Buoyancy Oscillations and Time Discretization

The goal of this investigation was to use the Euler-Forward (EF) and the Leap-Frog (LF) schemes to determine the vertical velocity and the vertical position of an air parcel undergoing buoyancy oscillations in the stable atmosphere. The plots below show the results of the two schemes. The attached file contains the C++ code used to generate output files named "EF_1s.txt", "EF_10s.txt", "LF_1s.txt", and "LF_10s.txt" where EF and LF denote the iterative method used to generate the output values, and 1s and 10s denote the size of the time-step used.

The Euler-Forward equations used in to update vertical velocity and vertical position are as follows:

$$z_{n+1}(t) \approx z_n(t) + w_n(t)\Delta t \tag{1}$$

$$w_{n+1}(t) \approx w_n(t) - N^2 z_n(t)\Delta t \tag{2}$$

The first iteration in the Leap-Frog computation used the result from Euler-Forward and then continued on using the following equations:

$$z_{n+1}(t) = z_{n-1}(t) + 2w_n(t)\Delta t \tag{3}$$

$$w_{n+1}(t) = w_{n-1}(t) - 2N^2 z_n(t)\Delta t \tag{4}$$

Results from the two discretization schemes at Δt = 10 seconds and Δt = 1 second were compared to the given analytical solution for vertical position, allowing us to calculated the absolute error in the calculated position. The following plots delineate these results, allowing us to compare the two schemes with each other and with the analytical solution. Below each figure is a brief discussion and interpretation of the results.

Figure 1: $z_p(t)$ for exact solution and numerical solutions using EF Scheme

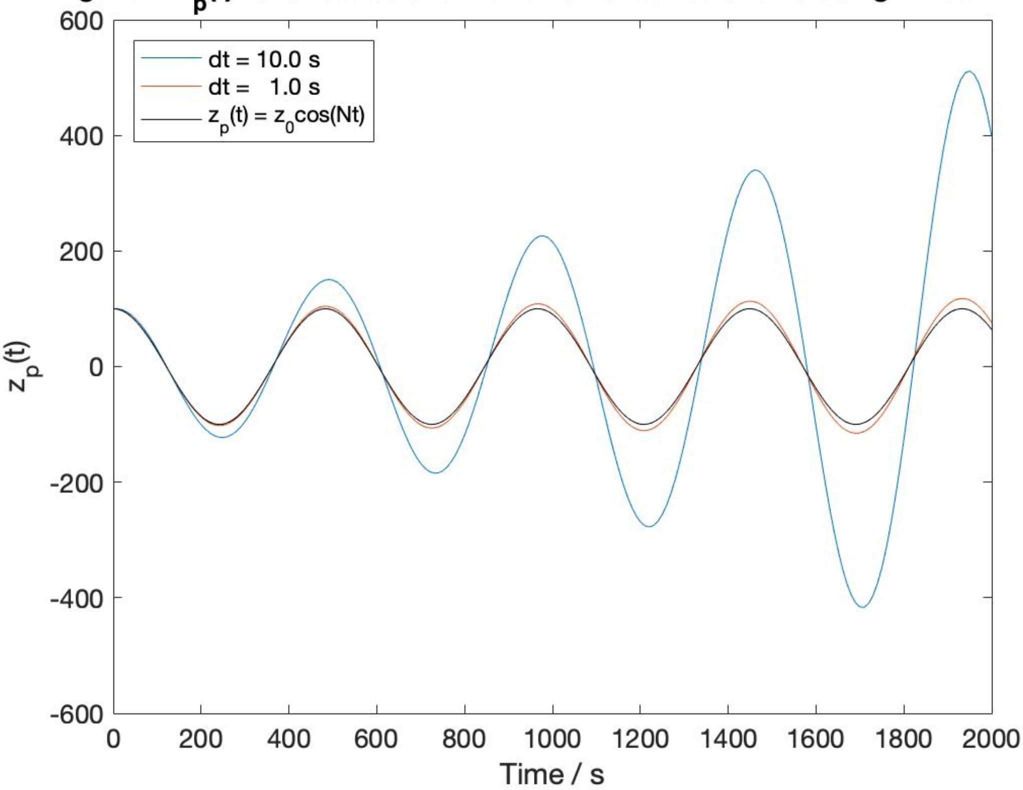

Figure 1 shows that, in the case of the Euler-Forward scheme, the smaller the time-step, the smaller the deviation from the analytical solution for vertical position. However, as we go further in time, the amplitude of the calculated solution does increase. This is due to an accumulation of errors from previous iterations which are used to predict future values in vertical position.

Figure 2: $w_p(t)$ for numerical solutions using EF Scheme

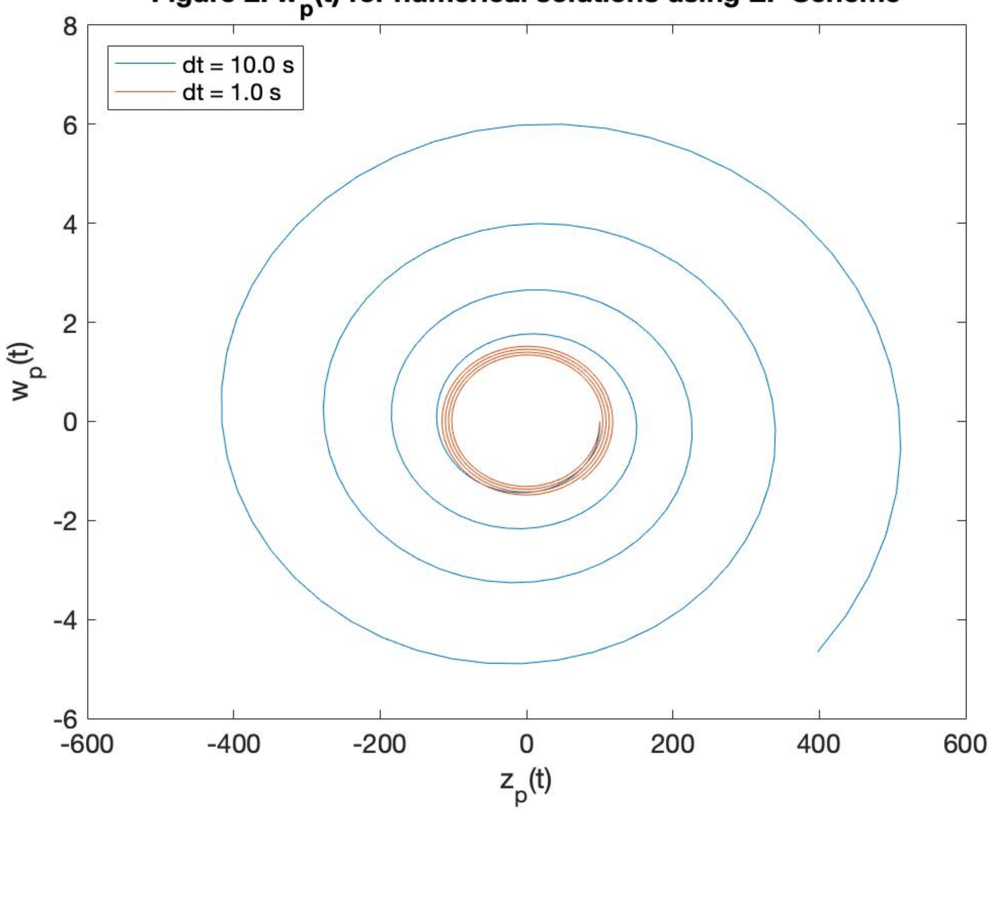

Figure 2 is the phase space of the system, generated using the Euler-Forward scheme. The ideal phase space would be represented by an oval, centred at (0,0). The figure above shows that the greater the time-step, the larger and faster the deviation from the oval. For Δt = 10 seconds, we see that the plot is spiralling outward with larger and faster-widening gaps as compared to the plot for Δt = 1 second.

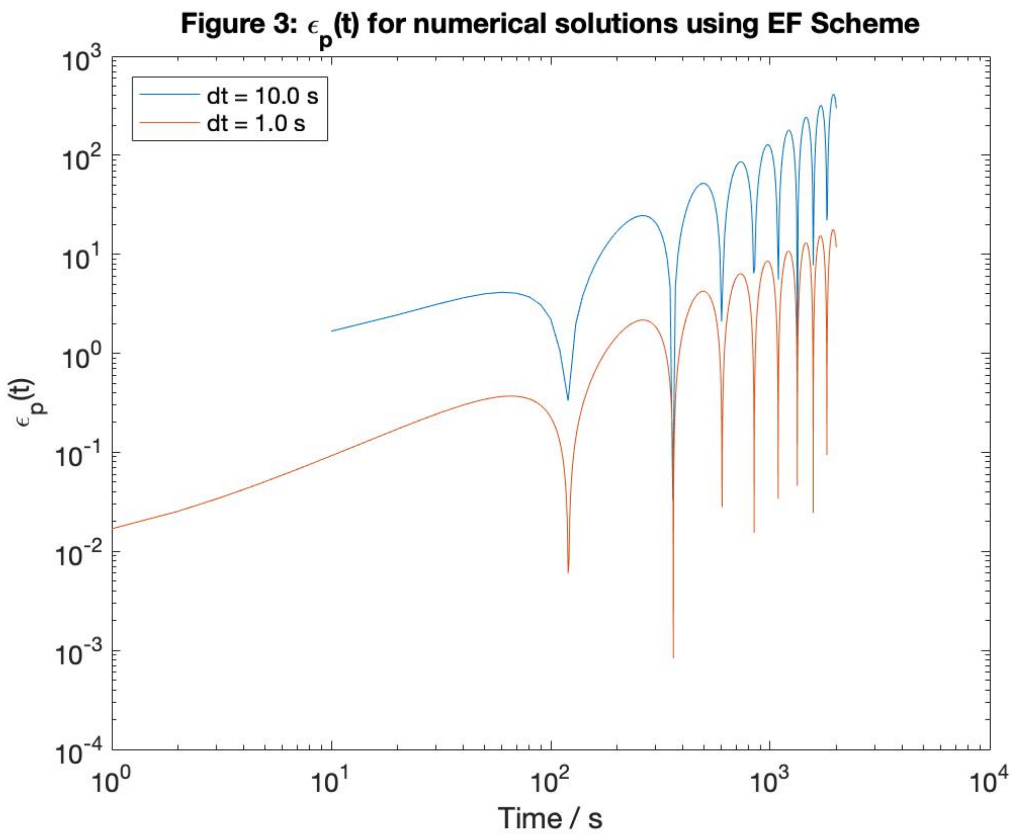

Figure 3 shows that the lowest errors at regular intervals of approximately 500 seconds, corresponding to moments in Figure 1 where the Euler-Forward approximation is closest to the point where its plot intersects with that of the analytical solution. This also corresponds to the time it would take for the air parcel to next return to its origin vertical position. The error for Δt = 10 seconds is consistently higher than that for Δt = 1 second due to the larger time-step which results in a weaker approximation and the accumulation of larger errors. The peaks in both errors increase with time; however, the rate of increase is presumably more pronounced in the case of the Euler-Forward approximation for Δt = 10 seconds.

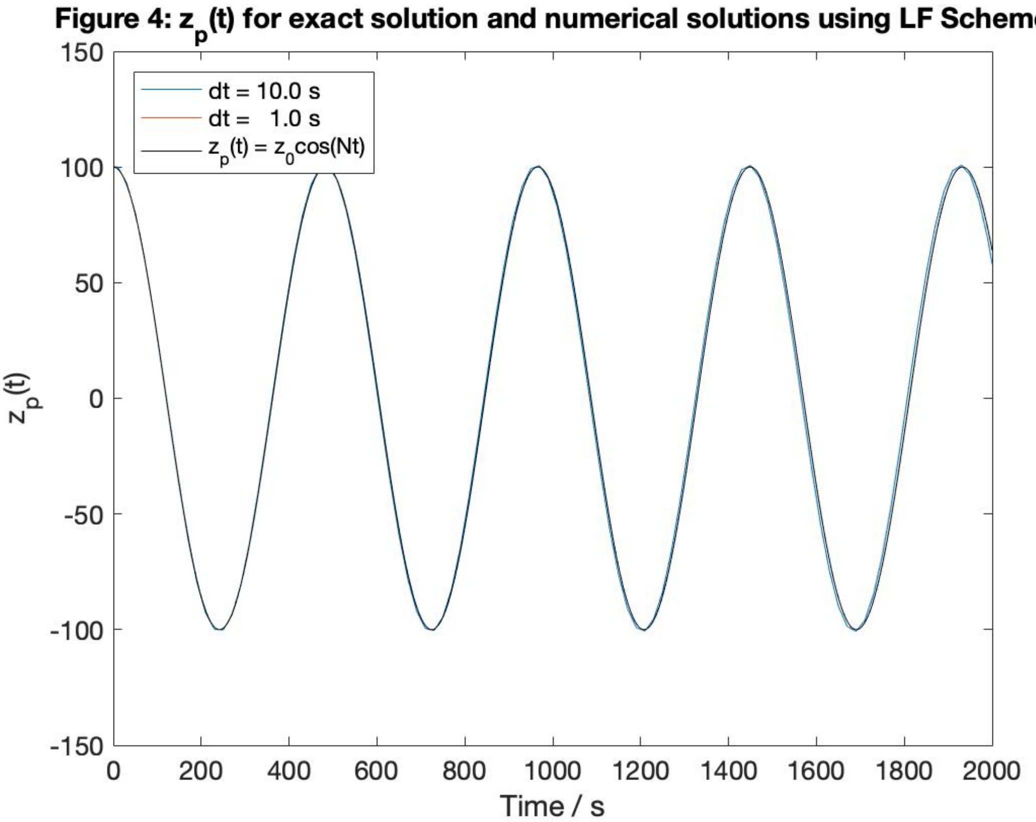

Figure 4 shows that the Leap-Frog scheme shows little deviation within 2000 seconds, even with a time-step as large as 10 seconds. This is because Leap-Frog is a multi-step method — each iteration requires knowledge of the values of the previous two — meaning that the Leap-Frog method is second-order accurate. The Leap-Frog scheme suggests that the calculated plot of vertical position has a shorter wavelength and that this difference accumulates and becomes more prominent as time progresses.

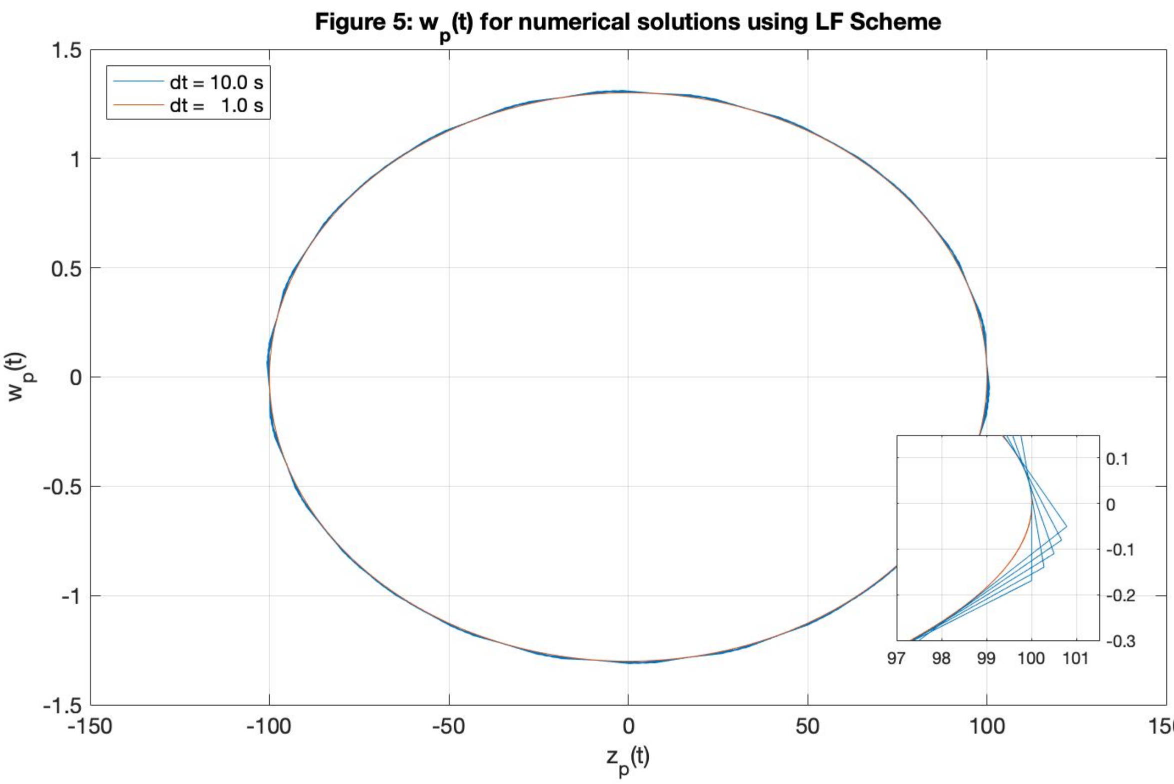

Figure 5 shows the phase diagram. As can be seen it is a lot closer to the ideal oval. The calculated solutions cause "correcting deviations" which prevent the solution from spiralling outward as was the case with the Euler-Forward plot (Figure 2). The zoomed inset shows this observation.

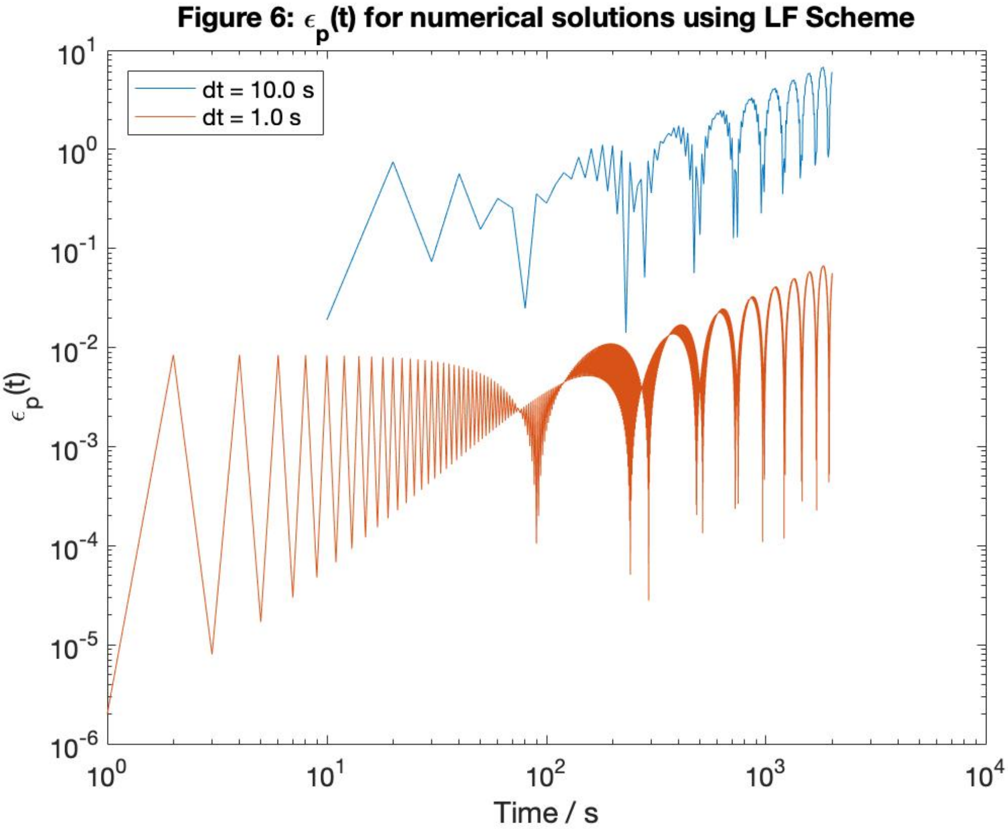

Figure 6 shows us that error associated with vertical position when calculated using the Leap-Frog scheme oscillates but is generally lower than the error associated with the Euler-Forward scheme. The overall shape is the same as was for Euler-Forward — there are peaks where error is greatest, corresponding to the points in Figure 4 which fall between the crest and the trough of each wave. There are dips in this plot where the error is smallest, corresponding to points in time in Figure 4 closest to where the calculated solution intersects with the analytical solution (near the peaks and troughs of the original cosine wave).