Problem: 2D Poisson equation

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1 Theory

Partial differential equations are a mainstay of physics, and as such, many methods have been developed for solving them numerically. The problem to be solved in this idiom is obtaining a numerical solution to Poisson's equation

$$\nabla^2 u(x,y) = v(x,y) \tag{1}$$

by the finite-difference Jacobi relaxation method, i.e. by making a grid with some grid cell area h^2 and iteratively solving

$$\begin{aligned} u_{i+1}(x,y) =& \frac{1}{4h^2}\Big(u_i(x+h,y) + u_i(x-h,y) \qquad (2) \\ &+ u_i(x,y-h) + u_i(x,y+h) - 4u_i(x,y)\Big) - \frac{h^2}{4}v(x,y) \qquad (3) \end{aligned}$$

until some desired convergence. In other words, each grid point is updated by considering it and its neighbours in the previous iteration as well as the source term.

2 Specifics

- The grid must be a unit square, i.e. the sides are of length 1. The grid spacing h is $1/M$, if the number of grid cells is M^2.
- You must use a handwritten version of the algorithm above. Other than that, anything goes.

- The source term is $v(x, y) = 6xy(1 - y) - 2x^3$
- The boundary conditions are $u(0, y) = 0, u(1, y) = y(1 - y), u(x, 0) = 0, u(x, 1) = 0$.
- The code is convergent when the maximum grid point absolute difference between two iterations is 10^{-6}. In other words, in pseudocode, when max(abs(current-previous))$< 1e - 6$.
- The exercise is considered complete when the convergent numerical solution has been checked against the analytical solution $u(x, y) = y(1 - y)x^3$ and found to be within acceptable bounds. This is done by considering the average error per grid point, which must be below 10^{-3}. Remember that the solution is unique only up to an additive constant, so you should normalize so that the highest absolute value of your solution is 1!

3 Notes

Be sure to be careful with the boundary conditions and take some care to set the array for $v(x, y)$ correctly. It is easy to make off-by-one errors that make convergence to the analytical solution impossible.