Simulation of a 2D single-mode Rayleigh-Taylor instability

1. Mathematical Problem

- Equations: 2D incompressible Navier Stokes in vorticity-streamfunction formulations with advection written in Jacobian form

$$\frac{\partial \omega}{\partial t} = -J(\psi, \omega) + \beta g \frac{\partial \theta}{\partial x} + \nu \nabla^2 \omega$$

$$\frac{\partial \theta}{\partial t} = -J(\psi, \theta) + \kappa \nabla^2 \theta$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \omega$$

- Constants: $g = 9.80665$ m/s, $\nu = 1.58 \times 10^{-6}$ m²/s, $\kappa = 1.58 \times 10^{-6}$ m²/s
- Domain: $-\frac{1}{2} L_1 \leq x \leq \frac{1}{2} L_1$ and $-\frac{1}{2} L_2 \leq z \leq \frac{1}{2} L_2$ where $L_1 = 0.25$ m, $L_2 = 1.00$ m
- Boundary conditions: periodic in x for all variables (ψ, ω, θ) and walls in z ($\psi = 0$, $\omega = 0$, $\theta(x, z = L_2) = 299$ K, $\theta(x, z = 0) = 300$ K)
- Initial condition: cosine perturbation centered in x and z

$$\theta(x,z) = \begin{cases} \theta_0 \; if \; z < \frac{1}{25}\cos(30x) \; and \; z < \frac{1}{27}\cos(30x) \\ \theta_0 + \Delta\theta \; if \; z < \frac{1}{25}\cos(30x) \; and \; z \geq \frac{1}{27}\cos(30x) \\ \theta_0 + 2\Delta\theta \; otherwise \end{cases} \quad with \; \theta_0 = 299 \, K, \; \Delta\theta = 0.5 \, K$$

Fluid starts from rest ($\psi = 0$, $\omega = 0$). Total integration time is 50 seconds

2. Numerical Discretization

- Spatial discretization: second-order differences for all spatial derivatives in all 3 equations. RUN A has 101 grid points in x-direction and 401 in z-direction ($\Delta x = \Delta z = 0.0025$ m), and RUN B has 201 grid points in x-direction and 801 in z-direction ($\Delta x = \Delta z = 0.00125$ m). Arakawa discretization for the Jacobian to mitigate non-linear instability.
- Time discretization: (Experiment #1) AB3 for advection, buoyancy, and diffusion with Euler-Forward for the first two time steps. (Experiment #2) RK4 for advection, buoyancy, and diffusion. Time step of 0.05 seconds used for RUN A (Experiments #1 and #2) and 0.01 seconds used for RUN B (Experiments #1 and #2).
- Poisson solver: SOR iterative solver with estimated optimal α and tolerance of 10^{-10}. ψ set to 0 across the domain as initial guess for solver.

3. References

- Simulation set-up inspired by (but not totally followed from) Section 5 of Young, Y.-N., Tufo, H., Dubey, A., and Rosner, R., 2001. On the miscible Rayleigh-Taylor instability: two and three dimensions. Journal of Fluid Mechanics, volume 447, pp. 377-408

- Simulation comparison with respect to choices for Atwood and Reynolds numbers for single mode perturbation as presented in Section 3 of Wie, T., Livescu, D., 2012. Late-time quadratic growth in single-mode Rayleigh-Taylor instability. Physical Review E 86, 046405