AOS 180 – Modeling Task 1

Buoyancy oscillations and time discretization

Due date: April 12^{th} 11:00AM

Problem description

The vertical position of an air parcel $z_p(t)$ undergoing buoyancy oscillation in the stable atmosphere satisfies the second-order ordinary differential equation (ODE)

$$\frac{d^2 z_p}{dt^2} = -N^2 z_p. \tag{1}$$

Here, z_p is measured with respect to the air parcel position of neutral stability and $N = \sqrt{(g/\theta) d\theta/dz}$ is the Brunt-Väisäla frequency. We are interested in the solution to this initial value problem with the initial conditions

$$z_p(t=0) = z_0 \qquad \left.\frac{dz_p}{dt}\right|_{t=0} = 0. \tag{2}$$

Physically, this set of initial conditions corresponds to a case in which we move the air parcel a distance z_0 from its equilibrium position and release it at the instant $t = 0$ with an initial vertical velocity $w = dz_p/dt = 0$. The analytical solution to this problem is given by

$$z_p(t) = z_0 \cos(Nt). \tag{3}$$

We are interested in investigating a simple numerical solution to the same problem.

Numerical approach

The best approach to solve numerically a second-order ODE is to reduce it to a system of two first-order ODEs. This is a mathematical convenience, but here it also has physical interpretation. We can write the governing equations as

$$\frac{dz_p}{dt} = w \tag{4}$$

$$\frac{dw}{dt} = -N^2 z_p. \tag{5}$$

The first equation is the definition of a new function $w(t)$, which in this case is the vertical velocity of the air parcel. The second equation is the governing equation (1) written in terms of $w(t)$. Conveniently, each one of the initial conditions given by (2) is appropriate for one of the two first-order ODEs in (4)–(5). So all you have to do is discretize both equations and solve them simultaneously by marching in time.

For the present investigation, we will use $g = 9.81\,\text{m/s}$, $\theta = 290\,\text{K}$, $d\theta/dz = 5\,\text{K/km}$, and $z_0 = 100\,\text{m}$, and integrate the evolution of the system for a total time $T_{tot} = 2000\,\text{s}$.

We will compare the numerical solution to the analytical one by defining a local (in time) discretization error as

$$\epsilon_{z_p}(t) \equiv \left| z_p^{\text{exact}}(t) - z_p^{\text{approx}}(t) \right|, \tag{6}$$

where $|\cdot|$ represents the absolute value.

a. Using the Euler-Forward (EF) scheme, integrate the set of ODEs marching in time from the initial condition up to $t = T_{tot}$. Because this is a very simple problem, we will save outputs every time step. Make sure you save the position and velocity of the air parcel obtained from the numerical integration as well as those obtained from the exact solution. Also save the local discretization error for both variables. Perform these calculations for $\Delta t = 10\,\mathrm{s}$ and $\Delta t = 1\,\mathrm{s}$.

b. Repeat the calculations of item **a** using the leapfrog scheme. In this case, you will need to use the EF scheme for the first iteration, and then use the leapfrog scheme for the rest. Note: use the leapfrog scheme as described in class or found in the textbook – sometimes for second-order ODEs, the word leapfrog is used for integrations in which velocity and position are updated on alternating time steps and this is not what we want here.

Report

Please include in your report:

i. Figures: I suggest you make 6 plots, 3 for each integration method. Plot 1 should display z_p as a function of t for the exact solution and the numerical solutions using EF for both $\Delta t = 10\,\mathrm{s}$ and $\Delta t = 1\,\mathrm{s}$. Plot 2 should show w_p as a function of z_p for the same two Δt's as in plot 1 – this is the *phase space* for our system[1]. Finally, plot 3 should show the errors in z_p (i.e., $\epsilon_{z_p}(t)$) for the two time steps as a function of t. For this last plot use logarithmic scale on both axes. Plots 4, 5, and 6 are identical but for the leapfrog integration. Note: when making figures, you should spend some time making sure your figures are clear, have all the labels and units, and convey the information in a useful way. Adjust the ranges for the axes according, choose colors, include a key, etc., Also, figures should look nice!

ii. Documentation: Please include in your report the equations used to update velocity and position for each one of the two discretization schemes.

iii. Discussion: Please briefly discuss the results you see in your plots. One or two sentences per plot is enough (not describing the plot, but interpreting the results). In particular, I would like you to comment on the effects of reducing Δt for both integration schemes. If you can, elaborate on the reason for the differences between the two methods.

Please upload a **.pdf** file with your report and a file with your code to the dropbox that will be available on the CCLE website for the course.

[1]The phase space of a dynamical system is a space in which all possible states of a system are represented, with each possible state corresponding to one unique point in the phase space. Our simple system (our air parcel) is completely characterized by position and velocity. The line you plot represents the trajectory of the system, and it shows all the possible states for the system for the initial conditions used. For different initial conditions, a different trajectory would be obtained.