AOS 180 – Modeling Task 2
A linear wave equation
Due date: April 19^{th}11:00AM

Problem description

We are interested in the solution of a simple one-dimensional linear wave equation given by

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \tag{1}$$

on the domain $0 \leq x \leq L_x$ for $t \geq 0$. Here $c > 0$ is a constant propagation velocity and the quantity u is arbitrary and it has no units. We will use an initial condition

$$u(x, t = 0) = u_0(x) = A \sin\left(\frac{2\pi x}{\lambda_x}\right), \tag{2}$$

where A is the amplitude and λ_x the wavenumber of the initial wave. As for the boundary conditions, we will use cyclic (or periodic) boundary condition. This means that the wave leaves the domain on one side and enters back on the other. Mathematically we write $u(x = 0, t) = u(x = L_x, t)$.

For this problem the analytical solution is given by

$$u(x, t) = u_0(x - ct) = A \sin\left(\frac{2\pi(x - ct)}{\lambda_x}\right). \tag{3}$$

We will also do one solution using a "square wave" as initial condition (click here). All you have to do is replace the initial condition by a discontinuous function that is defined using the *sign* function:

$$u(x, t = 0) = u_0(x) = \text{sign}\left[A \sin\left(\frac{2\pi x}{\lambda_x}\right)\right], \tag{4}$$

where

$$\text{sign}\,[x] = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}. \tag{5}$$

Numerical approach

We will use the forward in time and backward in space (FTBS) discretization scheme for this problem, so the recursive relation that you need to implement is already in your notes. We have a set of basic parameters for this simulation and then several experiments in which one or more parameters are changed. The basic parameters are listed below:

1. Domain size $L_x = 50\,\text{m}$ discretized with 51 grid points (including one point at each boundary).

2. Total time for integration is $T_{tot} = 200\,\mathrm{s}$ (two complete cycles of the wave) and the time step is $\Delta t = 2\,\mathrm{s}$.
3. The propagation velocity is $c = 0.5\,\mathrm{m/s}$.
4. The initial condition is a wave with amplitude $A = 1$ and wavelength $\lambda_x = 50\,\mathrm{m}$.

We will compare the numerical solution to the exact solution by defining a the discretization error as the maximum error at a given time

$$\epsilon_u(t) \equiv \max\left\{ \left| u^{\mathrm{exact}}(x_i, t) - u^{\mathrm{approx}}(x_i, t) \right| \right\}, \tag{6}$$

where $|\cdot|$ represents the absolute value.

As we will see next week, the errors in the numerical solution depend on two parameters: the Courant number $C = c\Delta t/\Delta x$ and the ratio between wavelength and grid spacing $\lambda/\Delta x$. We will discuss these errors in more details next week, but you can already explore this empirically by using your numerical model.

For these simulations we will generate two types of outputs. The first type consists of simple summary statistics that we will output as a function of time every single time step. My suggestion is that for each time step you write a line into this file containing the time step counter, the corresponding time, the solution error $\epsilon_u(t)$, and the amplitude of the wave (given by the maximum value of u at that time step). The second type of output is the output of the full solution at all points and the corresponding analytical solution for visual comparison. For this, I suggest you produce outputs every 5 or 10 time steps (instead of every single time step).

Note: this implementation is very similar to the one described in problem 5.1 of the textbook, and the book also has a sample code to solve problem 5.1. However, I recommend that you write your own code first, debug it, make it work, and then check the sample program included in the textbook to see how the authors suggest you write your code. I do not like how they treat (by actually not treating) the boundary condition in that sample code.

In the table below you will find the set of numerical experiments that I suggest you conduct. Of course, feel free to design a number of additional tests to get a better understanding of the behavior of your model. We are always running the model for $T_{tot} = 200\,\mathrm{s}$, but it may be interesting to run it longer to see the behavior after several cycles of the wave. Always keep track of the parameters used for each simulation as well as C and $\lambda/\Delta x$.

Exp.	Δt	λ_x	Main goal
#1	2 s	50 m	Base case for code testing with optimal C
#2	< 2 s	50 m	Assessing effects of reducing C (use at least 2 cases with $\Delta t = 1,\ 0.5\,\mathrm{s}$)
#3	> 2 s	50 m	Assessing effects of increasing C (use at least 2 cases with $\Delta t = 4,\ 8\,\mathrm{s}$)
#4	≤ 2 s	10 m	Assessing effects of reducing C using a smaller wavelengths ($\Delta t = 2,\ 1\,\mathrm{s}$)
#5	1 s	50 m	Solution for a square wave

Two notes of caution: (1) the fact that we are using periodic boundary conditions limits us to choose wavelengths that yield an integer number of waves in the domain, or the boundary condition will mishandle the wave. Even though this is not going to happen with the choices listed in the table, try one case to see what happens. (2) If wave amplitudes become too small, using single precision may cause trouble...

Report

Please include in your report:

i. Figures: Now you will design the figures on your own. In general, your plots should be designed to help you discuss the main observations you made of your model and to help illustrate the points you want to discuss (see item iii below). I have a few suggestions, though. (1) Plots comparing analytical solution and numerical solution at a given time step are useful in helping you interpret the results. Make many of those, at selected times when you want to understand what is happening. For your report, include only a few, illustrating things you would like to discuss. Make sure you use these to illustrate problems such as numerical diffusion and linear instability of the numerical scheme. (2) Plots comparing the numerical error and the amplitude of the solution as a function of time are useful to compare the behavior of the model for different conditions. Use these as your primary guide to the discussions. As a suggestion, plots of error look better on a log-log scale and plots of amplitude look better on a linear-log scale (changes in amplitude tend to have exponential behavior). (3) Plots of the amplitude as a function of time step number are also useful, because they help you estimate the decay/growth rate of amplitude per time step, an important quantity to characterize your numerical solution (you can estimate the rates by fitting exponential functions to the amplitude plots). These are also better on a linear-log scale.

ii. Documentation: Please include in your report the recursive equation used and how you implemented the boundary condition. Also include a table of your experiments containing the basic configuration for each run together with C and $\lambda_x/\Delta x$. Also include other things you think may be useful (e.g. the decay/growth rate for the amplitude, etc).

iii. Discussion: Please briefly discuss each one of your numerical experiments in your report. A few sentences per experiment is enough. In particular, I would like you to use experiments #1, #2, and #3 to discuss the effects of reducing C on the behavior of the solution (stable or unstable, error, amplitude decay/growth rate, error/time step versus error/time, etc.). Use experiment #4 to include a discussion of the effects of $\lambda_x/\Delta x$ on the solution. Finally, use experiment #5 to discuss how numerical errors can impact the shape of more complex waves (i.e. anything that is not a monochromatic wave).

I will upload to CCLE a simple Matlab/Python script to make one plot and one animation of the solution. This is meant to serve as model for you to develop your own scripts to prepare the figures for this report and the future ones as well. If you need more help with these scripts, let me know.

Please upload a **.pdf** file with your report and a file with your code to the dropbox that will be available on the CCLE website for the course.

Additional suggestions

- Switch the sign of c and your upwind scheme is now a "downwind" scheme. What happens?
- If you still want more fun, there are a bunch of problems for the linear wave equation in the list of problems for Chapter 5 of the textbook, many of which have plots with illustration of the solution in Chapter 4 of the textbook. Enjoy!