Analysis of Stress - Continued.

Advanced Mechanics of Solids ME202

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Outline

Stress transformation deals with determination of the different components of stress under a rotation of coordinate axes.

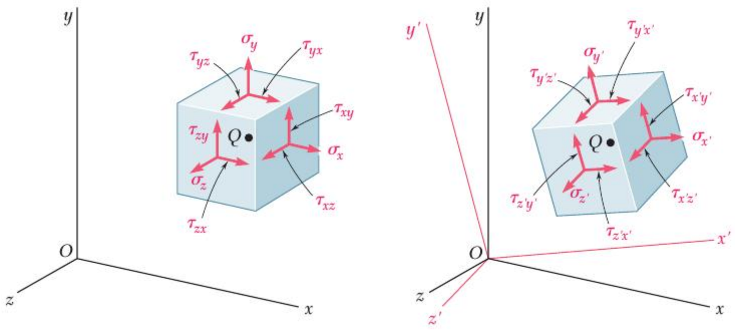

Figure: General state of stress at a point.

Stress Transformation

Definition: The matrix form of the stress tensor is different for different coordinate systems. However, the matrix of one coordinate system is related to the matrix of another coordinate system. The process of converting the stress matrix of one coordinate system to another coordinate system is called **stress transformation**

Stress Transformation

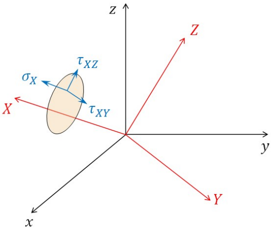

Figure: Stress components on plane perpendicular to transformed X-axis.

Table: Direction Cosines

	x	y	z
X	$l_x = \cos\theta_{xX}$	$l_y = \cos\theta_{yX}$	$l_z = \cos\theta_{zX}$
Y	$m_x = \cos\theta_{xY}$	$m_y = \cos\theta_{yY}$	$m_z = \cos\theta_{zY}$
Z	$n_x = \cos\theta_{xZ}$	$n_y = \cos\theta_{yZ}$	$n_z = \cos\theta_{zZ}$

Cauchy's stress formula

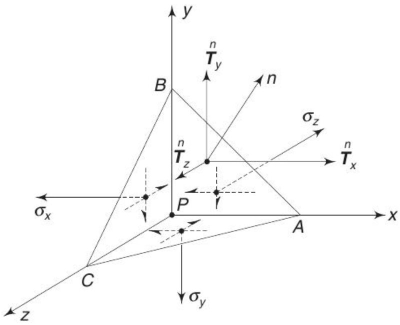

Figure: Tetrahedron at point P

For a plane with direction cosines n_x, n_y and n_z,

$$\overset{n}{\mathbf{T}} = \overset{n}{T}_x \mathbf{i} + \overset{n}{T}_y \mathbf{j} + \overset{n}{T}_z \mathbf{k}$$

$$\overset{n}{T}_x = \sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z$$

$$\overset{n}{T}_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z$$

$$\overset{n}{T}_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$

Stress Vector on X-plane

Hence, the stress vector on a plane perpendicular to X-axis (Conveniently referred as X-plane), having direction cosines l_x, l_y and l_z is given by,

$$\overset{X}{\mathbf{T}} = \overset{X}{T}_x\mathbf{i} + \overset{X}{T}_y\mathbf{j} + \overset{X}{T}_z\mathbf{k}$$

where,

$$\overset{X}{T}_x = \sigma_x l_x + \tau_{yx} l_y + \tau_{zx} l_z$$

$$\overset{X}{T}_y = \tau_{xy} l_x + \sigma_y l_y + \tau_{zy} l_z$$

$$\overset{X}{T}_z = \tau_{xz} l_x + \tau_{yz} l_y + \sigma_z l_z$$

Stress component Normal to X-Plane

$$\sigma_{XX} = \overset{X}{\mathbf{T}} \bullet (l_x\mathbf{i} + l_y\mathbf{j} + l_z\mathbf{k})$$

$$\sigma_{XX} = (\overset{X}{T}_x\mathbf{i} + \overset{X}{T}_y\mathbf{j} + \overset{X}{T}_z\mathbf{k}) \bullet (l_x\mathbf{i} + l_y\mathbf{j} + l_z\mathbf{k})$$

$$\sigma_{XX} = \overset{X}{T}_x l_x + \overset{X}{T}_y l_y + \overset{X}{T}_z l_z$$

$$\sigma_{XX} = (\sigma_x l_x + \tau_{yx} l_y + \tau_{zx} l_z) l_x + (\tau_{xy} l_x + \sigma_y l_y + \tau_{zy} l_z) l_y + (\tau_{xz} l_x + \tau_{yz} l_y + \sigma_z l_z) l_z$$

After rearranging, we get

$$\sigma_{XX} = l_x{}^2\sigma_x + l_y{}^2\sigma_y + l_z{}^2\sigma_z + 2l_x l_y \tau_{xy} + 2l_y l_z \tau_{yz} + 2l_z l_x \tau_{zx}$$

Stress component Normal to Y and Z-Planes

For Y-Plane

$$\sigma_{YY} = \overset{Y}{\mathbf{T}} \bullet (m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k})$$

$$\sigma_{YY} = (\overset{Y}{T}_x\mathbf{i} + \overset{Y}{T}_y\mathbf{j} + \overset{Y}{T}_z\mathbf{k}) \bullet (m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k})$$

$$\sigma_{YY} = {m_x}^2\sigma_x + {m_y}^2\sigma_y + {m_z}^2\sigma_z + 2m_x m_y\tau_{xy} + 2m_y m_z\tau_{yz} + 2m_z m_x\tau_{zx}$$

For Z-Plane

$$\sigma_{ZZ} = \overset{Z}{\mathbf{T}} \bullet (n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k})$$

$$\sigma_{ZZ} = (\overset{Z}{T}_x\mathbf{i} + \overset{Z}{T}_y\mathbf{j} + \overset{Z}{T}_z\mathbf{k}) \bullet (n_x\mathbf{i} + n_y\mathbf{j} + n_z\mathbf{k})$$

$$\sigma_{ZZ} = {n_x}^2\sigma_x + {n_y}^2\sigma_y + {n_z}^2\sigma_z + 2n_x n_y\tau_{xy} + 2n_y n_z\tau_{yz} + 2n_z n_x\tau_{zx}$$

Shear Stress Components on X-Plane

The shear stress component τ_{XY} is the component of the stress vector on a plane perpendicular to the X axis in the Y direction

$$\tau_{XY} = \overset{X}{\mathbf{T}} \bullet (m_x\mathbf{i} + m_y\mathbf{j} + m_z\mathbf{k})$$

$$\tau_{XY} = \overset{X}{T}_x m_x + \overset{X}{T}_y m_y + \overset{X}{T}_z m_z$$

$$\begin{aligned}\tau_{XY} = (\sigma_x l_x + \tau_{yx} l_y + \tau_{zx} l_z) m_x + (\tau_{xy} l_x + \sigma_y l_y + \tau_{zy} l_z) m_y \\ + (\tau_{xz} l_x + \tau_{yz} l_y + \sigma_z l_z) m_z\end{aligned}$$

$$\begin{aligned}\tau_{XY} = l_x m_x \sigma_x + l_y m_y \sigma_y + l_z m_z \sigma_z + (l_x m_y + l_y m_x)\tau_{xy} + (l_y m_z + l_z m_y)\tau_{yz} \\ + (l_z m_x + l_x m_z)\tau_{zx}\end{aligned}$$

Shear Stress Components

$$\tau_{YZ} = m_x n_x \sigma_x + m_y n_y \sigma_y + m_z n_z \sigma_z + (m_x n_y + m_y n_x)\tau_{xy} + (m_y n_z + m_z n_y)\tau_{yz} + (m_z n_x + m_x n_z)\tau_{zx}$$

$$\tau_{ZX} = n_x l_x \sigma_x + n_y l_y \sigma_y + n_z l_z \sigma_z + (n_x l_y + n_y l_x)\tau_{xy} + (n_y l_z + n_z l_y)\tau_{yz} + (n_z l_x + n_x l_z)\tau_{zx}$$

Transformation Equation in Matrix Form

Written in matrix form, the transformation equation is :

$$[\sigma]_{XYZ} = [T][\sigma]_{xyz}[T]^T$$

where, $[T]$ is the transformation matrix and $[T]^T$ is the transpose of the transformation matrix given by:

$$[T] = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix}$$

and

$$[T]^T = \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

Transformation Equation in Matrix Form

The transformation equation is :

$$[\sigma]_{XYZ} = [T][\sigma]_{xyz}[T]^T$$

$$\begin{bmatrix} \sigma_X & \tau_{XY} & \tau_{XZ} \\ \tau_{YX} & \sigma_Y & \tau_{YZ} \\ \tau_{ZX} & \tau_{ZY} & \sigma_Z \end{bmatrix} = \begin{bmatrix} l_x & l_y & l_z \\ m_x & m_y & m_z \\ n_x & n_y & n_z \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} l_x & m_x & n_x \\ l_y & m_y & n_y \\ l_z & m_z & n_z \end{bmatrix}$$

Example of transformation in x-y plane.

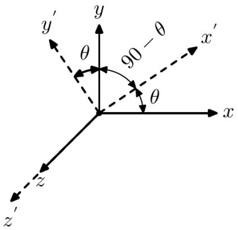

Figure: Transformation in x-y plane

The z and z' axes are same and $\theta_{x'x} = \theta, \theta_{x'y} = 90 - \theta, \theta_{x'z} = 90$; $\theta_{y'x} = 90 + \theta, \theta_{y'y} = \theta, \theta_{y'z} = 90$; $\theta_{z'x} = 90, \theta_{z'y} = 90, \theta_{z'z} = 0$;

Example of transformation in x-y plane.

Transformation matrix for this case can be obtained as:

$$[T] = \begin{bmatrix} \cos\theta & \cos(90-\theta) & \cos 90 \\ \cos(90+\theta) & \cos\theta & \cos 90 \\ \cos 90 & \cos 90 & \cos 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The matrix obtained above is applicable for transformation from Cartesian to polar coordinates.

$$\begin{bmatrix} \sigma_r & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \sigma_\theta & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \sigma_z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example-1

The resisting traction vectors on the Cartesian coordinate planes passing through a point are,

$$\overset{\mathbf{i}}{\mathbf{T}} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}; \quad \overset{\mathbf{j}}{\mathbf{T}} = 2\mathbf{i} - \mathbf{k}; \quad \overset{\mathbf{k}}{\mathbf{T}} = -2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

The unit of the traction is kPa. Then,
(a) write down the matrix of stress tensor in the Cartesian coordinate system,
(b) evaluate the matrix of a new coordinate system obtained by rotating the Cartesian coordinate system through an angle 30^0 in the anti-clockwise direction.

Solution:

The matrix of the stress tensor is obtained by writing the given traction vectors in the rows of the matrix as

$$[\sigma_{ij}] = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$

The transformation matrix

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos 30 & \sin 30 & 0 \\ -\sin 30 & \cos 30 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & 0.500 & 0 \\ -0.500 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example-1

Solution:

Stress tensor for the new coordinate system is given by

$$[\sigma_{ij}]_{new} = [T][\sigma_{ij}][T]^T$$

$$[\sigma_{ij}]_{new} = \begin{bmatrix} 0.866 & 0.500 & 0 \\ -0.500 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & -2 \\ 2 & 0 & -1 \\ -2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0.866 & -0.500 & 0 \\ 0.500 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.982 & -0.299 & -2.232 \\ -0.299 & -0.982 & 0.134 \\ -2.232 & 0.134 & 2 \end{bmatrix}$$

Principal Stresses and Principal planes

From the failure considerations of materials, following questions are important.

1. Are there any planes passing through the given point on which the resultant stresses are wholly normal (in other words, the resultant stress vector is along the normal)?
2. What is the plane on which the normal stress is a maximum and what is its magnitude?
3. What is the plane on which the tangential or shear stress is a maximum and what it is its magnitude?

Assume there is a plane **n** with direction cosines n_x, n_y and n_z on which the stress is fully normal.
Let σ be the magnitude of stress vector
Then, we have

$$\overset{n}{\mathbf{T}} = \sigma \mathbf{n}$$

$$\overset{n}{T}_x = \sigma n_x, \overset{n}{T}_y = \sigma n_y, \overset{n}{T}_z = \sigma n_z$$

Using Cauchy's formula,

$$\sigma_x n_x + \tau_{yx} n_y + \tau_{zx} n_z = \sigma n_x$$

$$\tau_{xy} n_x + \sigma_y n_y + \tau_{zy} n_z = \sigma n_y$$

$$\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = \sigma n_z$$

Principal Stresses and Principal planes

$$(\sigma_x - \sigma)n_x + \tau_{yx}n_y + \tau_{zx}n_z = 0$$

$$\tau_{xy}n_x + (\sigma_y - \sigma)n_y + \tau_{zy}n_z = 0$$

$$\tau_{xz}n_x + \tau_{yz}n_y + (\sigma_z - \sigma)n_z = 0$$

The above equations constitute a system of linear homogeneous simultaneous equations.
In order to have a non-trivial solution,(other than $n_x = n_y = n_z = 0$)

$$\begin{vmatrix} (\sigma_x - \sigma) & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & (\sigma_y - \sigma) & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & (\sigma_z - \sigma) \end{vmatrix} = 0$$

Principal Stresses and Principal planes

On expanding the Determinant, we get a cubic equation in σ as

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + ({\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}}^2 - {\tau_{yz}}^2 - {\tau_{zx}}^2)\sigma - (\sigma_x\sigma_y\sigma_z - \sigma_x{\tau_{yz}}^2 - \sigma_y{\tau_{zx}}^2 - \sigma_z{\tau_{xy}}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}) = 0$$

Or

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

Where,
$I_1 = (\sigma_x + \sigma_y + \sigma_z)$, $I_2 = (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - {\tau_{xy}}^2 - {\tau_{yz}}^2 - {\tau_{zx}}^2)$,
$I_3 = (\sigma_x\sigma_y\sigma_z - \sigma_x{\tau_{yz}}^2 - \sigma_y{\tau_{zx}}^2 - \sigma_z{\tau_{xy}}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx})$ are called **Stress Invariants** as their values don't change during a co-ordinate transformation.

Stress invariants can be calculated by

$$I_1 = (\sigma_x + \sigma_y + \sigma_z)$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{yz} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_x & \tau_{xz} \\ \tau_{xz} & \sigma_z \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{vmatrix}$$

Principal Stresses and Principal planes

- The cubic equation $\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$ has 3 real roots
- Each of this roots can be substituted to

$$(\sigma_x - \sigma)n_x + \tau_{yx}n_y + \tau_{zx}n_z = 0$$

$$\tau_{xy}n_x + (\sigma_y - \sigma)n_y + \tau_{zy}n_z = 0$$

$$\tau_{xz}n_x + \tau_{yz}n_y + (\sigma_z - \sigma)n_z = 0$$

  to get corresponding values of n_x, n_y and n_z
- In order to avoid trivial solution, the condition ${n_x}^2 + {n_y}^2 + {n_z}^2 = 1$ is used with any two of the above equations to obtain n_x, n_y and n_z

Stress Invariants

For any arbitrary co-ordinate system O_{xyz}

- $I_1 = (\sigma_x + \sigma_y + \sigma_z)$,
- $I_2 = (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - {\tau_{xy}}^2 - {\tau_{yz}}^2 - {\tau_{zx}}^2)$
- $I_3 = (\sigma_x\sigma_y\sigma_z - \sigma_x{\tau_{yz}}^2 - \sigma_y{\tau_{zx}}^2 - \sigma_z{\tau_{xy}}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx})$

For a coordinate system coinciding with principal axes

- $I_1 = (\sigma_1 + \sigma_2 + \sigma_3)$,
- $I_2 = (\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$
- $I_3 = (\sigma_1\sigma_2\sigma_3)$

Notes on Principal Planes

1. Principal planes are planes on which the resultant stress is normal.
2. The shear stress on a principal plane is zero.
3. Principal planes are planes on which the normal stress has an extreme value.
4. One principal plane is subjected to maximum value of principal stress. This plane is called major principal plane.
5. One principal plane is subjected to minimum value of principal stress. This plane is called minor principal plane.
6. There is a plane, which is a subjected to an intermediate stress.
7. There are three principal planes. These planes are mutually perpendicular to each other.
8. The principal planes can be form a set of three mutually perpendicular planes for writing the stress tensor.

Principal Stresses and Principal planes-Example-1

At a point P, the rectangular stress components are

$$\tau_{ij} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -2 & -3 \\ 1 & -3 & 4 \end{bmatrix}$$

all in units of kPa. Find the principal stresses and check for invariance.

Solution:

To obtain the cubic equation

$$\begin{vmatrix} 1-\sigma & 2 & 1 \\ 2 & -2-\sigma & -3 \\ 1 & -3 & 4-\sigma \end{vmatrix} = 0$$

$$(1-\sigma)[-(2+\sigma)(4-\sigma)-9]-2[2(4-\sigma)+3]+1[-6+(2+\sigma)] = 0$$

$$(1-\sigma)[\sigma^2-2\sigma-17]-2[11-2\sigma]+1[\sigma-4] = 0$$

$$\sigma^2-2\sigma-17-\sigma^3+2\sigma^2+17\sigma-22+4\sigma+\sigma-4 = 0$$

$$\sigma^3-3\sigma^2-20\sigma+43 = 0$$

Alternatively,

$$I_1 = (\sigma_x + \sigma_y + \sigma_z) = 3$$

$$I_2 = (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - {\tau_{xy}}^2 - {\tau_{yz}}^2 - {\tau_{zx}}^2) = -2-8+4-4-9-1 = -20$$

$$I_3 = (\sigma_x\sigma_y\sigma_z - \sigma_x{\tau_{yz}}^2 - \sigma_y{\tau_{zx}}^2 - \sigma_z{\tau_{xy}}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx})$$
$$= -8 - 9 + 2 - 16 - 12 = -43$$

Hence, the cubic equation ($\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$) is

$$\sigma^3 - 3\sigma^2 - 20\sigma + 43 = 0$$

Principal Stresses and Principal planes-Example-1

The solutions of $(\sigma^3 - 3\sigma^2 - 20\sigma + 43 = 0)$ are,

$$\sigma_1 = 5.25kPa$$
$$\sigma_2 = 1.95kPa$$
$$\sigma_3 = -4.2kPa$$

The stress invariants are,

$$I_1 = 5.25 + 1.95 - 4.2 = 3$$

$$I_2 = (5.25 \times 1.95) + (1.95 \times -4.2) + (-4.2 \times 5.25) = -20$$
$$I_3 = -(5.25 \times 1.95 \times 4.2) = -43$$

These agree with their earlier values

Principal Stresses and Principal planes-Example-2

With respect to the frame of reference O_{xyz}, the following state of stress exists. Determine the principal stresses and their associated directions.

$$\tau_{ij} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Principal Stresses and Principal planes-Example-2

Solution:

$$I_1 = (\sigma_x + \sigma_y + \sigma_z) = 1 + 1 + 1 = 3$$

$$I_2 = (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - {\tau_{xy}}^2 - {\tau_{yz}}^2 - {\tau_{zx}}^2) = 1+1+1-4-1-1 = -3$$

$$I_3 = (\sigma_x\sigma_y\sigma_z - \sigma_x{\tau_{yz}}^2 - \sigma_y{\tau_{zx}}^2 - \sigma_z{\tau_{xy}}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx})$$
$$= 1 - 1 - 1 - 4 + 4 = -1$$

Hence, the cubic equation $(\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0)$ is

$$\sigma^3 - 3\sigma^2 - 3\sigma + 1 = 0$$

Principal Stresses and Principal planes-Example-2

The solutions of ($\sigma^3 - 3\sigma^2 - 3\sigma - 1 = 0$) are,

$$\sigma_1 = -1$$

$$\sigma_2 = 3.7321$$

$$\sigma_3 = 0.2679$$

Directions of principal axes:
For $\sigma_1 = -1$,

$$(1+1)n_x + 2n_y + 1n_z = 0$$

$$2n_x + (1+1)n_y + 1n_z = 0$$

$$1n_x + 1n_y + (1+1)n_z = 0$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let A_1, B_1, C_1 respectively be the cofactors of the elements of first raw of the tensor

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Hence,

$$A_1 = (-1)^2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 2 \times 2 - 1 \times 1 = 3$$

Similarly, $B_1 = -1(4-1) = -3$ and $C_1 = 2 - 2 = 0$

Principal Stresses and Principal planes-Example-2

Direction cosines of can be obtained by,

$$n_x = \frac{A_1}{\sqrt{{A_1}^2 + {B_1}^2 + {C_1}^2}} = \frac{3}{\sqrt{3^2 + (-3)^2 + 0^2}} = 0.7071$$

$$n_y = \frac{-3}{\sqrt{3^2 + (-3)^2 + 0^2}} = -0.7071$$

$$n_z = 0$$

Principal Stresses and Principal planes-Example-2

For $\sigma_2 = 3.7321$,

$$\begin{bmatrix} -2.7321 & 2 & 1 \\ 2 & -2.7321 & 1 \\ 1 & 1 & -2.7321 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let A_1, B_1, C_1 respectively are obtained as
$A_1 = 6.464, B_1 = 6.464, C_1 = 4.732$

$$n_x = \frac{6.4641}{10.294} = 0.6280$$

$$n_y = \frac{6.4641}{10.294} = 0.6280$$

$$n_z = \frac{4.7320}{10.294} = 0.4597$$

For $\sigma_3 = 0.2679$,

$$\begin{bmatrix} 0.7320 & 2 & 1 \\ 2 & 0.7320 & 1 \\ 1 & 1 & 0.7320 \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let A_1, B_1, C_1 respectively are obtained as
$A_1 = -0.4641, B_1 = -0.4641, C_1 = 1.2679$

$$n_x = \frac{-0.4641}{1.4279} = -0.3251$$
$$n_y = \frac{-0.4641}{1.4279} = -0.3251$$
$$n_z = \frac{1.2679}{1.4279} = 0.8881$$

Octahedral Stresses

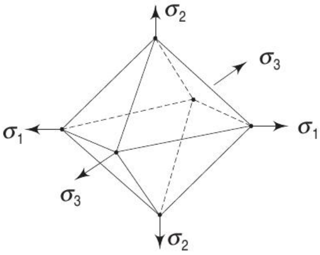

Figure: Octahedral planes

- A plane that is equally inclined to all the three principal axes is called an **octahedral plane**.
- Octahedral plane will have $n_x = n_y = n_z = \pm\frac{1}{\sqrt{3}}$
- There are eight such Octahedral planes
- The normal and shearing stresses on these planes are called the **octahedral normal stress** and **octahedral shearing stress** respectively.

Octahedral Stresses

- Octahedral Stresses in terms of Principal Stresses/Stress invariants

$$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}I_1$$

$${\tau_{oct}}^2 = \frac{1}{3}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = \frac{2}{9}(I_1^2 - 3I_2)$$

- Octahedral Stresses in terms of $\sigma_x, \tau xy, \cdots etc$ are

$$\sigma_{oct} = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$${\tau_{oct}}^2 = \frac{1}{3}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]$$

Hydrostatic and Deviatoric Stress Components

Any arbitrary state of stress can be resolved into a hydrostatic state and a state of pure shear.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_{Hyd} & 0 & 0 \\ 0 & \sigma_{Hyd} & 0 \\ 0 & 0 & \sigma_{Hyd} \end{bmatrix} + \begin{bmatrix} \sigma_x - \sigma_{Hyd} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_{Hyd} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_{Hyd} \end{bmatrix}$$

Hydrostatic Stress

- Hydrostatic stress is simply the average of the three normal stress components of any stress tensor.

$$\boldsymbol{\sigma}_{\text{Hyd}} = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

- It is a scalar quantity, although it is regularly used in tensor form as

$$\boldsymbol{\sigma}_{\text{Hyd}} = \begin{bmatrix} \sigma_{Hyd} & 0 & 0 \\ 0 & \sigma_{Hyd} & 0 \\ 0 & 0 & \sigma_{Hyd} \end{bmatrix}$$

- Hydrostatic stresses, being a function of I_1(First Invariant),do not change under coordinate transformations.

Deviatoric Stress

- Deviatoric stress is what's left after subtracting out the hydrostatic stress. The deviatoric stress will be represented by $\boldsymbol{\sigma}'$

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{\text{Hyd}}$$

- In Tensor notation

$$\boldsymbol{\sigma'} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} - \begin{bmatrix} \sigma_{Hyd} & 0 & 0 \\ 0 & \sigma_{Hyd} & 0 \\ 0 & 0 & \sigma_{Hyd} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_x - \sigma_{Hyd} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_{Hyd} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_{Hyd} \end{bmatrix}$$

Equations of Equilibrium

- The state of stress in a body varies from point to point.
- **Equations of Equilibrium** are the conditions to be satisfied by stress components when the body is in equilibrium.
- These equations are needed when the theory of elasticity is used to derive load-stress and load-deflection relations for a body.

Equations of Equilibrium

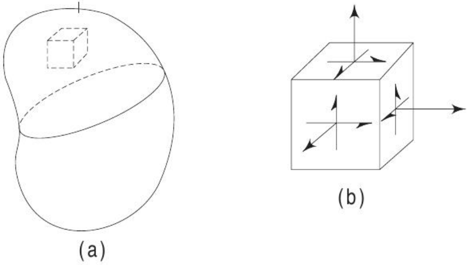

Figure: Isolated cubical element

Equations of Equilibrium

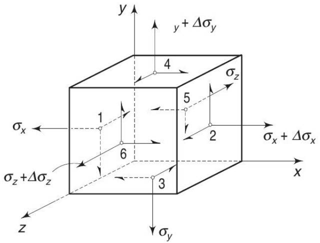

Figure: Variation of stresses

- Face-1: $\sigma_x, \tau_{xy}, \tau_{xz}$
- Face-2: $\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x,\ \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x,\ \tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}\Delta x,$
- Face-3: $\sigma_y, \tau_{yx}, \tau_{yz}$
- Face-4: $\sigma_y + \frac{\partial \sigma_y}{\partial y}\Delta y,$ $\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y,\ \tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}\Delta y$
- Face-5: $\sigma_z, \tau_{zx}, \tau_{zy}$
- Face-6: $\sigma_z + \frac{\partial \sigma_z}{\partial z}\Delta z,\ \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z,\ \tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}\Delta z$
- Body force components per unit volume are $\gamma_x, \gamma_y, \gamma_z$

For equilibrium in x-direction,

$$(\sigma_x+\frac{\partial\sigma_x}{\partial x}\Delta x)\Delta y\Delta z-\sigma_x\Delta y\Delta z+(\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}\Delta y)\Delta x\Delta z-\tau_{yx}\Delta x\Delta z$$
$$+(\tau_{zx}+\frac{\partial\tau_{zx}}{\partial z}\Delta z)\Delta x\Delta y-\tau_{zx}\Delta x\Delta y+\gamma_x\Delta x\Delta y\Delta z=0$$

Cancelling terms, dividing by $\Delta x, \Delta y, \Delta z$ and going to the limit, we get

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\gamma_x=0$$

Similarly, equating forces in the y and z directions respectively to zero, we get two more equations.

Equations of Equilibrium

On the basis of the fact that the cross shears are equal, we obtain the three differential equations of equilibrium as

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \gamma_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \gamma_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \gamma_z = 0$$