

03: Linear Algebra - Review

Matrices - overview

- Rectangular array of numbers written between square brackets
 - 2D array
 - Named as capital letters (A,B,X,Y)
- Dimension of a matrix are [Rows x Columns]
 - Start at top left
 - To bottom left
 - To bottom right
 - $R^{[r \times c]}$ means a matrix which has r rows and c columns

$$A = \begin{bmatrix} 1402 & 191 \\ 1371 & 821 \\ 949 & 1437 \\ 147 & 1448 \end{bmatrix}$$

- Is a [4 x 2] matrix

- Matrix elements
 - $A_{(i,j)}$ = entry in i^{th} row and j^{th} column

Handwritten annotations for matrix A:

- $A_{11} = 1402$ (blue)
- $A_{12} = 191$ (red)
- $A_{32} = 1437$ (pink)
- $A_{41} = 147$ (cyan)

- Provides a way to organize, index and access a lot of data

Vectors - overview

- Is an n by 1 matrix
 - Usually referred to as a lower case letter
 - n rows
 - 1 column
 - e.g.

$$y = \begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}$$

- Is a 4 dimensional vector
 - Refer to this as a vector R_4
- Vector elements
 - $v_i = i^{\text{th}}$ element of the vector
 - Vectors can be 0-indexed (C++) or 1-indexed (MATLAB)
 - In math 1-indexed is most common
 - But in machine learning 0-index is useful
 - Normally assume using 1-index vectors, but be aware sometimes these will (explicitly) be 0 index ones

Matrix manipulation

- **Addition**
 - Add up elements one at a time
 - Can only add matrices of the *same dimensions*
 - Creates a new matrix of the same dimensions of the ones added

$$\begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 0.5 \\ 2 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0.5 \\ 4 & 10 \\ 3 & 2 \end{bmatrix}$$

- **Multiplication by scalar**

- Scalar = real number
- Multiply each element by the scalar
- Generates a matrix of the same size as the original matrix

$$3 \times \begin{bmatrix} 1 & 0 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 6 & 15 \\ 9 & 3 \end{bmatrix}$$

- **Division by a scalar**

- Same as multiplying a matrix by $1/4$
- Each element is divided by the scalar

- **Combination of operands**

- Evaluate multiplications first

$$3 \times \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} / 3$$

- **Matrix by vector multiplication**

- $[3 \times 2]$ matrix * $[2 \times 1]$ vector
 - New matrix is $[3 \times 1]$
 - More generally if $[a \times b] * [b \times c]$
 - Then new matrix is $[a \times c]$
 - How do you do it?
 - Take the two vector numbers and multiply them with the first row of the matrix
 - Then add results together - this number is the first number in the new vector
 - The multiply second row by vector and add the results together
 - Then multiply final row by vector and add them together

$$\begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 16 \\ 4 \\ 7 \end{bmatrix}$$

$3 \times 2 \quad 2 \times 1$

$$\begin{aligned} 1 \times 1 + 3 \times 5 &= 16 \\ 4 \times 1 + 0 \times 5 &= 4 \\ 2 \times 1 + 1 \times 5 &= 7 \end{aligned}$$

- Detailed explanation

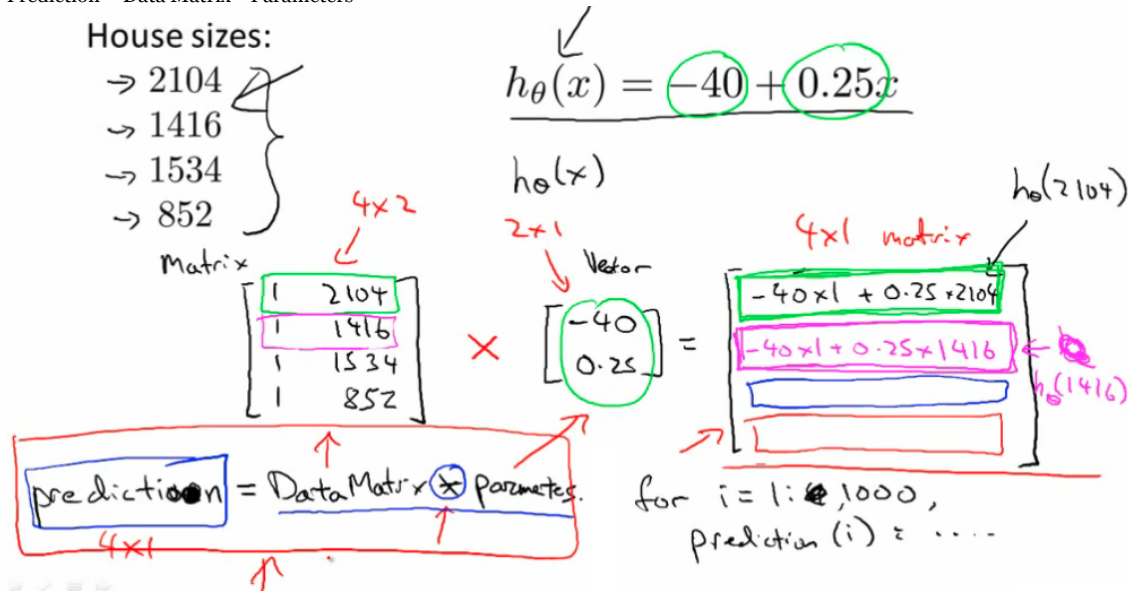
- $A * x = y$
 - A is $m \times n$ matrix
 - x is $n \times 1$ matrix
 - n must match between vector and matrix
 - i.e. inner dimensions must match
 - Result is an m-dimensional vector
- To get y_i - multiply A's i^{th} row with all the elements of vector x and add them up

- Neat trick

- Say we have a data set with four values
- Say we also have a hypothesis $h_\theta(x) = -40 + 0.25x$
 - Create your data as a matrix which can be multiplied by a vector

03_Linear_algebra_review

- Have the parameters in a vector which your matrix can be multiplied by
- Means we can do
 - Prediction = Data Matrix * Parameters



- Here we add an extra column to the data with 1s - this means our θ_0 values can be calculated and expressed

- The diagram above shows how this works

- This can be far more efficient computationally than lots of for loops
- This is also easier and cleaner to code (assuming you have appropriate libraries to do matrix multiplication)

• Matrix-matrix multiplication

- General idea
 - Step through the second matrix one column at a time
 - Multiply each column vector from second matrix by the entire first matrix, each time generating a vector
 - The final product is these vectors combined (not added or summed, but literally just put together)
- Details
 - $A \times B = C$
 - $A = [m \times n]$
 - $B = [n \times o]$
 - $C = [m \times o]$
 - With vector multiplications $o = 1$
 - Can only multiply matrix where columns in A match rows in B
- Mechanism
 - Take column 1 of B, treat as a vector
 - Multiply A by that column - generates an $[m \times 1]$ vector
 - Repeat for each column in B
 - There are o columns in B, so we get o columns in C
- Summary
 - The i^{th} column of matrix C is obtained by multiplying A with the i^{th} column of B
- Start with an example
- $A \times B$

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 5 & 2 \end{bmatrix}$$

- Initially
 - Take matrix A and multiply by the first column vector from B
 - Take the matrix A and multiply by the second column vector from B

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 11 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

- 2 x 3 times 3 x 2 gives you a 2 x 2 matrix

Implementation/use

- House prices, but now we have three hypothesis and the same data set
- To apply all three hypothesis to all data we can do this efficiently using matrix-matrix multiplication
 - Have
 - Data matrix
 - Parameter matrix
 - Example
 - Four houses, where we want to predict the prize
 - Three competing hypotheses
 - Because our hypothesis are one variable, to make the matrices match up we make our data (houses sizes) vector into a 4x2 matrix by adding an extra column of 1s

House sizes:

$$\begin{Bmatrix} 2104 \\ 1416 \\ 1534 \\ 852 \end{Bmatrix}$$

Have 3 competing hypotheses:

$$\begin{aligned} 1. & h_{\theta}(x) = -40 + 0.25x \\ 2. & h_{\theta}(x) = 200 + 0.1x \\ 3. & h_{\theta}(x) = -150 + 0.4x \end{aligned}$$

Matrix

$$\begin{bmatrix} 1 & 2104 \\ 1 & 1416 \\ 1 & 1534 \\ 1 & 852 \end{bmatrix}$$

Matrix

$$\begin{bmatrix} -40 & 200 & -150 \\ 0.25 & 0.1 & 0.4 \end{bmatrix} = \begin{bmatrix} 486 & 410 & 692 \\ 314 & 342 & 416 \\ 344 & 353 & 464 \\ 173 & 285 & 191 \end{bmatrix}$$

Prediction of first h_{θ}
Predictions of 2nd h_{θ}

- What does this mean
 - Can quickly apply three hypotheses at once, making 12 predictions
 - Lots of good linear algebra libraries to do this kind of thing very efficiently

Matrix multiplication properties

- Can pack a lot into one operation
 - However, should be careful of how you use those operations
 - Some interesting properties
- **Commutativity**
 - When working with raw numbers/scalars multiplication is commutative
 - $3 * 5 == 5 * 3$
 - This is not true for matrix
 - $A \times B \neq B \times A$
 - **Matrix multiplication is not commutative**
- **Associativity**
 - $3 \times 5 \times 2 == 3 \times 10 = 15 \times 2$
 - Associative property
 - **Matrix multiplications is associative**
 - $A \times (B \times C) == (A \times B) \times C$
- **Identity matrix**
 - 1 is the identity for any scalar
 - i.e. $1 \times z = z$
 - for any real number
 - In matrices we have an identity matrix called I
 - Sometimes called $I_{\{n \times n\}}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4 \times 4}$$

- See some identity matrices above
 - Different identity matrix for each set of dimensions
 - Has
 - 1s along the diagonals
 - 0s everywhere else
 - 1x1 matrix is just "1"
- Has the property that any matrix A which can be multiplied by an identity matrix gives you matrix A back
 - So if A is [m x n] then
 - A * I
 - I = n x n
 - I * A
 - I = m x m
 - (To make inside dimensions match to allow multiplication)
- Identity matrix dimensions are implicit
- Remember that matrices are not commutative AB != BA
 - Except when B is the identity matrix
 - Then AB == BA

Inverse and transpose operations

- **Matrix inverse**
 - How does the concept of "the inverse" relate to real numbers?
 - 1 = "identity element" (as mentioned above)
 - Each number has an inverse
 - This is the number you multiply a number by to get the identity element
 - i.e. if you have x, x * 1/x = 1
 - e.g. given the number 3
 - 3 * 3⁻¹ = 1 (the identity number/matrix)
 - In the space of real numbers **not everything has an inverse**
 - e.g. 0 does not have an inverse
 - What is the inverse of a matrix
 - If A is an m x m matrix, then A inverse = A⁻¹
 - So A * A⁻¹ = I
 - Only matrices which are m x m have inverses
 - Square matrices only!
 - Example
 - 2 x 2 matrix
- $$\underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 16 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 0.4 & -0.1 \\ -0.05 & 0.075 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A^{-1}A} = I_{2 \times 2}$$
- How did you find the inverse
 - Turns out that you can sometimes do it by hand, although this is very hard
 - Numerical software for computing a matrices inverse
 - Lots of open source libraries
 - If A is all zeros then there is no inverse matrix
 - Some others don't, intuition should be matrices that don't have an inverse are a singular matrix or a degenerate matrix (i.e. when it's too close to 0)
 - So if all the values of a matrix reach zero, this can be described as reaching singularity
- **Matrix transpose**
 - Have matrix A (which is [n x m]) how do you change it to become [m x n] while keeping the same values
 - i.e. swap rows and columns!
 - How you do it;
 - Take first row of A - becomes 1st column of A^T
 - Second row of A - becomes 2nd column...
 - A is an m x n matrix
 - B is a transpose of A
 - Then B is an n x m matrix
 - A_(i,j) = B_(j,i)

$$\underline{A} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 9 \end{bmatrix} \quad \underline{A^T} = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 9 \end{bmatrix}$$