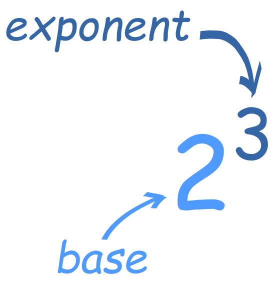
Working with Exponents and Logarithms

What is an Exponent?



The exponent of a number says **how many times** to use the number in a multiplication.

In this example: $\mathbf{2^3 = 2 \times 2 \times 2 = 8}$

(2 is used 3 times in a multiplication to get 8)

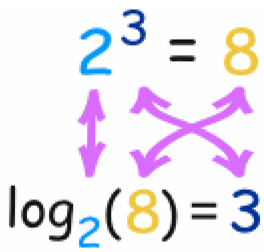
What is a Logarithm?

A Logarithm goes the other way.

It asks the question "what exponent produced this?":

$$2^{?} = 8$$

And answers it like this:

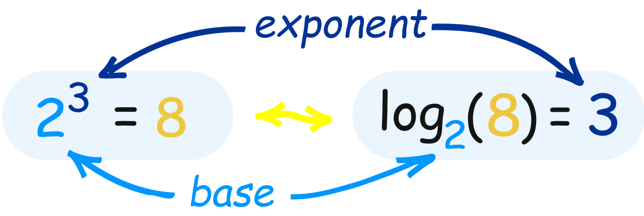


In that example:

- The Exponent takes **2 and 3** and gives **8** *(2, used 3 times in a multiplication, makes 8)*
- The Logarithm takes **2 and 8** and gives **3** *(2 makes 8 when used 3 times in a multiplication)*

A Logarithm says **how many** of one number to multiply to get another number

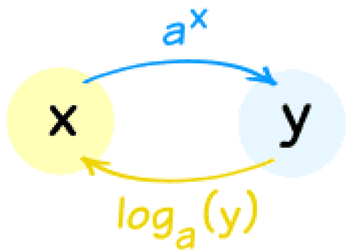
So a logarithm actually gives you the **exponent as its answer**:



(Also see how [Exponents, Roots and Logarithms] are related.)

Working Together

Exponents and Logarithms work well together because they "undo" each other (so long as the base "a" is the same):



They are "Inverse Functions"

Doing one, then the other, gets you back to where you started:

Doing $\mathbf{a^x}$ then $\mathbf{\log_a}$ gives you **x** back again: $\log_a(a^x) = x$

Doing $\mathbf{\log_a}$ then $\mathbf{a^x}$ gives you **x** back again: $a^{\log_a(x)} = x$

It is too bad they are written *so differently* ... it makes things look strange. So it may help to think of a^x as "up" and $\log_a(x)$ as "down":

going up, then down, returns you back again: down(up(x)) = x

going down, then up, returns you back again: up(down(x)) = x

Anyway, the important thing is that:

The Logarithmic Function is "undone" by the Exponential Function.

(and vice versa)

Like in this example:

Example, what is **x** in **$\log_3(x) = 5$**

Start with: $\log_3(x) = 5$

We want to "undo" the $\log_3$ so we can get "x ="

Use the Exponential Function (on both sides): $3^{\log_3(x)} = 3^5$

And we know that $3^{\log_3(x)} = x$, so: $x = 3^5$

Answer: $x = 243$

And also:

Example: Calculate y in **$y=\log_4(1/4)$**

Start with: $y = \log_4(1/4)$

Use the Exponential Function on both sides: $4^y = 4^{\log_4(\frac{1}{4})}$

Simplify: $4^y = 1/4$

Now a simple trick: $\mathbf{1/4 = 4^{-1}}$

So: $4^{\mathbf{y}} = 4^{\mathbf{-1}}$

And so: $\mathbf{y = -1}$

Properties of Logarithms

One of the powerful things about Logarithms is that they can **turn multiply into add**.

$$\log_a(m \times n) = \log_a m + \log_a n$$

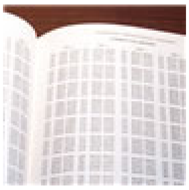
"the log of multiplication is the sum of the logs"

Why is that true? See Footnote.

Using that property and the Laws of Exponents we get these useful properties:

$\log_a(m \times n) = \log_a m + \log_a n$	*the log of multiplication is the sum of the logs*
$\log_a(m/n) = \log_a m - \log_a n$	*the log of division is the difference of the logs*
$\log_a(1/n) = -\log_a n$	this just follows on from the previous "division" rule, because $\log_a(1) = 0$
$\log_a(m^r) = r\ (\log_a m\ )$	*the log of m with an exponent r is r times the log of m*

Remember: the base "a" is always the same!



History: Logarithms were very useful before calculators were invented ... for example, instead of multiplying two large numbers, by using logarithms you could turn it into addition (much easier!)

And there were books full of Logarithm tables to help.

Let us have some fun using the properties:

Example: Simplify $\mathbf{\log_a(\ (x^2+1)^4\sqrt{x}\ )}$

Start with: $\log_a(\ (x^2+1)^4\sqrt{x}\ )$

Use $\mathbf{\log_a(mn) = \log_a m + \log_a n}$: $\log_a(\ (x^2+1)^4\ ) + \log_a(\ \sqrt{x}\ )$

Use $\mathbf{\log_a(m^r) = r\ (\ \log_a m\ )}$: $4\log_a(x^2+1) + \log_a(\ \sqrt{x}\ )$

Also $\mathbf{\sqrt{x} = x^{1/2}}$: $4\log_a(x^2+1) + \log_a(\ x^{1/2}\ )$

Use $\mathbf{\log_a(m^r) = r\ (\ \log_a m\ )}$ again: $4\log_a(x^2+1) + \tfrac{1}{2}\log_a(x)$

That is as far as we can simplify it ... we can't do anything with $\log_a(x^2+1)$.

Answer: $4\log_a(x^2+1) + \tfrac{1}{2}\log_a(x)$

Note: there is no rule for handling $\log_a(m+n)$ or $\log_a(m-n)$

We can also apply the logarithm rules "backwards" to combine logarithms:

Example: Turn this into one logarithm: $\mathbf{\log_a(5) + \log_a(x) - \log_a(2)}$

Start with: $\log_a(5) + \log_a(x) - \log_a(2)$

Use $\mathbf{\log_a(mn) = \log_a m + \log_a n}$: $\log_a(5x) - \log_a(2)$

Use $\mathbf{\log_a(m/n) = \log_a m - \log_a n}$: $\log_a(5x/2)$

Answer: $\log_a(5x/2)$

The Natural Logarithm and Natural Exponential Functions

When the base is **e** ("Euler's Number" = **2.718281828459**...) we get:

- The Natural Logarithm $\mathbf{\log_e(x)}$ which is more commonly written $\ln(x)$
- The Natural Exponential Function e^x

And the same idea that one can "undo" the other is still true:

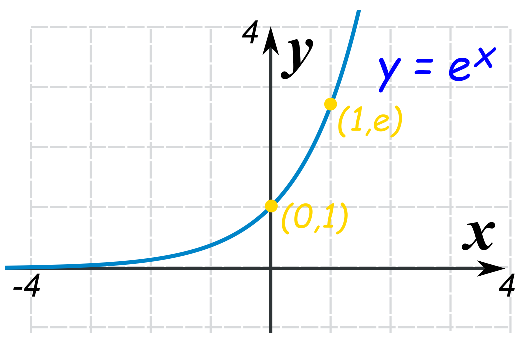
$$\ln(e^x) = x$$

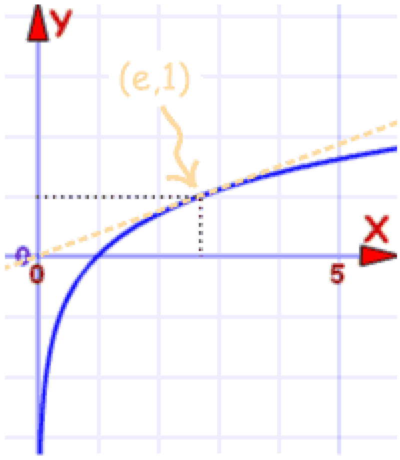
$$e^{(\ln x)} = x$$

And here are their graphs:

Natural Logarithm

Natural Exponential Function





Graph of $\mathbf{f(x) = ln(x)}$

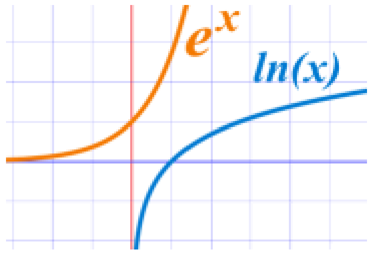
Passes through **(1,0)** and **(e,1)**

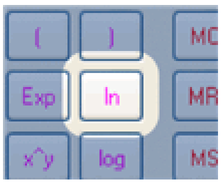
Graph of $\mathbf{f(x) = e^x}$

Passes through **(0,1)** and **(1,e)**

They are the **same curve** with x-axis and y-axis **flipped**.

Which is another thing to show you they are inverse functions.





On a calculator the Natural Logarithm is the "ln" button.

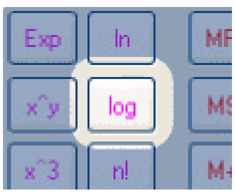
Always try to use Natural Logarithms and the Natural Exponential Function whenever possible.

The Common Logarithm

When the base is **10** you get:

- The Common Logarithm $\mathbf{log_{10}(x)}$, which is sometimes written as $\log(x)$

Engineers love to use it, but it is not used much in mathematics.



On a calculator the Common Logarithm is the "log" button.

It is handy because it tells you how "big" the number is in decimal (how many times you need to use 10 in a multiplication).

Example: Calculate $\log_{10} 100$

Well, 10 × 10 = 100, so when 10 is used **2** times in a multiplication you get 100:

$$\log_{10} 100 = 2$$

Likewise $\log_{10} 1{,}000 = 3$, $\log_{10} 10{,}000 = 4$, and so on.

Example: Calculate $\log_{10} 369$

OK, best to use my calculator's "log" button:

$$\log_{10} 369 = 2.567...$$

Changing the Base

What if we want to change the base of a logarithm?

Easy! Just use this formula:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

"x goes up, a goes down"

Or another way to think of it is that $\mathbf{\log_b a}$ is like a "conversion factor" (same formula as above):

$$\log_a x = \log_b x \,/\, \mathbf{\log_b a}$$

So now we can convert from any base to any other base.

Another useful property is:

$$\log_a x = 1 \,/\, \log_x a$$

See how "x" and "a" swap positions?

Example: Calculate $1 / \log_8 2$

$$1 \,/\, \log_8 2 = \log_2 8$$

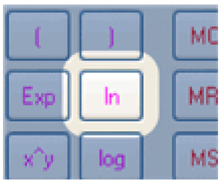
And 2 × 2 × 2 = 8, so when 2 is used **3** times in a multiplication you get 8:

$$1 / \log_8 2 = \log_2 8 = 3$$

But we use the Natural Logarithm more often, so this is worth remembering:

$$\log_a x = \ln x / \ln a$$

Example: Calculate $\log_4 22$



My calculator doesn't have a "**$\log_4$**" button ...

... but it does have an "**ln**" button, so we can use that:

$$\begin{aligned} \log_4 22 &= \ln 22 / \ln 4 \\ &= 3.09.../1.39... \\ &= 2.23 \textit{ (to 2 decimal places)} \end{aligned}$$

What does this answer mean? It means that 4 with an exponent of 2.23 equals 22. So we can check that answer:

Check: $4^{2.23} = 22.01$ (close enough!)

Here is another example:

Example: Calculate $\log_5 125$

$$\begin{aligned} \log_5 125 &= \ln 125 / \ln 5 \\ &= 4.83.../1.61... \\ &= 3 \textit{ (exactly)} \end{aligned}$$

I happen to know that 5 × 5 × 5 = 125, (5 is used **3** times to get 125), so I expected an answer of **3**, and it worked!

Real World Usage

Here are some uses for Logarithms in the real world:

Earthquakes

The magnitude of an earthquake is a Logarithmic scale.

The famous "Richter Scale" uses this formula:

$$M = \log_{10} A + B$$

Where **A** is the amplitude (in mm) measured by the Seismograph
and **B** is a distance correction factor

Nowadays there are more complicated formulas, but they still use a logarithmic scale.

Sound

Loudness is measured in Decibels (dB for short):

$$\text{Loudness in dB} = 10 \log_{10} (p \times 10^{12})$$

where **p** is the sound pressure.

Acidic or Alkaline

Acidity (or Alkalinity) is measured in pH:

$$pH = -\log_{10} [H^+]$$

where $\mathbf{H^+}$ is the molar concentration of dissolved hydrogen ions.
Note: in chemistry [ ] means molar concentration (moles per liter).

More Examples

Example: Solve $2 \log_8 x = \log_8 16$

Start with: $2 \log_8 x = \log_8 16$

Bring the "2" into the log: $\log_8 x^2 = \log_8 16$

Remove the logs (they are same base): $x^2 = 16$

Solve: $x = -4$ or $+4$

But ... but ... but ... you cannot have a log of a negative number!

So the −4 case is not defined.

Answer: 4

Check: use your calculator to see if this is the right answer ... also try the "−4" case.

Example: Solve $e^{-w} = e^{2w+6}$

Start with: $e^{-w} = e^{2w+6}$

Apply **ln** to both sides: $\ln(e^{-w}) = \ln(e^{2w+6})$

And **ln(e^w)=w**: $-w = 2w+6$

Simplify: $-3w = 6$

Solve: $w = 6/-3 = -2$

Answer: $w = -2$

Check: $e^{-(-2)} = e^2$ and $e^{2(-2)+6} = e^2$

Footnote: Why does **log(m × n) = log(m) + log(n)** ?

To see **why**, we will use $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$:

First, make **m** and **n** into "exponents of logarithms":

Then use one of the [Laws of Exponents](#)

Finally undo the exponents.

$a^{\log_a(x)} = x$

$$\begin{aligned} \log_a(m \times n) &= \log_a(a^{\log_a m} \times a^{\log_a n}) \\ &= \log_a(a^{\log_a m + \log_a n}) \\ &= \log_a m + \log_a n \end{aligned}$$

$x^m x^n = x^{m+n}$

$\log_a(a^x) = x$

It is one of those clever things we do in mathematics which can be described as *"we can't do it here, so let's go over **there**, then do it, then come back"*

Question 1 Question 2 Question 3 Question 4 Question 5
Question 6 Question 7 Question 8 Question 9 Question 10