# Chi-Square Distribution $\chi^2$

$$P(X^2 > \chi^2_{\alpha,k}) = \int_{\chi^2_{\alpha,k}}^{\infty} f(u)\,du = \alpha$$

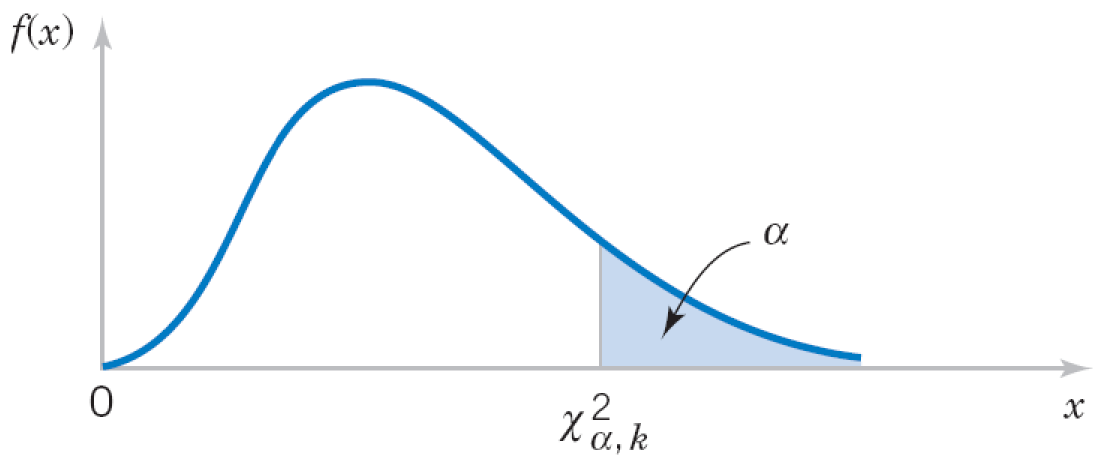


**Figure 4-22** Percentage point $\chi^2_{\alpha,k}$ of the $\chi^2$ distribution.



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# Inference on the Variance of a Normal Population

$$\frac{\sum(y_u - \bar{y})^2}{\sigma^2} \sim \chi^2_{n-1} \sim \frac{(n-1)\, s^2}{\sigma^2}$$

Laboratory technicians in an hospital are expected to be able to perform highly reproducible blood cell counts.

An applicant for a technician position is tested. He is given blood in a test tube and he is asked to provide 20 random cell counts from that tube using a standard procedure. The 20 cells counts represent a random sample which has a sample variance $s^2$ = 15.3 (cells)$^2$. If the variance of the cell counts exceeds 10 (cells)$^2$, the applicant will not be considered for the job. Is there evidence in the sample data to suggest that the applicant's laboratory skills are not satisfactory? Use $\alpha$=0.05 and assumes that the cell counts have a normal distribution.

# Inference on the Variance of a Normal Population

$$\frac{\sum(y_u - \bar{y})^2}{\sigma^2} \sim \chi^2_{n-1} \sim \frac{(n-1)\,s^2}{\sigma^2}$$

**Testing Hypotheses on the Variance of a Normal Distribution**

Null hypothesis: $H_0\colon \sigma^2 = \sigma_0^2$

Test statistic: $\chi_0^2 = \dfrac{(n-1)S^2}{\sigma_0^2}$

| Alternative Hypotheses | Rejection Criterion |
|---|---|
| $H_1\colon \sigma^2 \neq \sigma_0^2$ | $\chi_0^2 > \chi^2_{\alpha/2,n-1}$ or $\chi_0^2 < \chi^2_{1-\alpha/2,n-1}$ |
| $H_1\colon \sigma^2 > \sigma_0^2$ | $\chi_0^2 > \chi^2_{\alpha,\,n-1}$ |
| $H_1\colon \sigma^2 < \sigma_0^2$ | $\chi_0^2 < \chi^2_{1-\alpha,\,n-1}$ |

$\sum (y_u - \bar{y})^2$ = sum of squared residuals and $\sigma^2/(n-1)$ is a scale factor



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# Chi-Square Distribution with n-1 d.f.

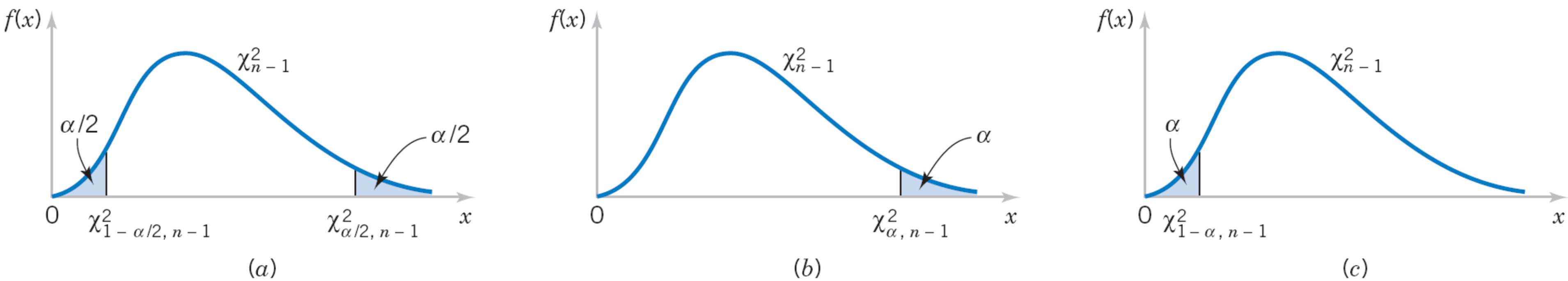


Figure 4-23 Distribution of the test statistic for $H_0$: $\sigma^2 = \sigma_0^2$ with critical region values for (a) $H_1$: $\sigma^2 \neq \sigma_0^2$, (b) $H_0$: $\sigma^2 > \sigma_0^2$, and (c) $H_0$: $\sigma^2 < \sigma_0^2$.



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# Inference about the Variance of a Normal Population

$$\frac{\sum(y_u - \bar{y})^2}{\sigma^2} \sim \chi^2_{n-1} \sim \frac{(n-1)\, s^2}{\sigma^2}$$

Laboratory technicians in an hospital are expected to be able to perform highly reproducible blood cell counts.

A job applicant for a technician position is tested. He is given blood in a test tube and he is asked to provide 20 random cell counts from that tube using a standard procedure. The 20 counts represent a random sample which has a sample variance $s^2$ = 153 (cells)$^2$. If the variance of the applicant's cell counts exceeds 100 (cells)$^2$, he will not be considered for the job. Is there evidence in the sample data to suggest that the applicant's laboratory skills are not satisfactory? Use $\alpha$=0.05 and assumes that the cell counts have a normal distribution.

$H_o$: $\sigma^2 = 100$
$H_1$: $\sigma^2 > 100$

$X_o^2$= 29.07 and Tables (Appendix) show that $X^2_{0.05,19}$= 30.14, therefore cannot reject $H_o$
(no evidence that the applicant's skills are not satisfactory)

# Inference on the Ratio of Variances of Two Normal Populations

## The Test Procedure with F ratio

Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ be a random sample from a normal population with mean $\mu_1$ and variance $\sigma_1^2$, and let $X_{21}, X_{22}, \ldots, X_{2n_2}$ be a random sample from a second normal population with mean $\mu_2$ and variance $\sigma_2^2$. Assume that both normal populations are independent. Let $S_1^2$ and $S_2^2$ be the sample variances. Then the ratio

$$F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2}$$

has an $F$ distribution with $n_1 - 1$ numerator degrees of freedom and $n_2 - 1$ denominator degrees of freedom.



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# F Distribution for Inference on the Ratio of Two Variances

## The *F* Distribution

We wish to test the hypotheses:

$$H_0: \sigma_1^2 = \sigma_2^2$$

$$H_1: \sigma_1^2 \neq \sigma_2^2$$

A student in bone engineering is interested in making a suspension of polymer microspheres. He found that two similar instruments in his lab can be used. One was designed by a company named Teckman and the other one by Abilent. The variability in the diameter of microspheres in the suspension is a critical characteristics and low variability is desirable for subsequent processing steps. Therefore the student is interested in determining if these instruments result in a significantly different microsphere diameter variability.

# Inference on the Ratio of Variances of Two Normal Populations

## The *F* Distribution

$$f_{1-\alpha,u,v} = \frac{1}{f_{\alpha,v,u}} \qquad (5\text{-}20)$$

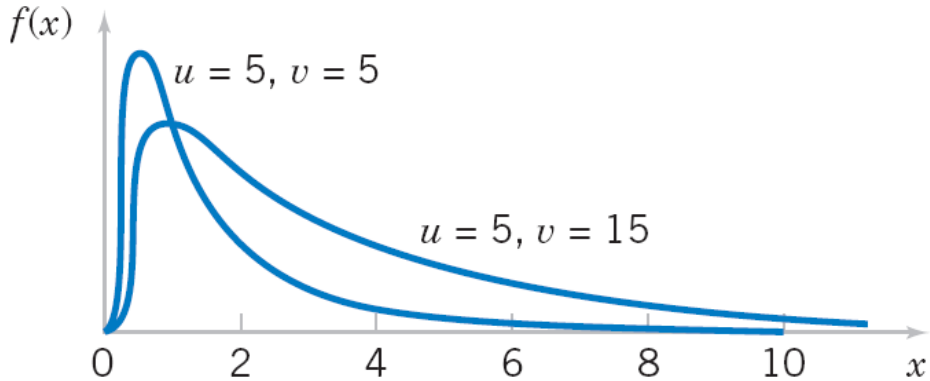


Figure 5-4 Probability density functions of two *F* distributions.

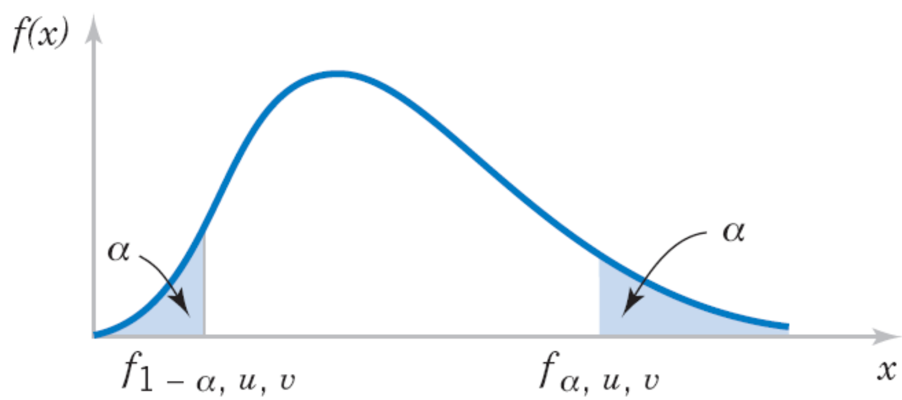


Figure 5-5 Upper and lower percentage points of the *F* distribution.



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# Inference on the Ratio of Variances of Two Normal Populations

## The Test Procedure

**Testing Hypotheses on the Equality of Variances of Two Normal Distributions**

Null hypothesis: $H_0\colon \sigma_1^2 = \sigma_2^2$

Test statistic: $$F_0 = \frac{S_1^2}{S_2^2} \qquad (5\text{-}21)$$

| Alternative Hypotheses | Rejection Criterion |
|---|---|
| $H_1\colon \sigma_1^2 \neq \sigma_2^2$ | $f_0 > f_{\alpha/2,n_1-1,n_2-1}$ or $f_0 < f_{1-\alpha/2,n_1-1,n_2-1}$ |
| $H_1\colon \sigma_1^2 > \sigma_2^2$ | $f_0 > f_{\alpha,n_1-1,n_2-1}$ |
| $H_1\colon \sigma_1^2 < \sigma_2^2$ | $f_0 < f_{1-\alpha,n_1-1,n_2-1}$ |

The critical regions are shown in Fig. 5-6.



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# Inference on the Ratio of Variances of Two Normal Populations

## The Test Procedure

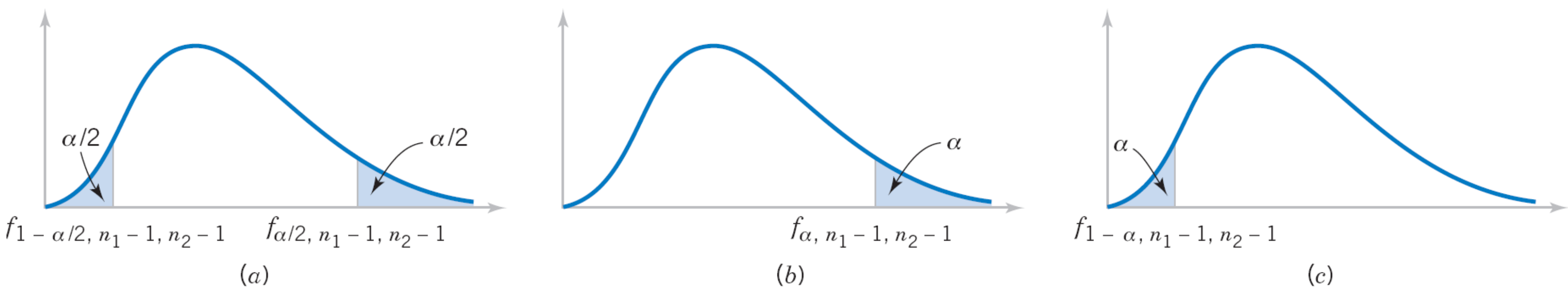


**Figure 5-6** The $F$ distribution for the test of $H_0$: $\sigma_1^2 = \sigma_2^2$ with critical region values for (a) $H_1$: $\sigma_1^2 \neq \sigma_2^2$, (b) $H_1$: $\sigma_1^2 > \sigma_2^2$, and (c) $H_1$: $\sigma_1^2 < \sigma_2^2$.



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## Example for the use of the F distribution for a two-sample variance test

A student in bone engineering is interested in making a suspension of polymer microspheres. He found that two similar instruments in his lab can be used. One was designed by a company named Teckman and the other one by Abilent. The variability in the diameter of microspheres in the suspension is a critical characteristics and low variability is desirable for subsequent processing steps. Therefore the student is interested in determining if these instruments result in a significantly different microsphere diameter variability. To test this, the student prepares 16 microsphere *suspensions* with each instrument ($n_1$=$n_2$=16). The sample standard deviations $s_1$=1.96 microns and $s_2$=2.14 microns. Use $\alpha$=0.05.

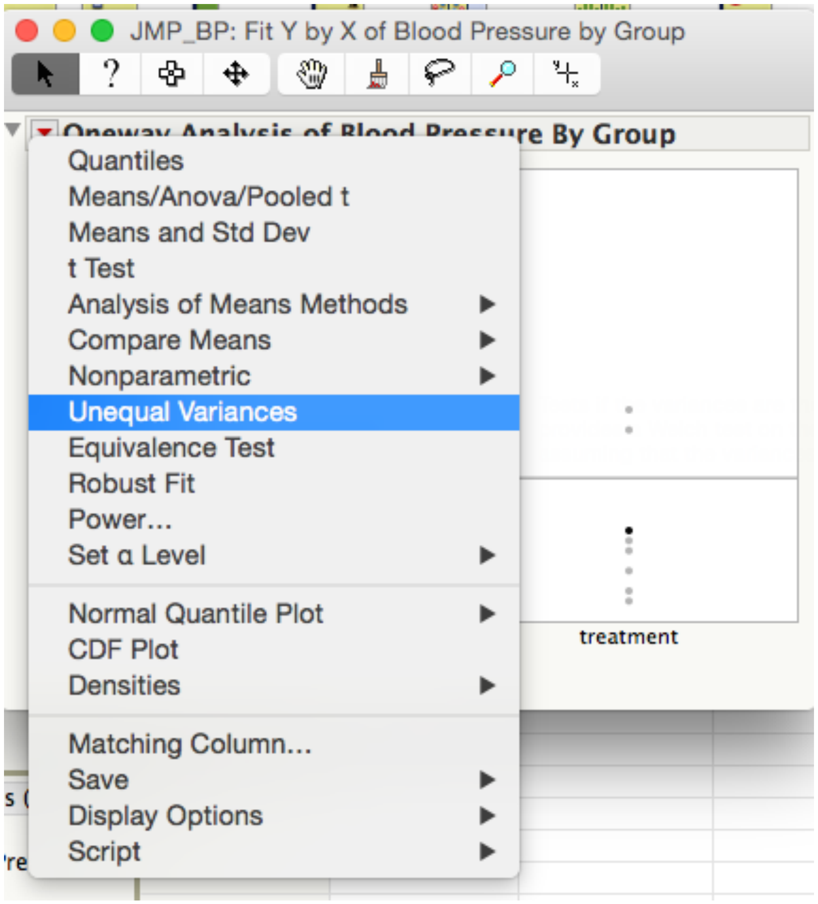
$$H_0: \sigma_1^2 = \sigma_2^2$$

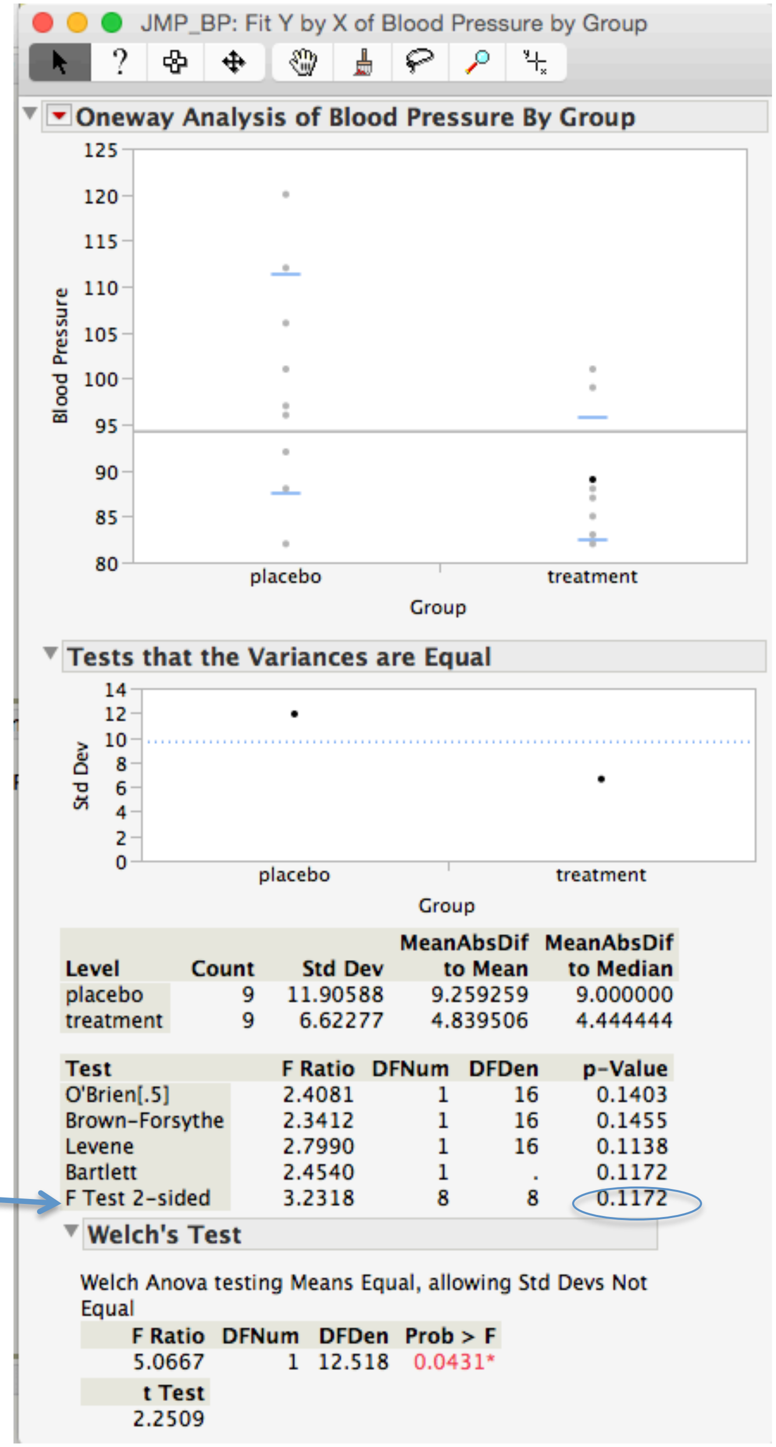
$$H_1: \sigma_1^2 \neq \sigma_2^2$$

$f_o = s_1^2/s_2^2 = 0.85$

$f_{0.975,15,15} = 0.35 < 0.85 < f_{0.025,15,15} = 2.86$ , we cannot reject $H_o$
(no strong evidence of differences between instruments in terms of variability in microsphere diameter)

F-test in JMP (fit Y by X) for BP example



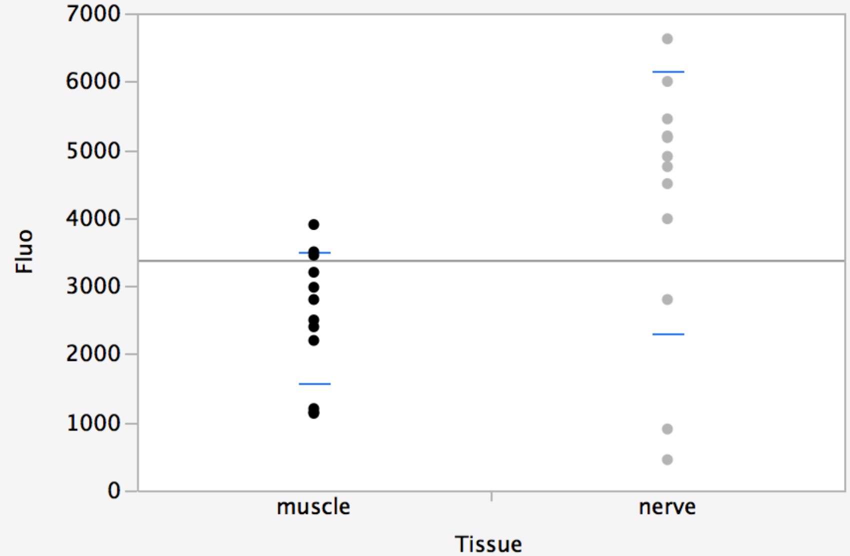




we cannot reject $H_o$
no strong evidence of difference in variance

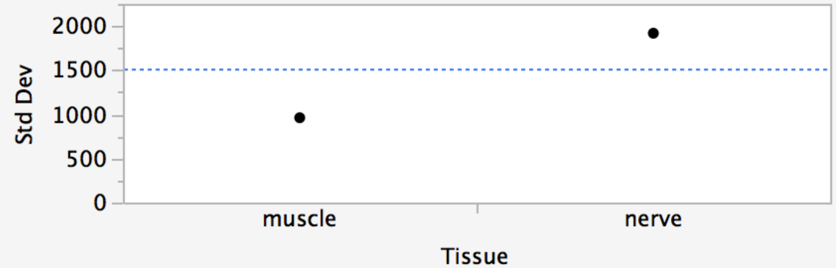
F-test in JMP (fit Y by X) for nerve vs muscle fluo (ex.2.1 textbook)

EX 2.1 nerve vs muscle fluo: Fit Y by X of Fluo by Tissue

**Oneway Analysis of Fluo By Tissue**




**Tests that the Variances are Equal**




| Level | Count | Std Dev | MeanAbsDif to Mean | MeanAbsDif to Median |
|---|---|---|---|---|
| muscle | 12 | 960.506 | 770.833 | 770.833 |
| nerve | 12 | 1917.992 | 1462.778 | 1330.417 |

| Test | F Ratio | DFNum | DFDen | p-Value |
|---|---|---|---|---|
| O'Brien[.5] | 3.0739 | 1 | 22 | 0.0935 |
| Brown-Forsythe | 1.5491 | 1 | 22 | 0.2264 |
| Levene | 3.5491 | 1 | 22 | 0.0729 |
| Bartlett | 4.6759 | 1 | . | 0.0306* |
| F Test 2-sided | 3.9874 | 11 | 11 | 0.0305* |

F-test: we reject $H_o$
(evidence of difference in variance)