

Conditions for Stability and Convergence of Set-Valued Stochastic Approximations: Applications to Approximate Value and Fixed point Iterations with Noise

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Abstract

The study of stochastic approximation algorithms (SAAs) with set-valued mean-fields has been popular during recent times due to their applications to many large scale model-free problems arising in stochastic optimization and control. The analysis of such algorithms requires the almost sure boundedness of the iterates, which can be hard to verify. In this paper we extend the ideas of Abounadi, Bertsekas and Borkar involving regular SAAs (with point-to-point maps), to develop easily verifiable sufficient conditions, based on Lyapunov functions, for both stability and convergence of SAAs with set-valued mean-fields. As an important application of our results, we analyze the stochastic approximation counterpart of approximate value iteration (AVI), an important dynamic programming method designed to tackle Bellman's curse of dimensionality. Our framework significantly relaxes the assumptions involved in the analysis of AVI methods. Further, our analysis covers both stochastic shortest path and infinite horizon discounted cost problems. Generalizing further, we present an SAA, together with an analysis of its stability and convergence for finding fixed points of contractive set-valued maps. To the best of our knowledge ours is the first SAA for finding fixed points of set-valued maps.

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1 Introduction

Stochastic approximation algorithms (SAAs) are an important class of iterative schemes that are used to solve problems arising in stochastic optimization, stochastic control, machine learning and financial mathematics, among others. SAAs constitute a powerful tool, due to their model free approach, to solving problems. The first stochastic approximation algorithm was developed by Robbins and Monro [16] in 1951 to solve the root finding problem. Important contributions to modern stochastic approximations theory were made by Benaïm (1996) [3], Benaïm and Hirsch (1996) [4], Borkar (1997) [8], Borkar and Meyn (1999) [10] and Benaïm, Hofbauer and Sorin (2005) [5], to name a few.

An important aspect of the analysis of SAAs lies in verifying the almost sure boundedness (stability) of the iterates. This can be hard in many applications. In this paper we present easily verifiable sufficient conditions for both stability and convergence of SAAs with set-valued mean-fields. Specifically, we consider the following iterative scheme:

$$x_{n+1} = x_n + a(n)(y_n + M_{n+1}), \quad (1)$$

where $x_n \in \mathbb{R}^d$ for all n ; $\{a(n)\}_{n \geq 0}$ is a step-size sequence; $y_n \in H(x_n)$ for all n such that $H : \mathbb{R}^d \rightarrow \{\text{subsets of } \mathbb{R}^d\}$ is a Marchaud map; and $\{M_{n+1}\}_{n \geq 0}$ is the noise sequence. *We present three different yet overlapping sets of (easily verifiable) conditions for stability (almost sure boundedness) and convergence (to a closed connected internally chain transitive invariant set of $\dot{x}(t) \in H(x(t))$) of (1).* The reader is referred to Section 4 for the analysis.

The problem of stability for SAAs with set-valued mean-fields has been previously studied by Ramaswamy and Bhatnagar [15]. They developed the first set of sufficient conditions for the stability and convergence of (1) by extending the ideas of Borkar and Meyn [10]. Their sufficient conditions are based on the “limiting properties” of an *associated scaled differential inclusion*. On the contrary, the conditions presented in this paper are based on “local properties” of the *associated differential inclusion*. We believe that the stability criterion presented here is applicable to scenarios that are in some sense “orthogonal” to those that readily use the stability criterion of [15]. Our work contributes to the literature of SAAs with set-valued mean-fields by presenting the only sets of sufficient conditions for stability and convergence of (1), after the ones presented by us in [15].

In this paper, we extend the Lyapunov function based stability approach of Abounadi, Bertsekas and Borkar [1] developed originally for the case of SAAs involving point-to-point maps, to include SAAs with set-valued mean-fields. Our stability criterion is dependent in part on the possibility of comparing various instances of an algorithm being analyzed. For the exact nature of this comparison, the reader is referred to assumption (A5) of Section 3. As stated earlier, we present three sets of assumptions that are qualitatively different yet overlapping, for stability and convergence. As a consequence, the framework developed herein is semantically rich enough to cover a multitude of scenarios encountered in reinforcement learning, stochastic optimization and other applications.

We answer the following important question: Does the mere existence of a Lyapunov function for $\dot{x}(t) \in H(x(t))$ imply the almost sure boundedness (stability) of (1)? We will show that the existence of a global/local Lyapunov function allows us to construct an “inward directing set”, see Proposition 2 for details. We then use this inward directing set to develop a partner projective scheme (to (1)). This scheme is shown to converge to a point inside the previously constructed inward directing set. In order to show the stability of (1), we compare the same to the aforementioned partner projective scheme. The exact nature of this comparison is outlined in (A5). It is imperative that (1) and its partner projective scheme are comparable. In other words, it seems that the mere existence of a Lyapunov function is *insufficient* to ensure stability. Additional assumptions such as (A5) of Section 3 are needed for this purpose. We demonstrate the verifiability of our assumptions by using our framework to comprehensively analyze two important problems: (i) approximate value iteration methods with possibly biased approximation errors and (ii) SAAs for finding fixed points of contractive set-valued maps. *It is worth noting that our analysis of approximate value iteration methods does not distinguish between biased and unbiased approximation errors.*

In Section 7, as an application of our main results, we present a complete analysis of approximate value iteration methods, an important class of dynamic programming algorithms, under significantly weaker set of assumptions. Value iteration is an important dynamic programming method used to numerically compute the optimal cost-to-go vector. However, it is well known that for many important applications, it suffers from *Bellman’s curse of dimensionality*. Approximate Value Iteration (AVI) methods endeavor to address Bellman’s curse of dimensionality by introducing an *approximation operator* that allows for approximations at every step of the classical value iteration method. If the approximation errors are allowed to be unbounded then the algorithm may not converge, see [6] for details. AVIs with bounded approximation errors have been previously studied in [6, 11, 13]. Bertsekas and Tsitsiklis [6] studied scenarios wherein the approximation errors are uniformly bounded over all states. Munos [13] extended the analysis of [6], allowing for approximation errors that are bounded in the weighted p-norm sense, for the infinite horizon discounted cost problem. However this convergence analysis requires that the transition probabilities or future state distributions be “smooth”, see [13] for details. For a detailed comparison of our results concerning AVI methods to those already present in literature, see Section 7.2.

An important contribution of this paper is in providing a convergence analysis of AVIs without the aforementioned restriction on transition probabilities or future distributions (cf. 7). Our analysis encompasses both the stochastic shortest path and the discounted cost infinite horizon problems. When analyzing stochastic iterative AVIs (see (10) in Section 7 for details on stochastic iterative AVIs), stability (almost sure boundedness) of the iterates is normally assumed to hold. As stated before, stability is a hard assumption to verify. Further, it is unclear if the introduction of an *approximation operator* leads to unstable iterates. Another important contribution of this paper is in showing stability of stochastic iterative AVIs, under weak, verifiable conditions. In Section 7,

it is shown that a stochastic iterative AVI converges to a *possibly* suboptimal cost-to-go vector J_∞ which belongs to a “small neighborhood” of the optimal vector, J^* . Further it is shown that the size of this neighborhood is directly proportional to the magnitude of the approximating errors, see Theorems 3 and 4 for details.

Thus, in Section 7 we provide a complete analysis (stability and convergence) of general AVI methods under weak albeit easily verifiable set of sufficient conditions. We eliminate all previous restrictions on the “smoothness” of transition probabilities and future distributions. We also allow for more general “operational noise” as compared to previous literature. An important aspect of our analysis is that it encompasses both stochastic shortest path and infinite horizon discounted cost problems. We provide a unified analysis for stability and convergence of AVI methods wherein the approximation errors are bounded with respect to multiple norms.

In Section 8, as yet another application of our framework, we develop and analyze, for the first time, general SAAs for finding fixed points of set-valued maps. Fixed point theory is an active area of research due to its applications in a multitude of areas. Our contribution on this front is in *analyzing stochastic approximation algorithms for finding fixed points of contractive set-valued maps*, see Section 8 for details. As mentioned before, we show that such algorithms are bounded almost surely and that they converge to a sample path dependent fixed point of the set-valued map under consideration. *To the best of our knowledge ours is the first SAA for finding fixed points of set-valued maps.*

1.1 Organization of this paper

In Section 2 we list the definitions and notations used in this paper. In Section 3 we present three sets of sufficient conditions for stability and convergence (to a closed connected internally chain transitive invariant set of the associated DI) of SAAs with set-valued mean-fields. Through Sections 4, 5 and 6 we present our main results, Theorems 1 and 2. In Section 7 we analyze stochastic iterative AVI methods, see Theorem 3 and 4. In Section 8 we develop and analyze a SAA for finding fixed points of contractive set-valued maps, see Theorem 5. A detailed discussion on assumption (A5), which is crucial to our analysis, is provided in Section 9. Finally Section 10 provides the concluding remarks.

2 Definitions and Notations

The definitions and notations encountered in this paper are listed in this section.

[Upper-semicontinuous map] We say that H is upper-semicontinuous, if given sequences $\{x_n\}_{n \geq 1}$ (in \mathbb{R}^n) and $\{y_n\}_{n \geq 1}$ (in \mathbb{R}^m) with $x_n \rightarrow x$, $y_n \rightarrow y$ and $y_n \in H(x_n)$, $n \geq 1$, then $y \in H(x)$.

[Marchaud Map] A set-valued map $H : \mathbb{R}^n \rightarrow \{\text{subsets of } \mathbb{R}^m\}$ is called *Marchaud* if it satisfies the following properties: **(i)** for each $x \in \mathbb{R}^n$, $H(x)$ is convex and compact; **(ii)** (*point-wise boundedness*) for each $x \in \mathbb{R}^n$, $\sup_{w \in H(x)} \|w\| < K(1 + \|x\|)$ for some $K > 0$; **(iii)** H is *upper-semicontinuous*.

Let H be a Marchaud map on \mathbb{R}^d . The differential inclusion (DI) given by

$$\dot{x} \in H(x) \quad (2)$$

is guaranteed to have at least one solution that is absolutely continuous. The reader is referred to [2] for more details. We say that $\mathbf{x} \in \sum$ if \mathbf{x} is an absolutely continuous map that satisfies (2). The *set-valued semiflow* Φ associated with (2) is defined on $[0, +\infty) \times \mathbb{R}^d$ as:

$\Phi_t(x) = \{\mathbf{x}(t) \mid \mathbf{x} \in \sum, \mathbf{x}(0) = x\}$. Let $B \times M \subset [0, +\infty) \times \mathbb{R}^d$ and define

$$\Phi_B(M) = \bigcup_{t \in B, x \in M} \Phi_t(x).$$

[Limit set of a solution & ω -limit-set] The limit set of a solution \mathbf{x} with $\mathbf{x}(0) = x$ is given by $L(x) = \bigcap_{t \geq 0} \overline{\mathbf{x}([t, +\infty))}$. Let $M \subseteq \mathbb{R}^d$, the ω -limit-set be defined by $\omega_\Phi(M) = \bigcap_{t \geq 0} \overline{\Phi_{[t, +\infty)}(M)}$. Similarly the *limit set* of a solution \mathbf{x} is given by $L(x) = \bigcap_{t \geq 0} \overline{\mathbf{x}([t, +\infty))}$.

[Invariant set] $M \subseteq \mathbb{R}^d$ is *invariant* if for every $x \in M$ there exists a trajectory, $\mathbf{x} \in \sum$, entirely in M with $\mathbf{x}(0) = x$, $\mathbf{x}(t) \in M$, for all $t \geq 0$. Note that the definition of invariant set used in this paper, is the same as that of positive invariant set in [5] and [9].

[Open and closed neighborhoods of a set] Let $x \in \mathbb{R}^d$ and $A \subseteq \mathbb{R}^d$, then $d(x, A) := \inf\{\|x - y\| \mid y \in A\}$. We define the δ -open neighborhood of A by $N^\delta(A) := \{x \mid d(x, A) < \delta\}$. The δ -closed neighborhood of A is defined by $\overline{N^\delta(A)} := \{x \mid d(x, A) \leq \delta\}$. The open ball of radius r around the origin is represented by $B_r(0)$, while the closed ball is represented by $\overline{B}_r(0)$.

[Internally chain transitive set] $M \subset \mathbb{R}^d$ is said to be internally chain transitive if M is compact and for every $x, y \in M$, $\epsilon > 0$ and $T > 0$ we have the following: There exists n and Φ^1, \dots, Φ^n that are n solutions to the differential inclusion $\dot{x}(t) \in H(x(t))$, points $x_1(=x), \dots, x_{n+1}(=y) \in M$ and n real numbers t_1, t_2, \dots, t_n greater than T such that: $\Phi_{t_i}^i(x_i) \in N^\epsilon(x_{i+1})$ and $\Phi_{[0, t_i]}^i(x_i) \subset M$ for $1 \leq i \leq n$. The sequence $(x_1(=x), \dots, x_{n+1}(=y))$ is called an (ϵ, T) chain in M from x to y .

[Attracting set & fundamental neighborhood] $A \subseteq \mathbb{R}^d$ is *attracting* if it is compact and there exists a neighborhood U such that for any $\epsilon > 0$, $\exists T(\epsilon) \geq 0$ with $\Phi_{[T(\epsilon), +\infty)}(U) \subset N^\epsilon(A)$. Such a U is called the *fundamental neighborhood* of A . In addition to being compact if the *attracting set* is also invariant then it is called an *attractor*. The *basin of attraction* of A is given by $B(A) = \{x \mid \omega_\Phi(x) \subset A\}$.

[Lyapunov stable] The above set A is Lyapunov stable if for all $\delta > 0$, $\exists \epsilon > 0$ such that $\Phi_{[0, +\infty)}(N^\epsilon(A)) \subseteq N^\delta(A)$.

3 Assumptions

Consider the following iteration in \mathbb{R}^d :

$$x_{n+1} = x_n + a(n) [y_n + M_{n+1}], \quad (3)$$

where $y_n \in H(x_n)$ for all $n \geq 0$ with $H : \mathbb{R}^d \rightarrow \{\text{subsets of } \mathbb{R}^d\}$, $\{a(n)\}_{n \geq 0}$ is the given step-size sequence and $\{M_{n+1}\}_{n \geq 0}$ is the given noise sequence.

We make the following assumptions ¹:

- (A1) $H : \mathbb{R}^d \rightarrow \{\text{subsets of } \mathbb{R}^d\}$ is a Marchaud map. For all $x \in \mathbb{R}^d$ $\sup_{z \in H(x)} \|z\| \leq K(1 + \|x\|)$, where $K > 0$ is the given Marchaud constant.
- (A2) The step-size sequence $\{a(n)\}_{n \geq 0}$ is such that $\forall n$ $a(n) \geq 0$, $\sum_{n \geq 0} a(n) = \infty$ and $\sum_{n \geq 0} a(n)^2 < \infty$.
- (A3) (i) For all $n \geq 0$ $\|M_{n+1}\| \leq D$, where $D > 0$ is some constant.
(ii) $\lim_{k \rightarrow \infty} \sum_{n=k}^{m_T(k)} a(n)M_{n+1} = 0$ for all T , where $m_T(k) := \min \{m \geq k \mid \sum_{n=k}^m a(n) \geq T\}$.
- (A4a) Associated with the differential inclusion (DI) $\dot{x}(t) \in H(x(t))$ is a compact set Λ , a bounded open neighborhood \mathcal{U} ($\Lambda \subseteq \mathcal{U} \subseteq \mathbb{R}^d$) and a function $V : \overline{\mathcal{U}} \rightarrow \mathbb{R}^+$ such that
 - (i) $\forall t \geq 0$ $\Phi_t(\mathcal{U}) \subseteq \mathcal{U}$ i.e., \mathcal{U} is strongly positively invariant.
 - (ii) $V^{-1}(0) = \Lambda$.
 - (iii) V is a continuous function such that for all $x \in \mathcal{U} \setminus \Lambda$ and $y \in \Phi_t(x)$ we have $V(x) > V(y)$, for any $t > 0$.
- (A5) Let $\{x_n\}_{n \geq 0}$ and $\{\hat{x}_n\}_{n \geq 0}$ be two sequences generated by (3) on a common probability space with the same noise sequence $\{M_{n+1}\}_{n \geq 0}$. Then $\sup_n \|x_n - \hat{x}_n\| < \infty$ a.s.

Note that it follows from Proposition 3.25 of Benaïm, Hofbauer and Sorin [5] that Λ contains a Lyapunov stable attracting set. Further there exists an attractor contained in Λ whose basin of attraction contains \mathcal{U} , with $V^{-1}([0, r])$ as fundamental neighborhoods for small values of r .

Note that assumptions on the noise, (A3), will be weakened to include general noise sequences later, see Section 6. Also, (A5) is similar to an assumption used by Abounadi et. al. [1] in the case of regular SAAs (with point-to-point maps). The reader is referred to Section 9 for more detailed discussions on this assumption.

We define open sets \mathcal{B}_a and \mathcal{C}_a such that the following conditions are satisfied: (a) $\Lambda \subseteq \overline{\mathcal{B}_a} \subseteq \mathcal{C}_a$ (b) $\overline{\mathcal{C}_a} \subseteq \mathcal{U}$ and (c) \mathcal{C}_a is a fundamental neighborhood of the attractor contained in Λ . Since V is continuous, it follows that $\mathcal{V}_r := \{x \mid V(x) < r\}$ is open relative to $\overline{\mathcal{U}}$, for all $r > 0$. Further, assumption (A4a)(ii) implies that $\bigcap_{r>0} \mathcal{V}_r = \Lambda$ and that $\mathcal{V}_r \subset \mathcal{U}$ for small values of r . It may be noted that the closure of \mathcal{V}_r is $\overline{\mathcal{V}_r} := \{x \mid V(x) \leq r\}$. Let $\mathcal{B}_a := \mathcal{V}_{R_b}$ and $\mathcal{C}_a := \mathcal{V}_{R_c}$, where $0 < R_b < R_c$. Here R_c is chosen small enough to ensure that $\overline{\mathcal{V}_{R_c}} \subseteq \mathcal{U}$ and the above conditions (b) and (c) hold. Note that condition (a) is automatically satisfied since $R_b < R_c$.

¹ Note that assumption (A4a) presents a set of Lyapunov conditions on the associated DI. Two more sets of alternative conditions, viz., (A4b) and (A4c) will also be presented subsequently in this section.

We now present an alternative to assumption (A4a).

(A4b) Associated with $\dot{x}(t) \in H(x(t))$ is a compact set Λ , a bounded open neighborhood \mathcal{U} and a function $V : \overline{\mathcal{U}} \rightarrow \mathbb{R}^+$ such that

- (i) $\forall t \geq 0 \ \Phi_t(\mathcal{U}) \subseteq \mathcal{U}$ i.e., \mathcal{U} is strongly positively invariant.
- (ii) $V^{-1}(0) = \Lambda$.
- (iii) V is an upper semicontinuous function such that for all $x \in \overline{\mathcal{U}} \setminus \Lambda$ and $y \in \Phi_t(x)$ we have $V(x) > V(y)$, where $t > 0$.
- (iv) V is bounded on $\overline{\mathcal{U}}$. In other words, $\sup_{x \in \mathcal{U}} \|V(x)\| < \infty$.

The difference between statements (A4a)(iii) and (A4b)(iii) contributes to the qualitative difference between assumptions (A4a) and (A4b). It follows from *Proposition 3.25* of Benaïm, Hofbauer and Sorin [5] that Λ contains an attractor set whose basin of attraction contains \mathcal{U} . As in the case of (A4a), we define open sets \mathcal{B}_b and \mathcal{C}_b satisfying the above stated (a) and (b). But first, we prove that sets of the form $\{x \mid V(x) < r\}$ are open relative to $\overline{\mathcal{U}}$, as expected.

Proposition 1. *For any $r < \sup_{u \in \mathcal{U}} V(u)$, the set $\mathcal{V}_r = \{x \mid V(x) < r\}$ is open relative to $\overline{\mathcal{U}}$. Further, $\overline{\mathcal{V}_r} = \{x \mid V(x) \leq r\}$.*

Proof. Suppose \mathcal{V}_r is not open relative to $\overline{\mathcal{U}}$, then there exists $\{x_n\}_{n \geq 0}$ such that $x_n \rightarrow x$, as $n \rightarrow \infty$, with $x_n \in \overline{\mathcal{U}} \setminus \mathcal{V}_r$ for every n and $x \in \mathcal{V}_r$. It follows from the boundedness of V , i.e., (A4b)(iv) that there exists $\{m(n)\}_{n \geq 0} \subseteq \{n \geq 0\}$ such that $V(x_{m(n)}) \rightarrow v$ as $m(n) \rightarrow \infty$ for some $v \in \mathbb{R}^+$. Since $x_{m(n)} \rightarrow x$, it follows from the upper semicontinuity of V that $v = V(x)$. Since $v < r$ and $\inf_{m(n)} V(x_{m(n)}) \geq r$, we get a contradiction.

To prove the second part of the proposition, it is enough to show that $V(x) = r$ for every $x \in \partial \mathcal{V}_r$. Since \mathcal{V}_r is open relative to $\overline{\mathcal{U}}$ and $x \in \partial \mathcal{V}_r$, we have $V(x) \geq r$. It is left to show that $V(x) \leq r$. Since $x \in \partial \mathcal{V}_r$, there exists $x_n \in \mathcal{V}_r$ such that $x_n \rightarrow x$. It follows from the upper semicontinuity and the boundedness of V that $V(x) \leq \limsup_{n \rightarrow \infty} V(x_n) \leq r$. \square

As in the case of (A4a), we have $\Lambda = \bigcap_{r>0} \mathcal{V}_r$ and $\mathcal{V}_r \subset \mathcal{U}$ for small values of r . We are now ready to define \mathcal{B}_b and \mathcal{C}_b . Define $\mathcal{C}_b := \mathcal{V}_r$ such that $\overline{\mathcal{V}_r} \subset \mathcal{U}$, possible for small values of r . Further, choose \mathcal{B}_b such that \mathcal{B}_b is open and $\Lambda \subset \mathcal{B}_b \subset \overline{\mathcal{B}_b} \subset \mathcal{C}_b$. This is possible since Λ is compact and \mathcal{C}_b is open.

Remark 1. *Let us suppose we are given a differential inclusion $\dot{x}(t) \in H(x(t))$, an associated attractor set \mathcal{A} and a strongly positive invariant neighborhood \mathcal{U} of \mathcal{A} . We define an upper-semicontinuous Lyapunov function V , as found in Remark 3.26, Section 3.8 of [5]. In other words, $V : \overline{\mathcal{U}} \rightarrow \mathbb{R}^+$ given by $V(x) := \max \{d(y, \mathcal{A})g(t) \mid y \in \Phi_t(x), t \geq 0\}$, where g is an increasing function such that $0 < c < g(t) < d$ for all $t \geq 0$. We claim that V satisfies (A4b). To verify this claim we consider the following. \mathcal{U} is strongly positive invariant and $V(x) \leq \sup_{u \in \mathcal{U}} d(u, \mathcal{A}) \times d$ for $x \in \mathcal{U}$, hence $\sup_{u \in \mathcal{U}} V(u) < \infty$. It follows from the*

upper semicontinuity of V that $\sup_{u \in \overline{\mathcal{U}}} V(u) < \infty$ i.e., (A4b)(iv) is satisfied. It is left to show that (A4b)(iii) is also satisfied. Fix $x \in \mathcal{U}$ and $t > 0$. It follows from the definition of a semi-flow that $\Phi_s(y) \subseteq \Phi_{t+s}(x)$ for any $y \in \Phi_t(x)$, where $s > 0$. Further,

$$V(x) \geq \max\{d(z, \mathcal{A})g(t+s) \mid z \in \Phi_s(y), s \geq 0\} \text{ and}$$

$$\max\{d(z, \mathcal{A})g(t+s) \mid z \in \Phi_s(y), s \geq 0\} > \max\{d(z, \mathcal{A})g(s) \mid z \in \Phi_s(y), s \geq 0\}.$$

The RHS of the above equation is $V(y)$ i.e., $V(x) > V(y)$.

We consider one final alternative to (A4a) and (A4b) below.

- (A4c) (i) \mathcal{A} is the global attractor of $\dot{x}(t) \in H(x(t))$.
(ii) $V : \mathbb{R}^d \rightarrow \mathbb{R}^+$ is an upper semicontinuous function such that $V(x) > V(y)$ for all $x \in \mathbb{R}^d \setminus \mathcal{A}$, $y \in \Phi_t(x)$ and $t > 0$.
(iii) $V(x) \geq V(y)$ for all $x \in \mathcal{A}$, $y \in \Phi_t(x)$ and $t > 0$.

As with (A4a) and (A4b), we define open sets \mathcal{B}_c and \mathcal{C}_c satisfying conditions (a) and (b), see below the statement of (A4a). Recall that $\mathcal{V}_r = \{x \mid V(x) < r\}$. Define $\mathcal{C}_c := \mathcal{V}_r$ and $\mathcal{B}_c := \mathcal{V}_s$ for appropriate r and s satisfying $\sup_{x \in \mathbb{R}^d} V(x) > r > s > \sup_{x \in \mathcal{A}} V(x)$. Suppose we are unable to find such an s then we may choose \mathcal{B}_c to be any open set satisfying the required properties once \mathcal{C}_c is fixed as mentioned before.

Remark 2. In Remark 1, we explicitly constructed a local Lyapunov function satisfying (A4b). Similarly, here, we construct a global Lyapunov function satisfying (A4c). Define the function $V : \mathbb{R}^d \rightarrow \mathbb{R}^+$ as $V(x) := \max\{d(y, \mathcal{A})g(t) \mid y \in \Phi_t(x), t \geq 0\}$, where $g(\cdot)$ is defined in Remark 1. This Lyapunov function, V , satisfies (A4c). The proof is similar to the one found in Remark 1.

Inward directing sets: Given a differential inclusion $\dot{x}(t) \in H(x(t))$, an open set \mathcal{O} is said to be an inward directing set with respect to the aforementioned differential inclusion, if $\Phi_t(x) \subseteq \mathcal{O}$, $t > 0$, whenever $x \in \overline{\mathcal{O}}$. Specifically, any solution to the DI with starting point at the boundary of \mathcal{O} is “directed inwards”, into \mathcal{O} .

Proposition 2. The open sets \mathcal{C}_a , \mathcal{C}_b and \mathcal{C}_c , constructed in accordance to assumptions (A4a), (A4b) and (A4c) respectively, are inward directing sets with respect to $\dot{x}(t) \in H(x(t))$.

Proof. Recall that the set \mathcal{C}_a is constructed such that $V(x) = R_c$ for every $x \in \partial\mathcal{C}_a$ and $V(y) < R_c$ for every $y \in \mathcal{C}_a$. Since $\overline{\mathcal{C}_a} \subseteq \mathcal{U}$, it follows from (A4a)(iii) that $V(x) > V(y)$, where $x \in \partial\mathcal{C}_a$, $y \in \Phi_t(x)$ and $t > 0$. In other words, $V(y) < R_c$ and $\Phi_t(x) \subset \mathcal{C}_a$ for $t > 0$ and $x \in \partial\mathcal{C}_a$. It is left to show that $\Phi_t(x) \subset \mathcal{C}_a$ for $x \in \mathcal{C}_a$ and $t > 0$. This follows directly from the observation that $R_c > V(x) > V(y)$ for every $y \in \Phi_t(x)$ and $t > 0$.

\mathcal{C}_b and \mathcal{C}_c can be similarly shown to be inward directing. \square

In what follows, we use assumptions (A1)-(A3), (A5) and the existence of an inward directing set with respect to the associated DI, to prove the stability

of (3). As a consequence of Proposition 2, we may verify one among (A4a), (A4b) and (A4c) to ensure the existence of such an inward directing set. It may be noted that these assumptions are qualitatively different. However, their primary role is to help us find one of the aforementioned inward directing sets. Depending on the nature of the iteration being analyzed, it may be easier to verify one rather than the others.

4 Analysis of the projective scheme

We begin this section with a minor note on notations. Since the roles of (A4a), (A4b) and (A4c) are indistinguishable, we shall refer to them generically as (A4). In a similar manner, $\mathcal{B}_{a/b/c}$ and $\mathcal{C}_{a/b/c}$ are generically referred to as \mathcal{B} and \mathcal{C} respectively. Note that $\bar{\mathcal{B}} \subseteq \mathcal{C}$. We also define the projection map, $\bigcap_{\mathcal{B}, \mathcal{C}} : \mathbb{R}^d \rightarrow \{\text{subsets of } \mathbb{R}^d\}$, as follows:

$$\bigcap_{\mathcal{B}, \mathcal{C}}(x) := \begin{cases} \{x\}, & \text{if } x \in \mathcal{C} \\ \{y \mid d(y, x) = d(x, \bar{\mathcal{B}}), y \in \bar{\mathcal{B}}\}, & \text{otherwise} \end{cases}.$$

As in [1], in order to prove stability, (3) is compared to the following projective scheme.

$$\begin{aligned} \tilde{x}(n+1) &= x_n + a(n) [y_n + M_{n+1}], \\ x_{n+1} &= z_n, \text{ where } z_n \in \bigcap_{\mathcal{B}, \mathcal{C}}(\tilde{x}_{n+1}), \end{aligned} \tag{4}$$

where $y_n \in H(x_n)$ and $x_0 \in \bigcap_{\mathcal{B}, \mathcal{C}}(\tilde{x}_0)$, with $\tilde{x}_0 \in \mathbb{R}^d$. Note that the initial point \tilde{x}_0 is first projected before starting the projective scheme. The above equation can be rewritten as

$$x_{n+1} = x_n + a(n) [y_n + M_{n+1}] + g_n, \tag{5}$$

where $g_n = z_n - (x_n + a(n) [y_n + M_{n+1}])$. Let us construct a linearly interpolated trajectory using (5). We begin by dividing $[0, \infty)$ into diminishing intervals using the step-size sequence. Let $t_0 := 0$ and $t_n := \sum_{m=0}^{n-1} a(m)$ for $n \geq 1$. The linearly interpolated trajectory $X_l(t)$ is defined as follows:

$$X_l(t) := \begin{cases} x_n, & \text{for } t = t_n \\ \left(1 - \frac{t-t_n}{a(n)}\right) x_n + \left(\frac{t-t_n}{a(n)}\right) \tilde{x}_{n+1}, & \text{for } t \in [t_n, t_{n+1}) \end{cases}. \tag{6}$$

The above constructed trajectory is right continuous with left-hand limits, *i.e.*, $X_l(t) = \lim_{s \downarrow t} X_l(s)$ and $\lim_{s \uparrow t} X_l(s)$ exists. Further the jumps occur exactly at those t_n 's for which the corresponding g_{n-1} 's are non-zero. We also define three piece-wise constant trajectories $X_c(\cdot)$, $Y_c(\cdot)$ and $G_c(\cdot)$ as follows: $X_c(t) := x_n$, $Y_c(t) := y_n$ and $G_c(t) := \sum_{m=0}^{n-1} g_m$ for $t \in [t_n, t_{n+1})$. The trajectories $X_c(\cdot)$, $Y_c(\cdot)$ and $G_c(\cdot)$ are also right continuous with left-hand limits. We define a linearly

interpolated trajectory associated with $\{M_{n+1}\}_{n \geq 0}$ as follows:

$$W_l(t) := \begin{cases} \sum_{m=0}^{n-1} a(m)M_{m+1} & \text{for } t = t_n \\ \left(1 - \frac{t-t_n}{a(n)}\right) W_l(t_n) + \left(\frac{t-t_n}{a(n)}\right) W_l(t_{n+1}), & \text{for } t \in [t_n, t_{n+1}) \end{cases}.$$

We define a few “left-shifted trajectories” using the above constructed trajectories. For $t \geq 0$,

$$\begin{aligned} X_l^n(t) &:= X_l(t_n + t), \\ X_c^n(t) &:= X_c(t_n + t), \\ Y_c^n(t) &:= Y_c(t_n + t), \\ G_c^n(t) &:= G_c(t_n + t) - G_c(t_n), \\ W_l^n(t) &:= W_l(t_n + t) - W_l(t_n). \end{aligned}$$

Lemma 1. $X_l^n(t) = X_l^n(0) + \int_0^t Y_c^n(\tau) d\tau + W_l^n(t) + G_c^n(t)$ for $t \geq 0$.

Proof. Fix $s \in [t_m, t_{m+1})$ for some $m \geq 0$. We have the following

$$\begin{aligned} X_l(s) &= \left(1 - \frac{s - t_m}{a(m)}\right) x_m + \left(\frac{s - t_m}{a(m)}\right) \tilde{x}_{m+1}, \\ X_l(s) &= \left(1 - \frac{s - t_n}{a(m)}\right) x_m + \left(\frac{s - t_m}{a(m)}\right) (x_m + a(m)[y_m + M_{m+1}]), \\ X_l(s) &= x_m + (s - t_m)[y_m + M_{m+1}]. \end{aligned}$$

Let us express $X_l^n(t)$ in the form of the above equation. Note that $t_n + t \in [t_{n+k}, t_{n+k+1})$ for some $k \geq 0$. Then we have the following:

$$X_l^n(t) = x_{n+k} + (t_n + t - t_{n+k})[y_{n+k} + M_{n+k+1}].$$

Unfolding x_{n+k} , in the above equation till x_n ($X_l^n(0)$), yields:

$$X_l^n(t) = X_l^n(0) + \sum_{l=n}^{n+k-1} (a(l)[y_l + M_{l+1}] + g_l) + (t_n + t - t_{n+k})[y_{n+k} + M_{n+k+1}]. \quad (7)$$

We make the following observations:

$$G_c^n(t) = \sum_{l=n}^{n+k-1} g_l,$$

$$W_l^n(t_{n+k} - t_n) = \sum_{l=n}^{n+k-1} a(l)M_{l+1},$$

$$W_l^n(t) = W_l^n(t_{n+k} - t_n) + (t_n + t - t_{n+k})M_{n+k+1} \text{ and}$$

$$\int_0^t Y_c^n(\tau) d\tau = \sum_{l=n}^{n+k-1} a(l)y_l + (t_n + t - t_{n+k})y_{n+k}.$$

As a consequence of the above observations, (7) becomes:

$$X_l^n(t) = X_l^n(0) + \int_0^t Y_c^n(\tau) d\tau + W_l^n(t) + G_c^n(t).$$

□

Fix $T > 0$. If $\{X_l^n([0, T]) \mid n \geq 0\}$ and $\{G_c^n([0, T]) \mid n \geq 0\}$ are viewed as subsets of $D([0, T], \mathbb{R}^d)$ equipped with the Skorohod topology, then we may use the Arzela-Ascoli theorem for $D([0, T], \mathbb{R}^d)$ to show that they are relatively compact, see Billingsley [7] for details. The Arzela-Ascoli theorem for $D([0, T], \mathbb{R}^d)$ states the following: A set $S \subseteq D([0, T], \mathbb{R}^d)$, is relatively compact if and only if the following conditions are satisfied:

$$\begin{aligned} & \sup_{x(\cdot) \in S} \sup_{t \in [0, T]} \|x(t)\| < \infty, \\ & \lim_{\delta \rightarrow 0} \sup_{x(\cdot) \in S} \sup_{t_1 \leq t \leq t_2, t_2 - t_1 \leq \delta} \min \{\|x(t) - x(t_1)\|, \|x(t_2) - x(t)\|\} = 0, \\ & \lim_{\delta \rightarrow 0} \sup_{x(\cdot) \in S} \sup_{t_1, t_2 \in [0, \delta]} \|x(t_2) - x(t_1)\| = 0 \text{ and} \\ & \lim_{\delta \rightarrow 0} \sup_{x(\cdot) \in S} \sup_{t_1, t_2 \in [T - \delta, T]} \|x(t_2) - x(t_1)\| = 0. \end{aligned}$$

If $\{X_l^n([0, T]) \mid n \geq 0\}$ and $\{G_c^n([0, T]) \mid n \geq 0\}$ are point-wise bounded and any two of their discontinuities are separated by at least Δ , for some fixed $\Delta > 0$, then the above four conditions will be satisfied, see [7] for details.

Lemma 2. $\{X_l^n([0, T]) \mid n \geq 0\}$ and $\{G_c^n([0, T]) \mid n \geq 0\}$ are relatively compact in $D([0, T], \mathbb{R}^d)$ equipped with the Skorohod topology.

Proof. Recall from (A3)(i) that $\|M_{n+1}\| \leq D$, $n \geq 0$. Since H is Marchaud, it follows that $\sup_{x \in \bar{\mathcal{C}}, y \in H(x)} \|y\| \leq C_1$ for some $C_1 > 0$ and that $\sup_{n \geq 0} \|\tilde{x}_{n+1} - x_n\| \leq C_2$.

$x_n\| \leq \left(\sup_{m \geq 0} a(m) \right) (C_1 + D)$. Further, $g_n \leq \|\tilde{x}_{n+1} - x_n\| + d(x_n, \partial \mathcal{B}) \leq C_2$ for some constant C_2 that is independent of n . In other words, we have that the sequences $\{X_l^n([0, T]) \mid n \geq 0\}$ and $\{G_c^n([0, T]) \mid n \geq 0\}$ are point-wise bounded. It remains to show that any two discontinuities are separated. Let $d := \min_{x \in \partial \mathcal{C}} d(x, \bar{\mathcal{B}})$ and $D_1 := D + \sup_{x \in \bar{\mathcal{C}}} \sup_{y \in H(x)} \|y\|$. Clearly $d > 0$. Define

$$m(n) = \max \left\{ j > 0 \mid \sum_{k=n}^{n+j} a(k) < d/D_1 \right\}.$$

If there is a jump at t_n , then $x_n \in \partial \mathcal{B}$. It follows from the definition of $m(n)$ that $x_k \in \mathcal{C}$ for $n \leq k \leq m(n)$. In other words, there are no discontinuities in the interval $[t_n, t_{n+m(n)})$ and $t_{n+m(n)} - t_n \geq \frac{d}{2D_1}$. If we fix $\Delta := \frac{d}{2D_1} > 0$, then any two discontinuities are separated by at least Δ . \square

Since T is arbitrary, it follows that $\{X_l^n([0, \infty)) \mid n \geq 0\}$ and $\{G_c^n([0, \infty)) \mid n \geq 0\}$ are relatively compact in $D([0, \infty), \mathbb{R}^d)$. Since $\{W_l^n([0, T]) \mid n \geq 0\}$ is point-wise bounded (assumption (A3)(i)) and continuous, it is relatively compact in $D([0, T], \mathbb{R}^d)$. It follows from (A3)(ii) that any limit of $\{W_l^n([0, T]) \mid n \geq 0\}$, in $D([0, T], \mathbb{R}^d)$, is the constant 0 function. Suppose we consider subsequences of $\{X_l^n([0, T]) \mid n \geq 0\}$ and $\left\{ X_l^n(0) + \int_0^T Y_c^n(\tau) d\tau + G_c^n(T) \mid n \geq 0 \right\}$ along which the aforementioned noise trajectories converge, then their limits are identical.

Consider $\{m(n)\}_{n \geq 0} \subseteq \mathbb{N}$ along which $\{G_c^{m(n)}([0, T])\}_{n \geq 0}$ and $\{X_l^{m(n)}([0, T])\}_{n \geq 0}$ converge in $D([0, T], \mathbb{R}^d)$. Further, let $g_{m(n)-1} = 0$ for all $n \geq 0$. Suppose the limit of $\{G_c^{m(n)}([0, T])\}_{n \geq 0}$ is the constant 0 function, then it can be shown that the limit of $\{X_l^{m(n)}([0, T])\}_{n \geq 0}$ is

$$X(t) = X(0) + \int_0^t Y(\tau) d\tau,$$

where $X(0) \in \bar{\mathcal{C}}$ and $Y(t) \in H(X(t))$ for $t \in [0, T]$. The proof of this is along the lines of the proof of *Theorem 2 in Chapter 5.2* of Borkar [9]. Suppose every limit of $\{G_c^{m(n)}([0, T])\}_{n \geq 0}$ is the constant 0 function whenever $g_{m(n)-1} = 0$ for all $n \geq 0$, then every limit of $\{X_l^{m(n)}([0, T])\}_{n \geq 0}$ is a solution to $\dot{X}(t) \in H(X(t))$. Suppose we show that the aforementioned statement is true for every $T > 0$. Then, along with (A5) the stability of (3) is implied. Note that the set $K := \{n \mid g_n = 0\}$ has infinite cardinality since any two discontinuities are at least $\Delta > 0$ apart.

Lemma 3. *Let $K = \{n \mid g_n = 0\}$. Without loss of generality let $\{X_l^n([0, T])\}_{n \in K}$ and $\{G_c^n([0, T])\}_{n \in K}$ be convergent, as $n \rightarrow \infty$, in $D([0, T], \mathbb{R}^d)$. Then $X_l^n(t) \rightarrow X(0) + \int_0^t Y(s) ds + G(t)$ for $t \in [0, T]$ and $G(\cdot) \equiv 0$.*

Proof. We begin by making the following observations:

- (a) $X(0) \in \bar{\mathcal{C}}$.
- (b) Any two discontinuities of $X(\cdot)$ are at least Δ apart.
- (c) $G(0) = 0$.
- (d) Solutions to $\dot{x}(t) \in H(x(t))$ with starting points in $\bar{\mathcal{C}}$ will not hit the boundary, $\partial\mathcal{C}$, later, i.e., they remain in the interior of \mathcal{C} . This observation is a consequence of Proposition 2.

It follows from the above observations that (i) $X(t) \in \mathcal{C}$ for small values of t , (ii) $\tau := \inf\{t \mid t > 0, X(t^+) \neq X(t^-)\}$ and $\tau > 0$. It follows from the nature of convergence that $\exists \tau'_n > \tau > \tau_n, n \geq 0$ such that

$$\tau'_n - \tau_n \rightarrow 0,$$

$$\|X_l^n(\tau'_n) - X(\tau^+)\| \rightarrow 0 \text{ and}$$

$$\|X_l^n(\tau_n) - X(\tau^-)\| \rightarrow 0.$$

For large values of n , $X_l^n(\cdot)$ has exactly one jump (point of discontinuity) at $\hat{\tau}_n \in [\tau_n, \tau'_n]$, let us call this point of discontinuity as $\hat{\tau}_n$. Let $\delta := \|X(\tau^+) - X(\tau^-)\| > 0$, then for large values of n we have

$$\|X_l^n(\hat{\tau}_n^+) - X_l^n(\hat{\tau}_n^-)\| \geq \delta/2.$$

Also, $X_l^n(\hat{\tau}_n^-)$ is not in \mathcal{C} and $X_l^n(\hat{\tau}_n^+)$ is in $\partial\mathcal{B}$. Further, since $\hat{\tau}_n^- - \tau_n \rightarrow 0$, as $n \rightarrow \infty$, it follows that

$$X_l^n(\hat{\tau}_n^-) - X(\tau^-) \rightarrow 0.$$

Hence, $X(\tau^-) \notin \mathcal{C}$. Similarly, we have that $X(\tau^+) \in \partial\mathcal{B}$. Observe that $X([0, \tau))$ is a solution to $\dot{x}(\cdot) \in H(x(\cdot))$ such that $X(0) \in \bar{\mathcal{C}}$, since $G(t) = 0$ for $t \in [0, \tau)$. Further, since \mathcal{C} is *inward directing*, we have that $X(t) \in \mathcal{C}$ for $t \in [0, \tau)$. Since $X(t) \in \mathcal{C}$ for $t < \tau$ and $X(\tau^-) \notin \mathcal{C}$ we have $X(\tau^-) \in \partial\mathcal{C}$.

Since $X(0) \in \bar{\mathcal{C}}$, we have that $V(X(0)) \leq R$, for some $0 < R < \infty$. As a consequence of our choice of \mathcal{C} (\mathcal{C} is $\mathcal{C}_a/\mathcal{C}_b/\mathcal{C}_c$ within the context of (A4a)/(A4b)/(A4c)) we have $V(x) = V(y)$ for any $x, y \in \partial\mathcal{C}$, hence we may fix $R := V(x)$ for any $x \in \partial\mathcal{C}$. Fix $\tau_0 \in (0, \tau)$, it follows from Proposition 2 that $V(X(\tau_0)) < R$. Let $t_n \uparrow \tau$ such that $t_n \in (\tau_0, \tau)$ for $n \geq 1$. Without loss of generality, $X(t_n) \rightarrow X(\tau^-)$ and $V(X(t_n)) \rightarrow V(X(\tau^-))$, as $t_n \rightarrow \tau$ (else we may choose a sub-sequence of $\{t_n\}_{n \geq 0}$ along which $V(X(t_n))$ is convergent). Thus, $\exists N$ such that $V(X(t_n)) > V(X(\tau_0))$ for $n \geq N$. Since $X([\tau_0, t_n])$ is a solution to $\dot{x}(t) \in H(x(t))$ with starting point $X(\tau_0)$, the aforementioned conclusion contradicts (A4a)(iii)/(A4b)(iii)/(A4c). In other words, $X(\tau^-) \in \mathcal{C}$ and $\notin \partial\mathcal{C}$. Thus we have shown that there is no jump at τ , i.e., $X(\tau^+) = X(\tau^-)$. \square

Suppose (A4a) / (A4b) holds, then it follows from Proposition 3.25 of Benaim, Hofbauer and Sorin [5] that there is an attracting set $\mathcal{A} \subseteq \Lambda$ such that $\bar{\mathcal{C}}_a / \bar{\mathcal{C}}_b$ is within the basin of attraction. Suppose (A4c) holds, then \mathcal{A} is the globally attracting set of $\dot{x}(t) \in H(x(t))$.

Lemma 4. *The projective stochastic approximation scheme given by (4) converges to the attractor \mathcal{A} .*

Proof. We begin by noting that T of Lemma 3 is arbitrary. Since $G \equiv 0$, after a certain number of iterations of (4), there are no projections, i.e., $\tilde{x}_n = x_n$ for $n \geq N$. Here N could be sample path dependent. Further, it follows from Lemma 3 that the projective scheme given by (4) tracks a solution to $\dot{x}(t) \in H(x(t))$. In other words, the projective scheme given by (4) converges to a limit point of the DI, $\dot{x}(t) \in H(x(t))$.

The iterates given by (4) are within \mathcal{C} after sometime and they track a solution to $\dot{x}(t) \in H(x(t))$. Since \mathcal{C} is within the basin of attraction of \mathcal{A} , the iterates converge to \mathcal{A} . \square

5 Main results: stability and convergence

In this section we show that the iterates given by (3) are stable (bounded almost surely) and converge to a closed connected internally chain transitive invariant set associated with $\dot{x}(t) \in H(x(t))$. We show the stability of (3), by comparing the iterates generated by (3) to those generated by (4). As stated in (A5), we assume that the noise sequence is exactly the same for both (3) and (4).

Theorem 1. *Under (A1)-(A5), the iterates given by (3) are stable (bounded almost surely) and converge to a closed, connected, internally chain transitive, invariant set associated with $\dot{x}(t) \in H(x(t))$.*

Proof. For notational convenience we use $\{\hat{x}_n\}_{n \geq 0}$ for the iterates generated by the projective scheme and $\{x_n\}_{n \geq 0}$ for the iterates generated by (3). Recall that $\hat{x}_n \rightarrow \mathcal{A}$, where $\mathcal{A} \subset \mathcal{B}$, see Lemma 4. In other words there exists N , possibly sample path dependent, such that $\hat{x}_n \in \mathcal{B}$ for all $n \geq N$. In other words,

$\sup_{n \geq N} \|\hat{x}_n\| \leq \sup_{x \in \mathcal{B}} \|x\| < \infty$. It follows from (A5) that $\sup_{n \geq N} \|x_n\| < \infty$. Almost surely, there exists N , possibly sample path dependent, such that $\hat{x}_n \in \mathcal{B}$ for all $n \geq N$. This directly leads to the stability of (3).

The iterates given by (3) satisfy assumptions (A1)-(A3). Further, we have shown above that they are also stable. It follows from *Theorem 3.6 and Lemma 3.8* of Benaïm [5] that the iterates converge to a closed connected internally chain transitive invariant set associated with $\dot{x}(t) \in H(x(t))$. \square

6 General noise sequences

The restriction (A3) on the noise sequence, is rather strict since it only allows for “bounded noise”. In this section, we show that Theorem 1 will continue to hold even when the noise sequence is more generally a square integrable Martingale difference sequence.

(A3a) $(M_n, \mathcal{F}_n)_{n \geq 1}$ is a square integrable Martingale difference sequence $\left(E[M_{n+1} | \mathcal{F}_n] = 0 \text{ and } EM_{n+1}^2 < \infty, n \geq 0 \right)$ such that

$$E[\|M_{n+1}\|^2 | \mathcal{F}_n] \leq K(1 + \|x_n\|^2), \text{ where } n \geq 0 \text{ and } K > 0.$$

In the above, $\mathcal{F}_0 := \sigma\langle x_0 \rangle$ and $\mathcal{F}_n := \sigma\langle x_0, x_1, \dots, x_n, M_1, \dots, M_n \rangle$ for $n \geq 1$. Further without loss of generality, we may assume that K in (A3a) and (A1) are equal (otherwise we can use maximum of the two constants).

In the analysis of the projective scheme given by (4), assumption (A3) is used in Lemma 2. Specifically, (A3)(i) is used to show that any two discontinuities of $\{X_n^l([0, T])\}_{n \geq 0}$ and $\{G_c^n([0, T])\}_{n \geq 0}$ are separated by at least $\Delta > 0$. We show that the aforementioned property holds when (A3) is replaced by (A3a). First, we prove an auxiliary result.

Lemma 5. *Let $\{t_{m(n)}, t_{l(n)}\}_{n \geq 0}$ be such that $t_{l(n)} > t_{m(n)}$, $t_{m(n+1)} > t_{l(n)}$ and $\lim_{n \rightarrow \infty} t_{l(n)} - t_{m(n)} = 0$. Fix an arbitrary $c > 0$ and consider the following:*

$$\psi_n := \left\| \sum_{i=m(n)}^{l(n)-1} a(i)M_{i+1} \right\|.$$

Then $P(\{\psi_n > c\} \text{ i.o.}) = 0$ within the context of the projective scheme given by (4).

Proof. We need to show that $\sum_{n \geq 0} P(\psi_n > c) < \infty$. It follows from Chebyshev's inequality that

$$P(\psi_n > c) \leq \frac{E\psi_n^2}{c^2} = \frac{E \left[\left\| \sum_{i=m(n)}^{l(n)-1} a(i)M_{i+1} \right\|^2 \right]}{c^2}.$$

Since $\{M_{n+1}\}_{n \geq 0}$ is a Martingale difference sequence, we get:

$$P(\psi_n > c) \leq \frac{\sum_{i=m(n)}^{l(n)-1} a(i)^2 E[\|M_{i+1}\|^2]}{c^2}. \quad (8)$$

Within the context of the projective scheme given by (4), almost surely $\forall n, x_n \in \bar{\mathcal{C}}$, i.e., $\sup_{n \geq 0} \|x_n\| \leq C_1 < \infty$ a.s. It follows from (A3a) that $E[\|M_{n+1}\|^2] \leq K(1 + E\|x_n\|^2)$. Hence, $E[\|M_{n+1}\|^2] \leq K(1 + C_1^2)$. Equation (8) becomes

$$P(\psi_n > c) \leq \frac{\sum_{i=m(n)}^{l(n)-1} a(i)^2 K(1 + C_1^2)}{c^2}.$$

Since $t_{l(n)} > t_{m(n)}$ and $t_{m(n+1)} > t_{l(n)}$, we have $\sum_{n \geq 0} \sum_{i=m(n)}^{l(n)-1} a(i)^2 \leq \sum_{n \geq 0} a(n)^2$.

Finally we get,

$$\sum_{n \geq 0} P(\psi_n > c) \leq \frac{\left(\sum_{n \geq 0} a(n)^2\right) K(1 + C_1^2)}{c^2} < \infty.$$

□

Let us consider the scenario in which we cannot find Δ , the separation between any two points of discontinuity. In other words, there exists $\{t(m(n)), t(l(n))\}_{n \geq 0}$ such that $t_{l(n)} > t_{m(n)}$, $t_{m(n+1)} > t_{l(n)}$ and $\lim_{n \rightarrow \infty} t_{l(n)} - t_{m(n)} = 0$. Without loss of generality we assume that there are no jumps between $t(m(n))$ and $t(l(n))$. Note that $X_l(t_{m(n)}^+) \in \partial\mathcal{B}$ and $X_l(t_{l(n)}^-) \notin \mathcal{C}$ for all $n \geq 0$. We have

$$X_l(t_{l(n)}^-) = X_l(t_{m(n)}) + \sum_{i=m(n)}^{l(n)-1} a(i) (y_i + M_{i+1}).$$

We have that $\sup_{n \geq 0} \|y_n\| \leq D'$ for some $0 < D' < \infty$. The above equation becomes

$$\begin{aligned} \|X_l(t_{l(n)}^-) - X_l(t_{m(n)})\| &\leq \sum_{i=m(n)}^{l(n)-1} a(i) D' + \left\| \sum_{i=m(n)}^{l(n)-1} a(i) M_{i+1} \right\|, \\ d \leq \|X_l(t_{l(n)}^-) - X_l(t_{m(n)})\| &\leq (t_{l(n)} - t_{m(n)}) D' + \left\| \sum_{i=m(n)}^{l(n)-1} a(i) M_{i+1} \right\|. \end{aligned}$$

Since $t_{l(n)} - t_{m(n)} \rightarrow 0$, for large n $\left\| \sum_{i=m(n)}^{l(n)-1} a(i) M_{i+1} \right\| > d/2$. This directly contradicts Lemma 5. Hence we can always find $\Delta > 0$ separating any two points of discontinuity.

In Lemma 4, (A3) is used to ensure the convergence of (4) to the attractor \mathcal{A} . In Theorem 1, (A3) is used to ensure the convergence of (3) to a closed connected internally chain transitive invariant set of the associated DI . Specifically, it is (A3)(ii) that ensures these convergences. Let us define $\zeta_n := \sum_{k=0}^{n-1} a(k)M_{k+1}$, $n \geq 1$. If $\{\zeta_n\}_{n \geq 1}$ converges, then it trivially follows that the Martingale noise sequence satisfies (A3)(ii). To show convergence, it is enough to show that the corresponding quadratic variation process converges almost surely. In other words, we need to show that $\sum_{n \geq 0} a(n)^2 E(\|M_{n+1}\|^2 | \mathcal{F}_n) < \infty$ a.s or $\sum_{n \geq 0} E(a(n)^2 \|M_{n+1}\|^2) < \infty$. Consider the following

$$\sum_{n \geq 0} a(n)^2 E\|M_{n+1}\|^2 = \sum_{n \geq 0} a(n)^2 E[E[\|M_{n+1}\|^2 | \mathcal{F}_n]] \leq \sum_{n \geq 0} a(n)^2 K(1 + E\|x_n\|^2). \quad (9)$$

Convergence of the quadratic variation process in the context of Lemma 4 follows from (9) and $E\|x_n\|^2 \leq \sup_{x \in \bar{\mathcal{C}}} \|x\|^2$. In other words,

$$\sum_{n \geq 0} a(n)^2 E[\|M_{n+1}\|^2] \leq \sum_{n \geq 0} a(n)^2 K(1 + \sup_{x \in \bar{\mathcal{C}}} \|x\|^2) < \infty.$$

Similarly, for convergence in Theorem 4, it follows from (9) and stability of the iterates ($\sup_{n \geq 0} \|x_n\| < \infty$ a.s.) that

$$\sum_{n \geq 0} a(n)^2 E[\|M_{n+1}\|^2 | \mathcal{F}_n] \leq \sum_{n \geq 0} a(n)^2 K(1 + \sup_{n \geq 0} \|x_n\|^2) < \infty \text{ a.s.}$$

In other words, both in Lemma 4 and Theorem 1, assumption (A3)(ii) is satisfied. The following generalized version of Theorem 1 is a direct consequence of the observations made above.

Theorem 2. *Under (A1), (A2), (A3a), (A4) and (A5), the iterates given by (3) are stable (bounded almost surely) and converge to a closed connected internally chain transitive invariant set associated with $\dot{x}(t) \in H(x(t))$.*

7 Application: Approximate value iteration methods

In this section we present an analysis of

$$J_{n+1} = ATJ_n,$$

where $J_n \in \mathbb{R}^d$ for all n , T is the Bellman operator, and A is the approximation operator. Specifically, we consider the following stochastic approximation counterpart of the above scheme:

$$J_{n+1} = J_n + a(n)[TJ_n - J_n + \epsilon_n + M_{n+1}], \text{ where} \quad (10)$$

- (i) T is the Bellman operator.
- (ii) $\{a(n)\}_{n \geq 0}$ is the step-size sequence, satisfying (A2).

- (iii) $\epsilon_n = ATJ_n - TJ_n$ is the approximation error at stage n .
- (iv) $\{M_{n+1}\}_{n \geq 0}$ is the Martingale difference noise sequence.

Let us call (10) as a stochastic iterative AVI. It is worth noting that we do not distinguish between stochastic shortest path and infinite horizon discounted cost problems. The definition of the Bellman operator T changes appropriately. We make the following assumptions:

- (AV1) The Bellman operator T is contractive with respect to some weighted max-norm, $\|\cdot\|_\nu$, i.e., $\|Tx - Ty\|_\nu \leq \alpha\|x - y\|_\nu$ for some $0 < \alpha < 1$.
- (AV2) T has a unique fixed point J^* and J^* is the unique globally asymptotically stable equilibrium point of $\dot{J}(t) = TJ(t) - J(t)$.
- (AV3) $\|\epsilon_n\|_\nu \leq \epsilon$ for some fixed $\epsilon > 0$.

Recall the definition of the weighted max-norm: Given $\nu = (\nu_1, \dots, \nu_d)$ such that $\nu_i > 0$ for $1 \leq i \leq d$, $\|x\|_\nu = \max \left\{ \frac{|x_i|}{\nu_i} \mid 1 \leq i \leq d \right\}$, where $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$. Later in this section, we analyze (10) where the approximation errors are bounded in the general weighted p-norm sense (weighted Euclidean norms). (AV1) is readily satisfied in many applications, see *Section 2.2* of Bertsekas and Tsitsiklis [6] for details. First, let us consider a couple of technical lemmas.

Lemma 6. $B^\epsilon := \{y \mid \|y\|_\nu \leq \epsilon\}$ is a convex compact subset of \mathbb{R}^d , where $\epsilon > 0$.

Proof. First we show that B^ϵ is convex. Given $y_1, y_2 \in B^\epsilon$ and $y = \lambda y_1 + (1 - \lambda)y_2$, where $\lambda \in (0, 1)$, we need show that $y \in B^\epsilon$. $\|y\|_\nu \leq \lambda\|y_1\|_\nu + (1 - \lambda)\|y_2\|_\nu = \lambda\epsilon + (1 - \lambda)\epsilon$.

Now we show that B^ϵ is compact. Define $\nu_{\max} := \max \{\nu_i \mid 1 \leq i \leq d\}$. Since $\|y\|_\infty \leq \nu_{\max}\|y\|_\nu$, it follows that B^ϵ is a bounded set. It is left to show that B^ϵ is closed. Let $y_n \rightarrow y$ and $y_n \in B^\epsilon$ for every n . Since $\liminf_{n \rightarrow \infty} \|y_n\|_\nu \geq \|y\|_\nu$, it follows that $y \in B^\epsilon$. \square

Lemma 7. The set-valued map \tilde{T} given by $\tilde{T}x \mapsto Tx + B^\epsilon$ is a Marchaud map.

Proof. Since B^ϵ is a compact convex set, it follows that $\tilde{T}x$ is compact and convex. Since T is a contraction map, we have:

$$\|Tx\|_\nu \leq \|T0\|_\nu + \|x - 0\|_\nu.$$

Let $\nu_{\min} := \min \{\nu_1, \nu_2, \dots, \nu_d\}$ and $\nu_{\max} := \max \{\nu_1, \nu_2, \dots, \nu_d\}$, then observe that $\|z\|_\nu \leq \|z\|/\nu_{\min}$, where $z \in \mathbb{R}^d$ and $\|\cdot\|$ is the standard Euclidean norm. Also, observe that

$$\|z\| = \sqrt{z_1^2 + \dots + z_d^2} \leq (d \nu_{\max}) \|z\|_\nu.$$

Hence we have

$$\begin{aligned} \|Tx\| &\leq (d \nu_{\max}) \|Tx\|_\nu \leq (d \nu_{\max}) (\|T0\|_\nu + \|x\|_\nu) \text{ and that} \\ \|Tx\| &\leq K'(1 + \|x\|), \end{aligned} \tag{11}$$

where $K' := \left(\frac{d \nu_{max}}{\nu_{min}} \right) \vee (d \nu_{max} \|T0\|_\nu)$. We have that $\sup_{z \in \tilde{T}x} \|z\| \leq \|Tx\| + \sup_{z \in B^\epsilon} \|z\|$. It follows from Lemma 6 that $K := K' \vee \sup_{z \in B^\epsilon} \|z\|$ is finite, hence $\sup_{z \in \tilde{T}x} \|z\| \leq K(1 + \|x\|)$.

We now show that \tilde{T} is upper semicontinuous. Let $x_n \rightarrow x$, $y_n \rightarrow y$ and $y_n \in \tilde{T}x_n$ for $n \geq 0$. Since T is continuous, we have that $Tx_n \rightarrow Tx$. Hence, $(y_n - Tx_n) \rightarrow (y - Tx)$ and $\epsilon \geq \liminf_{n \rightarrow \infty} \|y_n - Tx_n\|_\nu \geq \|y - Tx\|_\nu$. In other words, $y \in \tilde{T}x$. \square

Let us define H_ν , the Hausdorff metric with respect to the weighted max-norm as follows: Given $A, B \subseteq \mathbb{R}^d$, $H_\nu(A, B) := \max_{x \in A} d_\nu(x, B) \vee \max_{y \in B} d_\nu(y, A)$, where $d_\nu(x, B) := \min\{\|x - y\|_\nu \mid y \in B\}$ and $d_\nu(y, A) := \min\{\|x - y\|_\nu \mid x \in A\}$. Given $x, y \in \mathbb{R}^d$, there exist $x^* \in \tilde{T}x$ and $y^* \in \tilde{T}y$ such that $\|x^* - y^*\|_\nu = H_\nu(\tilde{T}x, \tilde{T}y)$. For any $x_0 \in \tilde{T}x$ and $y_0 \in \tilde{T}y$ we have

$$\|x_0 - y_0\|_\nu \leq \|x_0 - x^*\|_\nu + \|x^* - y^*\|_\nu + \|y^* - y_0\|_\nu = 2\epsilon + H(\tilde{T}x, \tilde{T}y). \quad (12)$$

For any $z \in \mathbb{R}^d$ we have

$$\nu_{min} \|z\|_\nu \leq \|z\| \leq (d \nu_{max}) \|z\|_\nu. \quad (13)$$

Consider the set of all equilibrium points of \tilde{T} : $\mathcal{A} := \{J \mid \|TJ - J\|_\nu \leq \epsilon\}$. For small values of ϵ , it follows from the above set of inequalities that \mathcal{A} belongs to a small neighborhood of J^* . Similarly for small values of ϵ , the mean-field \tilde{T} is a minor perturbation of T . Recall that J^* is the globally asymptotic stable equilibrium point of $\dot{J}(t) = TJ(t) - J(t)$. It follows from the upper semicontinuity of attractors that for small values of ϵ there exists \mathcal{A}' within a small neighborhood of J^* such that \mathcal{A}' is the global attractor of the perturbed system $\dot{J}(t) \in \tilde{T}J(t) - J(t)$. We will show that (10) converges to $\mathcal{A} \cap \mathcal{A}'$ and that $\mathcal{A} \cap \mathcal{A}' \neq \emptyset$.

We may construct a global Lyapunov function for the attractor \mathcal{A}' of $\dot{J}(t) \in \tilde{T}J(t) - J(t)$ as illustrated in Remark 2. In other words, (10) satisfies (A4c). Hence we can find sets \mathcal{B} and \mathcal{C} such that $\mathcal{A}' \subseteq \mathcal{B}$ and $\bar{\mathcal{B}} \subseteq \mathcal{C}$. Let us consider the following projective approximate value iteration:

$$\begin{aligned} J_{n+1} &= \hat{J}_n + a(n) \left(T\hat{J}_n - \hat{J}_n + \hat{\epsilon}_n + M_{n+1} \right), \\ \hat{J}_{n+1} &= \bigcap_{\mathcal{B}, \mathcal{C}} (J_{n+1}). \end{aligned} \quad (14)$$

It is worth noting that the noise sequences in (10) and (14) are identical and that $\hat{\epsilon}_n \leq \epsilon$ for all n . Following the analysis in Section 4, we conclude that $\hat{J}_n \rightarrow \mathcal{A}'$. We are now ready to analyze (10).

Theorem 3. *Under (AV1)-(AV3), (A2), and (A3a), (10) is stable and converges to some point in $\{J \mid \|TJ - J\|_\nu \leq \epsilon\}$, where ϵ is the norm-bound on the approximation errors.*

Proof. We start by showing that (10) satisfies assumption (A5). Earlier we showed that $\hat{J}_n \rightarrow \mathcal{A}'$. This implies that there exists N , possibly sample path dependent, such that $\hat{J}_n \in \bar{\mathcal{B}}$ for all $n \geq N$. For $k \geq 0$ and $n \geq N$,

$$\|J_{n+k+1} - \hat{J}_{n+k+1}\|_\nu \leq \|J_{n+k} - \hat{J}_{n+k} + a(n+k) \left((TJ_{n+k} + \epsilon_{n+k}) - (T\hat{J}_{n+k} + \hat{\epsilon}_{n+k}) - (J_{n+k} - \hat{J}_{n+k}) \right)\|_\nu.$$

Grouping terms of interest in the above inequality we get:

$$\|J_{n+k+1} - \hat{J}_{n+k+1}\|_\nu \leq (1-a(n+k))\|J_{n+k} - \hat{J}_{n+k}\|_\nu + a(n+k)\|(TJ_{n+k} + \epsilon_{n+k}) - (T\hat{J}_{n+k} + \hat{\epsilon}_{n+k})\|_\nu.$$

As a consequence of (12) the above equation becomes

$$\|J_{n+k+1} - \hat{J}_{n+k+1}\|_\nu \leq (1-a(n+k))\|J_{n+k} - \hat{J}_{n+k}\|_\nu + a(n+k) \left(2\epsilon + H_\nu(\tilde{T}J_{n+k}, \tilde{T}\hat{J}_{n+k}) \right). \quad (15)$$

We now consider the following two cases:

Case 1. $2\epsilon \leq (1-\alpha)\|J_{n+k} - \hat{J}_{n+k}\|_\nu$:

In this case (15) becomes

$$\|J_{n+k+1} - \hat{J}_{n+k+1}\|_\nu \leq (1-a(n+k))\|J_{n+k} - \hat{J}_{n+k}\|_\nu + a(n+k) \left((1-\alpha)\|J_{n+k} - \hat{J}_{n+k}\|_\nu + \alpha\|J_{n+k} - \hat{J}_{n+k}\|_\nu \right).$$

Simplifying the above equation, we get

$$\|J_{n+k+1} - \hat{J}_{n+k+1}\|_\nu \leq \|J_{n+k} - \hat{J}_{n+k}\|_\nu.$$

Case 2. $2\epsilon > (1-\alpha)\|J_{n+k} - \hat{J}_{n+k}\|_\nu$:

In this case (15) becomes

$$\|J_{n+k+1} - \hat{J}_{n+k+1}\|_\nu \leq (1-a(n+k))\frac{2\epsilon}{1-\alpha} + a(n+k) \left(2\epsilon + \alpha\frac{2\epsilon}{1-\alpha} \right).$$

Simplifying the above equation, we get

$$\|J_{n+k+1} - \hat{J}_{n+k+1}\|_\nu \leq \frac{2\epsilon}{1-\alpha}.$$

We may thus conclude the following:

$$\|J_{n+k+1} - \hat{J}_{n+k+1}\|_\nu \leq \|J_{n+k} - \hat{J}_{n+k}\|_\nu \vee \left(\frac{2\epsilon}{1-\alpha} \right).$$

Applying the above set of arguments to $\|J_{n+k} - \hat{J}_{n+k}\|_\nu$ and proceeding recursively to $\|J_N - \hat{J}_N\|_\nu$ we may conclude that for any $n \geq N$

$$\|J_n - \hat{J}_n\|_\nu \leq \|J_N - \hat{J}_N\|_\nu \vee \left(\frac{2\epsilon}{1-\alpha} \right). \quad (16)$$

In other words, we have $\sup_{n \geq 0} \|J_n - \hat{J}_n\|_\nu < \infty$ a.s. It follows from (13) that

$\sup_{n \geq 0} \|J_n - \hat{J}_n\| < \infty$ a.s. (in the Euclidean norm). The above arguments are inspired by Abounadi, Bertsekas and Borkar [1] and Jaakkola, Jordan and Singh [12].

We have shown that (10) satisfies (A1), (A2), (A3), (A4c) and (A5). It follows from Theorem 2 that the iterates given by (10) track a solution to $\dot{J}(t) \in \tilde{T}J(t) - J(t)$ and that $J_n \rightarrow \mathcal{S}$, where \mathcal{S} is a closed connected internally chain transitive invariant set of $\dot{J}(t) \in \tilde{T}J(t) - J(t)$. Since \mathcal{A}' is a global attractor of $\dot{J}(t) \in \tilde{T}J(t) - J(t)$, it follows that $\mathcal{S} \subseteq \mathcal{A}'$.

Now we show that $J_n \rightarrow \mathcal{A}$, the set of equilibrium points of $\tilde{T}J - J$, implying that $J_n \rightarrow \mathcal{A} \cap \mathcal{A}'$ and $\mathcal{A} \cap \mathcal{A}' \neq \emptyset$. Before proceeding further, we consider the following theorem from Aubin and Cellina [2]:

[Theorem 2, Chapter 6 [2]] *Let F be an upper semicontinuous map from a closed subset $K \subset X$ to X with compact convex values and $x(\cdot)$ be a solution trajectory of $\dot{x}(t) \in F(x(t))$ that converges to some x^* in K . Then x^* is an equilibrium of F .*

We have already shown that $\sup_{n \geq 0} \|J_n\| < \infty$ a.s. In other words, there exists a large compact convex set $\mathcal{K} \subseteq \mathbb{R}^d$, possibly sample path dependent, such that $J_n \in \mathcal{K}$ for all $n \geq 0$. Further, \mathcal{K} can be chosen such that the “tracking solution” of $\dot{x}(t) \in \tilde{T}J(t) - J(t)$ is also inside \mathcal{K} , asymptotically. It follows that the conditions of the above stated theorem are satisfied, hence every limit point of (10) is an equilibrium point of $\tilde{T}J - J$. In other words, $J_n \rightarrow \{J \mid \|TJ - J\|_\nu \leq \epsilon\}$. \square

Remark 3. *As the bound on the approximation errors decreases, the size of the limiting set of the corresponding approximate value iteration given by (10) also decreases. Specifically, as $\epsilon \downarrow 0$, $\{J \mid \|TJ - J\|_\nu \leq \epsilon\} \downarrow J^*$.*

7.1 Approximation errors bounded in the weighted p-norm sense

The approximation errors encountered hitherto in this section are bounded in the weighted max-norm sense. As stated earlier, these errors are often a consequence of approximation operators that are used to counter *Bellman’s curse of dimensionality*. In a model-free setting, typically one is given data of the form (x_n, v_n) , where v_n s are unbiased estimates of the objective function at x_n ’s. Note that the role of an approximation operator may be played by a supervised learning algorithm. This algorithm would return a good fit g from within a class of functions \mathcal{T} . The objective for these algorithms would be to minimize the approximation errors. Previously, we considered approximation operators that minimize errors in the weighted max-norm sense. *This is relevant to large-scale applications where it may not be possible to approximate over all the states uniformly.*

In many applications the approximation operators work by minimizing the errors in the ℓ_1 and ℓ_2 norms, see Munos [13] for details. In this section, we consider the general case of approximation errors being bounded in the weighted p-norm sense. Specifically, we analyze (10) with $\|\epsilon_n\|_{\omega,p} \leq \epsilon$ for some fixed $\epsilon > 0$. Recall the definition of the weighted p-norm of a given $z \in \mathbb{R}^d$:

$$\|z\|_{\omega,p} = \left(\sum_{i=1}^d |\omega_i z_i|^p \right)^{1/p},$$

where $\omega = (\omega_1, \dots, \omega_d)$ is such that $\omega_i > 0$, $1 \leq i \leq d$ and $p \geq 1$.

Recall that the Bellman operator T is contractive with respect to $\|\cdot\|_\nu$. First, we establish the relationship between the weighted p-norm and the weighted max-norm. Again, fix $z \in \mathbb{R}^d$. We have that $|z_i| \leq \nu_{\max} \|z\|_\nu$, where ν_{\max} is defined in Lemma 6. Hence $\|z\|_{\omega,p} \leq (d \omega_{\max} \nu_{\max}) \|z\|_\nu$, where $\omega_{\max} := \max \{\omega_i \mid 1 \leq i \leq d\}$. Similarly, $\|z\|_\nu \leq \frac{\|z\|_{\omega,p}}{\nu_{\min}}$. We have the following inequality:

$$\nu_{\min} \|z\|_\nu \leq \|z\|_{\omega,p} \leq (d \omega_{\max} \nu_{\max}) \|z\|_\nu. \quad (17)$$

In this section we consider the following stochastic iterative AVI scheme:

$$J_{n+1} = J_n + a(n) (TJ_n - J_n + \tilde{\epsilon}_n + M_{n+1}) \quad (18)$$

where $\|\tilde{\epsilon}_n\|_{\omega,p} \leq \epsilon$ for all $n \geq 0$. If we define $\bar{T}J := TJ + \{x \mid \|x\|_{\omega,p} \leq \epsilon\}$, as in the previous subsection, we can show that \bar{T} is a Marchaud map. Further, we may state an identical theorem for (18).

Theorem 4. *Under (AV1)-(AV3), (A2), and (A3a), (18) is stable and converges to some point in $\{J \mid \|TJ - J\|_{\omega,p} \leq \epsilon\}$, where ϵ is the norm-bound on the approximation errors.*

Proof. As in the proof of Theorem 3, we need to show that (18) satisfies (A5). It follows from (17) that

$$\|\epsilon_n\|_{\omega,p} \leq \epsilon \implies \|\epsilon_n\|_\nu \leq \epsilon / \nu_{\min}.$$

We are now in the setting of Theorem 3, and it follows that (18) satisfies (A5). With regards to convergence, using similar arguments, we can show that (18) converges to the set of equilibrium points of $\bar{T}J - J$ given by $\{J \mid \|\bar{T}J - J\|_{\omega,p} \leq \epsilon\}$. As in the previous subsection, $\{J \mid \|\bar{T}J - J\|_{\omega,p} \leq \epsilon\} \downarrow J^*$ as $\epsilon \downarrow 0$. \square

7.2 Comparison with previous literature

An important contribution in understanding the convergence of approximate value iteration (AVI) methods has been due to Munos in 2005 [13]. This paper analyzed AVI methods (for the infinite horizon discounted case problem) wherein the approximation errors are bounded in the weighted p-norm sense, a significant improvement over [6] that only considered max norms. However, in [13], the basic procedure considered is a numerical AVI scheme where complete knowledge of the ‘system model’, i.e., the transition probabilities is assumed. In addition, in [13], convergence is shown under one of the following two assumptions on transition probabilities. Let μ be a distribution over the state space.

(MA1) $\exists C > 0$ such that for all states x, y and policy π , we have $P^\pi(x, y) \leq C\mu(y)$.

(MA2) \exists a distribution ρ and co-efficients $c(m) \forall m \geq 1$ and policies π_1, \dots, π_m such that $\rho P^{\pi_1} \dots P^{\pi_m} \leq c(m)\mu$. The smoothness constant C of the discounted future state distribution is given by $C := (1-\gamma)^2 \sum_{m \geq 1} m\gamma^{m-1}c(m)$,

where γ is the discount factor.

These smoothness requirements on the transition probabilities are strong and will not hold for a large class of systems.

On the contrary, we consider a stochastic approximation counterpart of AVI that involves an ‘operational noise’ component in addition, that we model as a general martingale difference noise sequence. An algorithm such as (10) arises for instance in the Q-learning procedure (where the Bellman operator T is in fact the Q-Bellman operator) or when the same is a Bellman operator corresponding to a given stationary policy, see Chapter 5 of [6]. In such a model, no information is assumed known about the system transition probabilities, i.e., the setting in such a case is ‘model-free’. Thus our convergence analysis works in the case of AVI schemes for which (a) information on the transition probabilities is not known, (b) no such restrictions as (MA1) or (MA2) are imposed on the transition probabilities, and (c) there is a measurement error (albeit bounded) that may arise for instance from the use of function approximation. We require that J^* be the unique globally asymptotically stable equilibrium of $\dot{J}(t) = TJ(t) - J(t)$, where T is the Bellman operator that is a contraction map. Since J^* is the unique solution to the equation $TJ = J$, it is natural to expect that the aforementioned requirement holds true.

It is also important to note that our analysis works for both stochastic shortest path and infinite horizon discounted cost problems.

Value iteration is an important reinforcement learning algorithm. As stated in Section 1, in the case of large-scale problems, AVI methods are used to obtain suboptimal solutions (arising from say the use of function approximation techniques). Showing that the stochastic approximation counterpart is bounded almost surely can be hard in many reinforcement learning applications. Thus, one of the most significant contributions of this paper, not addressed in previous literature, lies in the development of easily verifiable sufficient conditions for the almost sure boundedness of AVI methods involving set-valued dynamics.

8 Application: Finding fixed points of set-valued maps

In Section 7 we showed that the stochastic iterative AVI given by (10) converges to a vector that belongs to a small neighborhood of the optimal cost-to-go vector J^* . To do this we started by observing that the fixed points of the perturbed Bellman operator belong to a small neighborhood of J^* as a consequence of the upper semicontinuity of attractor sets. Then we showed that (10) converges to a fixed point of the perturbed Bellman operator, thereby showing that (10) converges to a small neighborhood of J^* . In this section, we generalize the ideas of Section 7 to develop and analyze a SAA for finding fixed points of contractive set-valued maps.

Suppose that we are given a set-valued map $T : \mathbb{R}^d \rightarrow \{\text{subsets of } \mathbb{R}^d\}$. We present sufficient conditions under which the following stochastic approximation algorithm is bounded a.s. and converges to a fixed point of T :

$$x_{n+1} = x_n + a(n) [y_n + M_{n+1}], \quad (19)$$

where

- (i) $y_n \in Tx_n - x_n$ for all $n \geq 0$.
- (ii) $\{a(n)\}_{n \geq 0}$ is the given step-size sequence satisfying (A2).
- (iii) $\{M_{n+1}\}_{n \geq 0}$ is the Martingale difference noise sequence satisfying (A3a).

Definitions: Given a metric space (\mathbb{R}^d, ρ) , we define the Hausdorff metric with respect to ρ as follows:

$$H_\rho(A, B) := \left(\min_{x \in A} \rho(x, B) \right) \vee \left(\min_{y \in B} \rho(y, A) \right),$$

where $A, B \subset \mathbb{R}^d$ and $\rho(u, C) := \min\{\rho(u, v) \mid v \in C\}$ for any $u \in \mathbb{R}^d$ and $C \subseteq \mathbb{R}^d$. We call a set-valued map T as contractive if and only if $H_\rho(Tx, Ty) \leq \alpha \rho(x, y)$, where $x, y \in \mathbb{R}^d$ and $0 < \alpha < 1$. We say that T is of *bounded diameter* if and only if $\text{diam}(Tx) \leq D$, where $x \in \mathbb{R}^d$ and $0 < D < \infty$. We define $\text{diam}(A) := \sup\{\rho(z_1, z_2) \mid z_1, z_2 \in A\}$ for any $A \subset \mathbb{R}^d$. We impose the following restrictions on (19):

- (AF1) T is a Marchaud map that is of *bounded diameter* and *contractive* with respect to some metric ρ .
- (AF2) The metric ρ is such that $\|x - y\| \leq C \rho(x, y)$ for $x, y \in \mathbb{R}^d$, $C > 0$.
- (AF3) Let $F := \{x \mid x \in Tx\}$ denote the set of fixed points of T . There exists a compact subset $F' \subseteq F$ along with a strongly positive invariant bounded open neighborhood.

OR

F is the unique global attractor of $\dot{x}(t) \in Tx(t) - x(t)$.

Since T is assumed to be contractive with respect to ρ , it follows from *Theorem 5* of Nadler [14] that T has at least one fixed point. Assumption (AF2) is readily satisfied by the popular metric norms such as the weighted p-norms and the weighted max-norms among others. Assumption (AF3) is imposed to ensure that (19) satisfies (A4b) or (A4c). Specifically, (AF3) is imposed to ensure the existence of an inward directing set associated with $\dot{x}(t) \in Tx(t) - x(t)$, see Proposition 2 for details. In other words, we can find bounded open sets \mathcal{C}_F and \mathcal{B}_F such that \mathcal{C}_F is inward directing and $\overline{\mathcal{B}_F} \subset \mathcal{C}_F$.

As in Section 7, we compare (19) with it's projective counterpart given by:

$$\begin{aligned} \tilde{x}_{n+1} &= \hat{x}_n + a(n)(y_n + M_{n+1}), \\ \hat{x}_{n+1} &\in \bigcap_{\mathcal{B}_F, \mathcal{C}_F}(\tilde{x}_{n+1}), \end{aligned} \tag{20}$$

where $y_n \in T\hat{x}_n - \hat{x}_n$, $\{M_{n+1}\}_{n \geq 0}$ is identical for both (19) and (20) and $\bigcap_{\mathcal{B}_F, \mathcal{C}_F}(\cdot)$ is the projection operator defined at the beginning of Section 4. The analysis of the above projective scheme proceeds in an identical manner as in Section 4. Specifically, we may show that every limit point of the projective scheme (20) belongs to $\overline{\mathcal{B}_F}$. The following theorem is immediate.

Theorem 5. *Under (AF1)-(AF3) and (A3a), the iterates given by (19) are bounded almost surely. Further, any limit point of (19) (as $n \rightarrow \infty$) is a fixed point of the set-valued map T .*

Proof. The proof of this theorem proceeds in a similar manner to that of Theorem 3. We only provide an outline here to avoid repetition. We begin by showing that (19) is bounded almost surely (stable) by comparing it to (20). Since the limit points of (20) belong to $\overline{\mathcal{B}_F}$, there exists N , possibly sample path dependent, such that $\hat{x}_n \in \overline{\mathcal{C}_F}$ for all $n \geq N$. For $k \geq 0$ we have the following inequality:

$$\rho(x_{n+k+1}, \hat{x}_{n+k+1}) \leq (1-a(n+k))\rho(x_{n+k}, \hat{x}_{n+k}) + a(n+k)(2D + H_\rho(Tx_{n+k}, T\hat{x}_{n+k})),$$

where $\text{diam}(Tx) \leq D$ for every $x \in \mathbb{R}^d$. Recall that $0 < \alpha < 1$ is the contraction parameter of the set-valued map T . We consider two possible cases.

Case 1. $2D \leq (1 - \alpha)\rho(x_{n+k}, \hat{x}_{n+k})$: In this case, it can be shown that

$$\rho(x_{n+k+1}, \hat{x}_{n+k+1}) \leq \rho(x_{n+k}, \hat{x}_{n+k}).$$

Case 2. $2D > (1 - \alpha)\rho(x_{n+k}, \hat{x}_{n+k})$: In this case, it can be shown that

$$\rho(x_{n+k+1}, \hat{x}_{n+k+1}) \leq \frac{2D}{1 - \alpha}.$$

We conclude the following:

$$\rho(x_n, \hat{x}_n) \leq \left(\frac{2D}{1 - \alpha} \right) \vee \rho(x_N, \hat{x}_N), \quad n \geq N.$$

It follows from the above inequality and (AF2) that (19) satisfies assumption (A5). Hence, we get that $\{x_n\}_{n \geq 0}$ is bounded almost surely (stable).

Since the iterates are stable, it follows from [Theorem 2, Chapter 6, [2]] that every limit point of (19) is an equilibrium point of the set-valued map $x \mapsto Tx - x$. In other words, if x^* is a limit point of (19), then $0 \in Tx^* - x^*$, i.e., $x^* \in Tx^*$. Hence we have shown that every limit point of (19) is a fixed point of the set-valued map T . \square

Remark 4. It is assumed that T is of bounded diameter, see (AF1). The primary task of this assumption is in showing the almost sure boundedness of (19). Specifically, it is used to show that (A5) is satisfied. Depending on the problem at hand, one may wish to do away with this “bounded diameter” assumption. For example, we may have $\sup_{n \geq 0} \text{diam}(Tx_n) < \infty$ a.s., then the bounded diameter assumption can be dispensed with.

Since T is Marchaud, it is point-wise bounded, i.e., $\sup_{z \in Tx} \|z\| \leq K(1 + \|x\|)$, where $K > 0$. In other words, $\text{diam}(Tx) \leq 2K(1 + \|x\|)$. In theory, the point-wise boundedness of T allows for unbounded diameters, i.e., $\text{diam}(Tx) \uparrow \infty$ as $\|x\| \uparrow \infty$. Our bounded diameter assumption prevents this scenario from happening. In applications that use “approximate operators”, it is often reasonable to assume that the errors (due to approximations) are bounded. Then the “associated set-valued map” is naturally of bounded diameter. The reader is referred to Section 7 for an example of this setting.

9 A note on assumption (A5)

In this section, we investigate the following question: Under what conditions is the stability of a SAA guaranteed, provided it is known a priori that the

corresponding projective scheme is convergent. In *Lemma 2.1* of Abounadi et. al. [1] this question is answered for simple iterative schemes. Here, we make an investigation for the setting of SAAs with set-valued mean-fields. We prove the following lemma.

Lemma 8. *Let \mathcal{B} and \mathcal{C} be open bounded subsets of \mathbb{R}^d such that $\overline{\mathcal{B}} \subset \mathcal{C}$. Consider the algorithm*

$$x_{n+1} \in G_n(x_n, \xi_n), \quad n \geq 0.$$

We make the following assumptions:

1. $\{\xi_n\}_{n \geq 0}$ is a random sequence that constitutes noise.
2. G_n is of bounded diameter and contractive (in the first co-ordinate with second co-ordinate fixed) with respect to some metric ρ , for $n \geq 0$. In other words, $H_\rho(G_n(x, \xi), G_n(y, \xi)) \leq \alpha \rho(x, y)$ for some $0 < \alpha < 1$ and $\sup_{u, v \in G_n(x, \xi)} \rho(u, v) \leq D$, where $0 < D < \infty$ and $x \in \mathbb{R}^d$.
3. There exists $C > 0$ such that $\|x - y\| \leq C \rho(x, y)$.
4. The sequence $\{\tilde{x}_n\}_{n \geq 0}$ generated by

$$\tilde{x}_{n+1} \in G_n\left(\bigcap_{\mathcal{B}, \mathcal{C}}(\tilde{x}_n), \xi_n\right)$$

converges to some vector $x^ \in \mathcal{B}$.*

Then $\{x_n\}_{n \geq 0}$ is bounded almost surely.

Proof. Since $\tilde{x}_n \rightarrow x^*$ as $n \rightarrow \infty$, there exists N such that $\tilde{x}_n \in \mathcal{B}$ for all $n \geq N$. For $k \geq 0$ we have the following:

$$\rho(x_{n+k+1}, \tilde{x}_{n+k+1}) \leq 2D + \rho(x_{n+k}, \tilde{x}_{n+k}).$$

The details of the above inequality are identical to the inequality given by (12). Unfolding the right hand side down to stage n we get the following:

$$\rho(x_{n+k+1}, \tilde{x}_{n+k+1}) \leq (1 + \alpha + \dots + \alpha^k) 2D + \rho(x_n, \tilde{x}_n),$$

$$\rho(x_{n+k+1}, \tilde{x}_{n+k+1}) \leq \frac{2D}{1 - \alpha} + \rho(x_n, \tilde{x}_n).$$

In other words, we have

$$\sup_{n \geq N+1} \rho(x_n, \tilde{x}_n) \leq \frac{2D}{1 - \alpha} + \rho(x_N, \tilde{x}_N).$$

Hence we have that $\{x_n\}_{n \geq 0}$ is bounded almost surely. \square

10 Conclusions

In this paper we presented a Lyapunov function based stability criterion for SAAs with set-valued mean-fields. Specifically three different yet overlapping sets of sufficient conditions were presented for stability and convergence of SAAs with set-valued mean-fields. As an important application of our framework, we show that AVI methods can be analyzed under significantly relaxed set of assumptions than previous literature. Specifically, we showed that the stochastic iterative AVI is bounded almost surely and converges to a vector belonging to a small neighborhood of the optimal J^* . Further, we showed that this neighborhood depends on the errors made by the approximation operators. Finally, we developed and analyzed a SAA for finding fixed points of contractive set-valued maps. To the best of our knowledge ours is the first algorithm for finding fixed points of set-valued maps.

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