow the absolute priority rules, and (*e*) that the firm's asset value is observable and traded continuously. All of these assumptions can be generalized to reflect actual market conditions, except for the last one.

The last assumption is crucial to the theory. For the arbitrage argument to be valid, the last assumption must hold—the firm's asset value process must be traded (continuously) in frictionless and competitive markets (and thus observable). For this to be true, all of the firm's assets must be traded either directly or indirectly.

Let us consider the first possibility, the direct trading of the firm's assets. For almost all firms this is never true. Indeed, a firm's physical facilities (factories, offices), its accounts receivable, patents, etc., are not traded. Next, let us consider the second possibility, the indirect trading of the firm's assets. These could be traded indirectly via the balance sheet accounting identity if all of the firm's liabilities and equity trade. Again, for almost all firms this is never true. Indeed, the firm's trade credit (accounts payable), lines of credit, bank loans, pension fund obligations, etc., are not traded. Hence, this assumption is not satisfied in practice.

Whether or not the structural model should be applied in practice because of the violation of this assumption is a critical theoretical and empirical issue. We return to this issue in a subsequent section after we have introduced a different class of models termed reduced form models, which relax this assumption. For the remainder of this section we discuss only extensions to the first four assumptions.

3.2.1. Stochastic Interest Rates. Merton's (1974) model assumes that interest rates are constant. Stochastic interest rates, however, generate a fundamental risk inherent in all fixed income securities. As such, for a realistic credit risk model this assumption needs to be relaxed. Fortunately, it is easy to embed the firm's asset value process into a Heath et al. (1992) stochastic term structure model (see Longstaff & Schwartz 1995 and Lando 2004, p. 18). Expression 10 for the debt's value changes, but the logic is otherwise identical.

3.2.2. General Semimartingale Processes. The assumption that the firm's asset value process follows a geometric Brownian motion can be easily relaxed to the more general class of semimartingales. For example, the literature has already considered a jump diffusion model (see Zhou 2001 and Lando 2004). Stochastic volatilities (see Fouque et al. 2000) or Levy processes (see Cont & Tankov 2004) are also easy extensions. Conceptually, this is equivalent to applying the well-studied generalizations of the Black-Scholes model for pricing call options to this application. As with the stochastic interest rate generalization, expression 10 for the debt's value changes, but the logic is otherwise identical.

3.2.3. More Complex Liability Structures. Of course, few (if any) firms have the simple liability structure as given in Table 1. The balance sheet of a modern corporation is much more complex, containing various types of liabilities—some traded, some not. For example, such liabilities include accounts receivable, trade credit, bank loans, and publicly traded debt. Furthermore, debt issues commonly contain coupon payments, covenants, and embedded options (e.g., call or sinking fund provisions). Lastly, a firm's capital structure is not static, as in Merton's original model, but is dynamic—changing across time. These complications considerably complicate the analysis. Various generalizations of Merton's model have appeared in the literature to address these complications. These generalizations include Black & Cox (1976), Geske (1977), Mason & Bhattacharya (1981), Leland (1994), Anderson & Sundaresan (1996), and Leland & Toft (1996), among others.

3.2.4. Recovery Rates. In Merton's model, recovery rates are determined by the liability's absolute priorities. However, it has been well documented that in bankruptcy proceedings, these absolute priority rules are commonly disregarded (see Eberhart et al. 1990 and Weiss 1990). Extensions that overcome this limitation have also been studied (see Longstaff & Schwartz 1995 and Anderson & Sundaresan 1996).

Of course, one can combine any or all of these extensions into a single but more complex model.

4. REDUCED FORM MODELS

Reduced form models were introduced by Jarrow & Turnbull (1992, 1995) to overcome the nontradability and nonobservability of the firm's asset value process, critical assumptions underlying the structural approach. We illustrate the reduced form class of models using the Lando (1998) model. As in our presentation of the structural model, for simplicity, we consider only zero-coupon bonds. More complex fixed income securities and credit derivatives are discussed in a subsequent section.

4.1. The Cox Process Model

We consider a continuous-time, continuous-trading model with time horizon $[0, \tau^*]$. Given are a filtered probability space $[\Omega, F, (F_t)_{t\in[0,\tau^*]}, P]$ and a vector state variable process $\mathbf{X}_t$ that is F_t measurable. $\mathbf{X}_t$ represents the state variables in the economy relevant to credit risk, for example, interest rates, housing prices, gross national product, inflation, etc. Let F_t^X represent the filtration generated by the vector valued process $\mathbf{X}_t$.

4.1.1. The Default-Free Bond Market. We assume that traded in the economy are default-free zero-coupon bonds of all maturities and a money market account. The money market account earns interest continuously at the default-free spot rate of interest r_t, which is adapted to $(F_t)_{t\in[0,\tau^*]}$. We initialize the money market account with a dollar at time 0 and denote its time t value by[8]

$$B_t = e^{\int_0^t r_s ds}. \tag{18}$$

We let the time t value of a default-free zero-coupon bond paying a dollar at time T be denoted $p(t,T)$.

4.1.2. The Risky Debt Market. Next, let us consider a given firm whose balance sheet contains a collection of liabilities and equity. Some of the firm's liabilities may or may not trade. Similarly, the firm's equity may or may not trade. For the subsequent theory, we only need one of its liabilities, a zero-coupon bond with maturity $T \leq \tau^*$ and a face value of K dollars, to trade continuously in a frictionless and competitive market. We let D_t denote the time t value of the firm's zero-coupon bond.

To introduce credit risk, we assume that with positive probability the firm may default on one of its liabilities prior to the zero-coupon bond's maturity date. If the firm defaults, then with positive probability it may not be able to pay back the entire face value of the zero-coupon bond. Formally, we let τ denote the random default time for the firm—a

[8]Of course, we assume the necessary measurability and integrability such that the following expression is well defined.

stopping time generated by the point process $N_t = 1_{\tau \geq t}$, which is adapted to the filtration $(F_t)_{t\in[0,\tau^*]}$. We assume that the point process N_t is a Cox process with intensity $\lambda_t = \lambda_t(\mathbf{X}_t) \geq 0$.[9] Intuitively, a Cox process is a point process that, conditional upon the information set generated by the state variables process $\mathbf{X}_t$ over the entire trading horizon $[0, \tau^*]$, $F^X_{\tau^*}$, behaves like a standard Poisson process.

The default intensity $\lambda_t(\mathbf{X}_t)$ can be interpreted as the probability of default over a small time interval $[t, t + \Delta]$. An important characterization of the default time is that it is totally inaccessible;[10] that is, default comes as a surprise to the market. This is in contrast to the default time in a structural model based on a diffusion process. In the structural model, the default time is predictable;[11] in other words, default is anticipated and does not come as a surprise.

If the firm defaults prior to the zero-coupon bond's maturity date, we assume that the bond receives a recovery payment, less than or equal to the face value promised:

$$D_\tau = R_\tau \text{ if } \tau \leq T, \tag{19}$$

where $0 \leq R_\tau \leq K$ is F_τ measurable.

4.1.3. The Firm's Default Probability and Recovery Rate. It follows directly from the Cox process assumption that the probability that the debt defaults before its maturity date is given by

$$P\{\tau \leq T\} = 1 - E\left(e^{-\int_0^T \lambda_s ds}\right). \tag{20}$$

This follows from expression 44 in the appendix (Section 8), modified for the statistical probability. For example, if λ_t follows a square-root process as given by

$$d\lambda_t = \kappa(\mu - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t, \tag{21}$$

where κ, μ, $\sigma \geq 0$ are constants with $\kappa\mu > \sigma^2/2$ and W_t is a standard Brownian motion adapted to $(F_t)_{t\in[0,\tau^*]}$, then[12]

$$P\{\tau \leq T\} = 1 - e^{\alpha(T)+\beta(T)\lambda_0}, \text{ where}$$

$$\alpha(T) = \frac{2\kappa\mu}{\sigma^2}\ln\left(\frac{2\gamma e^{(\gamma+\kappa)T/2}}{2\gamma + (\gamma+\kappa)(e^{\gamma T} - 1)}\right),$$

$$\gamma = \sqrt{\kappa^2 + 2\sigma^2}, \text{ and}$$

$$\beta(T) = \frac{-2(e^{\gamma T} - 1) + e^{\gamma T}(\gamma - \kappa)}{2\gamma + (\gamma+\kappa)(e^{\gamma T} - 1)}.$$

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[9]The technical details regarding measurability and integrability of the intensity process are omitted from the text but contained in Lando (1998).

[10]Intuitively, a totally inaccessible stopping time is one that cannot be written as a predictable stopping time. For an exact definition of a totally inaccessible stopping time see Protter (2004).

[11]A stopping time τ is predictable if there is an increasing sequence of stopping times that approach τ from below; see Protter (2004).

[12]The proof is contained in Lando (2004, p. 293).

In the literature, three constant percentage recovery rate processes have been frequently used.[13]

4.1.3.1. *Recovery of face value.*

$$R_\tau \equiv \delta K, \text{where } \delta \in [0,1].$$

This corresponds to the percentage recovery rate used in Merton's (1974) structural model.

4.1.3.2. *Recovery of Treasury.*

$$R_\tau \equiv \delta K p(\tau, T), \text{where } \delta \in [0,1].$$

This recovery rate is due to Jarrow & Turnbull (1995). It states that on the default date the debt is worth some constant percentage of an otherwise equivalent, but default-free, zero-coupon bond.

4.1.3.3. *Recovery of market value.*

$$R_\tau \equiv \delta D_{\tau-}, \text{where } \delta \in [0,1]$$

and $D_{\tau-} \equiv \lim_{t \to \tau, t \leq \tau} D_t$ is the value of the debt issue an instant before default.

This recovery rate is due to Madan & Unal (1998) and Duffie & Singleton (1999). It states that on the default date the debt is worth some constant fraction of its value an instant before default, at time $\tau-$.

4.1.4. Valuing the Firm's Debt. We assume that there exists an equivalent probability measure Q such that

$$\frac{p(t,T)}{B_t} \text{ and } \frac{D_t}{B_t} \text{ are } Q\text{-martingales.} \tag{22}$$

By the first fundamental theorem of asset pricing (see Harrison & Pliska 1981) this implies that these markets are arbitrage free. Although it is not necessary for the subsequent analysis, we assume that the default-free bond markets are complete. However, we do not necessarily assume that the market for the firm's debt is complete. Using the second fundamental theorem of asset pricing (see Harrison & Pliska 1981, 1983), this implies that the martingale measure Q given in expression 22 need not be unique. Nonetheless, all of these martingale measures yield the same value for the firm's risky debt price. The martingale condition implies that

$$D_t = \tilde{E}\left(R_\tau e^{-\int_t^\tau r_s ds} 1_{\{\tau \leq T\}} + K e^{-\int_t^T r_s ds} 1_{\{T < \tau\}} | F_t\right), \tag{23}$$

where $\tilde{E}(\cdot)$ corresponds to expectation under the martingale probability Q.

Expression 23 shows that the debt's value is equal to its expected discounted payoff across the two possibilities regarding default: (*a*) Default occurs prior to maturity, in

[13]There is an abuse of notation in using the same symbol for three different percentage recovery rates. Nonetheless, this should cause no confusion in the subsequent text.

which case the debt receives its recovery value R_τ, or (*b*) the debt does not default before maturity, in which case it receives its face value K.

Under the Cox process assumption, using expressions 46 and 48 from the appendix, this expression simplifies to

$$D_t = \int_t^T \tilde{E}\left(R_s \tilde{\lambda}_s e^{-\int_t^s (r_u + \tilde{\lambda}_u) du} | F_t\right) ds + K\tilde{E}\left(e^{-\int_t^T (r_u + \tilde{\lambda}_u) du} | F_t\right), \tag{24}$$

where $\tilde{\lambda}_t \equiv \lambda_t \mu_t$ is the intensity process under the martingale probability Q and $\mu_t(\omega) \geq 0$ is a suitably integrable and nonnegative $(F_t)_{t\in[0,\tau^*]}$ adapted process.[14] First, we note that the intensity process λ_t under the statistical probability P is replaced by the intensity process $\tilde{\lambda}_t$ under the martingale probability Q. The difference between these two intensity processes represents a proportional jump risk premium, $\mu_t \geq 0$, for the point process N_t. This is distinct from the risk premia in the economy due to the evolution of $\mathbf{X}_t$, for instance for the interest rate risk embedded in r_t. These other risk premia are important and explicitly included via the martingale measure in expression 24.

As easily recognized, if $\mu_t \equiv 1$, then the two intensity processes are equivalent and there is no jump risk premium in the economy. This would be the case if default risk were diversifiable and unsystematic (see Jarrow et al. 2005 for sufficient conditions and related discussion). Whether or not jump risk is diversifiable is currently an open empirical question (see Driessen 2005). If jump risk is diversifiable, however, then the estimation of the intensity process simplifies considerably because historical default intensity estimations can be used in expression 24 without an adjustment for a jump risk premium. Default intensity process estimation using martingale methods is currently an active research area as well (see Chava & Jarrow 2004, Campbell et al. 2006, and Bharath & Shumway 2008).

Second, the omission of the default time τ from expression 24 implies that the computation of the debt's value is analogous to computing the value of an interest rate derivative under a default-free interest rate model [an HJM model (Heath et al. 1992)], where the default-free spot rate process r_t is replaced by a pseudo-default-free spot rate process $\left(r_u + \tilde{\lambda}_u\right)$. This computational simplification is one advantage of using reduced form models as contrasted with structural models.

For the different constant percentage recovery rates, expression 24 simplifies further to

$$D_t = \begin{cases} K\tilde{E}\left((1+\delta\tilde{\lambda}_s)e^{-\int_t^T (r_u+\tilde{\lambda}_u)du} | F_t\right), \text{if recovery of face value,} \\ K\left[\delta p(t,T) + (1-\delta)\tilde{E}\left(e^{-\int_t^T (r_s+\tilde{\lambda}_s)ds} | F_t\right)\right], \text{if recovery of Treasury,} \\ K\tilde{E}\left(e^{-\int_t^T \left(r_u + \tilde{\lambda}_u(1-\delta)\right)du} | F_t\right), \text{if recovery of market value.} \end{cases} \tag{25}$$

The proof for the recovery of face value case is just simple algebra and expression 24. The proof for the recovery of market value is given by expression 49 in the appendix. The proof for the recovery of Treasury is given in the appendix as well.

These representations of the debt value take particularly simple forms, each having a different implication for the risk structure of interest rates. The recovery of market value

[14]The details of the integrability conditions can be found in Bremaud (1981, p. 165).

has an especially nice representation, where the debt is priced as if it is default free, but with the discounting factor equal to the default-free spot rate of interest plus the expected loss rate under the martingale probability $\left(r_u + \tilde{\lambda}_u(1-\delta)\right)$. It is an open research question as to which representation is best (see, e.g., Bakshi et al. 2001).

4.1.5. Credit Risk Spreads. Recall that the credit risk spread, $s(t,T)$, is given by expression 12:

$$s(t,T) = \frac{\ln\left(\frac{K}{D_t}\right)}{(T-t)} - r.$$

Substitution of expression 25 gives the credit spreads for the different maturity zero-coupon bonds. It can be shown that for each of these recovery rate processes, as long as there is a positive probability of default over $(t,t+dt)$, the credit spread remains nonzero as the bond matures:

$$\lim_{(T-t)\to 0} s(t,T) > 0 \text{ if } \tilde{\lambda}_t > 0.$$

This characteristic of a reduced form model is in contrast to that exhibited by Merton's (1974) simple structural model (see expression 13 above) and is more consistent with the behavior of market credit risk spreads (see Duffie & Singleton 2003).

4.1.6. Default Contagion. This section considers default contagion within the context of the reduced form model. By assumption, we consider a collection of n firms each with arbitrary liability structures. Some of each firm's liabilities may or may not trade. The same is true for the firms' equities. Here, we are concerned with the firms' default process. We let τ_i denote the default time for firm i generated by the Cox process $N_i(t) = 1_{\tau_i \geq t}$ with intensity $\lambda_i(t) = \lambda_i(t,\mathbf{X}_t) \geq 0$.

We assume that, conditioned upon the state variables process $\mathbf{X}_t$ over the entire trading horizon $[0,\tau^*]$, $F^X_{\tau^*}$, the default events—the jumps in the point processes $N_i(t)$—are independent across firms. This is termed the conditional independence assumption. The conditional independence assumption implies that default correlations across firms result from the firms' default intensities depending upon the common state variables $\mathbf{X}_t$. We note that this conditional independence assumption, plus some additional conditions relating to the firms' liabilities, is sufficient to imply that the default jump risk is diversifiable in a large portfolio (see Jarrow et al. 2005).

Consider the time horizon $[0,T]$. Default contagion is characterized by determining the firm's joint default probability before time T. Under the above structure, this joint default distribution function is given by

$$P\{(\tau_1 \leq T),\ldots,(\tau_n \leq T)\} = E\left(\left(1-e^{-\int_0^T \lambda_1(u)du}\right)\cdots\left(1-e^{-\int_0^T \lambda_n(u)du}\right)\right). \tag{26}$$

This follows from expression 44 in the appendix and from the conditional independence assumption.

Given specific evolutions for the intensity processes, this expression can be computed either analytically or numerically. For example, one could use the square-root process in expression 21 with different parameters for each firm but a single Brownian motion representing the common state variable. This joint default distribution function also enables one to compute the probability that any subcollection of firms will default by time T.

4.1.7. Risk Management. Next, we consider a portfolio consisting of these n firms' liabilities with portfolio weights $w_i \geq 0$, where $\sum_{i=1}^{n} w_i = 1$. The dollar loss at time T given default for this portfolio over $[0,T]$ is defined by

$$L \equiv \sum_{i=1}^{n} w_i 1_{\{\tau_i \leq T\}} L_i \geq 0,$$

where the loss on firm i's liability is given by

$$L_i \equiv [1 - R_i(\tau_i)] e^{\int_{\tau_i}^{T} r_s ds}, \text{ if } \tau_i \leq T,$$

with $R_i\,(\tau_i)$ the recovery payoff on firm i's liability at the default date τ_i. As indicated, the loss on firm i's liability is defined to be one minus the recovery payoff, transformed to time T by investing in the money market account.

Risk management statistics are computed using the distribution function for these losses given by

$$P\{L \leq u\} \text{ for all } u \in [0, \infty) \tag{27}$$

under the statistical probability. Using standard probability methods, this distribution can be computed given two inputs: (a) the joint default distribution function in expression 26 and (b) the joint distribution function for the percentage losses given default:

$$P\{L_1 \leq u_1, \ldots, L_n \leq u_n | (\tau_1 \leq T), \ldots, (\tau_n \leq T)\}. \tag{28}$$

This conditional loss distribution can be determined given processes for the default-free spot rate r_t and each firm's recovery rate (see, e.g., Duffie & Singleton 2003, chapter 13).

4.2. Extensions

The reduced form model has been extended in many ways. To consider credit rating migration, Jarrow et al. (1997) introduced a Markov chain model where the states correspond to credit ratings, the worst state being default; see also Kijima & Komoribayashi (1998) and Lando & Skodebert (2002).

The issue of default correlation for pricing credit derivatives on baskets (e.g., CDO) was first handled with Cox processes (Lando 1998). As noted above, the use of Cox processes induces default correlations across firms through common state variables that drive the default intensities, but when conditioning on the state variables defaults are assumed to be independent across firms. However, this is not the only mechanism through which default correlations can be generated.

Default correlations are also possible through competitive industry considerations. For example, if a car maker defaults, it is more likely that a supplier of car parts may default due to lost business. This type of default contagion is a form of counterparty risk and it was first studied in the context of a reduced form model by Jarrow & Yu (2005). Recent evidence supports the importance of industry dependent counterparty risk (see Jorion & Zhang 2007). Counterparty risk in a reduced form model, an issue in and of itself, was previously studied by Jarrow & Turnbull (1995, 1997) and Duffie & Huang (1996). The reduced form model's abstract structure under very general settings is now well understood, and the extensions just discussed plus others can be found in textbook form (see Bielecki & Rutkowski 2002, Duffie & Singleton 2003, and Lando 2004).

An important new topic is the modeling of recovery rates. The existing constant percentage recovery rate processes (recovery of face value, Treasury, and market value) are useful for valuing predefault debt. However, they are of little use for pricing defaulted debt. The pricing of defaulted debt requires a stochastic model for the recovery rate process. Recent extensions studying stochastic recovery rates include Bakshi et al. (2001), Guo et al. (2007), and Guo et al. (2008). The pricing of defaulted debt is relevant because it trades in distressed debt markets (see Altman 1998). The understanding of distressed debt pricing is an open research area.

5. INCOMPLETE INFORMATION[15]

An important contribution to the credit risk literature was clarifying the relation between structural and reduced form models using the information sets employed in their construction. To set up the argument, let us denote

$$\chi_T \equiv \begin{cases} 1 \text{ if the firm defaults before } T \\ \quad 0 \quad \text{otherwise} \end{cases}.$$

Structural models use the management's information set, denoted $(G_t)_{t\in[0,\tau^*]}$, whereas reduced form models use the market's, denoted $(F_t)_{t\in[0,\tau^*]}$. The manager has access to the firm's asset values, but the market does not. For valuing credit risk, the important quantity to determine is the (normalized) conditional martingale default probability over some small horizon $[0,h]$ as $h \to 0$ using the proper information set:

$$\lim_{h\to 0}\frac{\tilde{E}(\chi_h|G_t)}{h} = 0 \quad \text{for the structural model, and}$$
$$\lim_{h\to 0}\frac{\tilde{E}(\chi_h|F_t)}{h} = \tilde{\lambda}(t) > 0 \quad \text{for the reduced form model.} \tag{29}$$

In the structural model this probability is zero when the default time is predictable. In the reduced form model it is strictly positive and equals the default intensity because the default time is inaccessible (see Lando 2004, p. 128).

Therefore, to understand the relation between structural and reduced form models, one can fix a default indicator, say letting

$$\chi_T \equiv 1_{\{V(T)\leq K\}}$$

from a structural model, and study how the martingale default probability changes as the information set changes.

The first paper studying this relation was by Duffie & Lando (2001), who viewed the market as having the management's information set plus accounting induced noise. Formally, we first consider the expanded information set consisting of the manager's information plus the noise process, denoted $(H_t)_{t\in[0,\tau]}$. The manager's information set consists of just the asset value process, and it satisfies $(G_t)_{t\in[0,T]} \subset (H_t)_{t\in[0,\tau]}$. This containment may be strict. The

[15]This first part of this section is based on Jarrow & Protter (2004).

market's information set is a subset of this expanded information set: $(F_t)_{t\in[0,T]} \subset (H_t)_{t\in[0,\tau]}$. Note that the market's information set need not be a subset of the manager's.

Duffie & Lando show that these different information sets can generate the relations as in expression 29. The idea is very intuitive. With perfect information, the manager can anticipate default. However, the market cannot perfectly anticipate default due to the noise in observing the asset value process. In fact, the noise in the asset value observations makes default a surprise.

Rather than adding noise to the manager's information set, one can alternatively view the market as having just less information than management. This approach was first studied by Cetin et al. (2004). Here, the market's information set is a reduction of the manager's: $(F_t)_{t\in[0,T]} \subset (G_t)_{t\in[0,\tau]}$. Again, Cetin et al. could show that these different information sets can generate the relations as in expression 29. The intuition is the same as in Duffie & Lando. Related papers include Kusuoka (1999), Blanchet-Scalliet & Jeanblanc (2004), and Jarrow et al. (2007).

This line of research has an important implication for the debate between using structural versus reduced form models in the pricing and hedging of credit-risky securities. As noted earlier, because the firm's asset value is not observable by the market, based on this line of research structural models should not be used to value or hedge fixed income securities. The reason is simple: The stochastic processes' parameters underlying the structural model cannot be estimated based on the market's information. Also, the market prices and hedges financial instruments based on its information. This does not imply, however, that structural models are not useful in practice. For internal risk management of a firm's liabilities, by the firm's managers, the structural model is the appropriate choice. Also, it is appropriate for the internal management of a sovereign nation's credit risk (see Gray et al. 2007).

Consistent with this reasoning, incomplete information can also affect correlations in credit risk spreads. A default by one firm may cause another firm's default intensity to increase as the market learns the reasons for the realized default. For example, in the Enron default, accounting fraud was recognized as playing a key component. Prior to Enron's default, accounting fraud was viewed as highly unlikely. This heightened awareness of accounting fraud may have increased the likelihood of default for other similar firms. This learning affects credit spreads as the market's estimate of the likelihood of default changes. This information learning effect corresponds to the concept of frailty in hazard rate estimation, see Schonbucher (2004), Duffie et al. (2006), and Chava et al. (2006).

6. CREDIT DERIVATIVES

This section studies the pricing of credit derivatives. Using a broad definition, a credit derivative is any financial security contract whose cash flows depend on the default process of one or more credit entities. A credit entity could be a person, a corporation, a government, or a trust. Under this definition, credit derivatives include equity (common, preferred), all corporate liabilities (e.g., debt, bank loans, credit lines, leases, trade credit), personal loans, government debt, counterparty risk in security transactions, all asset-backed securities (e.g., residential mortgages, commercial mortgages, student loans, car loans, bank loans, credit card debt), CDS, basket credit derivatives (e.g., first-to-default swap, *n*th-to-default swaps), CDO, default guarantees (e.g., FDIC deposit insurance), etc. As evidenced, this list is enormous, documenting why credit risk modeling is so important.

We study the pricing and hedging of these credit derivatives using only reduced form models because, as argued in the previous section, these are better suited for this application. In the context of reduced form models, the primary traded securities will be default-free zero coupon bonds and the risky zero-coupon bonds for the different credit entities. We assume that these primary securities trade continuously in a frictionless and competitive market.

The secondary or derivative securities are all other financial securities (traded or not) whose cash flows depend on the evolution of the term structure of the credit-risky zero-coupon bonds. We impose sufficient conditions such that the market is arbitrage free and such that a unique martingale measure can be identified to price the credit derivatives.

6.1. Single Entity Credit Derivatives

We first study credit derivatives whose cash flows depend only on one credit entity's default process. We briefly recall the notation. Consider a firm with default time τ where $N_t = 1_{\tau \geq t}$ is a Cox process with intensity $\lambda_t = \lambda_t(\mathbf{X}_t) \geq 0$ depending upon the state variables process $\mathbf{X}_t$.

We let $D(t,T)$ denote the time t value of the firm's zero-coupon bond with maturity $T \leq \tau^*$ and a face value of one dollar. We assume that there exists an equivalent probability Q such that $\frac{p(t,T)}{B_t}$ and $\frac{D(t,T)}{B_t}$ are martingales. We do not assume that the market is complete.

If the market is complete, then the martingale probability is unique, and hedging follows the standard procedures used in option pricing (see Musiela & Rutkowski 2005). If the market is incomplete, then to price credit derivatives we select a unique martingale probability Q from the set of possible martingale probabilities by assuming either that the market is in equilibrium or that enough derivatives trade such that their market prices uniquely determine Q (see Jacod & Protter 2006 and Schweizer & Wissel 2008 in this regard). In the incomplete case, complete hedging is not possible, but risk minimizing hedging can be employed (see Musiela & Rutkowski 2005, chapter 4).

Consider a random cash flow received at time T, denoted Y_T, that is F_T adapted with $\tilde{E}(|Y_T|) < \infty$. Then, the time t value of this cash flow is given by

$$Y_t = \tilde{E}\left(Y_T e^{-\int_t^T r_u du} | F_t \right) \tag{30}$$

using the selected probability Q, where $\tilde{\lambda}_t \equiv \lambda_t \mu_t$ is the intensity process under Q and $\mu_t\,(\omega) \geq 0$ is the default jump risk premium. To illustrate the pricing methodology, we consider three credit-risky instruments: a zero-coupon bond, a coupon bond, and a CDS.

6.1.1. Zero-Coupon Bonds. The zero-coupon bonds are the primary securities. Let us assume that the recovery rate on a T-maturity zero-coupon bond is given by

$$D(\tau,T) = R(\tau,T) \in [0,1] \text{ if } \tau \leq T, \tag{31}$$

where $R(\tau,T)$ is F_τ measurable. Repeating expression 24 for convenience, we note that

$$D(t,T) = \tilde{E}\left(e^{-\int_t^T (r_u + \tilde{\lambda}_u) du} | F_t \right) + \int_t^T \tilde{E}\left(R(s,T) \tilde{\lambda}_s e^{-\int_t^s (r_u + \tilde{\lambda}_u) du} | F_t \right) ds. \tag{32}$$

6.1.2. Coupon Bonds. This section studies a coupon bond on the firm. A coupon bond has a coupon rate $C \in [0,1]$, a face value that we assume to be one dollar, and a maturity date T. The bond pays the coupon rate C times the notional at intermediate dates $\{t_1,\ldots,t_n \equiv T\}$ but only up to the default time τ. If default happens prior to the maturity date, we assume the bond pays the recovery rate $R_\tau \in [0,1]$, which is F_τ measurable. If default does not happen, the face value is repaid at time T.

Denote the time $t \leq t_1$ value of the coupon bond as v_t. Then,

$$\begin{aligned} v_t &= \tilde{E}\left(\sum_{k=1}^{m} C1_{\{\tau>t_k\}}e^{-\int_t^{t_k} r_u du} + 1_{\{\tau>T\}}e^{-\int_t^{T} r_u du}|F_t\right) \\ &\quad +\tilde{E}\left(R_\tau 1_{\{\tau\leq T\}}e^{-\int_t^{\tau} r_u du}|F_t\right) \\ &= \sum_{k=1}^{m} C\tilde{E}\left(e^{-\int_t^{t_k}(r_u+\tilde{\lambda}_u)du}|F_t\right) + \tilde{E}\left(1e^{-\int_t^{T}(r_u+\tilde{\lambda}_u)du}|F_t\right) \\ &\quad + \int_{\mathrm{t}}^{\mathrm{T}} \tilde{E}\left(R_s\tilde{\lambda}_s e^{-\int_t^{s}(r_u+\tilde{\lambda}_u)du}|F_t\right)ds. \end{aligned} \tag{33}$$

If the bond is priced at par, then the C satisfies $v_t = 1$.

In general, by comparing expression 32 with expression 33, we see that

$$v_t \neq \sum_{k=1}^{m} CD(t,t_k) + D(t,T). \tag{34}$$

We note, however, that if either

1. $R(\tau,T) = R_\tau = \delta$, a constant (recovery of face value); or
2. $R(\tau,T) = \delta p(\tau,T)$, $R_\tau = \delta\left(\sum_{k=1}^{m} Cp(\tau,t_k) + p(\tau,T)\right)$, where δ is a constant (recovery of Treasury); or
3. $R(\tau,T) = \delta R(\tau-,T)$, $R_\tau = \delta R_{\tau-}$, where δ is a constant (recovery of market value); then

$$v_t = \sum_{k=1}^{m} CD(t,t_k) + D(t,T). \tag{35}$$

The proof of this statement follows directly from expressions 32 and 33 and algebraic manipulation. Necessary and sufficient conditions for the satisfaction of this equality can be found in Jarrow (2004).

Expression 35 is an important result. Similar to default-free zero coupon bonds, it facilitates the estimation of the credit risk yield curve using coupon bonds and smoothing procedures (see Jarrow 2002, chapter 16).

6.1.3. Credit Default Swaps. This section studies CDS on corporate debt. A CDS has a maturity date $T \leq \tau^*$ and a notional value, which we assume to be one dollar. The protection seller agrees to pay the protection buyer the difference between the face value of the debt and the recovery value at default, if the firm defaults before the maturity date. In return, the protection buyer pays a constant dollar spread times the notional at fixed intermediate dates until the swap's maturity or the default date, whichever comes first.

Consider a coupon bearing bond, as in the previous section, with recovery rate $R_\tau \leq 0$. The protection seller pays, at default,

$$L_\tau 1_{\{\tau\leq T\}} = \begin{cases} 1 - R_\tau & \text{if} \quad \tau \leq T \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

The protection buyer pays a constant dollar spread c times the notional at the intermediate dates $\{t_1,\ldots,t_n \equiv T\}$, but only up to the default time τ. The time $t \leq t_1$ value of the CDS to the protection seller is therefore

$$\tilde{E}\left(\sum_{k=1}^{m} c1_{\{\tau>t_k\}}e^{-\int_t^{t_k} r_u du} - L(\tau)1_{\{\tau\leq T\}}e^{-\int_t^{\tau} r_u du}|F_t\right)$$
$$= c\sum_{k=1}^{m}\tilde{E}\left(e^{-\int_t^{t_k}(r_u+\tilde{\lambda}_u)du}|F_t\right) - \int_t^T \tilde{E}\left(L_s\tilde{\lambda}_s e^{-\int_t^s (r_u+\tilde{\lambda}_u)du}|F_t\right)ds. \tag{37}$$

The proof uses expressions 46 and 48 in the appendix.

The market clearing CDS rate c makes this value zero:

$$c_{mkt} = \frac{\int_t^T \tilde{E}\left(L_s\tilde{\lambda}_s e^{-\int_t^s (r_u+\tilde{\lambda}_u)du}|F_t\right)ds}{\tilde{E}\left(e^{-\int_t^{t_k}(r_u+\tilde{\lambda}_u)du}|F_t\right)}. \tag{38}$$

Note that if the loss rate is a constant and both $\tilde{\lambda}_s, r_s$ follow an affine process, then an analytic solution for the CDS rate can be obtained (see Lando 2004). Otherwise, numerical methods need to be employed to evaluate this expression.

Note that the protection seller's position is similar to that of being long a coupon bond, except that (*a*) the value of the bond is not exchanged at time t (the start date) and (*b*) the notional (face) value's cash flow is not included if the bond does not default. This difference explains why a swap has zero value, but a coupon bond's value is strictly positive (e.g., if priced at par, its value is unity). Except for this difference, the CDS are synthetic coupon bonds.

Thus, CDS provide a convenient mechanism for shorting a coupon bond. Before the trading of CDS, shorting of coupon bonds was best accomplished by using repurchase agreements (to obtain possession of the shorted instrument). Given the illiquidity of the coupon bond market, the corporate debt repurchase agreement market was not an efficient mechanism for taking short positions. This observation partially explains the popularity and exponential growth of the CDS market since its inception.

6.2. Basket Credit Derivatives

This section studies the first-to-default swap and credit-risky securities.

6.2.1. First-to-Default Swaps. This section studies a simple basket credit derivative on n firms termed a first-to-default swap. This swap has a maturity date $T \leq \tau^*$ and a notional value, which we assume to be one dollar. The protection seller agrees to pay the protection buyer a cash flow if any of the firms default before the maturity date—but only one payment. This payment can depend on the firm defaulting. In return, the protection buyer pays a constant dollar spread times the notional at fixed intermediate dates until the swap's maturity or the first default date, whichever comes first.

We consider a collection of n firms. We let τ_i denote the default time for firm i generated by the Cox process $N_i(t) = 1_{\tau\geq t}$ with intensity $\lambda_i(t) = \lambda_i(t,\mathbf{X}_t) \geq 0$. We assume that conditioned upon $F_{\tau^*}^X$ the point processes $N_i(t)$ are independent across firms.

Let $D_i(t,T)$ denote the time t value of the firm i's zero-coupon bond with maturity $T \leq \tau^*$ and a face value of one dollar. We assume that there exists an equivalent probability Q such that $\frac{p(t,T)}{B_t}$ and $\frac{D_i(t,T)}{B_t}$ are martingales for all i. We do not necessarily assume that the market is complete. If the market is complete, then there is a unique martingale probability. If the market is incomplete, then to uniquely price credit derivatives we select a unique martingale probability Q from the set of all possible martingale probabilities by assuming either that the market is in equilibrium or that enough derivatives trade such that their market prices uniquely determine Q.

Let $\hat{\tau} \equiv \min\{\tau_1, ..., \tau_n\}$ denote the time of the first firm's default, and let $\hat{i} \equiv \arg\ \min\{\tau_1, ..., \tau_n\}$ denote the first defaulting firm. The protection seller's payoff is

$$L_{\hat{i}}(\hat{\tau})1_{\{\hat{\tau}\leq T\}} = \begin{cases} L_{\hat{i}}(\hat{\tau}) & \text{if} \quad \hat{\tau} \leq T \\ 0 & \text{otherwise}, \end{cases} \tag{39}$$

where $L_i(\hat{\tau})$ represents the payment made on the swap if firm i is the first to default at time $\hat{\tau}$; $L_i(\hat{\tau})$ is assumed to be $F_{\hat{\tau}}$ measurable.

The protection buyer pays a constant dollar spread c times the notional at the intermediate dates $\{t_1,...,t_n \equiv T\}$, but only up to the $\min\{\hat{\tau},T\}$. The time $t \leq t_1$ value of the first-to-default swap to the protection seller is therefore

$$\begin{aligned} &\tilde{E}\left(\textstyle\sum_{k=1}^{m} c1_{\{\hat{\tau}>t_k\}}e^{-\int_t^{t_k} r_u du} - L_{\hat{i}}(\hat{\tau})1_{\{\hat{\tau}\leq T\}}e^{-\int_t^{t_k} r_u du}|F_t\right) \\ &\quad = c\sum_{k=1}^{m}\tilde{E}\left(e^{-\int_t^{t_k}\left(r_u+\sum_{j=1}^{n}\tilde{\lambda}_j(u)\right)du}|F_t\right) \\ &\quad -\sum_{i=1}^{n}\int_t^T \tilde{E}\left(L_i(s)\tilde{\lambda}_i(s)e^{-\int_t^s\left(r_u+\sum_{j=1}^{n}\tilde{\lambda}_j(u)\right)du}|F_t\right)ds, \end{aligned} \tag{40}$$

where $\tilde{\lambda}_i(t) \equiv \lambda_i(t)\mu_t$ is the intensity process under the martingale probability Q and $\mu_t\,(\omega) \geq 0$ is the jump risk premium. The proof of this expression is in the appendix.

The market clearing swap rate c makes this value zero:

$$c_{mkt} = \frac{\sum\limits_{i=1}^{n}\int\limits_t^T \tilde{E}\left(L_i(s)\tilde{\lambda}_i(s)e^{-\int_t^s\left(r_u+\sum_{j=1}^{n}\tilde{\lambda}_j(u)\right)du}|F_t\right)ds}{\sum\limits_{k=1}^{m}\tilde{E}\left(e^{-\int_t^{t_k}\left(r_u+\sum_j^{=1n}\tilde{\lambda}_j(u)\right)du}|F_t\right).} \tag{41}$$

Note that if the payment rate $L_i\,(s)$ is a constant and both $\tilde{\lambda}_s, r_s$ follow an affine process, then an analytic solution for this swap rate can be obtained. Otherwise, as with CDS, numerical methods need to be employed.

6.2.2. Credit-Risky Securities. The previous examples illustrate the use of the reduced form model to price credit-risky securities. Due to length considerations, this review has not discussed the vast array of other credit derivatives and credit-risky securities. We leave this study to the many textbooks on the topic, including Bielecki & Rutkowski (2002), Bluhm et al. (2003), Duffie & Singleton (2003), Lando (2004), and Bluhm & Overbeck (2007).

7. DIRECTIONS FOR FUTURE RESEARCH

This paper has reviewed the class of credit risk models used in financial economics—the structural and reduced form models. The abstract structure of these models are well understood. With respect to the usage of these two models, we argue that the reduced form, and not the structural model, is the preferred choice for the pricing and hedging of traded credit-risky securities. The structural model is most useful for internal (corporate and sovereign) risk management. The remaining open questions in credit risk modeling relate to the estimation and implementation of realistic default contagion models, recovery rate models, and the inclusion of liquidity risk.

With respect to realistic default contagion models, as evidenced by the current credit crisis, the models used in the industry prior to 2007 were just too simple. The industry predominately used structural models modified by copulas (see Section 3 above). These models are static in nature and they do not capture the dynamic structure of credit risk. Furthermore, estimation of the model's inputs (prior to 2007) was based on a sample history with an expanding and healthy economy, where few defaults occurred. This sample estimation bias caused the industry to underestimate default risk for both subprime borrowers and financial institutions. Because capital adequacy ratios were also based on these simple models, they too were severely underestimated. The industry's underestimation of credit risk was a significant cause of the financial crisis. To avoid a repeat performance, the sophistication of default contagion models and the estimation procedures used need to be improved. This improvement requires significant research to discover those dynamic models that capture default contagion, yet whose parameters can be estimated, and whose values can be computed.

Prior to 2007, the standard recovery rate models used by industry were those models (see Section 4 above) where recovery rates are constants, independent of the health of the economy. Again, as evidenced by the crisis, this assumption was too simple. When the health of the economy declines, recovery rates also decline. The simple models used by industry did not capture this correlation, thereby underestimating the risk of a loss. For the future, stochastic recovery rate models are needed to better capture the dynamic nature of losses in default. This literature is just in its infancy.

Finally, credit risk and liquidity risk often go hand-in-hand. When credit risk is large and many financial contracts are defaulting, markets become skittish. When markets become skittish, liquidity dries up, prices for credit-risky securities fall, and lending rates soar. The existing credit risk models do not capture this dependence. Again, missing this dependence resulted in the market underestimating credit risk. Accurately capturing this dependence is important for valuation, hedging, and capital determination. The simultaneous modeling of both credit and liquidity risk is an important and relatively unexplored area that deserves much more attention.

8. APPENDIX

8.1. Loss Distribution Functions

This section proves that expression 16 can be computed given two inputs: (*a*) the joint default distribution function in expression 15 and (*b*) the joint distribution function for the percentage losses conditional upon all of the firms' defaulting as in expression 17.

Proof. For simplicity, define

$$\chi_i \equiv \begin{cases} 1 \text{ if the firm defaults at } T \\ 0 \quad \text{otherwise} \end{cases}$$

and $Y_i \equiv w_i \chi_i L_i K_i \in [0, \infty)$. Then,

$$P(L \le u) = P\left(\sum_{i=1}^{n} Y_i \le u\right) = \int_0^\infty \dots \int_0^\infty 1_{\left\{\sum_{i=1}^{n} Y_i \le u\right\}} dF(Y_1, \dots Y_n)$$

where $F(u_1, \dots u_n) = P(Y_1 \le u_1, \dots, Y_n \le u_n)$ for $(u_1, \dots, u_n) \in \mathbb{R}_+^n$.

Define $Z_i \equiv w_i L_i K_i$. Then,

$$P(Y_1 \le u_1, \dots, Y_n \le u_n) = P(\chi_1 Z_1 \le u_1, \dots, \chi_n Z_n \le u_n)$$

$$= \sum_{\chi_1, \dots, \chi_n : \chi_i \in \{0,1\} \text{for all } i} P(\chi_1 Z_1 \le u_1, \dots, \chi_n Z_n \le u_n | \chi_1, \dots, \chi_n) P(\chi_1, \dots, \chi_n).$$

Lastly, rearrange indices as necessary so that the first m firms have $\chi_i = 1$ and the firms from m + 1 to n have $\chi_i = 0$. Then,

$$P(\chi_1 Z_1 \le u_1, \dots, \chi_n Z_n \le u_n | \chi_1 = 1, \dots, \chi_m = 1, \chi_{m+1} = 0, \dots, \chi_m = 0)$$

$$= P(Z_1 \le u_1, \dots, Z_m \le u_m, 0 \le u_{m+1}, \dots, 0 \le u_n | \chi_1 = 1, \dots, \chi_m = 1, \chi_{m+1} = 0, \dots, \chi_m = 0)$$

$$= P(Z_1 \le u_1, \dots, Z_m \le u_m | \chi_1 = 1, \dots, \chi_m = 1, \chi_{m+1} = 0, \dots, \chi_m = 0)$$

$$= P\left(L_1 \le \frac{u_1}{w_1 K_1}, \dots, L_m \le \frac{u_m}{w_m K_m} | \chi_1 = 1, \dots, \chi_m = 1, \chi_{m+1} = 0, \dots, \chi_m = 0\right).$$

This completes the proof of the statement.

8.2. Cox Processes

Given is a filtered probability space $[\Omega, F, (F_t)_{t \in [0, \tau^*]}, Q]$ and a state variables process X_t that is F_t measurable. Let F_t^X represent the filtration generated by X_t. Let a default occur with random time τ. Let N_t = $1_{\tau \ge t}$ represent a Cox process with intensity $\tilde{\lambda}_t = \tilde{\lambda}_t(X_t)$. For this appendix we let $\tilde{E}(\cdot | F_t) = \tilde{E}_t(\cdot)$.
Given $F_{\tau^*}^X$, N_t behaves like a Poisson process with intensity $\tilde{\lambda}_t$.

$$Q_t(\tau > T | F_{\tau^*}^X) = \tilde{E}_t(1_{\tau > T} | F_{\tau^*}^X) = e^{-\int_t^T \tilde{\lambda}_u du}, \tag{42}$$

$$Q_t(\tau > T) = \tilde{E}_t[\tilde{E}_t(1_{\tau > T} | F_T^X)] = \tilde{E}_t\left(e^{-\int_t^T \tilde{\lambda}_u du}\right), \tag{43}$$

$$Q_t(\tau \le T) = 1 - Q_t(\tau > T) = 1 - \tilde{E}_t\left(e^{-\int_t^T \tilde{\lambda}_u du}\right), \tag{44}$$

$$Q_t(\tau \in [s, s + dt] | F_{\tau^*}^X) = \frac{dQ_t(\tau \le T | F_{\tau^*}^X)}{ds} = \tilde{\lambda}_s e^{-\int_t^s \tilde{\lambda}_u du}. \tag{45}$$

8.3. Cash Flow 1

Random cash flow $Y_T \in F_T$ at time T, but only if no default:

$$\tilde{E}_t\left(Y_T 1_{\tau>T} e^{-\int_t^T r_u du}\right) = \tilde{E}_t\left(Y_T e^{-\int_t^T r_u du} e^{-\int_t^T \tilde{\lambda}_u du}\right). \tag{46}$$

Proof.

$$\begin{aligned}
\tilde{E}_t\left(Y_T 1_{\tau>T} e^{-\int_t^T r_u du}\right) &= \tilde{E}_t\left(\tilde{E}_t\left(Y_T 1_{\tau>T} e^{-\int_t^T r_u du} | F_T^X\right)\right) \\
&= \tilde{E}_t\left(Y_T e^{-\int_t^T r_u du} \tilde{E}_t\left(1_{\tau>T} | F_T^X\right)\right) \\
&= \tilde{E}_t\left(Y_T e^{-\int_t^T r_u du} e^{-\int_t^T \tilde{\lambda}_u du}\right).
\end{aligned}$$

8.4. Cash Flow 2

Random payment rate of $y_s ds$ at time s over $[0,T]$, but only if no default:

$$\tilde{E}_t\left(\int_t^T y_s 1_{\tau>s} e^{-\int_t^s r_u du} ds\right) = \tilde{E}_t\left(\int_t^T y_s e^{-\int_t^s r_u du} e^{-\int_t^s \tilde{\lambda}_u du} ds\right). \tag{47}$$

Proof.

$$\begin{aligned}
\tilde{E}_t\left(\int_t^T y_s 1_{\tau>s} e^{-\int_t^s r_u du} ds\right) &= \tilde{E}_t\left(\tilde{E}_t\left(\int_t^T y_s 1_{\tau>s} e^{-\int_t^s r_u du} ds | F_T^X\right)\right) \\
&= \tilde{E}_t\left(\int_t^T y_s e^{-\int_t^s r_u du} \tilde{E}_t\left(1_{\tau>s} | F_T^X\right) ds\right) \\
&= \tilde{E}_t\left(\int_t^T y_s e^{-\int_t^s r_u du} e^{-\int_t^s \tilde{\lambda}_u du} ds\right).
\end{aligned}$$

8.5. Cash Flow 3

Random cash flow $Y_\tau \in F_\tau$ at time τ, but only if default during $[0,T]$:

$$\tilde{E}_t\left(Y_\tau 1_{\tau \leq T} e^{-\int_t^\tau r_u du}\right) = \tilde{E}_t\left(\int_t^T Y_s e^{-\int_t^s r_u du} \tilde{\lambda}_s e^{-\int_t^s \tilde{\lambda}_u du} ds\right). \tag{48}$$

Proof.

$$\begin{aligned}
\tilde{E}_t\left(Y_\tau 1_{\tau \leq T} e^{-\int_t^\tau r_u du}\right) &= \tilde{E}_t\left(\int_t^T \tilde{E}_t\left(Y_s e^{-\int_t^s r_u du} 1_{\tau=s} | F_T^X\right) ds\right) \\
&= \tilde{E}_t\left(\int_t^T Y_s e_t^{-\int_t^s r_u du} \tilde{E}\left(1_{\tau=s} | F_T^X\right) ds\right) \\
&= \tilde{E}_t\left(\int_t^T Y_s e^{-\int_t^s r_u du} \tilde{\lambda}_s e^{-\int_t^s \tilde{\lambda}_u du} ds\right).
\end{aligned}$$

8.6. Cash Flow 4

This proof is based on Hughston & Turnbull (2001).

Random cash flow $Y_T \in F_T$ at time T, but only if no default. If default $\tau \leq T$, then the payment is $\delta_\tau V_{\tau\text{-}}$, where $\delta_\tau \in F_\tau$.

$$\begin{aligned} V_t &= \tilde{E}_t\left(Y_T 1_{\tau>T} e^{-\int_t^T r_u du} + \delta_\tau V_{\tau-} 1_{t<\tau\leq T} e^{-\int_t^\tau r_u du}\right) \\ &= \tilde{E}_t\left(Y_T e^{-\int_t^T r_u du} e^{-\int_t^T \tilde{\lambda}_u (1-\delta_u) du}\right). \end{aligned} \tag{49}$$

Proof. Define $K_t = e^{\int_0^T r_u du}$. Then, $\dfrac{K_T}{K_t} = e^{\int_t^T r_u du}$.

Define $\Lambda_t = e^{-\int_t^T \tilde{\lambda}_u du}$. Then, $\dfrac{\Lambda_T}{\Lambda_t} = e^{-\int_t^T \tilde{\lambda}_u du}$.

Recall $N_t = 1_{\tau\geq t}$, so $1 - N_t = 1_{\tau<t}$.

$$\frac{V_t 1_{\tau>t}}{K_t} = \tilde{E}_t\left(\frac{Y_T 1_{\tau>T}}{K_t} + \delta_\tau \frac{V_{\tau-}}{K_\tau} 1_{t<\tau\leq T}\right)$$

$$= \tilde{E}_t\left(\frac{Y_T 1_{\tau>T}}{K_t} + \int_t^T \delta_s \frac{V_{s-}}{K_s} 1_{\tau=s} ds\right).$$

But $V_{s-} = 1_{\tau>s-} V_{s-}$ on $\{\tau = s\}$. Thus,

$= \tilde{E}_t\left(\frac{Y_T 1_{\tau>T}}{K_t} + \int_t^T \delta_s \frac{V_{s-} 1_{\tau>s-}}{K_s} 1_{\tau=s} ds\right)$. Using expression 48, we can write this as

$$= \tilde{E}_t\left(\frac{Y_T}{K_t}\frac{\Lambda_T}{\Lambda_t} + \int_t^T \delta_s \frac{V_{s-} 1_{\tau>s-}}{K_s}\frac{\Lambda_s}{\Lambda_t} \tilde{\lambda}_s ds\right).$$

Define $\tilde{V}_t = \dfrac{V_{s-} 1_{\tau>s-} \Lambda_s}{K_s}$. Note that $V_T 1_{\tau>T} = Y_T$.

Also $\Lambda_s = \Lambda_{s-}$ and $B_s = B_{s-}$.

We rewrite the last equation as

$$\tilde{V}_t + \int_0^t \delta_s \tilde{V}_{s-} \tilde{\lambda}_s ds = \tilde{E}_t\left(\tilde{V}_T + \int_0^T \delta_s \tilde{V}_{s-} \tilde{\lambda}_s ds\right).$$

Define $M_t = \tilde{V}_t + \int_0^t \delta_s \tilde{V}_{s-} \tilde{\lambda}_s ds$. The last equation shows that M_t is a martingale; that is $M_t = \tilde{E}_t(M_T)$. Rewrite as

$\tilde{V}_t = M_t - \int_0^t \delta_s \tilde{V}_{s-} \tilde{\lambda}_s ds$. Or,

$d\tilde{V}_t = dM_t - \delta_t \tilde{V}_{t-} \tilde{\lambda}_t dt$.

Define $\dfrac{d\tilde{M}_t}{\tilde{M}_t} = \dfrac{dM_t}{V_{t-}}$. Note that $\tilde{M}_t$ is also a martingale.

We can rewrite this as

$$\frac{d\tilde{V}_t}{\tilde{V}_t} = \frac{d\tilde{M}_t}{\tilde{M}_t} - \delta_t V_{t-}\tilde{\lambda}_t dt.$$

Using Ito's formula, we get

$d \log \tilde{V}_t = d \log \tilde{M}_t - \delta_t \tilde{\lambda}_t dt$. The solution is

$\tilde{V}_t = \tilde{M}_t e^{-\int_0^t \delta_s \tilde{\lambda}_s ds}$ because $\tilde{V}_0 = \tilde{M}_0$.

Substituting back, we get

$\tilde{M}_t = V_t e^{-\int_0^t [r_s + (1-\delta_s)\tilde{\lambda}_s] ds}$. As a martingale,

$\tilde{M}_t = \tilde{E}_t(\tilde{M}_T)$, or

$V_t 1_{\tau>t} e^{-\int_0^t [r_s+(1-\delta_s)\tilde{\lambda}_s]ds} = \tilde{E}_t(V_T 1_{\tau>T} e^{-\int_0^T [r_s+(1-\delta_s)\tilde{\lambda}_s]ds})$. Or,

$V_t 1_{\tau>t} = \tilde{E}_t(Y_T e^{-\int_t^T [r_s+(1-\delta_s)\tilde{\lambda}_s]ds})$.

8.7. Proof of Expression 25 for Recovery of Treasury

$$\begin{aligned}
D_t &= K\tilde{E}\left(\delta p(\tau, T) e^{-\int_t^\tau r_s ds} 1_{\{\tau \le T\}} + e^{-\int_t^T r_s ds} 1_{\{T<\tau\}} | F_t\right) \\
&= K\tilde{E}\left(\delta \tilde{E}\left(1 e^{-\int_\tau^T r_s ds} | F_\tau\right) e^{-\int_t^\tau r_s ds} 1_{\{\tau \le T\}} + e^{-\int_t^T r_s ds} 1_{\{T<\tau\}} \middle| F_t\right) \\
&= K\tilde{E}\left(\delta e^{-\int_t^T r_s ds} 1_{\{\tau \le T\}} + e^{-\int_t^T r_s ds} 1_{\{T<\tau\}} | F_t\right) \\
&= K\tilde{E}\left(\delta e^{-\int_t^T r_s ds} + (1-\delta) e^{-\int_t^T r_s ds} 1_{\{T<\tau\}} | F_t\right) \\
&= K\left[\delta p(t, T) + (1-\delta)\tilde{E}\left(e^{-\int_t^T r_s ds} 1_{\{T<\tau\}} | F_t\right)\right].
\end{aligned}$$

Using expression 47 gives

$$= K\left[\delta p(t, T) + (1-\delta)\tilde{E}\left(e^{-\int_t^T (r_s + \tilde{\lambda}_s) ds} \middle| F_t\right)\right].$$

8.8. Proof of Expression 40

This proof is based on Lando (2004).

Consider firms $i = 1,\ldots,n$. We let τ_i denote the default time for firm i generated by the Cox process $N_i(t) = 1_{\tau_i \ge t}$ with intensity $\tilde{\lambda}_i(t) = \tilde{\lambda}_i(t, \mathbf{X}_t) \ge 0$ under the martingale measure Q. We assume that, conditioned upon $F_{\tau^*}^X$, the point processes $N_i(t)$ are independent across firms. For this appendix we let $\tilde{E}(\cdot | F_t) = \tilde{E}_t(\cdot)$.

Proof. Define $\hat{\tau} \equiv \min\{\tau_1,..., \tau_n\}$, $\hat{i} \equiv \arg\ \min\{\tau_1,..., \tau_n\}$.

The protection buyer pays

$\tilde{E}_t\left(\tilde{E}\left(1_{\{\hat{\tau}>T\}}e^{-\int_t^{\hat{\tau}} r_u du}|F_{\tau^*}^X\right)\right) = \tilde{E}_t\left(\tilde{E}\left(1_{\{\tau_1>T\}}\cdots 1_{\{\tau_n>T\}}e^{-\int_t^T r_u du}|F_{\tau^*}^X\right)\right)$, and by conditional independence

$$= \tilde{E}_t\left(\tilde{E}\left(1_{\{\tau_1>T\}}|F_{\tau^*}^X\right)\cdots\tilde{E}\left(1_{\{\tau_n>T\}}\middle|F_{\tau^*}^X\right)e^{-\int_t^T r_u du}\right)$$

using 43

$$= \tilde{E}_t\left(e^{-\int_t^T \tilde{\lambda}_1(u)du}\cdots e^{-\int_t^T \tilde{\lambda}_n(u)du}e^{-\int_t^T r_u du}\right) = \tilde{E}_t\left(e^{-\int_t^T r_u du}e^{-\sum_{i=1}^n \int_t^T \tilde{\lambda}_i(u)du}\right).$$

To compute the protection seller's payment, note that

$$Q_t\left(i = \hat{i}, \tau_i \in [s, s+dt)|F_{\tau^*}^X\right) = Q_t\left(\tau_i \in [s, s+dt\right), \tau_j > s \ for\ i \neq j|F_{\tau^*}^X\right),$$

and by conditional independence

$$= Q_t\left(\tau_i \in [s, s+dt)|F_{\tau^*}^X\right)Q_t(\tau_j > s \ for\ i \neq j|F_{\tau^*}^X\right)$$

$$= Q_t\left(\tau_i \in [s, s+dt)|F_{\tau^*}^X\right)\prod_{i\neq j} e^{-\int_t^s \tilde{\lambda}_j(u)du}$$

$$= \tilde{\lambda}_i(s)e^{-\int_t^s \tilde{\lambda}_i(u)du}\prod_{i\neq j} e^{-\int_t^s \tilde{\lambda}_j(u)du} = \tilde{\lambda}_i(s)e\prod_{j=1}^n e^{-\int_t^s \tilde{\lambda}_j(u)du}.$$

Finally, $\tilde{E}_t\left(L\hat{i}(\hat{\tau})1_{\{\hat{\tau}\leq T\}}e^{-\int_{t\hat{\tau}} r_u du}|F_{\tau^*}^X\right) =$

$$\sum_{i=1}^n \tilde{E}_t\left(L_i(\tau_i)1_{\{\tau_i\leq T\}}e^{-\int_t^{\tau_t} r_u du}|i = \hat{i}, F_{\tau^*}^X)Q_t(i = \hat{i}|F_{\tau^*}^X\right)$$

$$= \sum_{i=1}^n \int_t^T \tilde{E}_t\left(L_i(s)e^{-\int_t^s r_u du}|i = \hat{i}, \tau_i \in [s, s+dt), F_{\tau^*}^X\right)$$

$$\times Q_t\left(i = \hat{i}, \tau_i \in [s, s+dt)|F_{\tau^*}^X\right)ds.$$

By the measurability of $L_{\hat{i}}(\hat{\tau})$ and $e^{-\int_t^{\hat{\tau}} r_u du}$ we have

$$= \int_t^T L_i(s)e^{-\int_t^s r_u du}\sum_{i=1}^n Q\left(i = \hat{i}, \tau_i \in [s, s+dt)|F_{\tau^*}^X\right)ds$$

$$= \sum_{i=1}^{n} \int_{t}^{T} L_i(s)\tilde{\lambda}_i(s) e^{-\int_t^s r_u du} \prod_{j=1}^{n} e^{-\int_t^s \tilde{\lambda}_j(u) du} ds$$

$$= \sum_{i=1}^{n} \int_{t}^{T} L_i(s)\tilde{\lambda}_i(s) e^{-\int_t^s \left(r_u + \sum_{j=1}^{n} \tilde{\lambda}_j(u)\right) du} ds.$$

Taking expectations $\tilde{E}(\cdot|F_t) = \tilde{E}_t(\cdot)$ gives the result.

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