AOS 180 – Modeling Task 5
2D Poisson's equation
Due date: May 10th 11:00AM

Problem description

We are interested in the solution of Poisson's equation relating streamfunction ψ to vorticity ζ in two dimensions

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \zeta(x, y), \tag{1}$$

on the domain $0 \le x \le L_x$ and $0 \le y \le L_y$.

For the vorticity field we will use the same function we used in Modeling Task 4, except that here we will repeat it twice. The vorticity field is

$$\zeta(x, y) = \frac{1}{2}A\left[\cos\left(d_1\pi\right) + 1\right] + \frac{1}{2}A\left[\cos\left(d_2\pi\right) + 1\right], \tag{2}$$

where A is the amplitude and $d_i(x, y)$ is given by

$$d_i = \min\left\{1, \frac{1}{r}\sqrt{\left(x - x_{0,i}\right)^2 + \left(y - y_{0,i}\right)^2}\right\}. \tag{3}$$

Here, each term on the right hand side of Eq. (2) is a vortex with radius r. The vortices are centered at $(x_{0,1}, y_{0,1})$ and $(x_{0,2}, y_{0,2})$.

Recall that the relationship between the streamfunction and the velocity is given by

$$(u, v) = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right). \tag{4}$$

Thus, given a vorticity field $\zeta(x, y)$, we can use Eqs. (1) and (4) to calculate the corresponding velocity field given by $u(x, y)$ and $v(x, y)$.

In this problem, we will use two different sets of boundary conditions, as this will give us more flexibility going forward. For the first model setup (let's label this "RUN A"), we will use periodic boundary conditions in the x and y directions. This means $\psi(x = 0, y) = \psi(x = L_x, y)$ and $\psi(x, y = 0) = \psi(x, y = L_y)$. For the second model setup (labelled "RUN B"), but we will use periodic boundary conditions in the x direction and Dirichlet boundary conditions in the y direction $\psi(x, y = 0) = \psi(x, y = L_y) = 0$. Note that when we impose $\psi(x, y = 0) = \psi(x, y = L_y) = 0$ along the boundary, we are really forcing the derivative of ψ in the direction tangent to the boundary to be zero. In this case, this means $\partial\psi/\partial x = 0$ along both boundaries, which essentially means $v(x, y = 0) = v(x, y = L_y) = 0$. Physically, this boundary condition represents a wall that cannot be penetrated by the fluid (this is usually called the "no-penetration" boundary condition at the solid wall). This information may be useful in interpreting your results and making sure they are correct.

Numerical approach

We will use the centered second-order finite difference scheme to discretize the Poisson equation and we will obtain an iterative solution using SOR. We will use a fairly refined and robust approach to decide when to stop the iterations. If you consider the general notation for the linear system $[A][x] = [f]$, the criteria is given by

$$\epsilon \equiv \frac{\|R\|_\infty}{\|A\|_\infty \|x\|_1 + \|f\|_\infty} \leq tol. \tag{5}$$

Here, $\|\cdot\|_\infty$ is the infinite or maximum norm, $\|\cdot\|_1$ is the ℓ_1 norm (or 1-norm), R is the residual, and *tol* is the tolerance you will set for the solution. For our discretized 2D Poisson's equation, this criterium translates into

$$\epsilon = \frac{\|R_{i,j}\|_\infty}{(2\Delta x^{-2} + 2\Delta y^{-2}) \|\psi_{i,j}\|_1 + \|\zeta_{i,j}\|_\infty} \leq tol. \tag{6}$$

For the present solution we will use the following basic configuration: $L_x = L_y = 100\,\mathrm{m}$, $A = 1\,\mathrm{s}^{-1}$, $r = 25\,\mathrm{m}$, and the centers of the two vortices $(x_{0,1}, y_{0,1}) = (50\,\mathrm{m}, 25\,\mathrm{m})$ and $(x_{0,2}, y_{0,2}) = (50\,\mathrm{m}, 75\,\mathrm{m})$. We will use 51 grid points in each direction, start with an initial guess $\psi = 0$, and set the tolerance $tol = 10^{-7}$. Of course, none of these values make any sense in the context of QG, but at the moment this is not important.

After you set up the vorticity field and before you start your iterations, you will have to make the average vorticity within the domain to be zero due to the solvability condition. Strictly, this is required for the case with periodic boundary conditions in both directions (RUN A), but let's do the same for RUN B as well. This may be a bit trickier than you expect due to roundoff errors. Remove the average of your vorticity field and plot the new average. How close to zero is the new average? If you are baffled, read up on Kahan summation algorithm here. You can implement this in your code or you can use brute force to solve this problem (i.e. remove the average many times until you reach the desired precision). Because we are using $tol = 10^{-7}$ on the solution, we should also require the average vorticity to be smaller than *tol* before we start our calculations.

Hint: there is a very useful discussion about how to implement SOR for solving Poisson equations on the textbook. An important note when you are designing your algorithm: be careful with your periodic boundary conditions, as they must be enforced at the appropriate location in every single iteration.

Post processing: This time we will do some extra work with the solution. We do not have the analytical solution to this problem (even though we could probably obtain one if we wanted). However, you can always calculate the numerical Laplacian of the solution and compare it to the vorticity field. Please do that for your final solution, and compare it to the vorticity field after removal of the average (i.e., after you solve for ψ, you can calculate $\nabla^2\psi$ and compare the result with the original ζ). In addition, in order to interpret your solution physically, it is better to calculate the velocity field (u, v) from your streamfunction ψ. Please do so using a centered discretization for the derivatives.

Report

Please include in your report:

i. Figures: Figures that briefly document your your solution and support your discussion (see below). I suggest you include at least a plot for the convergence of the iterations (i.e. ϵ as a function of iteration counter), and color plots or contour plots for the following: $\zeta(x, y)$, $\psi(x, y)$, $\nabla^2\psi(x, y)$, $u(x, y)$, $v(x, y)$.

ii. Documentation: Please include in your report the recursive equation used and how you implemented the boundary conditions. Also include a justification for your choice for the over-relaxation coefficient α.

iii. Discussion: Please briefly discuss your results. I am interested in the contrast between the two simulations both from a numerical convergence perspective and from a physical perspective (i.e., what are the differences between the velocity fields? are they consistent with the boundary conditions you are specifying?).

Please upload a **.pdf** file with your report and **.pdf** file with your code to the dropbox that will be available on the CCLE website for the course. If you also want to upload your original code, that is fine. Please, do not upload word files or other formats, as they do not work well in other operational systems.

Additional suggestions

I am asking you to report and discuss results for only 2 simulations. I suggest you also explore a few more things, even if you choose not to include them in your report.

- Try Gauss-Seidel and/or Jacobi iterations and compare their convergence with the one obtained from SOR.
- Another test you can do is to use the boundary condition $\psi = 0$ on the entire boundary (i.e. no periodic conditions in any direction). This may actually be useful in the near future.
- Vary the SOR coefficient within the range $0 < \alpha < 2$. Is the value you get for α_{opt} using the theoretical result really optimal?