A Linear Wave Equation

The goal of this investigation was to use the Forward-in-Time-Backward-in-Space (FTBS) scheme to determine the solution of a one-dimensional linear wave equation on a fixed domain. The plots below show the results of the scheme using variable sizes of time-steps and various wavelengths, thereby delineating he effects of changing the Courant's number (C). Also shown are the resultant numerical errors and amplitudes of the solutions as functions of time. The attached file contains the C++ code used to generate output files named "summary_statistics.csv", "full_solution.csv", and "exact_solution.csv".

The FTBS equation used to update the solution of the wave is as follows:

$$u_i^{n+1} = u_i^n - C(u_i^n - u_{i-1}^n) \tag{1}$$

The interior nodes of the solution were solved for recursively using equation (1). The right boundary was solved for using the numerical solution for the node to the left of it. Due to the periodic nature of the boundary condition, the left boundary was equated to the solved value for the right boundary. Thus, a full advancement of the solution in time was achieved.

Experiment #1 serves as the base case for code testing with optimal C. Here, Δt is 2 seconds and the wavelength, λ_x, is 50 m. In this experiment, the exact solution and the calculated solution are the same.

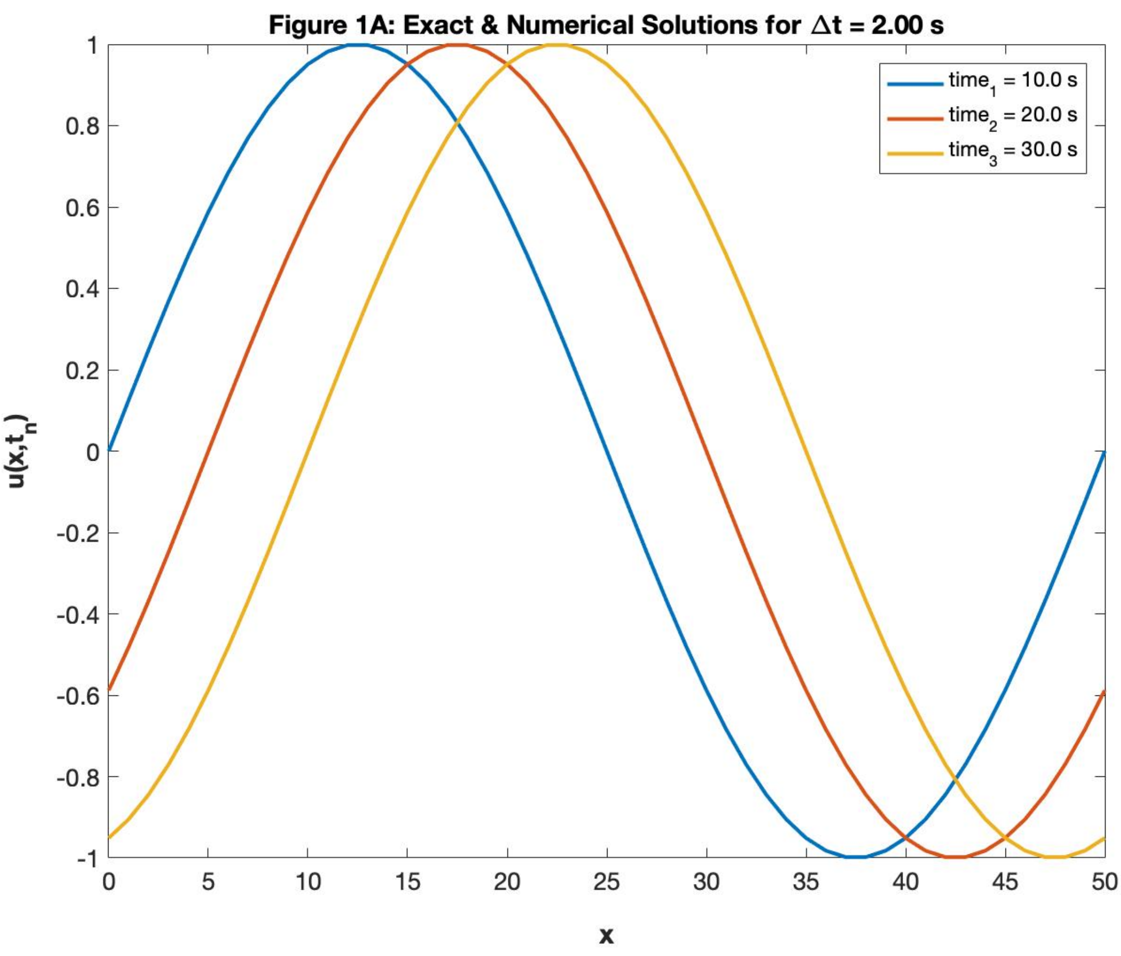

Figure 1A shows how the wave propagates with time, maintaining an amplitude of 1 with each time-step. Using the cyclic boundary condition, it can be deduced that the time taken for the wave to return to its initial condition is λ_x / c = 50 / 0.5 = 100 seconds.

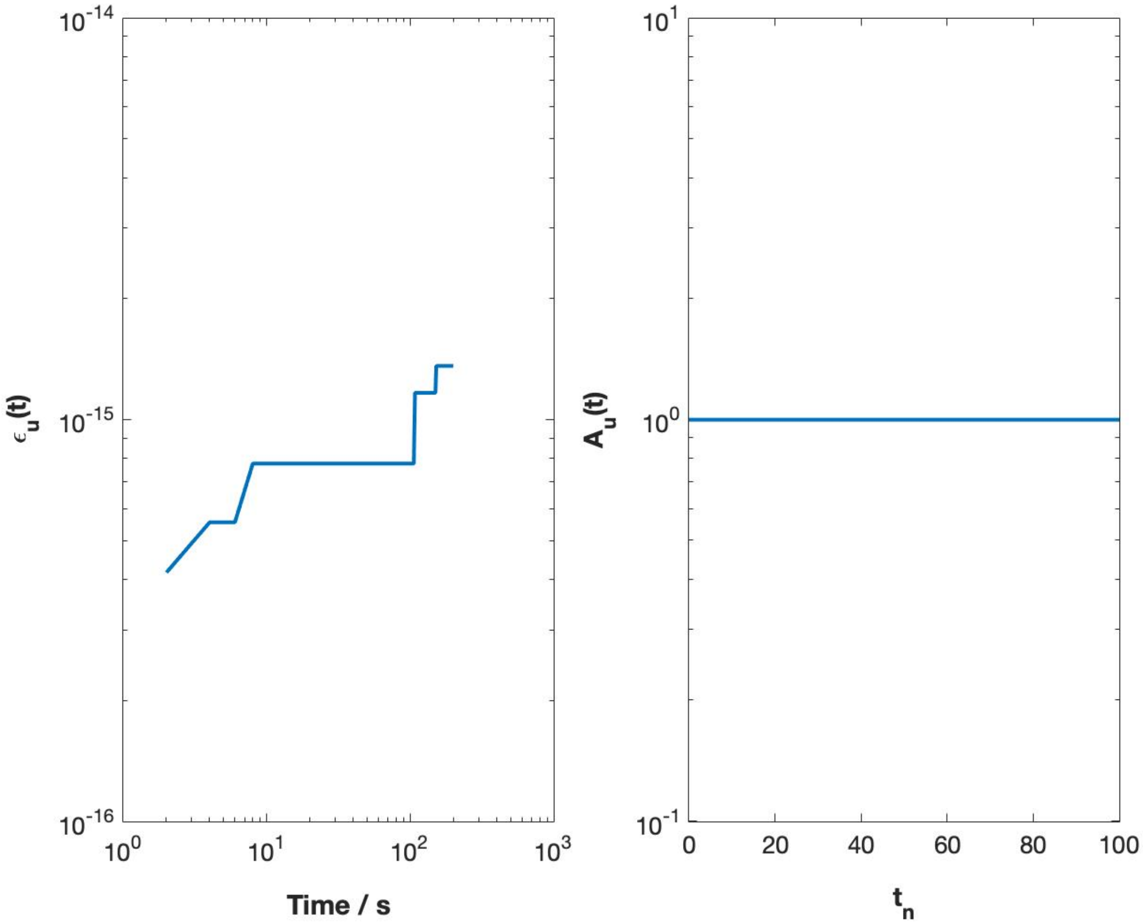

Figure 1B tells us that the error associated with the numerical solution in this experiment is minimal and is the result of small round-off errors accumulating in the computation for each time-step. As they have such a small magnitude, we can see that the overall amplitude of the wave remains unchanged during the 200 second period of observation.

Experiment #2 allows us to investigate the effect of changing C by reducing the size of Δt. C is directly proportional to Δt so a decrease in Δt will result in a corresponding decrease in the value of C. We can compare simulations for Δt = 0.25, 0.5, 0.75, and 1.0 seconds to see how and to what extent the numerical solutions deviate from the analytical solution.

We can do this while keeping in mind that our approximate equation is equivalent to:

$$\left(\frac{\partial u}{\partial t}\right)_i^n + c\left(\frac{\partial u}{\partial x}\right)_i^n = \boxed{\frac{1}{2}c\,\Delta x\left(1 - c\frac{\Delta x}{\Delta t}\right)\left(\frac{\partial^2 u}{\partial x^2}\right)} + \vartheta\left[\Delta t^2, \Delta x^2\right] \tag{2}$$

The leading order error term is boxed above and is equal to 0 when 1 - C = 0 (i.e when C = 1). In experiment #1, we saw that when Δx = 1m, Δt = 2 seconds, and c = 0.5 m/s, we have C = 1, in which case there is no numerical diffusion. Experiment #2 effectively shows us what happens when C < 1 (i.e, when there is artificial diffusion in the numerical solution).

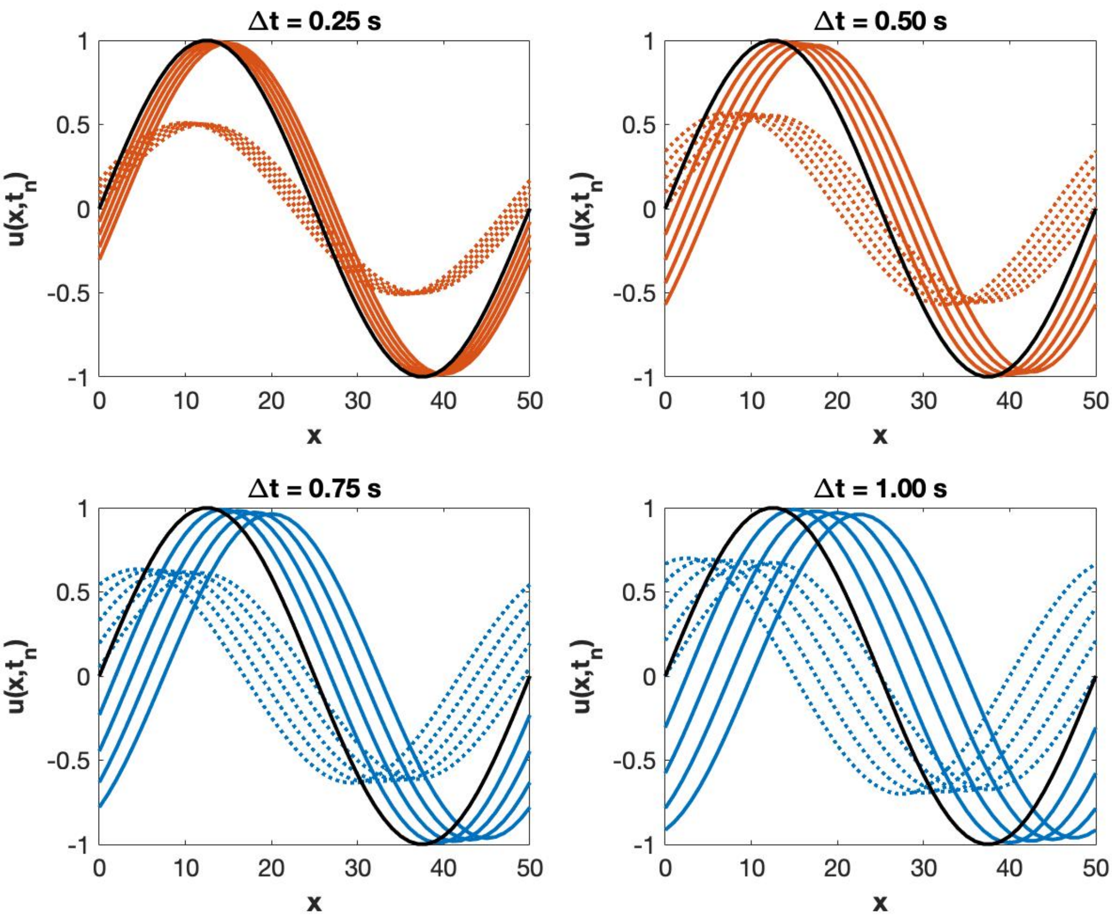

Figure 2A captures the trends observed when the simulation is run for the above listed values of Δt. The solid lines show the numerical result during the first few time-steps observed while the dotted lines represent the numerical solution obtained during the last few time-steps of the simulation. The waves in all four cases are consistently being dampened; however, this occurs to varying degrees. The smaller the Δt, the more pronounced the dampening effect. This is because the smaller the Δt, the smaller the Courant's number, C, and the greater the artificial diffusion that is manifested in the calculations.

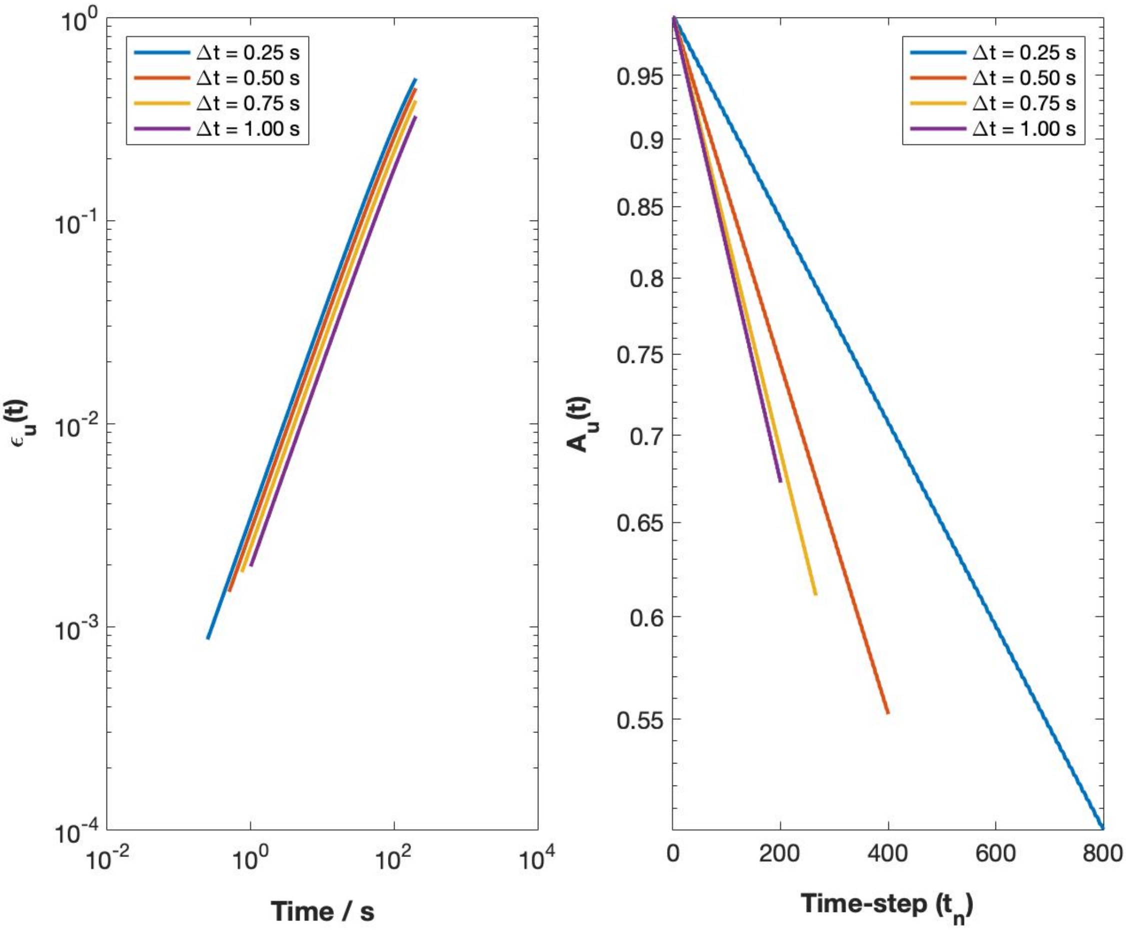

Figure 2B shows how error and amplitude vary over time and time-steps respectively. For all four Δt values, the associated error increases exponentially as we progress through time. The greatest error consistently occurs in the calculations for when Δt is 0.25 seconds. Accordingly, the amplitude for each simulation decreases exponentially, with the greatest overall decrease occurring when Δt is 0.25 seconds. These observations further explain the importance of choosing an appropriately large Δt to reduce the effect of "numerical diffusion" in the leading order error term.

Experiment #3 explains that Δt cannot be arbitrarily large either. By increasing the size of Δt, we are also increasing the value of C. When C > 1, we introduce "negative diffusion" which amplifies the amplitude. The greater the value of C, or the greater the value of Δt, the faster this occurs.

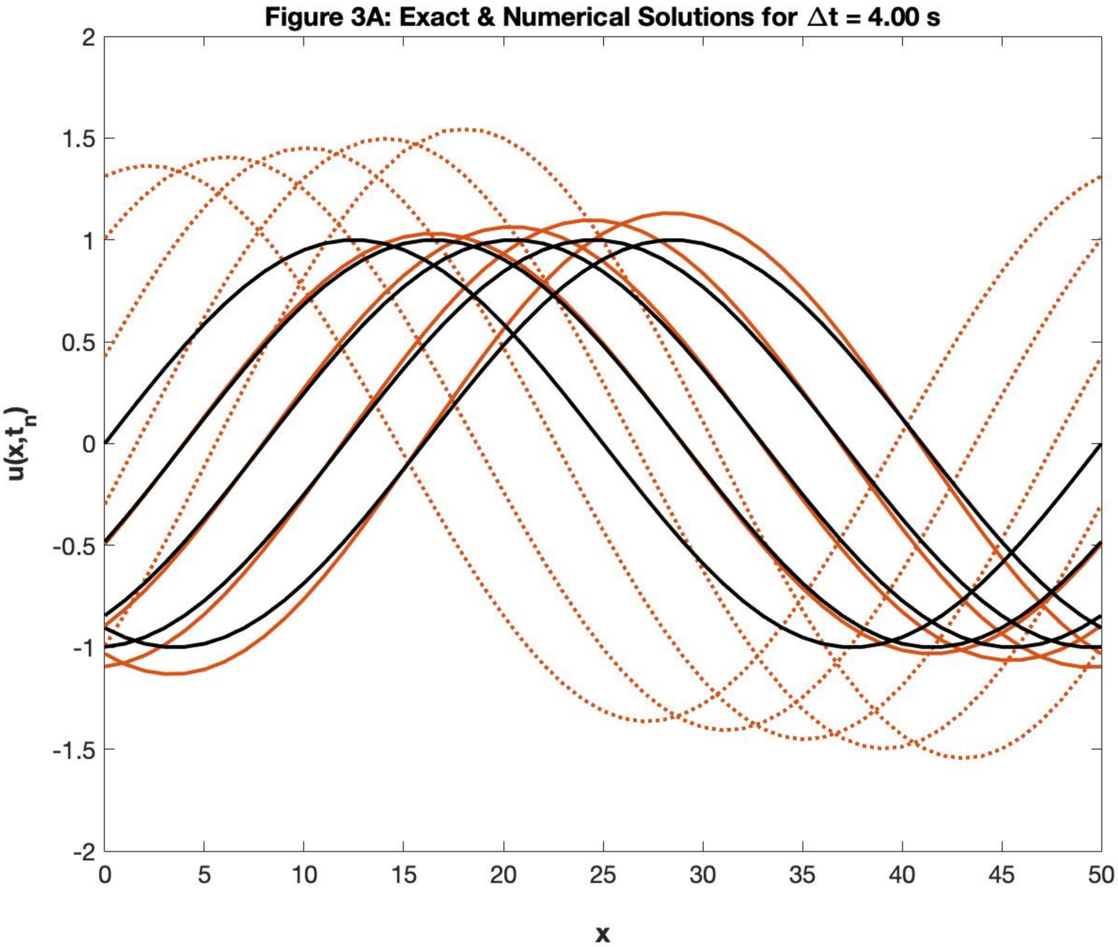

Figure 3A illustrates the case when Δt = 4.00 seconds. The solid red lines show gradual but clear departure from the exact solution, represented by the solid black waves. The dotted lines represent the numerical solution for the wave at arbitrary points in the future, chosen to illustrate the effect of the increasing amplitude before the solution expands ("blows up") beyond comprehension.

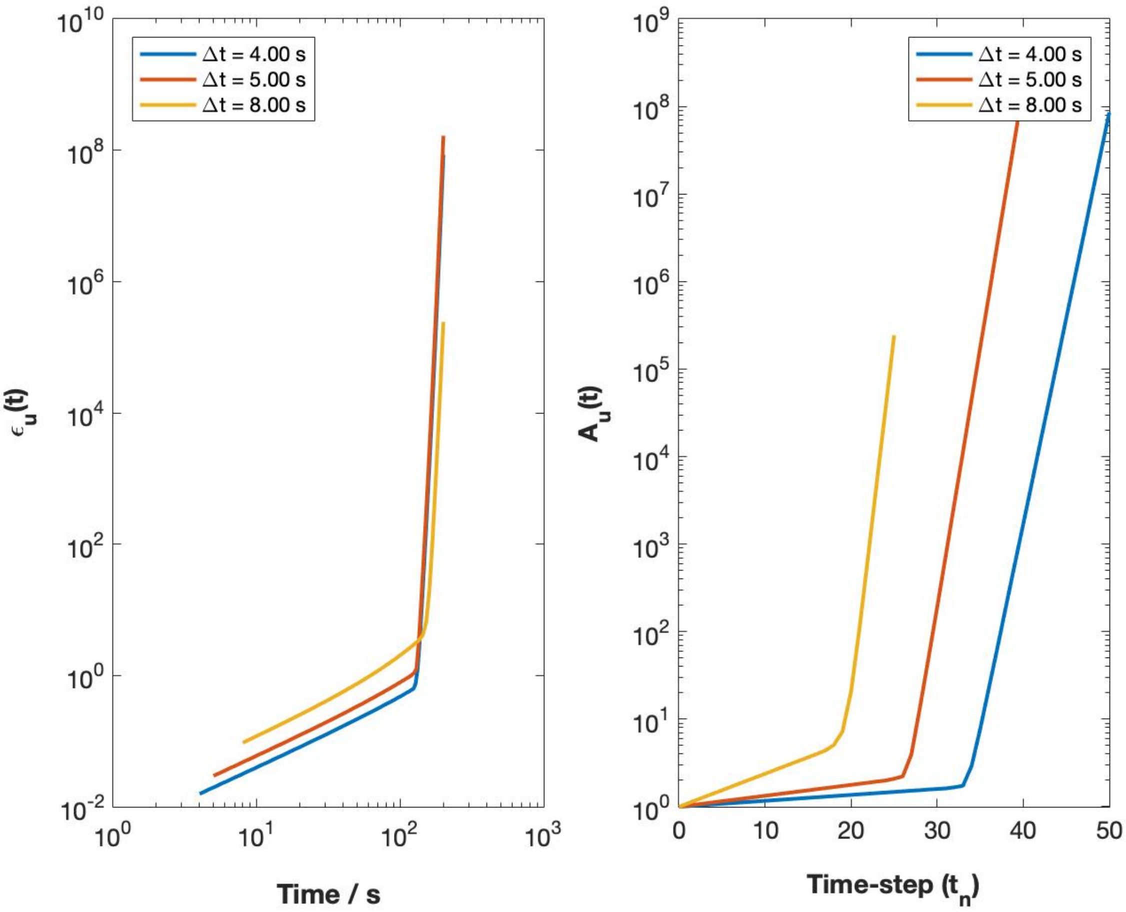

Figure 3A allows us to very clearly trace the particular moment in time and the exact time-step for when the solution expands uncontrollably. We can see that this moment occurs at different time-steps, depending on the size of Δt and, accordingly, on the size of C. This "blowing up" moment in all cases occurs once the error has grown to 1.

When completing a stability analysis of the FTBS scheme, we can define a changing function, **λ,** that relates the solution at t to a solution at t + Δt.

$$\lambda \equiv e^{-i\omega\Delta t} \tag{3}$$

Thus, our discrete equation (1) can be written as:

$$u_i^n e^{-i\omega\Delta t} = u_i^n - C\left(u_i^n - u_i^n e^{-ik\Delta x}\right) \tag{4}$$

The amplification factor, $|\lambda|^2$ can then be solved for in terms of C, k, and Δx.

$$|\lambda|^2 = 1 + 2C(1 - C)(\cos(k\Delta x) - 1) \tag{5}$$

As we want to ensure that $|\lambda|^2 = 1$, we can impose restrictions on these other terms. The Courant-Friedrichs-Lewy condition is that we want $|C| \leq 1$. Physically, this means that we want $c\Delta t \leq \Delta x$, or the distance travelled by the solution in one time-step must be smaller than the grid size.

Experiment #4 plays with changes in wavelength. By reducing the wavelength to 10 m, we can see the importance of grid size in relation to the distance travelled by the solution in one time-step.

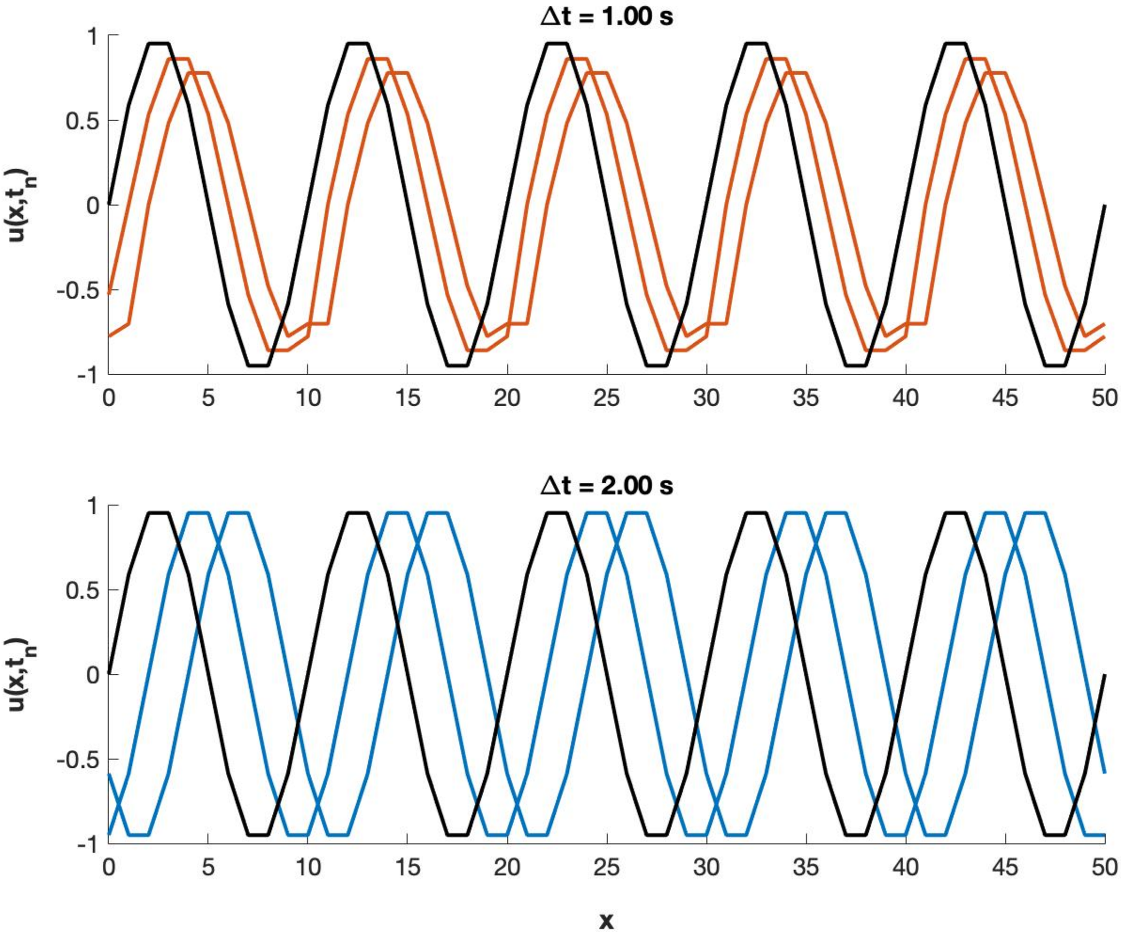

Figure 4A shows two trials differing in the size of Δt. We find that, due to the smaller wavelength, the resolution of the solution is more coarse. When $\Delta t = 2.00$ seconds, we have $C = 1$. Here amplification factor is 1 and the solution retains a constant amplitude. However, when $\Delta t = 1.00$ seconds, we see the effect of dampening return. The errors indicate this as well.

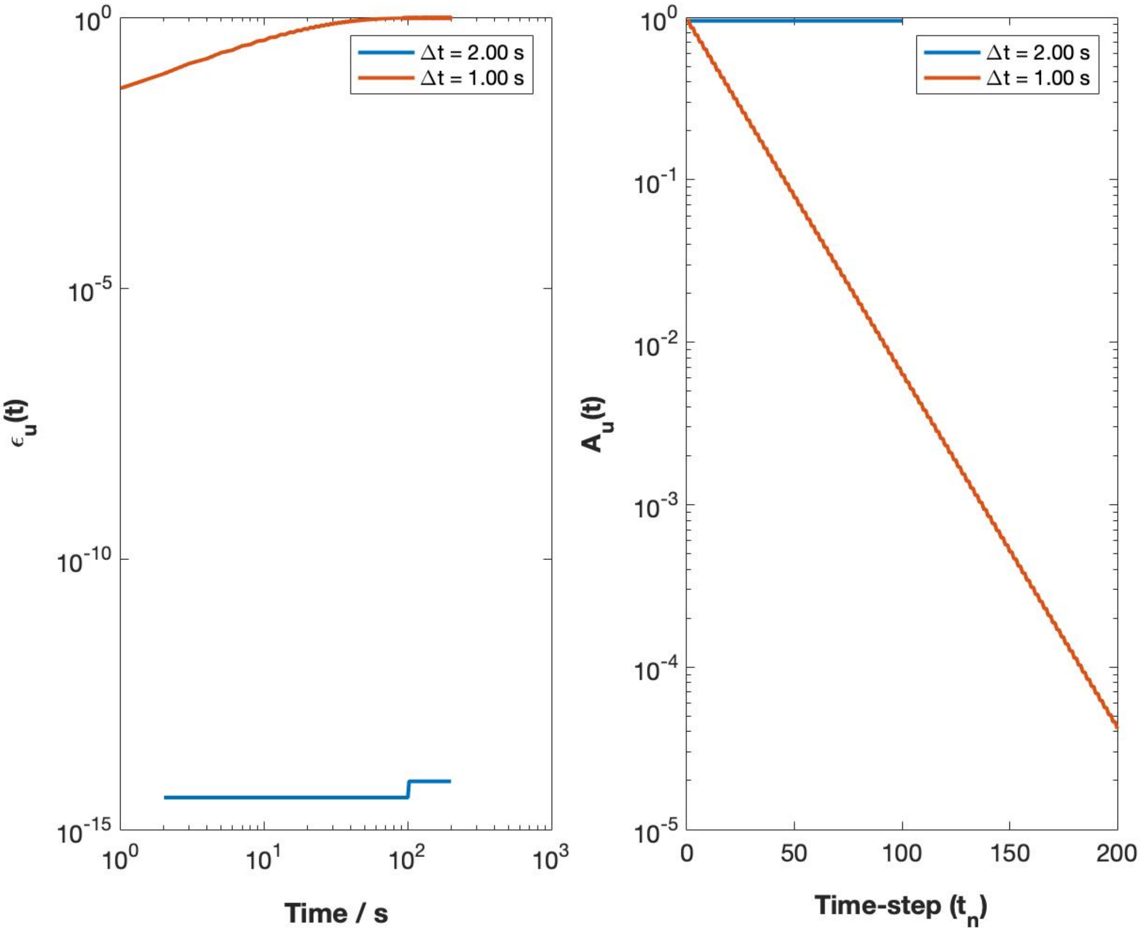

Figure 4B clearly shows the difference in the errors for the two Δt values. The error associated with Δt = 2.00 seconds is almost 15 orders of magnitude lower than that for Δt = 1.00 seconds. Accordingly, there is not observable change in amplitude of the solution when Δt = 2.00 seconds. However, there is a clear exponential decay in the size of the amplitude when Δt = 1.00 seconds.

Finally, experiment #5 tested the case of the "square wave". Here, the initial conditions and analytical solution were derived from using a sign function which equates all values above 0 to +1 and all values below 0 to -1.

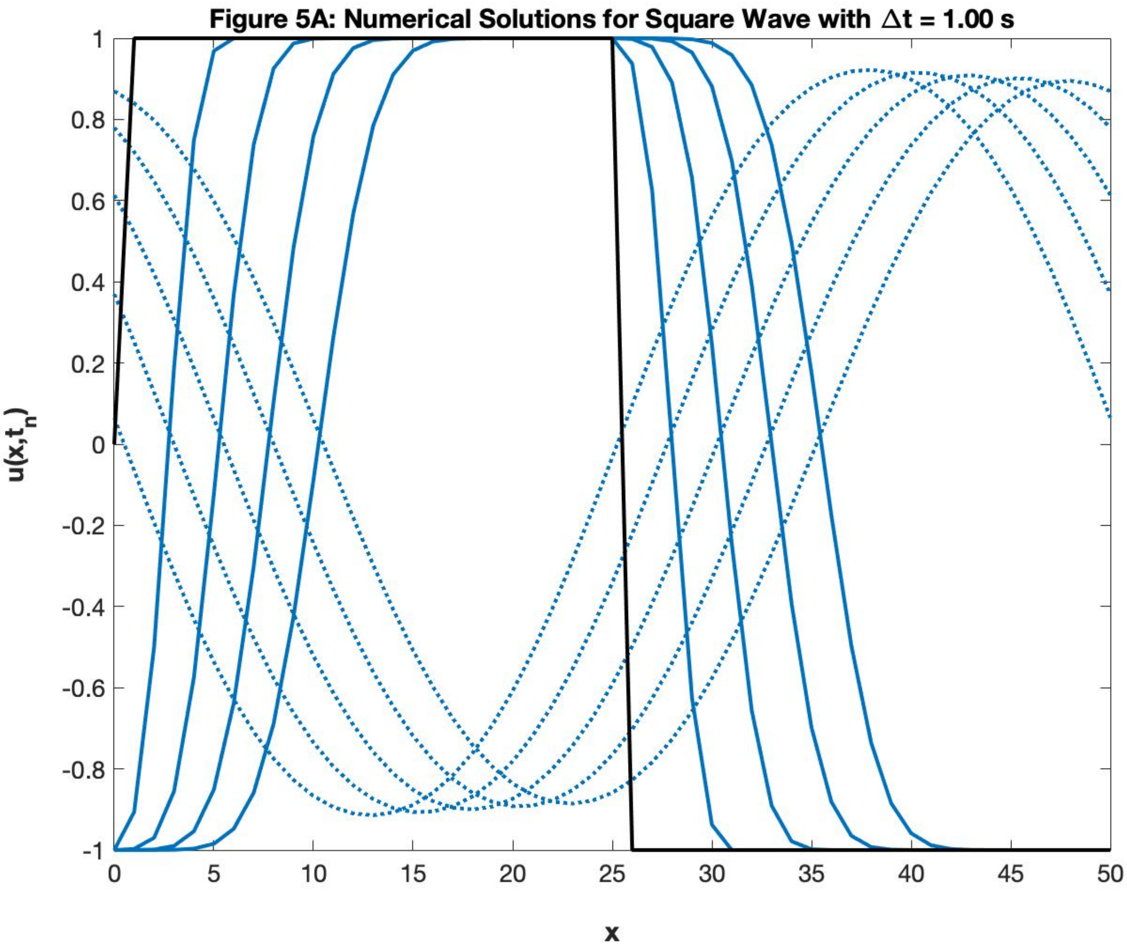

Figure 5A shows that the numerical solution immediately begins to depart from the square wave initial condition. The ‘edges’ of the wave erode to add a sinusoidal curvature. The dotted lines, which represent the wave in an arbitrary state in the future, show that once the wave has achieved a sinusoidal shape with amplitude 1, the dampening process begins to act on the amplitude.

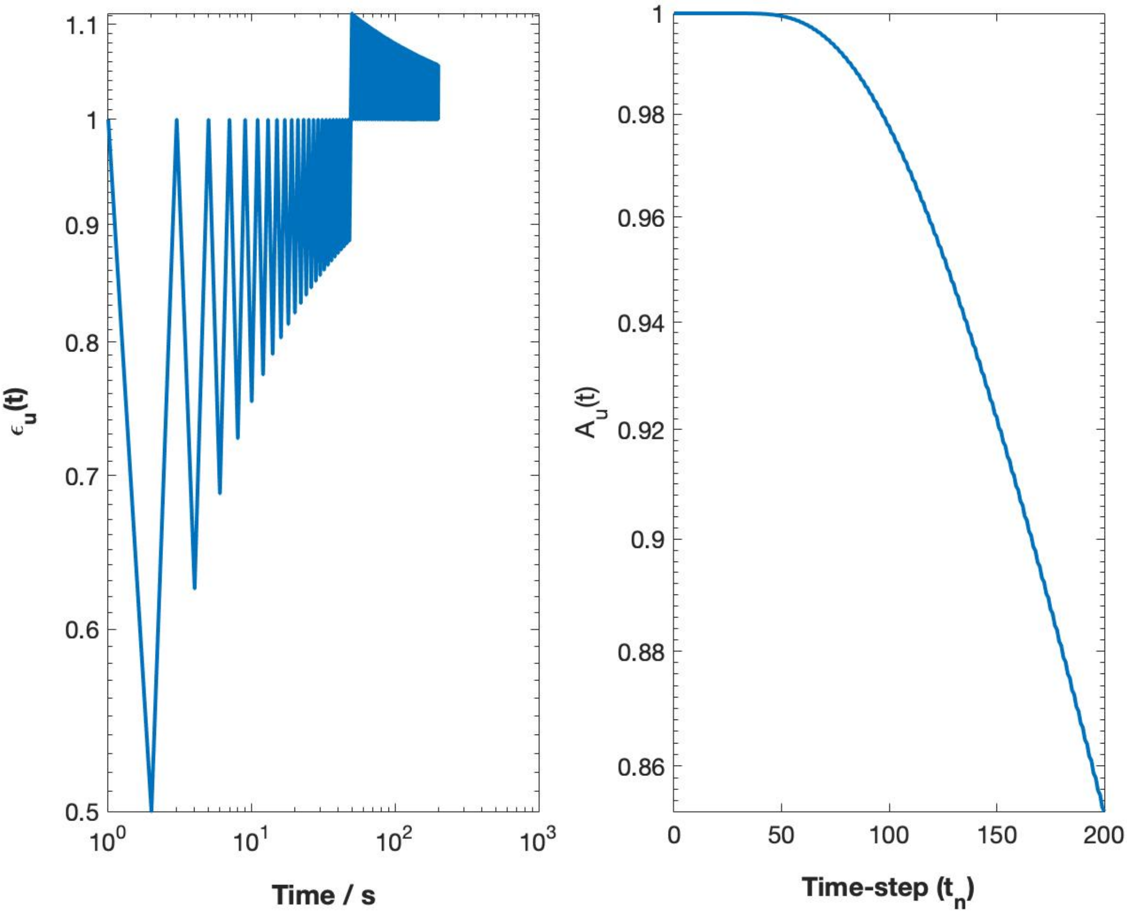

Figure 5B denotes the errors associated with this simulation. The change in the numerical solution is two-fold. First, there is a change in the shape of the curve from square to sinusoidal. Here, the amplitude is preserved. As such, the maximum error that is obtained is 1. Once the solution is perfectly sinusoidal, the secondary effect of dampening takes place. From this point on, the lowest error is 1. Errors thereafter fluctuate due to the exact solution being a square wave. We see that this second effect also corresponds to a rapid exponential decay in the amplitude of the numerical solution.

The below table encapsulates key statistics associated with each run of each experiment. C was calculated directly from its definition: $C = c\ \Delta t / \Delta x$. $\lambda_x / \Delta x$ denotes the ratio of the wavelength to grid size, which was important particularly in the discussion of experiment #4.

Experiment	**Trial**	**C**	**λ_x / Δx**
#1	$\Delta t = 2.00$ s	1	50
#2	$\Delta t = 0.25$ s	0.125	50
	$\Delta t = 0.50$ s	0.25	50
	$\Delta t = 0.75$ s	0.375	50
	$\Delta t = 1.00$ s	0.5	50

#3	Δt = 4.00 s	2	50
	Δt = 5.00 s	2.5	50
	Δt = 8.00 s	4	50
#4	Δt = 1.00 s	0.5	5
	Δt = 2.00 s	1	5
#5	Δt = 1.00 s	0.5	50