

## 2d Hughe's Flow using 5th WENO Advection and Fast Sweep

Now that both the 5th order WENO advection and 5th order Fast sweep algorithm is up and running, I will solve a synthetic problem to ensure that they are working as intended together.

### Problem Definition

I will be solving the problem where there is a rectangular slab (The one which I solved in my past with lower order schemes and see how good does the result using this new higher scheme is, comparing both of them

```
%-----Domain Parameter-----%
% Grid Points
global Nx;
global Ny;
Nx = 400;
Ny = 200;

%--Important Parameter
INFINITE = 10^2;
eta = 10^-6;
error_eta = 10^-9;

% To track the error for each iterations
iteration_error = [];

% Dimension of Domain
Lx = 100;
Ly = 50;

% Domain
x = linspace(0,Lx,Nx);
y = linspace(0,Ly,Ny);
[X,Y] = meshgrid(x,y);

% Defining velocity
global speed discomfort cost x_velocity y_velocity;

% Space Interval
global h;
h = Lx/Nx;

% Total Time
t_total = 10;
% Time Interval
dt = 0.01;

% --- Obstacle modelling ----%
obstacle = false(Ny,Nx);

%-----Initial Condition-----%
```

```

% Second Initial Condition (Discontineous)
rho_square = 0.2*ones(size(X));
rho_square( X>=10 & X<=20 & Y>=15 & Y<=35 ) = 0.6;

%--Function to compute Velocity using Density
function speed = makeSpeed(rho)
    global Nx Ny;
    speed = zeros(Ny,Nx);
    for i = 1:1:Nx
        for j = 1:1:Ny
            speed(j,i) = 1 - rho(j,i);
        end
    end
end

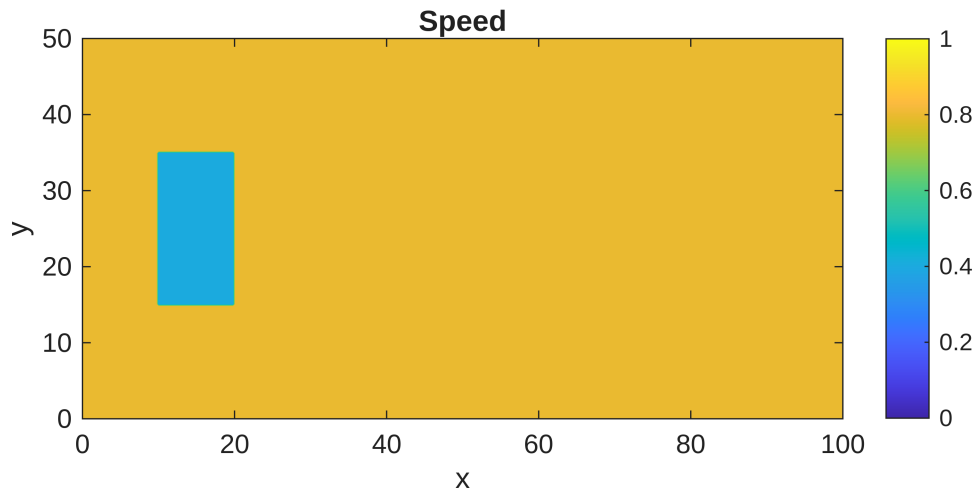
%--Function to compute Discomfort using Density
function discomfort = makeDiscomfort(rho)
    global Nx Ny;
    discomfort = zeros(Ny,Nx);
    for i = 1:1:Nx
        for j = 1:1:Ny
            discomfort(j,i) = rho(j,i)*rho(j,i);
        end
    end
end

%--Function to compute Cost function using Speed and Discomfort
function cost = makeCost(speed,discomfort)
    global Nx Ny;
    cost = zeros(Ny,Nx);
    for i = 1:1:Nx
        for j = 1:1:Ny
            cost(j,i) = 1/speed(j,i) + discomfort(j,i);
        end
    end
end

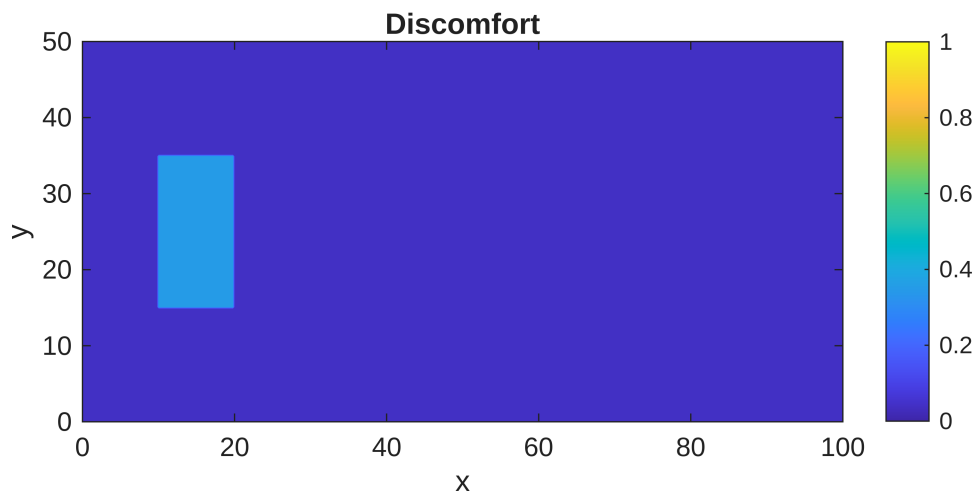
speed = makeSpeed(rho_square);
discomfort = makeDiscomfort(rho_square);
cost = makeCost(speed,discomfort);

% Plotting Speed
figure(1)
contourf(X,Y,speed,'LineStyle','none');
title('Speed');
clim([0,1]);
axis equal tight;
colorbar;
xlabel('x'); ylabel('y');

```



```
% Plotting Discomfort
figure(2)
contourf(X,Y,discomfort,'LineStyle','none');
title('Discomfort');
clim([0,1]);
axis equal tight;
colorbar;
xlabel('x'); ylabel('y');
```

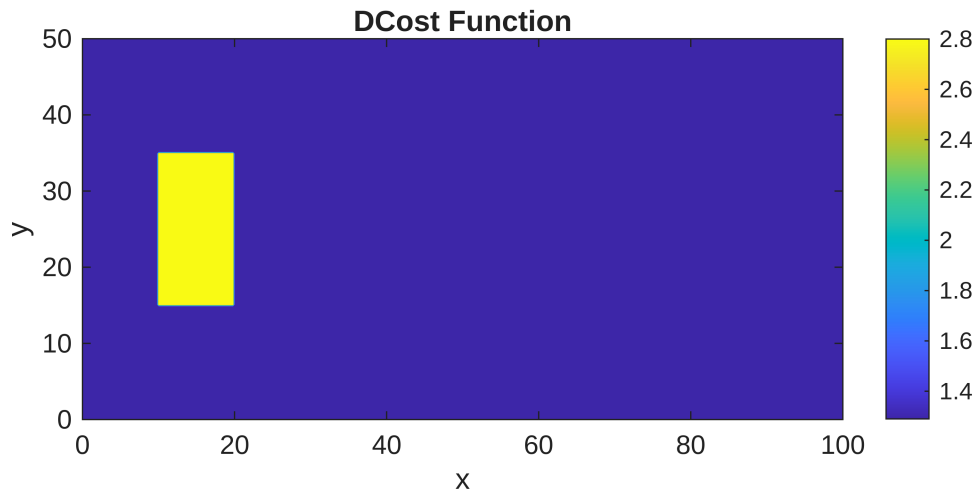


```
% Plotting Cost Function
figure(3);
```

```

contourf(X,Y,cost,'LineStyle','none');
title('DCost Function');
axis equal tight;
colorbar;
xlabel('x'); ylabel('y');

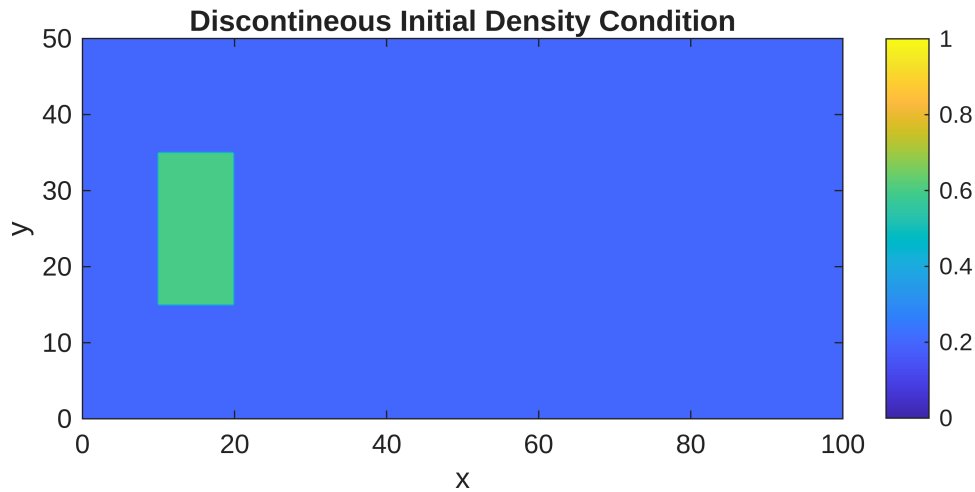
```



```

% Plotting the Initial Condition
figure(4);
contourf(X,Y,rho_square,'LineStyle','none');
title('Discontineous Initial Density Condition');
clim([0,1]);
axis equal tight;
colorbar;
xlabel('x'); ylabel('y');

```



```

%-- Function that will calculate the flux depending on the u,v and Density.
% X_velocity and y_velocity will be found using the potential.
function flux = makeFlux(rho,x_velocity,y_velocity)
    global Nx Ny;

    % Initializing Flux
    flux_x = zeros(Ny,Nx);
    flux_y = zeros(Ny,Nx);
    flux = zeros(2, Ny, Nx);

    for i = 1:1:Nx
        for j = 1:1:Ny
            flux_x(j,i) = rho(j,i) * x_velocity(j,i);
            flux_y(j,i) = rho(j,i) * y_velocity(j,i);

            flux(1,j,i) = flux_x(j,i);
            flux(2,j,i) = flux_y(j,i);
        end
    end
end

rho = rho_square;

%--Preallocating the memory
rho_n = zeros(Ny,Nx);
rho_1 = zeros(Ny,Nx);
rho_2 = zeros(Ny,Nx);
rho_next = zeros(Ny,Nx);

```

```

%-----MAIN SOLVER-----%
for t = 1:dt:t_total

    % Computing Cost For the density rho
    speed = makeSpeed(rho);
    discomfort = makeDiscomfort(rho);
    cost = makeCost(speed,discomfort);

    %-- Calculate Initial Phi guess using Lower order Fast Sweep
    phi_first_order =
First_order_Godunov_Fast_Sweep_Algorithm(cost,h,Nx,Ny,obstacle,INFINITE);
    %-- Calculate The final phi using higher order scheme and above initial
guess
    phi =
weno_fast_sweep(cost,h,Nx,Ny,obstacle,eta,error_eta,INFINITE,phi_first_order,i
teration_error);

    %--Calculate the Components of Velocity
    [phi_x, phi_y] = gradient(phi, h, h); % partial derivatives of phi in x
and y
    mag_phi = sqrt(phi_x.^2 + phi_y.^2);
    mag_phi = max(mag_phi, eta); % To be safe with denominator
    for i = 1:1:Nx
        for j = 1:1:Ny
            x_velocity(j,i) = speed(j,i) * (phi_x(j,i))/mag_phi(j,i);
            y_velocity(j,i) = speed(j,i) * (phi_y(j,i))/mag_phi(j,i);
        end
    end

    % -- Making rho_n and RHS of rho_n
    rho_n = rho;
    semi_discretize_form_rho_n = makeRHS(rho_n);

    % ----- Makign the RHO_1 and then RHS of rho_1 ----- %
    for i = 4:1:Nx-2
        for j = 4:1:Ny-2
            rho_1(j,i) = rho_n(j,i) + dt *
semi_discretize_form_rho_n(j-3,i-3);
        end
    end
    % -- Extrapolation (My Intuition)
    rho_1(:,3) = rho_1(:,4); rho_1(:,2) = rho_1(:,3); rho_1(:,1) =
rho_1(:,2); % Right Edge
    rho_1(3,:) = rho_1(4,:); rho_1(2,:) = rho_1(3,:); rho_1(1,:) =
rho_1(2,:); % Bottom Edge
    rho_1(Ny-1,:) = rho_1(Ny-2,:); rho_1(Ny,:) =
rho_1(Ny-1,:); % Top Edge

```

```

rho_1(:,Ny-1) = rho_1(:,Ny-2); rho_1(:,Ny) =
rho_1(:,Ny-1); % Left Edge
% -- RHS of discretised form using rho_1
semi_discretize_form_rho_1 = makeRHS(rho_1);

%----- Making the RHO_2 and RHS of RHO_2 -----%
for i = 4:1:Nx-2
    for j = 4:1:Ny-2
        rho_2(j,i) = 3/4 * rho_n(j,i) + 1/4 * (rho_1(j,i) + dt *
semi_discretize_form_rho_1(j-3,i-3));
    end
end
% -- Extrapolation (My Intuition)
rho_2(:,3) = rho_2(:,4); rho_2(:,2) = rho_2(:,3); rho_2(:,1) =
rho_2(:,2); % Right Edge
rho_2(3,:) = rho_2(4,:); rho_2(2,:) = rho_2(3,:); rho_2(1,:) =
rho_2(2,:); % Bottom Edge
rho_2(Ny-1,:) = rho_2(Ny-2,:); rho_2(Ny,:) =
rho_2(Ny-1,:); % Top Edge
rho_2(:,Ny-1) = rho_2(:,Ny-2); rho_2(:,Ny) =
rho_2(:,Ny-1); % Left Edge
% -- RHS of discretised form using rho_2
semi_discretize_form_rho_2 = makeRHS(rho_2);

%-----Making the rho_next-----%

for i = 4:1:Nx-2
    for j = 4:1:Ny-2
        rho_next(j,i) = 1/3 * rho_n(j,i) + 2/3 * (rho_2(j,i) + dt *
semi_discretize_form_rho_2(j-3,i-3));
    end
end
% -- Extrapolation (My Intuition)
rho_next(:,3) = rho_next(:,4); rho_next(:,2) = rho_next(:,3);
rho_next(:,1) = rho_next(:,2); % Right Edge
rho_next(3,:) = rho_next(4,:); rho_next(2,:) = rho_next(3,:);
rho_next(1,:) = rho_next(2,:); % Bottom Edge
rho_next(Ny-1,:) = rho_next(Ny-2,:); rho_next(Ny,:) =
rho_next(Ny-1,:); % Top Edge
rho_next(:,Ny-1) = rho_next(:,Ny-2); rho_next(:,Ny) =
rho_next(:,Ny-1); % Left Edge

%--Plotting the rho_next;
if mod(round(t/dt),10) == 0
    figure(6);
    contourf(x, y, rho_next, 20, 'LineColor', 'none');
    colorbar;

```

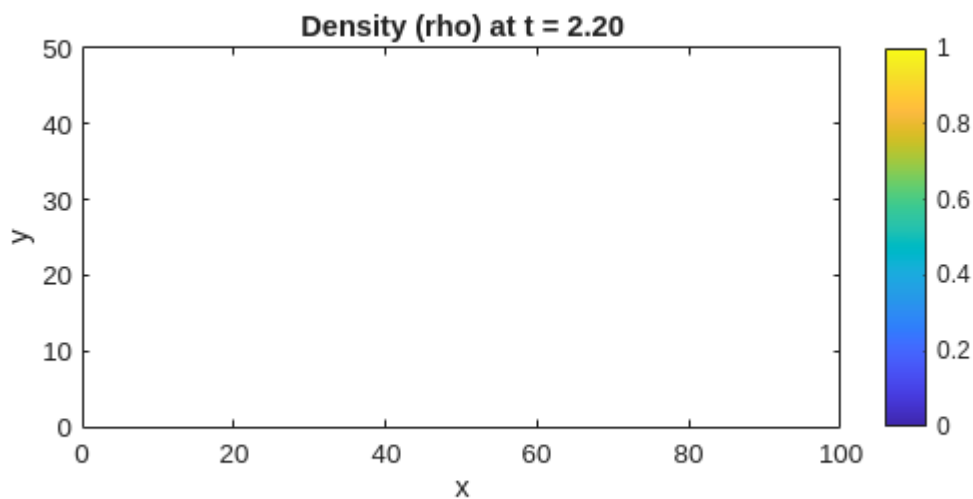
```

        clim([0 1]);
        title(sprintf('Density (rho) at t = %.2f', t));
        xlabel('x'); ylabel('y');
        axis equal tight;
        drawnow;
    end

    % Substituting back the rho value \
    rho = rho_next;
end

```

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## First order Godunov Fast Sweep Algorithm (Older Version)

```

function phi =
First_order_Godunov_Fast_Sweep_Algorithm(c,h,Nx_dash,Ny_dash,obstacle_dash,IN
FINITE)
    %-----
    % Initialization
    %-----
    phi = INFINITE*ones(Ny_dash, Nx_dash);    % Initial potential field
    phi(:,Nx_dash) = 0;                        % Exit Boundary

    % Set phi inside obstacle to Inf
    phi(obstacle_dash) = INFINITE;

    %-----
    % Fast Sweeping Iterations
    %-----

```



```

for sweep = 1:20
    % == Sweep 1: i→, j→
    for i = 2:Nx_dash-1
        for j = 2:Ny_dash-1
            if ~obstacle_dash(j, i)
                phi = update_phi(phi, c, i, j, h);
            end
        end
    end
    % == Sweep 2: i←, j→
    for i = Nx_dash-1:-1:2
        for j = 2:Ny_dash-1
            if ~obstacle_dash(j, i)
                phi = update_phi(phi, c, i, j, h);
            end
        end
    end
    % == Sweep 3: i←, j←
    for i = Nx_dash-1:-1:2
        for j = Ny_dash-1:-1:2
            if ~obstacle_dash(j, i)
                phi = update_phi(phi, c, i, j, h);
            end
        end
    end
    % == Sweep 4: i→, j←
    for i = 2:Nx_dash-1
        for j = Ny_dash-1:-1:2
            if ~obstacle_dash(j, i)
                phi = update_phi(phi, c, i, j, h);
            end
        end
    end

    phi(:,1) = phi(:,2); % Right
    phi(1, :) = phi(2, :); % Top
    phi(Ny_dash, :) = phi(Ny_dash-1,:); % Bottom

end

end

% Subfunction: Update one grid point using Godunov scheme

function phi = update_phi(phi, c_, i, j, dx)
    a = min(phi(j, i-1), phi(j, i+1)); % x-direction neighbors
    b = min(phi(j-1, i), phi(j+1, i)); % y-direction neighbors
    c = c_(j, i); % Local inverse speed
    if abs(a - b) >= c * dx
        phi(j, i) = min(a, b) + c * dx;
    end
end

```

```

else
    inside = 2 * (c * dx)^2 - (a - b)^2;
if inside >= 0
    phi(j, i) = (a + b + sqrt(inside)) / 2;
end
end
end
end

```

## WENO FAST SWEEP

```

function phi =
weno_fast sweep(c,h,Nx_dash,Ny_dash,obstacle_dash,eta,error_eta,INFINITE,initial_guess,iteration_error)
    % Phi is initialised with infinity
    phi = initial_guess;
    phi_old = INFINITE * ones(size(phi));

    % Iterations
    iterations = 0;
    loop_safety = 0;

    % Exit
    phi(:,Nx_dash) = 0;

    % Iterations
    while (sum(abs(phi(:)-phi_old(:))) > error_eta && loop_safety<100)

        %fprintf("Iteration = %d ",iterations);
        err = sum(abs(phi(:)-phi_old(:)));
        %fprintf("Error = %0.10f\n",err);

        phi_old = phi;
        iterations = iterations + 1;
        iteration_error(iterations) = err;
        loop_safety = loop_safety + 1;

        for sweep = 1:4
            switch sweep
                case 1
                    ix = 3:1:Nx_dash-2; jy = 3:1:Ny_dash-2;
                case 2
                    ix = 3:1:Nx_dash-2; jy = Ny_dash-2:-1:3;
                case 3
                    ix = Nx_dash-2:-1:3; jy = 3:1:Ny_dash-2;
                case 4
                    ix = Nx_dash-2:-1:3; jy = Ny_dash-2:-1:3;
            end
        end
    end

```

```

for i = ix
    for j = jy
        %if the point is outside the obstacle
        if(~obstacle_dash(j,i))
            %-----Calculating the phix_min-----_%

            r_back = (eta + (phi(j,i) - 2*phi(j,i-1) +
phi(j,i-2))^2)/(eta + (phi(j,i+1) - 2*phi(j,i) + phi(j,i-1))^2);
            r_front = (eta + (phi(j,i) - 2*phi(j,i+1) +
phi(j,i+2))^2)/(eta + (phi(j,i+1) - 2*phi(j,i) + phi(j,i-1))^2);

            % Calculate the w
            w_front = 1/(1+2*(r_front^2));
            w_back = 1/(1+2*(r_back^2));

            % Dell phi by Dell x plus and minus
            phix_minus = (1-w_back)*((phi(j,i+1)-phi(j,i-1)) /
(2*h)) + w_back*((3*phi(j,i) - 4*phi(j,i-1) + phi(j,i-2))/(2*h));
            phix_plus = (1-w_front)*((phi(j,i+1)-phi(j,i-1)) /
(2*h)) + w_front*((-3*phi(j,i)+4*phi(j,i+1)-phi(j,i+2))/(2*h));

            phix_min = min((phi(j,i) - h*phix_minus), (phi(j,i) +
h*phix_plus));

            %-----calculating the phiy_min-----_%

            r_back = (eta + (phi(j,i) - 2*phi(j-1,i) +
phi(j-2,i))^2) / (eta + (phi(j+1,i) - 2*phi(j,i) + phi(j-1,i))^2);
            r_front = (eta + (phi(j,i) - 2*phi(j+1,i) +
phi(j+2,i))^2) / (eta + (phi(j+1,i) - 2*phi(j,i) + phi(j-1,i))^2);

            % Calculate the w
            w_front = 1 / (1 + 2*(r_front^2));
            w_back = 1 / (1 + 2*(r_back^2));

            % Dell phi by Dell x plus and minus
            phiy_minus = (1-w_back) * ((phi(j+1,i)-phi(j-1,i)) /
(2*h)) + w_back * ((3*phi(j,i) - 4*phi(j-1,i) + phi(j-2,i))/(2*h));
            phiy_plus = (1-w_front) * ((phi(j+1,i) - phi(j-1,i)) /
(2*h)) + w_front * ((-3*phi(j,i) + 4*phi(j+1,i) - phi(j+2,i))/(2*h));

            phiy_min = min((phi(j,i) - h*phiy_minus), (phi(j,i)
+ h*phiy_plus));

            if abs(phix_min - phiy_min) >= c(j,i)*h

```

```

        phi(j,i) = min(phiy_min,phix_min) + c(j,i)*h;
    else
        phi(j,i) = ((phix_min + phiy_min) +
(2*(c(j,i)*h)^2 - (phix_min - phiy_min)^2)^0.5)/2;
    end
    % If the point is inside the obstacle
    else
        phi(j,i) = INFINITE;
    end
end
end
end

%-----Extrapolation-----%

% Bottom edge %
%phi(2,:) = phi(3,:);
%phi(1,:) = phi(3,:);
% Top edge %
%phi(Ny_dash,:) = phi(Ny_dash-2,:);
%phi(Ny_dash-1,:) = phi(Ny_dash-2,:);
% Right edge %
%phi(:,Nx_dash-1) = phi(:,Nx_dash-2);
% Left edge %
%phi(:,2) = phi(:,3);
%phi(:,1) = phi(:,3);

end
end
end

```

## RHS of Semi Discretize Form

```

function semi_discrete_form = makeRHS(rho)
    global Nx Ny x_velocity y_velocity h;
    semi_discrete_form = zeros(Ny-5,Nx-5);

    %--Flux for the given rho distribution
    flux = makeFlux(rho,x_velocity,y_velocity);
    flux_x = squeeze(flux(1,:,:));
    flux_y = squeeze(flux(2,:,:));

    % Calculating the maximum speed
    alpha_x = max(abs(x_velocity(:)));
    alpha_y = max(abs(y_velocity(:)));

    % -- Calculating the value of f+ and f- at {i+2,i+1,i,i-1,i-2,i-3} -- %
    fx_negative_whole = zeros(Ny,Nx);
    fx_positive_whole = zeros(Ny,Nx);
    fy_negative_whole = zeros(Ny,Nx);

```

```

fy_positive_whole = zeros(Ny,Nx);

fx_negative_half = zeros(Ny-5,Nx-5);
fx_positive_half = zeros(Ny-5,Nx-5);
fy_negative_half = zeros(Ny-5,Nx-5);
fy_positive_half = zeros(Ny-5,Nx-5);

for j = 1:1:Ny
    for i=1:1:Nx
        fx_positive_whole(j,i) = 1/2 * (flux_x(j,i) + alpha_x*rho(j,i));
        fx_negative_whole(j,i) = 1/2 * (flux_x(j,i) - alpha_x*rho(j,i));
        fy_positive_whole(j,i) = 1/2 * (flux_y(j,i) + alpha_y*rho(j,i));
        fy_negative_whole(j,i) = 1/2 * (flux_y(j,i) - alpha_y*rho(j,i));
    end
end

% -- Computing the RHS of Semi Discrete form.
for i = 4:1:Nx-2
    for j = 4:1:Ny-2

        %--X--%

        % fx{+}_[i+1/2,j]
        fx_plus_x_half_forward = stencil( ...
            fx_positive_whole(j,i-2), ...
            fx_positive_whole(j,i-1), ...
            fx_positive_whole(j,i), ...
            fx_positive_whole(j,i+1), ...
            fx_positive_whole(j,i+2));

        % fx{+}_[i-1/2,j]
        fx_plus_x_half_backward = stencil( ...
            fx_positive_whole(j,i-3), ...
            fx_positive_whole(j,i-2), ...
            fx_positive_whole(j,i-1), ...
            fx_positive_whole(j,i), ...
            fx_positive_whole(j,i+1));

        % fx{-}_[i+1/2,j]
        fx_minus_x_half_forward = stencil( ...
            fx_negative_whole(j,i+2), ...
            fx_negative_whole(j,i+1), ...
            fx_negative_whole(j,i), ...
            fx_negative_whole(j,i-1), ...
            fx_negative_whole(j,i-2));

        % fx{-}_[i-1/2,j]
        fx_minus_x_half_backward = stencil( ...
            fx_negative_whole(j,i+1), ...
            fx_negative_whole(j,i), ...

```

```

        fx_negative_whole(j,i-1), ...
        fx_negative_whole(j,i-2), ...
        fx_negative_whole(j,i-3));

%--Y--%

% fy{+}_[i,j+1/2]
fy_plus_y_half_forward = stencil( ...
    fy_positive_whole(j-2,i), ...
    fy_positive_whole(j-1,i), ...
    fy_positive_whole(j,i), ...
    fy_positive_whole(j+1,i), ...
    fy_positive_whole(j+2,i));

% fy{+}_[i,j-1/2]
fy_plus_y_half_backward = stencil( ...
    fy_positive_whole(j-3,i), ...
    fy_positive_whole(j-2,i), ...
    fy_positive_whole(j-1,i), ...
    fy_positive_whole(j,i), ...
    fy_positive_whole(j+1,i));

% fy{-}_[i,j+1/2]
fy_minus_y_half_forward = stencil( ...
    fy_negative_whole(j+2,i), ...
    fy_negative_whole(j+1,i), ...
    fy_negative_whole(j,i), ...
    fy_negative_whole(j-1,i), ...
    fy_negative_whole(j-2,i));

% fy{-}_[i,j-1/2]
fy_minus_y_half_backward = stencil( ...
    fy_negative_whole(j+1,i), ...
    fy_negative_whole(j,i), ...
    fy_negative_whole(j-1,i), ...
    fy_negative_whole(j-2,i), ...
    fy_negative_whole(j-3,i));

    fx_positive_half = fx_plus_x_half_forward +
fx_minus_x_half_forward;
    fx_negative_half = fx_plus_x_half_backward +
fx_minus_x_half_backward;
    fy_positive_half = fy_plus_y_half_forward +
fy_minus_y_half_forward;
    fy_negative_half = fy_plus_y_half_backward +
fy_minus_y_half_backward;

% Computing RHS of Semi Discretized form
    semi_discrete_form(j-3,i-3) = - 1/h * (fx_positive_half -
fx_negative_half) - 1/h * (fy_positive_half - fy_negative_half);

```

```
        end
    end
end
```

## WENO Advection Stencil

```
% To compute f half. Order of input matters
function f = stencil(f1,f2,f3,f4,f5)

    eta = 10^-6;

    % Compute S values
    s_1 = ((1/3) * f1) + ((-7/6) * f2) + ((11/6) * f3);
    s_2 = ((-1/6) * f2) + ((5/6) * f3) + ((1/3) * f4);
    s_3 = ((1/3) * f3) + ((5/6) * f4) + ((-1/6) * f5);

    % Compute Beta
    beta_1 = 13/12 * (f1 - 2*f2 + f3)^2 + 1/4 * (f1 - 4*f2 + 3*f3)^2;
    beta_2 = 13/12 * (f2 - 2*f3 + f4)^2 + 1/4 * (f2 - f4)^2;
    beta_3 = 13/12 * (f3 - 2*f4 + f5)^2 + 1/4 * (3*f3 - 4*f4 + f5)^2;

    % Definign Gamma
    gamma_1 = 1/10;
    gamma_2 = 3/5;
    gamma_3 = 3/10;

    % Computing weights bar
    w_1_dash = ( gamma_1 ) ./ ((eta + beta_1)^2);
    w_2_dash = ( gamma_2 ) ./ ((eta + beta_2)^2);
    w_3_dash = ( gamma_3 ) ./ ((eta + beta_3)^2);

    % Computing Weights
    w_1 = ( w_1_dash ) / (w_1_dash + w_2_dash + w_3_dash);
    w_2 = ( w_2_dash ) / (w_1_dash + w_2_dash + w_3_dash);
    w_3 = ( w_3_dash ) / (w_1_dash + w_2_dash + w_3_dash);

    % Computing Flux
    f = w_1 * s_1 + w_2 * s_2 + w_3 * s_3;
end
```