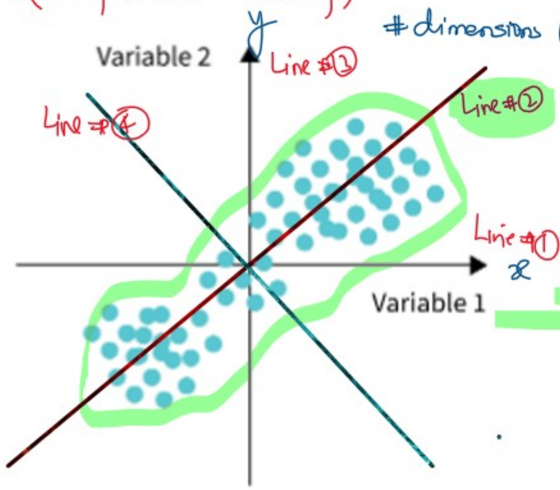
# PCA

13 April 2024 22:35

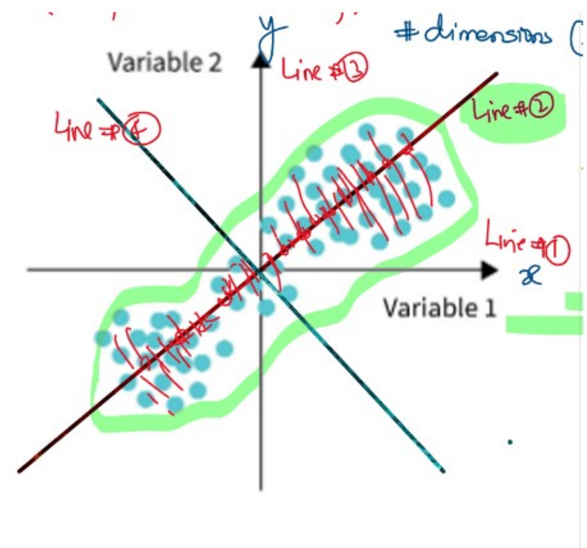
**Principal Component Analysis**

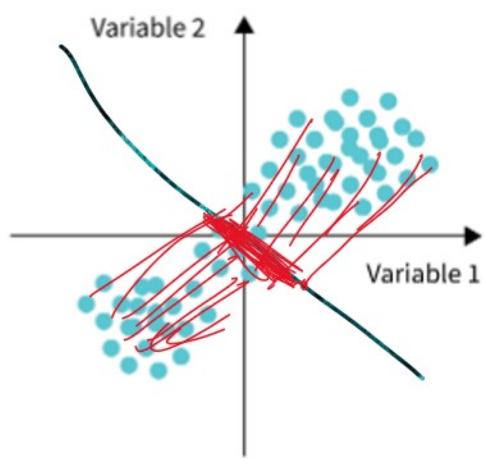
↳ (unsupervised learning)

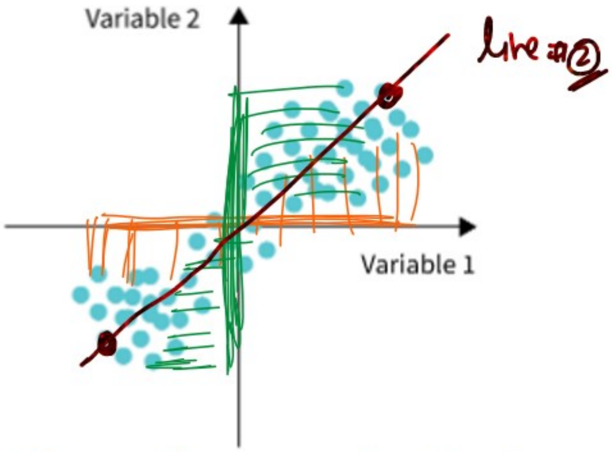


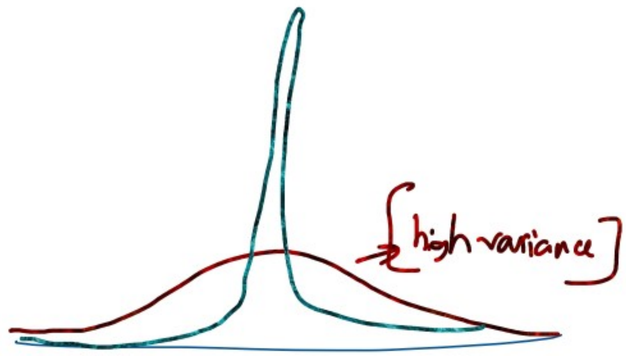

which line do you think exhibits the highest variance in the given data distribution?

# Highest variance is observed along the line # 2.






PCA uses an orthogonal transformations to convert a set of observations possibly correlated variables into a set of values called linearly uncorrelated (also known as Principal components)

**Application:** It is often used as a dimensionality reduction technique.

# Working principle for PCA

Steps involved in PCA are:

#1. standardize the dataset

standardization formula = $\left(\frac{X-\mu}{\sigma}\right)$

F1 # Salary → 25000 — 100000

F2 # Age — 20 — 80

#2 Calculate the covariance matrix for the features in the dataset.

#3. Calculate eigen values and eigen vectors for the covariance matrix.

#4. sort eigen values and their corresponding eigen vectors.

#5. Pick 'k' eigen values and form a matrix of eigen vectors

#5. Pick 'k' eigen values and form a matrix of eigen vectors

#6. Transform the original matrix.

## Eigen Values and Eigen vectors

For any square matrix $A_{n \times n}$:

$n$ = no. of rows = no. of columns.

* An eigen vector of A is a non zero vector $v$ such that

$$\boxed{A v = \lambda v}$$

$\lambda$ → eigen value.

A is a square matrix

$v$ → eigen vector

$v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times1}$ : Null vector or zero vectors

$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{3\times1}$ : Non-zero vector

* An eigen value of A is a scalar '$\lambda$' such that the equation $Av = \lambda v$ has a $\underbrace{\text{non-trivial solution}}_{\text{(non-zero solution)}}$

Examples Consider the matrix:

$$A = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}_{2\times2}.$$

Sol:

$$Av = \lambda v$$

$$\Rightarrow A v - \lambda v = 0 \quad \text{→ Null vector}$$

$$\Rightarrow (A - \lambda I) v = 0$$

$I$ → Identity Matrix $I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Rightarrow$$

$$(A - \lambda I) \qquad I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\boxed{(A-\lambda I)}\, v = 0$

$v$ → non-zero

$$\boxed{|A - \lambda I| = 0}$$

characteristics equation.

$$\begin{bmatrix} -4 & 8 \end{bmatrix} = \lambda \begin{bmatrix} 0 & 1 \end{bmatrix}$$

...ies equation

$$\Rightarrow \left| \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \left| \begin{bmatrix} (2-\lambda) & 2 \\ -4 & (8-\lambda) \end{bmatrix} \right| = 0$$

$$\Rightarrow (2-\lambda)(8-\lambda) - (-8) = 0$$

$$\Rightarrow 16 - 2\lambda - 8\lambda + \lambda^2 + 8 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda + 24 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda - 4\lambda + 24 = 0$$

$$\Rightarrow \lambda(\lambda - 6) - 4(\lambda - 6) = 0$$

$$(\lambda - 4)(\lambda - 6) = 0$$

$$\boxed{\lambda = 4, \quad \lambda = 6}$$ eigen values

For $\lambda = 4$,

$$A\vartheta = \lambda\vartheta$$

$$\Rightarrow \underbrace{\underset{2\times2}{\begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}}}_{2\times1} = 4 \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\vartheta_1 + 2\vartheta_2 \\ -4\vartheta_1 + 8\vartheta_2 \end{bmatrix} = \begin{bmatrix} 4\vartheta_1 \\ 4\vartheta_2 \end{bmatrix} \rightarrow 2\vartheta_1 + 2\vartheta_2 = 4\vartheta_1$$

$$\Rightarrow \begin{bmatrix} 2\vartheta_1 + 2\vartheta_2 \\ -4\vartheta_1 + 8\vartheta_2 \end{bmatrix} = \begin{bmatrix} 4\vartheta_1 \\ 4\vartheta_2 \end{bmatrix} \Rightarrow 2\vartheta_1 + 2\vartheta_2 = 4\vartheta_1$$

$$\Rightarrow \cancel{2}\vartheta_2 = \cancel{2}\vartheta_1$$

$$\boxed{\Rightarrow \vartheta_1 = \vartheta_2}$$

$$-4\vartheta_1 + 8\vartheta_2 = 4\vartheta_2$$

$$\Rightarrow 8\vartheta_2 - 4\vartheta_2 = 4\vartheta_1$$

$$\Rightarrow 4\vartheta_2 = 4\vartheta_1$$

$$\boxed{\Rightarrow \vartheta_1 = \vartheta_2}$$

$$\boxed{\lambda = 4}$$

Corresponding to this

$$\vartheta = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix}$$

$\because \vartheta_1 = \vartheta_2$

$$\vartheta = \begin{bmatrix} \vartheta_1 \\ \vartheta_1 \end{bmatrix}$$

$\vartheta_1 = 1 \quad \vartheta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vartheta_1 = -1 \quad \vartheta = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

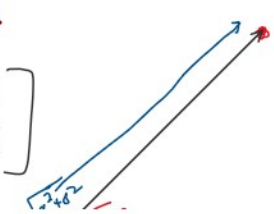
$\vartheta_1 = 7 \quad \vartheta = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

$\boxed{A\vartheta = \lambda\vartheta}$ has to be satisfied all the time.

$$\underset{A}{\begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}} \underset{\vartheta}{\begin{bmatrix} 7 \\ 7 \end{bmatrix}} = \begin{bmatrix} 28 \\ 28 \end{bmatrix} = \underset{\lambda}{4} \underset{\vartheta}{\begin{bmatrix} 7 \\ 7 \end{bmatrix}}$$

$\lambda = 4$

$$\underset{A}{\begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}} \underset{\vartheta}{\begin{bmatrix} -1 \\ -1 \end{bmatrix}} = \begin{bmatrix} -4 \\ -4 \end{bmatrix} = \underset{\lambda}{4} \underset{\vartheta}{\begin{bmatrix} -1 \\ -1 \end{bmatrix}}$$



$\vartheta =$

$A\vartheta =$

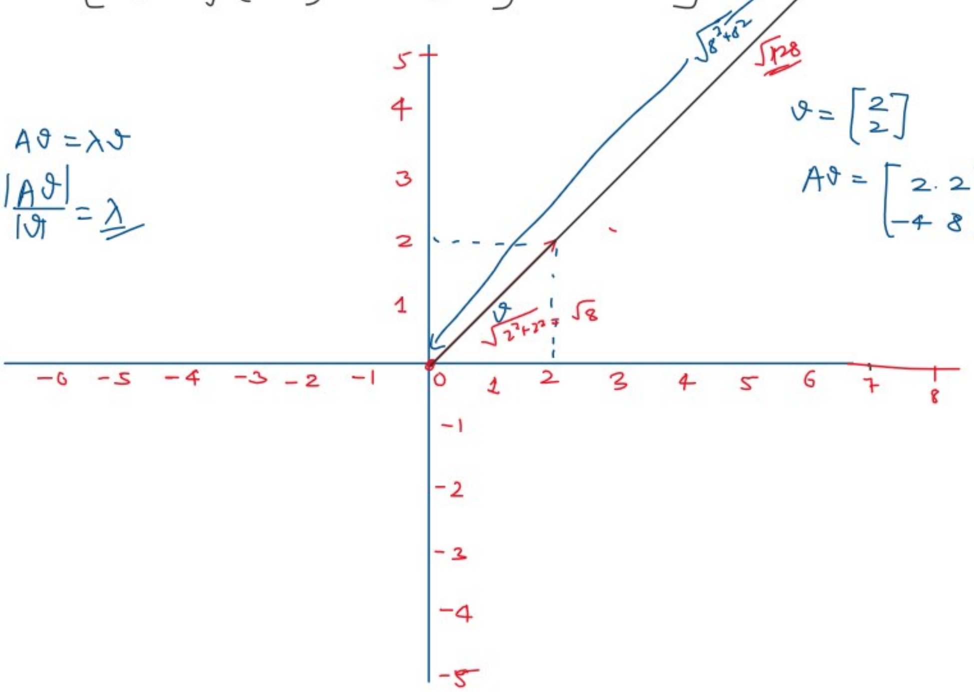
$$\begin{bmatrix} -4 & 8 \end{bmatrix}\begin{bmatrix} -1 \end{bmatrix} = \begin{bmatrix} -4 \end{bmatrix} - \quad \begin{bmatrix} -1 \end{bmatrix}$$

$v =$

$Av =$

$$Av = \lambda v$$

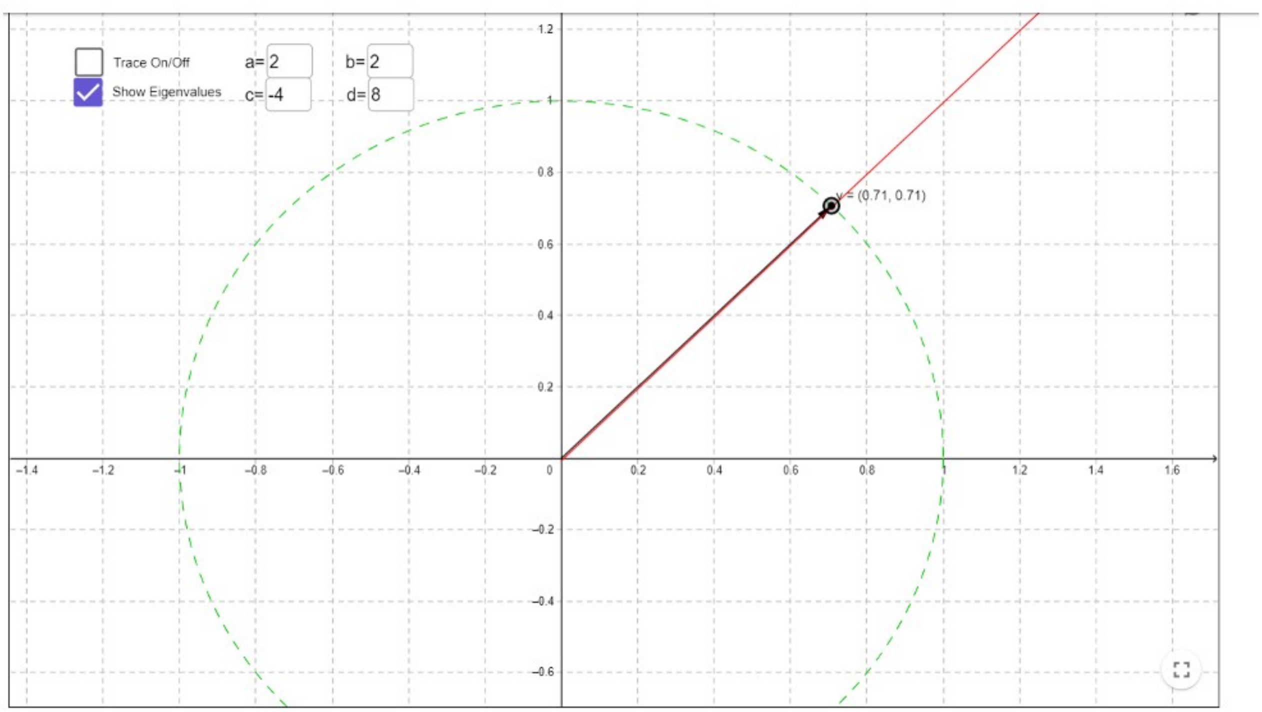
$$\frac{|Av|}{|v|} = \underline{\underline{\lambda}}$$

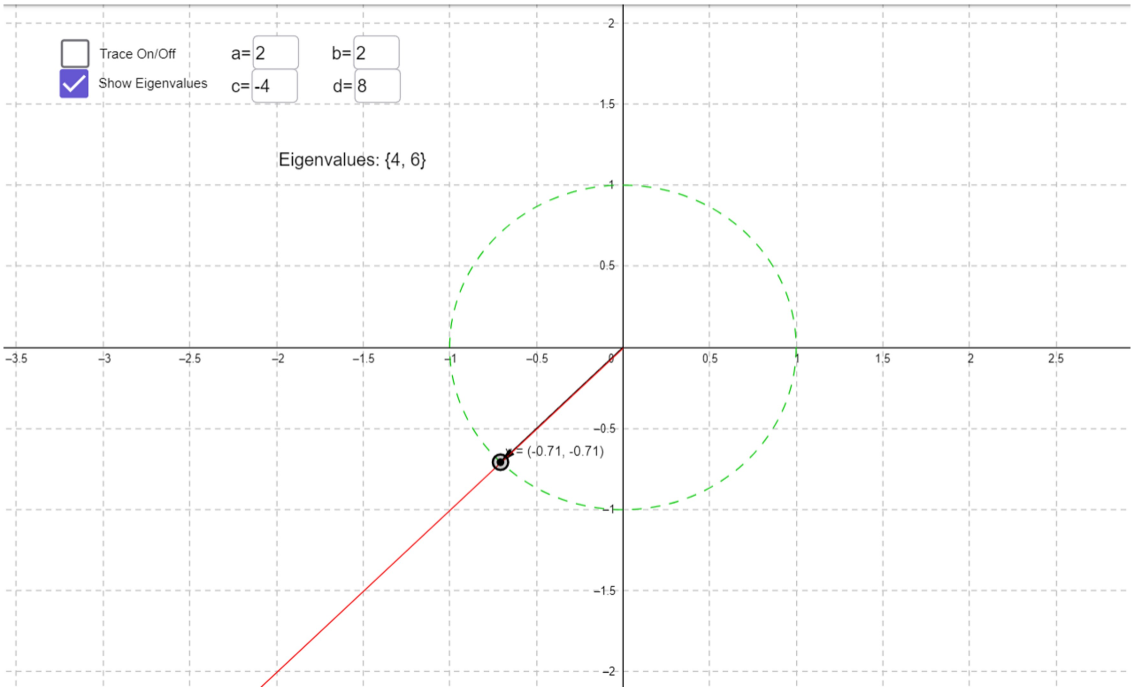


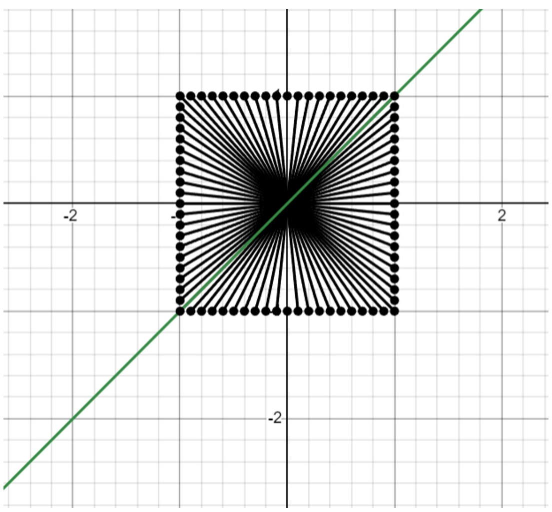

$$v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

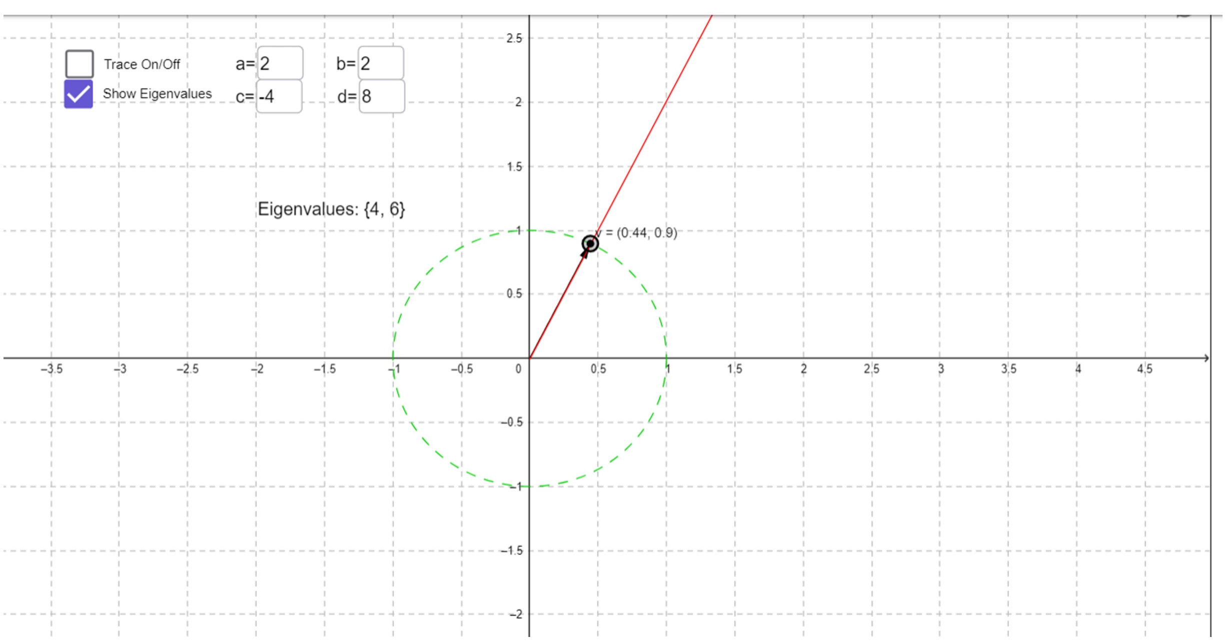
$$Av = \begin{bmatrix} 2 & 2 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

$$\frac{\sqrt{128}}{\sqrt{8}} = \sqrt{\frac{\cancel{128}\,16}{\cancel{8}}} = \boxed{4}$$




Trace On/Off
Show Eigenvalues
a= 2
b= 2
c= -4
d= 8
Eigenvalues: {4, 6}
= (-0.71, -0.71)


-2
2
-2


Trace On/Off
Show Eigenvalues
a= 2
b= 2
c= -4
d= 8
Eigenvalues: {4, 6}
= (0.44, 0.9)

For $\lambda = 6$

$$\begin{bmatrix}2 & 2\\ -4 & 8\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = 6\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

$\Rightarrow 2v_1 + 2v_2 = 6v_1$

$\Rightarrow \cancel{2}v_2 = \cancel{4}^{\,2}v_1$

$\Rightarrow \boxed{v_2 = 2v_1}$ ✓

$-4v_1 + 8v_2 = 6v_2$

$\Rightarrow -4v_1 = 6v_2 - 8v_2$

$\Rightarrow -4v_1 = \cancel{-}2v_2$

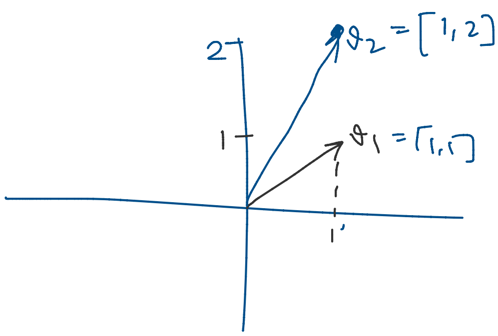
$\boxed{\Rightarrow v_2 = 2v_1}$ ✓

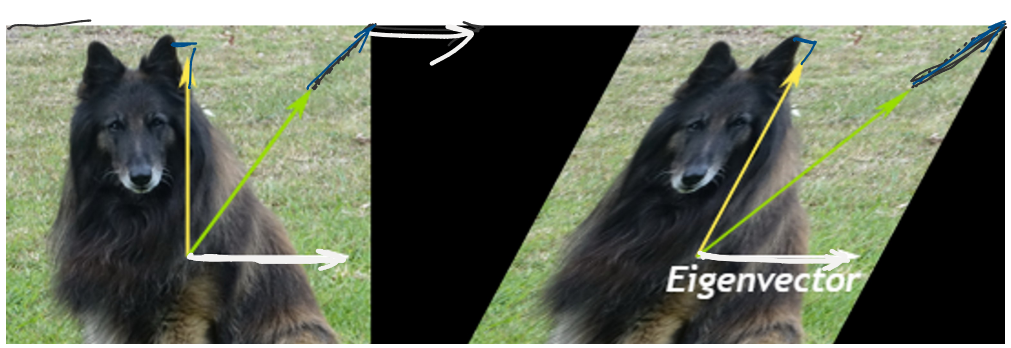
$\boxed{\begin{matrix}v_1 = 1\\ v_2 = 2\end{matrix}}$

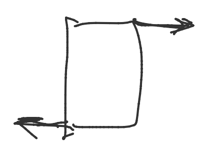
$v_1 = 10$
$v_2 = 20$

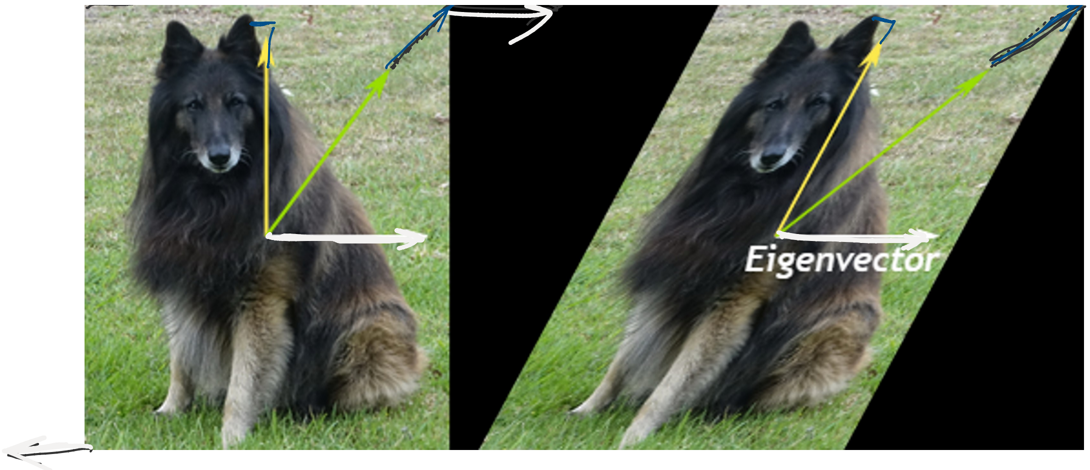
$\begin{bmatrix}10\\ 20\end{bmatrix} \quad \begin{bmatrix}5\\ 10\end{bmatrix} \quad \begin{bmatrix}6\\ 12\end{bmatrix}$

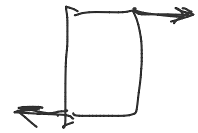
$\boxed{\lambda = 6}$



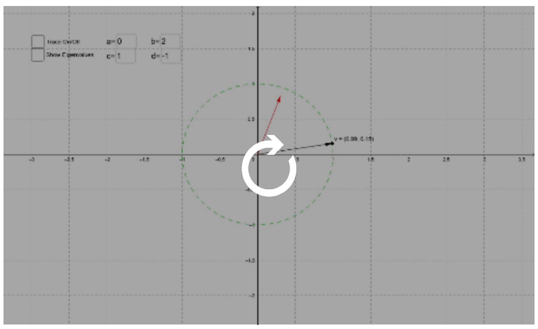



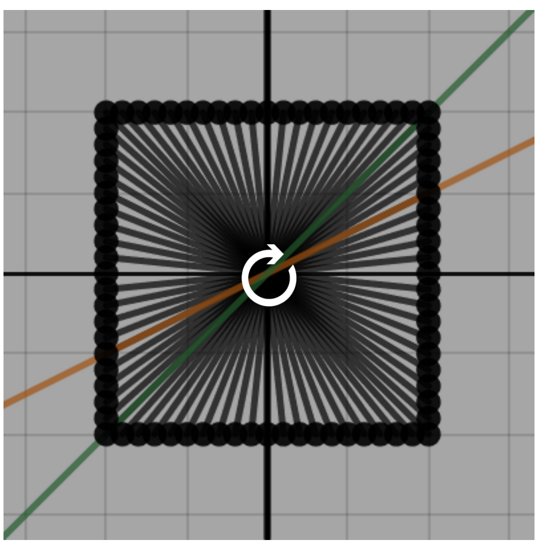


[Eigenvalue-Eigenvector Visualization](#)



[Eigenvectors](#)



| X1 | X2 | X3 | X4 |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 5 | 6 | 7 |
| 1 | 4 | 2 | 3 |
| 5 | 3 | 2 | 1 |
| 8 | 1 | 2 | 2 |

Mean: 4 3 3 3.4

Std. deviation 3 1.58 1.73 2.30