2D Poisson Equation

The goal of this investigation was to solve the following Poisson equation using successive over relaxation (SOR):

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta(x, y) \tag{1}$$

In this method, a finite difference scheme was used to discretize the Poisson equation. The following is the centered, second-order recursive equation used, with has a leading order error of $\mathcal{O}(\Delta x^2, \Delta y^2)$.

$$\left(\nabla^2\psi\right)_{i,j} = \frac{1}{\Delta x^2}\left(\psi_{i-1,j} - 2\psi_{i,j} + \psi_{i+1,j}\right) + \frac{1}{\Delta y^2}\left(\psi_{i,j-1} - 2\psi_{i,j} + \psi_{i,j+1}\right) \tag{2}$$

To iterate, we defined a residual term, $R = \nabla^2\psi - \zeta$. In essence, this means that the residual is an error term which tends to zero as our solution for the Laplacian of stream function, ψ, converges to the given vorticity, ζ. Using R, the recursive equation to approximate the stream function at each node was:

$$\psi_{i,j} = \psi_{i,j} + \frac{\alpha_{optimal}}{2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)} \cdot R_{i,j} \tag{3}$$

Here, the optimal α is calculated using a series of equations which are dependent on Δx and Δy.

$$\beta = \frac{\Delta x}{\Delta y} \tag{4}$$

$$\sigma = \frac{1}{1+\beta^2}\left(\cos\left(\frac{\pi}{nx}\right) + \beta^2\cos\left(\frac{\pi}{ny}\right)\right) \tag{5}$$

$$\alpha_{optimal} = \frac{2}{1+\sqrt{1-\sigma^2}} \tag{6}$$

SOR convergence is sensitive to choice of α. As numerical convergence depends on machine number round-off and/or truncation errors, a tolerance level of 10^{-7} was also defined. The criteria used to confirm convergence was ε, defined as:

$$\varepsilon = \frac{max\{\left|R_{i,j}\right|\}}{2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) \cdot \sum \psi_{i,j} + max\{\left|\zeta_{i,j}\right|\}} \tag{7}$$

Given an initial condition for vorticity, stream function can be calculated. This in turn can be used to produce a velocity field with longitudinal and latitudinal velocities, u and v.

$$u = \frac{\psi_{i,j-1} - \psi_{i,j+1}}{2\Delta y} \text{ and } v = \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x} \tag{8}$$

To analyze the numerical method, the resultant stream function can be used to calculate a numerical vorticity:

$$\zeta_{numerical} = \frac{1}{\Delta x^2}\left(\psi_{i-1,j} - 2\,\psi_{i,j} + \psi_{i+1,j}\right) + \frac{1}{\Delta y^2}\left(\psi_{i,j-1} - 2\,\psi_{i,j} + \psi_{i,j+1}\right) \tag{9}$$

This numerical vorticity was compared to the initial vorticity to determine the degree of accuracy of SOR.

In this investigation, two sets of boundary conditions were used. The first, RUN A, used periodic boundary conditions in both *x* and *y* directions. RUN B used Dirichlet boundary conditions in the *y*-direction wherein the $v = 0$ along both boundaries.

Outlined in this report are plots of some of the parameters defined above for the two runs on a 100 x 100 m domain with centres of two identical, counter-clockwise rotating vortices at (50, 25) and (50, 75).

RUN A took 2929 iterations to converge such that $\varepsilon < 10^{-7}$.

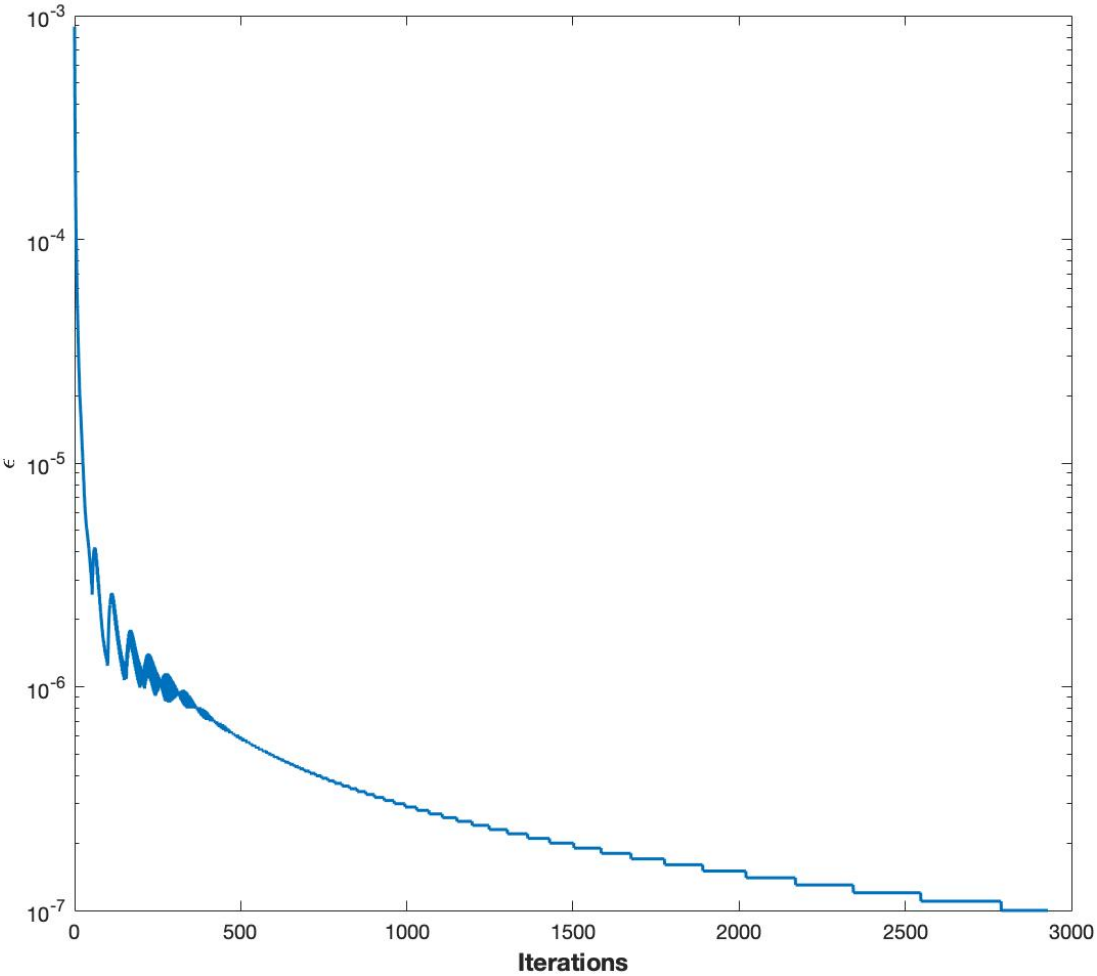


Figure 1A shows that the rate convergence of ε is greatest initially. There is a period of time where the value of ε fluctuates slightly. After this point, convergence follows a step-like pattern. As ε is directly proportional to the maximum absolute value of the residual, we would expect epsilon to decrease as residual does. In addition, ε is also dependent on ψ. As this run begins with an initial guess where $\psi = 0$ at all the nodes across the domain, ε begins at its largest value, 1. After this point, successive iterations reduce the value of ε as shown above.

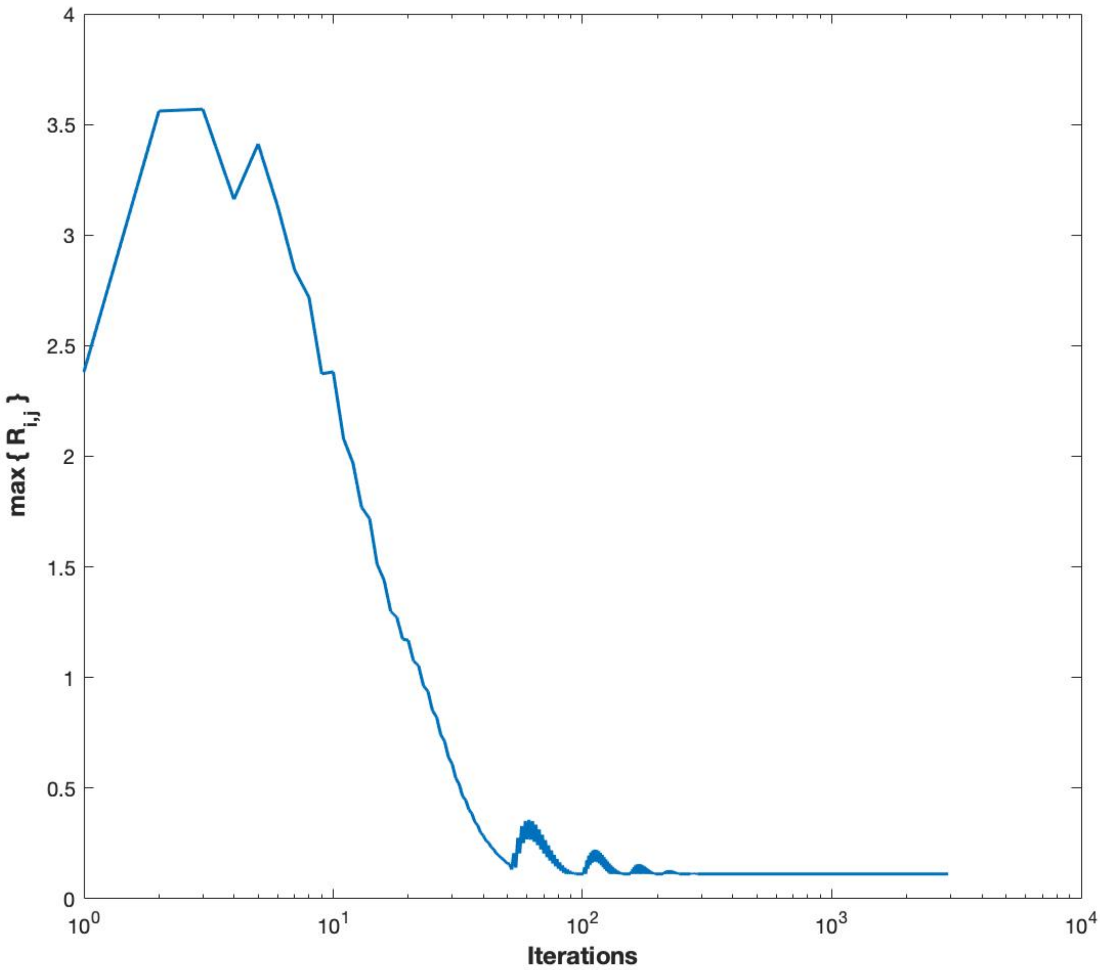


Figure 1B shows the convergence as reflected by the maximum value of the residual across the domain. As can be seen, this maximum value actually increases in the first iteration and then quickly decreases after successive iterations. There are a few small scale increases in the maximum residual at and around the 100th iteration which coincides with the region of fluctuation in the plot of ε.

The evolution of ψ can also be traced.

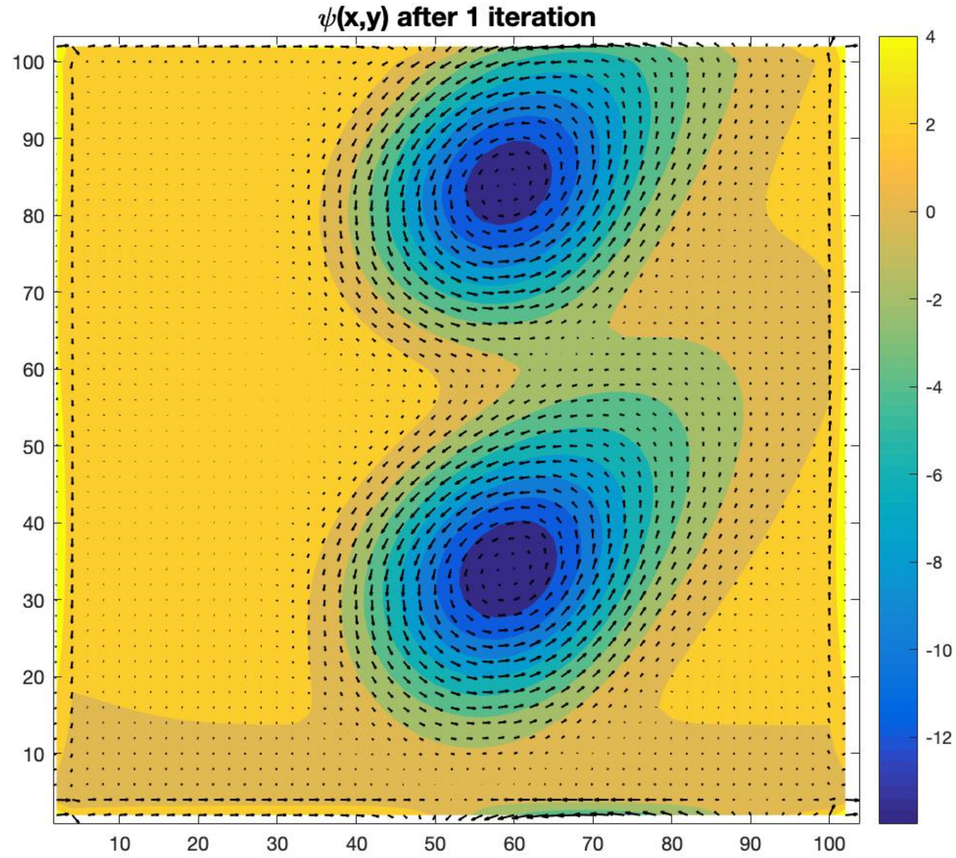


Figure 2A shows ψ across the domain after the very first SOR iteration. Quiver plots overlaid on top show the velocity field, calculated using centered difference schemes in the *x* and *y* directions. We can see that ht centers of the vortices are captured accurately; however, the stream function is elongated on the northeast corners of the vortices. This is more clearly shown by the velocity vectors which are greater in magnitude all around the vortex expect on this corner. This occurs because the SOR algorithm approximates residual and ψ from left to right starting from $y = 0$. This means that the initial iteration will underestimate the values at the bottom left corner (neat $y = 0$ and $x = 0$) due to there being little information available aside from the initial guess. As successive points above and to the right are calculated, more information in addition to the initial guess is being incorporated.

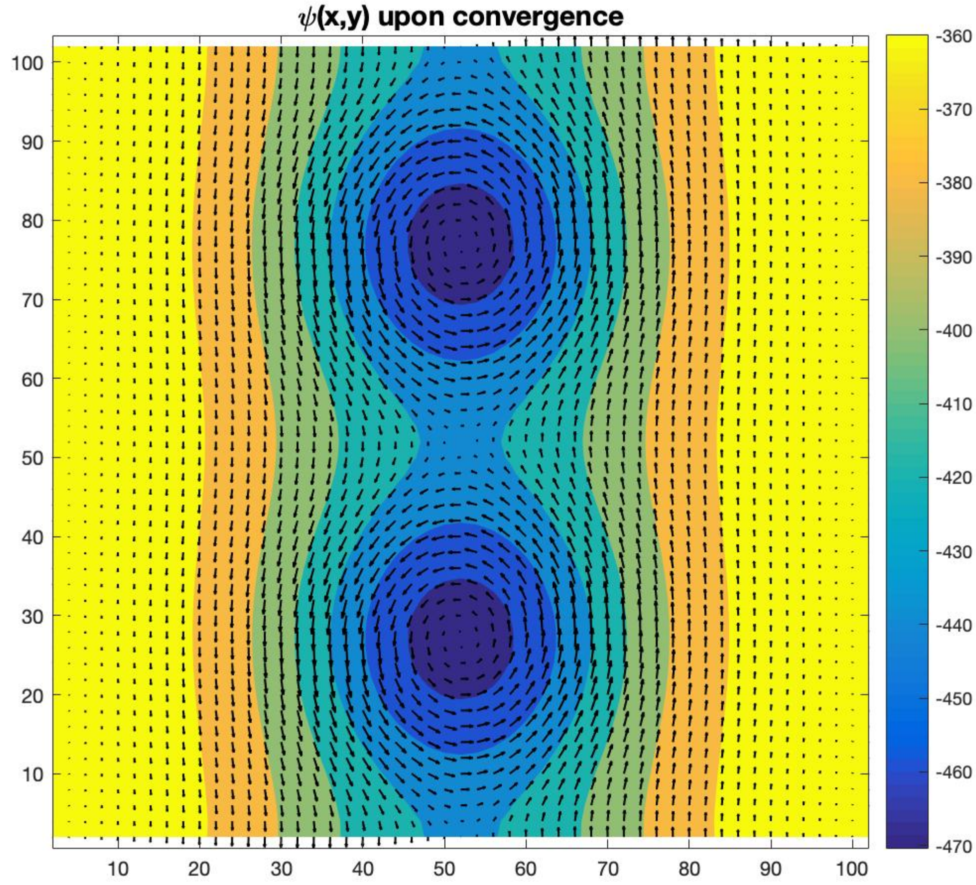


Figure 2B, in comparison, shows what ψ evolves to upon convergence. Here, there is perfect symmetry along the central north-south axis. This is as predicted as the vortices are modelled using a symmetrical cosine function. Due to the periodic boundary conditions, each vortex exerts the same effect on the other which is why this symmetry is maintained. This is also the reason why we have velocity vectors which point into and out of the domain along $y = 0$ and $y = 100$. The values are also much smaller than those in Figure 2A. This is because we started with an initial guess where $\psi = 0$ across the domain. As each iteration incrementally changes the values of ψ across the domain, the values of ψ deviate more and more from 0 until convergence.

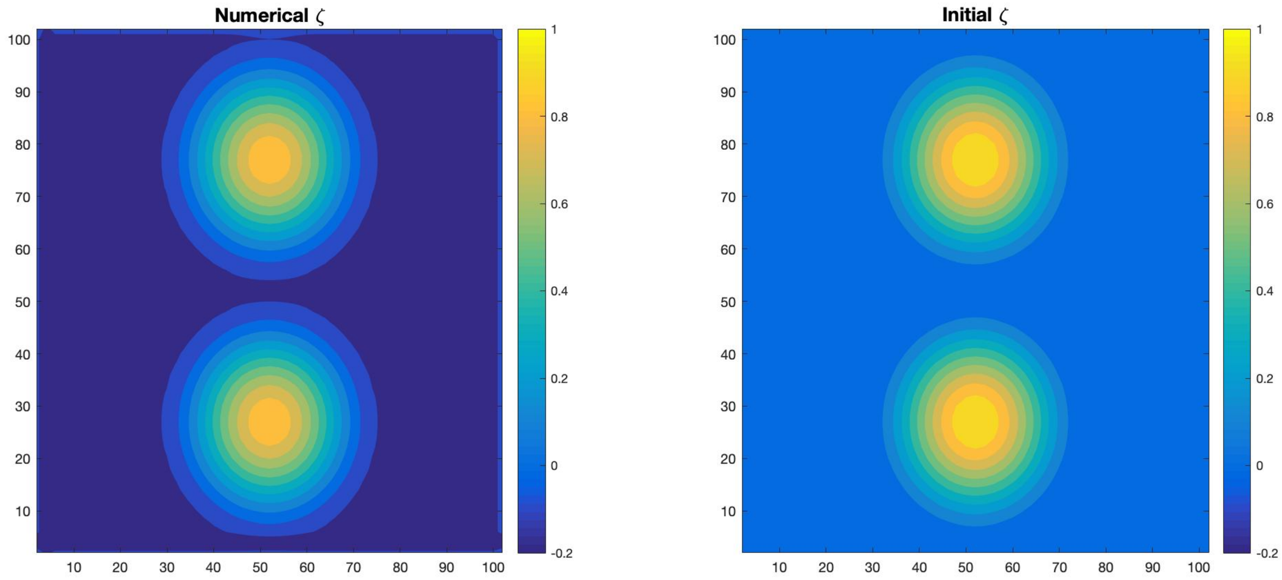
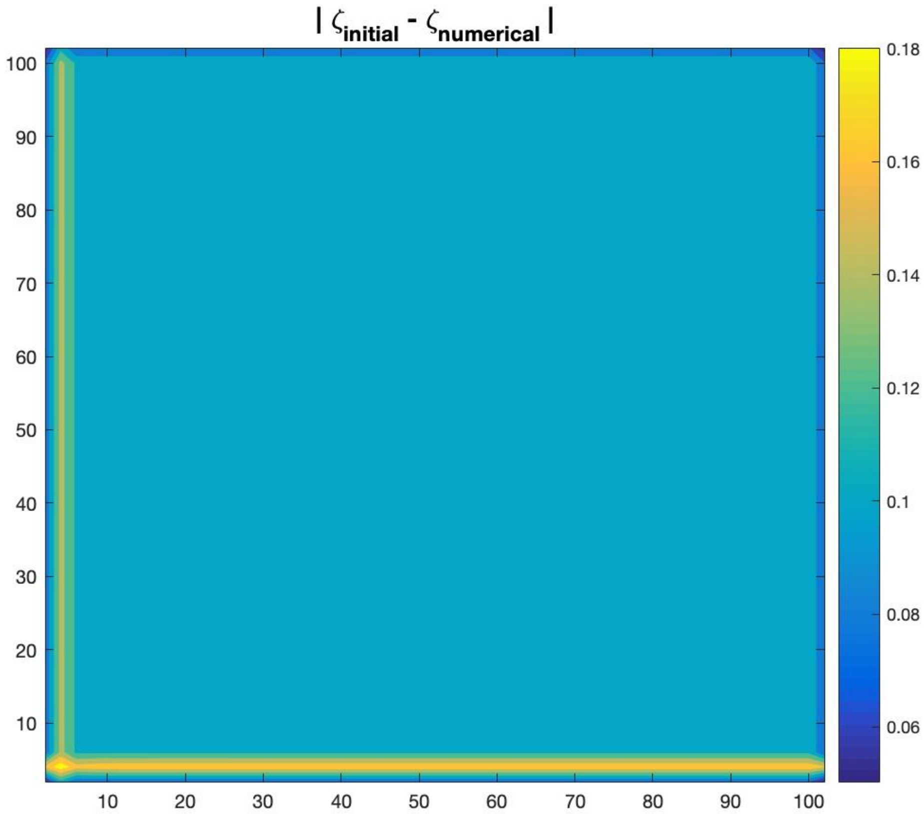


Figure 3A shows the initial vorticity field on the right and the numerical vorticity on the left, calculated using the Laplacian of the stream function obtained upon convergence. Clearly, both fields are similar in their depiction of the vortices and sizes. While the initial ζ has no negative values within the domain, the numerical ζ does.



The differences between the initial and calculated ζ-fields are more apparent in **Figure 3B**. To a large extent, the absolute difference is around 0.1 across the domain. On the y = 100 and x = 100 boundaries, we have slightly larger differences. Along the x = 0 and y = 0 boundaries, the differences are a lot smaller in magnitude. This is because of the way in which SOR iterations progress. As we begin from the bottom left corner, we use the initial guess to calculate a better value at the point (0,0). Thereafter, we progress across the domain and calculate (0,:) using the information from the calculation immediately before and the guesses for all the points which are yet to be iterated. This is why we see a bright yellow stripe of relatively large difference in vorticity along the boundary where $y = 0$. Following this, all the points along (:,0) would be the least accurate in their respective rows as they are the first in the row to be iterated. This is why we see relatively high error along the boundary where $x = 0$. The intersection of these two lines is at (0,0), where error is expected to be greatest as it is the first point in the domain to be iterated and therefore has the least up-to-date information available for the calculation. Errors are lowest along $y = 100$ and $x = 100$ for the contrapositive reason. The intersection of these is (100,100) which is the last point in the domain to be iterated. It has the most up-to-date information available for the calculation and so the smallest difference in numerical and initial ζ.

RUN B took 141 iterations to converge such that $\varepsilon < 10^{-7}$. Due to the existence of walls on the y boundaries, it makes sense that convergence is faster in RUN B as the solutions to calculate are bounded. In a periodic boundary condition (RUN A), there are infinite solutions to be considered.

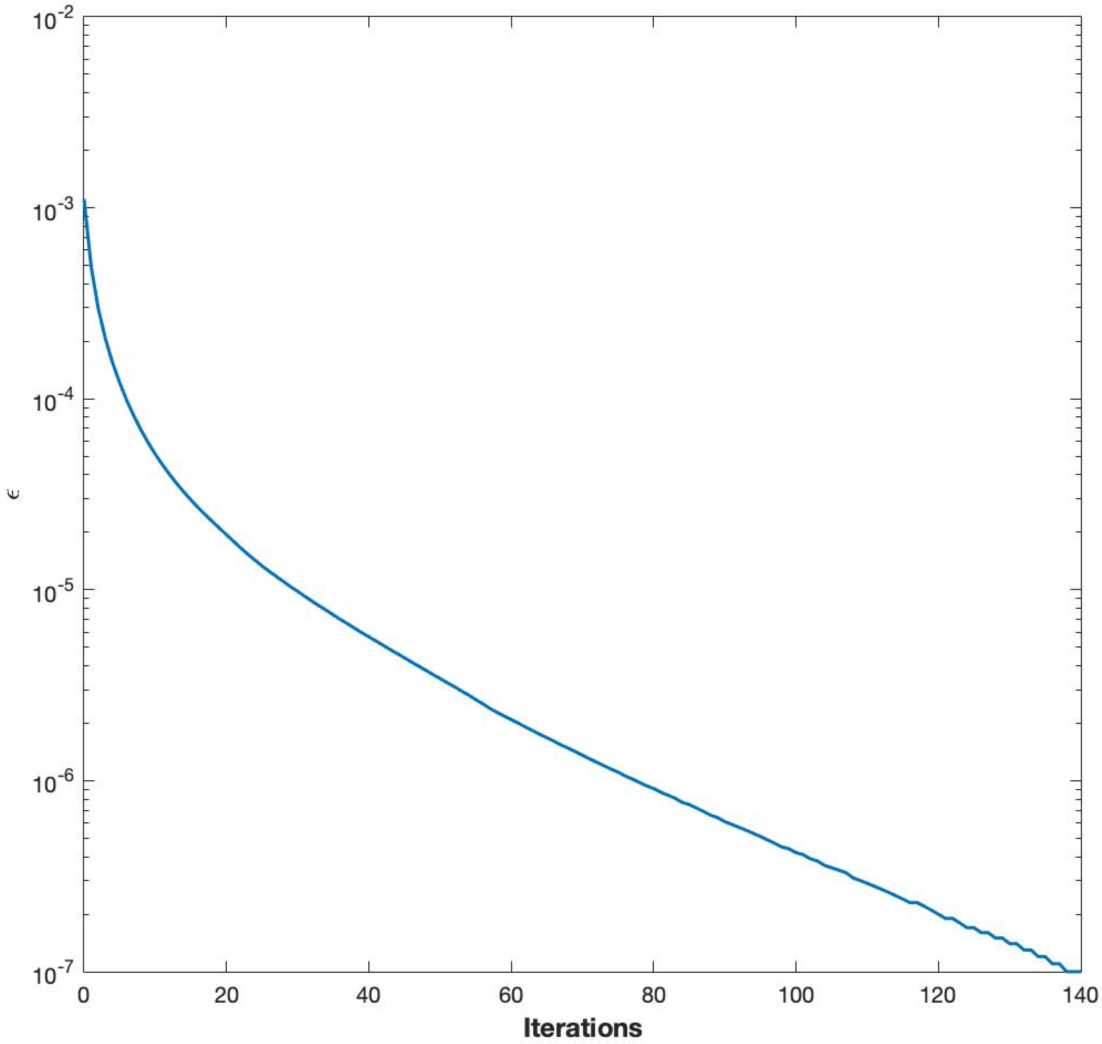
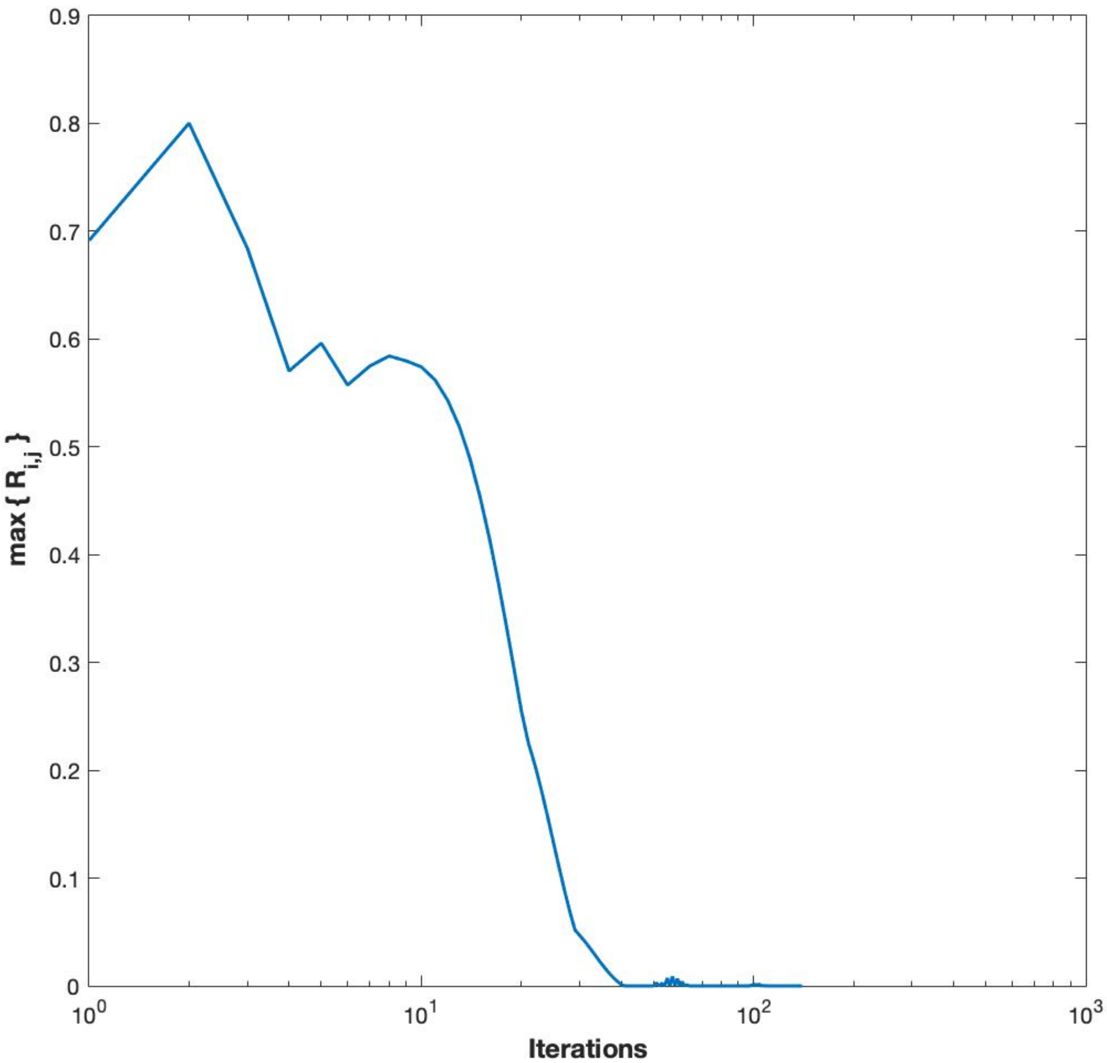


Figure 4A shows that ε is continually converging or decreasing. As there exists a unique solution, the convergence is direct with no regions of fluctuation was was observed in RUN A. The rate of convergence is greatest initially, but as the solution evolves from the initial guess to the final solution, we see that the rate of convergence decreases as there are smaller and smaller changes made per iteration. There is also a step-like decrease in the value of ε towards the end. This is typical of convergence which depends on machine precision and how it deals with round-off errors.



As expected, the convergence of residual is similarly fast. However, **Figure 4B** shows that there is an initial increase in maximum residual. This is known to happen in SOR iterations as error may sometime increase before it starts reducing.

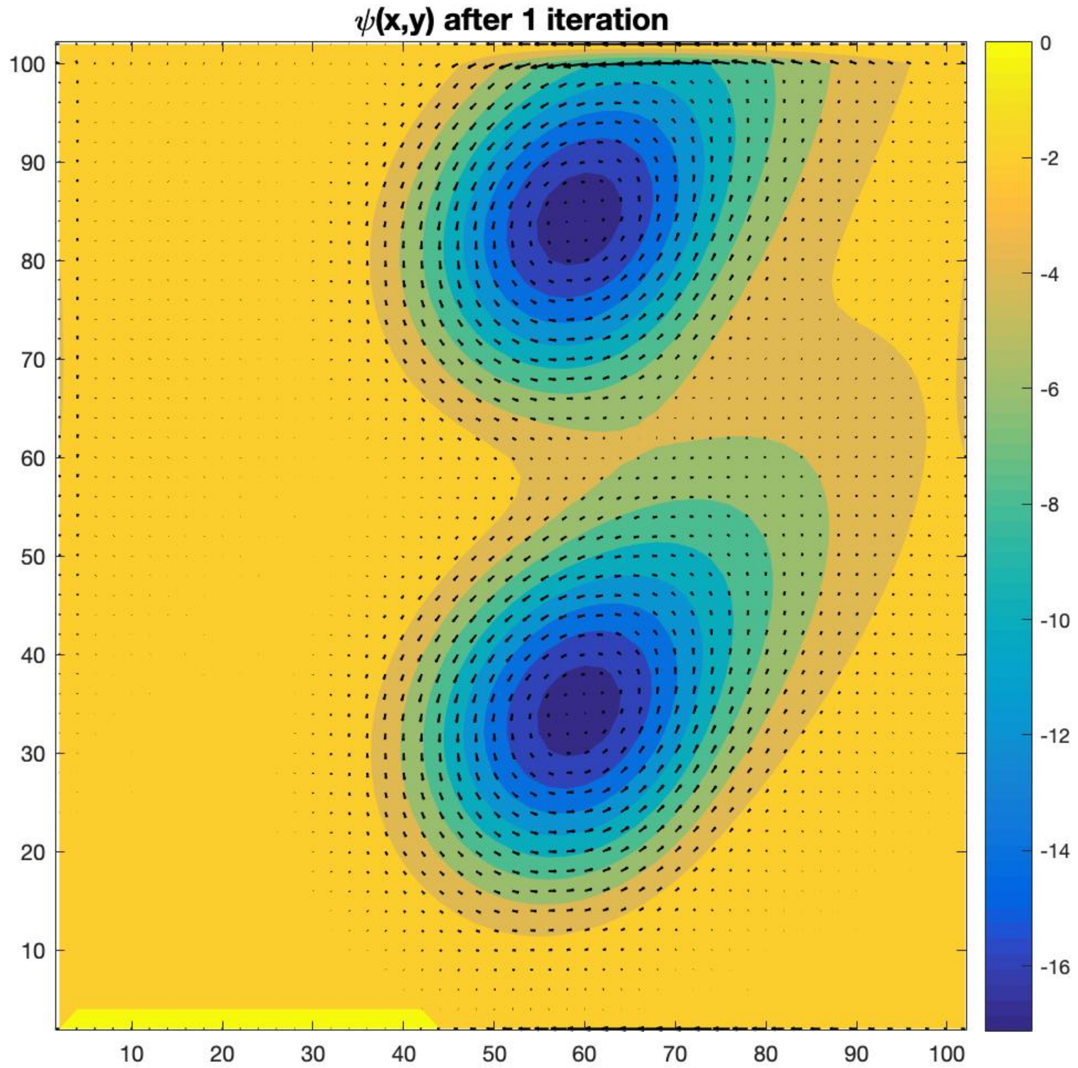


Figure 5A shows ψ across the domain after the very first SOR iteration. It is similar to Figure 2A in the way in which the vortices look to be elongated in the north-east direction. The difference is that we can clearly see the effect of the wall which abruptly forces velocity in the *y*-direction to be 0. An indication that the wall has been successfully captured is the fact that the velocity field strictly limits flow to be in the *x*-direction only at the boundaries $y = 0$ and $y = 100$.

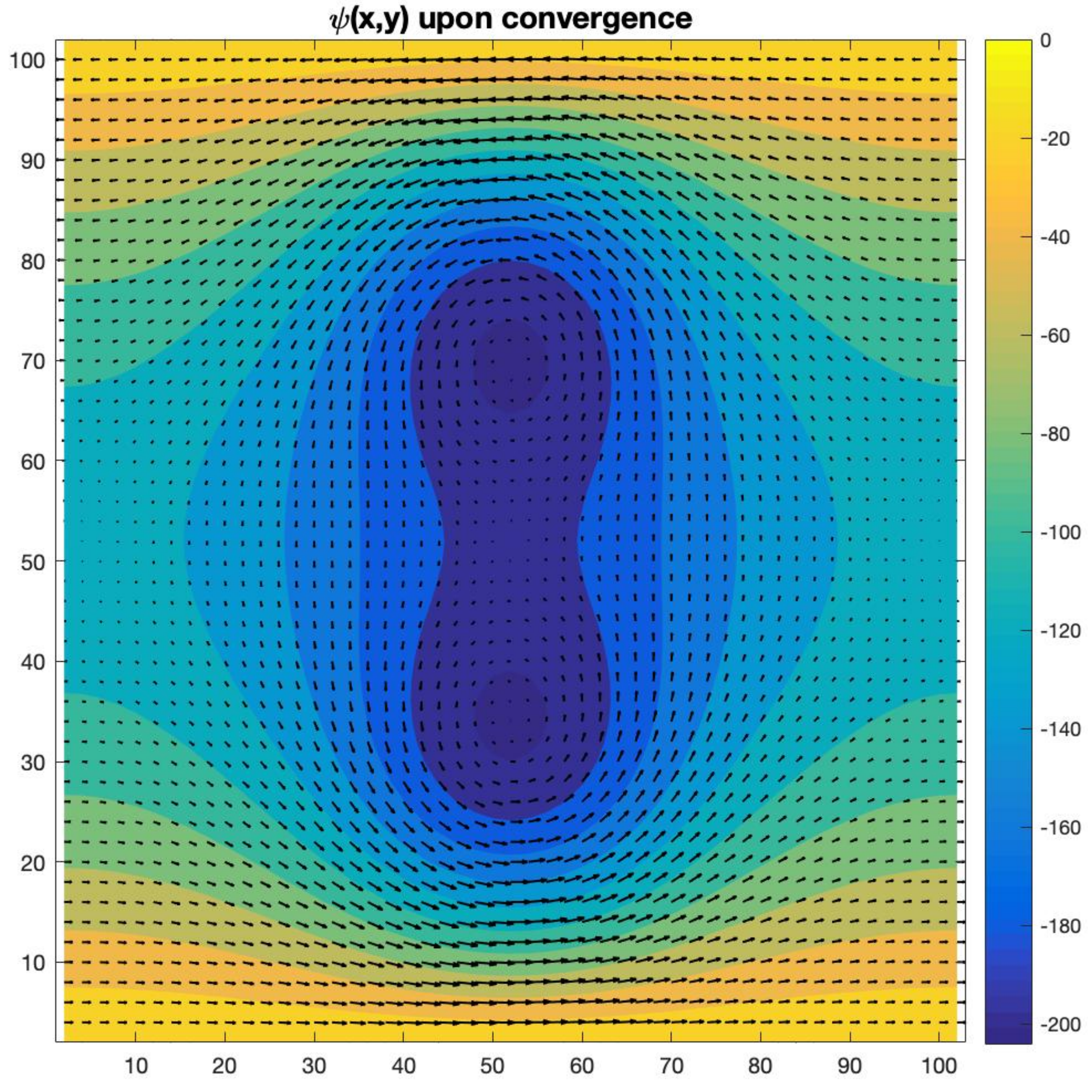


Figure 5B shows ψ across the domain upon convergence. Here, we can see that the vortexes are interacting strictly within the bounds of the two walls along $y = 0$ and $y = 100$. The velocity field shows that velocities are greatest between the vortex center and the wall that it is closest to due to the restriction imposed on airflow in this area. The overall effect is of an oval-shaped vortex rotating counter-clockwise with effectively two centers which are at the centers of the original two vortices.

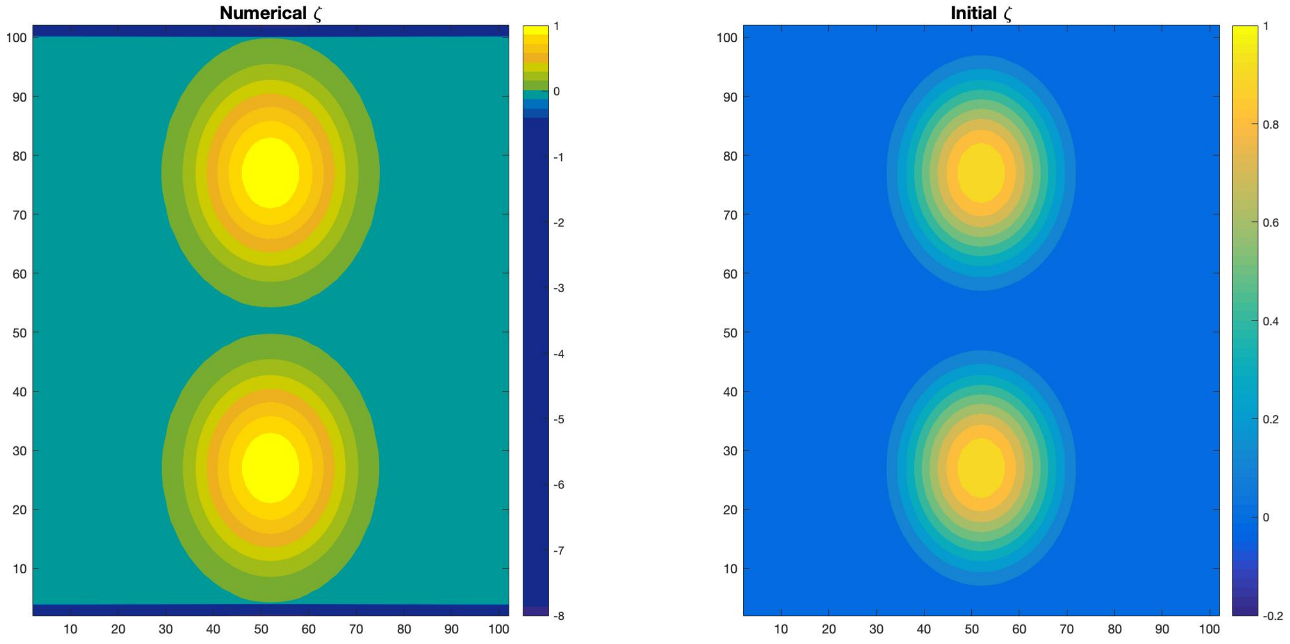


Figure 6A shows the initial and numerical vorticity fields. Once again, both fields are similar in their depiction of the vortices and sizes. This is difficult to gauge as there exist highly negative values for vorticity along the $y = 0$ and $y = 100$ boundaries where the walls exist. This skews the distribution of colours in the color bar.

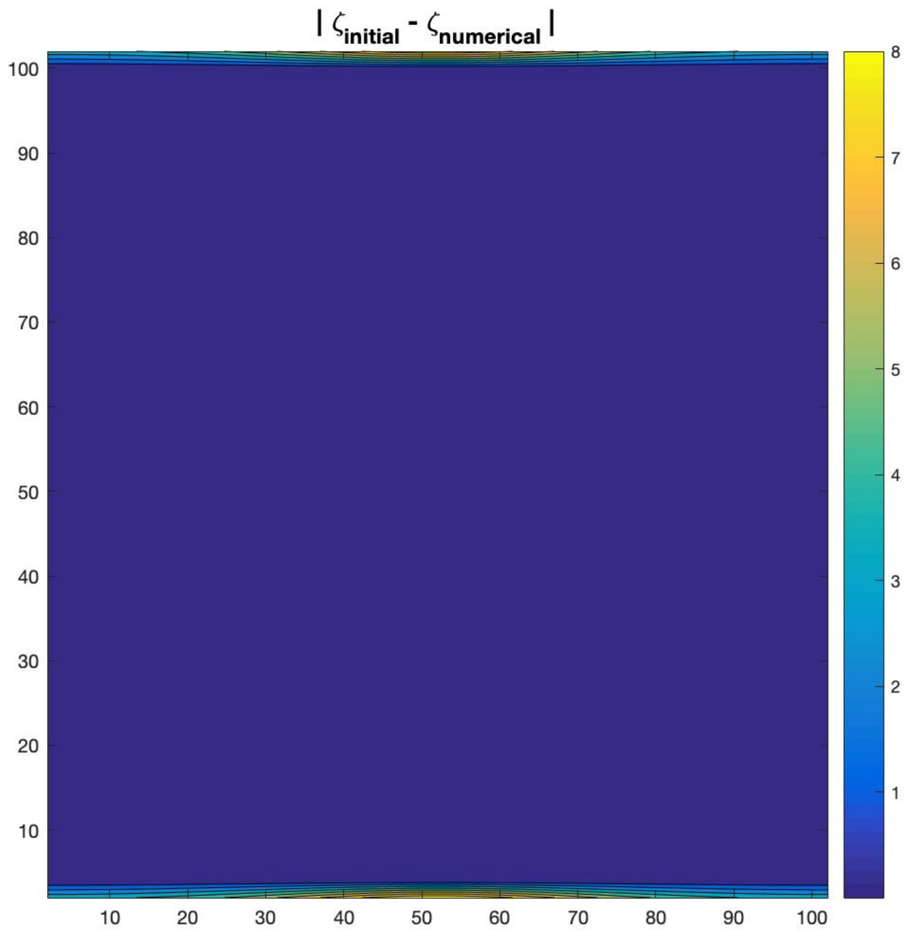


Figure 6B more clearly shows the difference between the numerical and initial vorticity. We see that the absolute difference is near 0 across the domain except for on the boundary walls where the difference is highest in the center of the wall and relative lower at the edges. This is as expected as, due to the walls, the nodes along these boundaries are not iterated.