Performance of AWGN channel using Monte Carlo Simulation

Arush S. Sharma, *Dept. of Electrical and Computer Engineering, University of Arizona,*

***Abstract*—This report documents the task that is required in the Project 1 of the Graduate Course ECE 535A in the context of performing a simulation study of transmission of a bipolar signal over the AWGN channel to estimate the probability of error of optimal detector with varying signal to noise ratio.**

***Index Terms*—AWGN, Monte Carlo, SNR, BER.**

I. Introduction

In Digital Communication, the possibility of error occurring in the bit stream transferred from source to destination over a communication channel always exist due to the presence of noise, interference, synchronization errors or distortion from external sources. In this case study the source transmits bipolar signal over Additive White Gaussian Noise (AWGN) channel wherein a noise is added by simulating different Signal to Noise Ratio (SNR) values. The receiver then decodes the received signal following the maximum-likelihood criterion which is explained in Section II. The probability of error or Bit Error Rate (BER) is computed by taking the ratio of number of bit errors divided by the total number of transmitted bits. The source code is written in C programming language, although technically the file is saved as .cpp extension due to the requirement of generating random numbers following Gaussian distribution which is possible by invoking "random" header file which is a part of C++ 11 standard library. In this simulation, the source sends 8000 bipolar bits to destination in total. Monte Carlo simulation is then performed 1000 times with different time-based seed to have more randomness and the data is then averaged. The rest of the section is as follows. Section II explains the principle of the Maximum Likelihood Detector. Section III describes the performance of BER with varying SNR which is achieved through Monte Carlo simulation. Section IV compares the BER result of Monte Carlo simulation with the theoretical expression stated in the Equation 1. Section V summarizes the task(s) involved in the project. Finally the report is concluded with the derivation of Q-function.

$$P_b = Q\left(\sqrt{\frac{E_b}{N_o/2}}\right) \tag{1}$$

arushsharma@email.arizona.edu

II. Maximum likelihood Detector

Assuming that the source wishes to transmit sequence of bits which are independent of each other, i.e., memoryless system, the detector would like to make a decision on transmitted signal based on the observation of received vector, r. With this goal in mind, the decision rule based on the computation of the posterior probabilities is defined as [1]

$$P(signal\ s_m\ was\ transmitted \mid r),\ m = 1, 2, ..., M \tag{2}$$

The decision criteria is based on selecting the signal corresponding to the maximum of the set of posterior probabilities $P(s_m \mid r)$. This criterion is called the maximum a posteriori probability (MAP) criterion. Using Bayes' rule, the posterior probabilities may be expressed as

$$P(s_m \mid r) = \frac{p(r|s_m)p(s_m)}{p(r)} \tag{3}$$

where $p(r \mid s_m)$ is the conditional pdf of the observed vector given s_m, and $P(s_m)$ is the a priori probability of the mth signal being transmitted.

Some simplification occurs in the MAP criterion when the M signals are equally probable events, i.e., $P(s_m) = 1/M$. Furthermore the denominator in Equation 3 is independent of which signal is transmitted. Thus the decision rule based on finding the signal that maximizes $P(s_m|r)$ is equivalent to finding the signal that maximizes $p(r|s_m)$ which is usually called the likelihood function. The decision criterion based on the maximum of $p(r|s_m)$ over the M signals is called the maximum-likelihood (ML) criterion. The detector based on MAP criterion makes the same decision as that of ML criterion as long as the signals are equally probable. In case of an AWGN channel, the likelihood function $p(r|s_m)$ is given as:

$$p(r|s_m) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[\frac{-(r-s_m)^2}{2\sigma^2}\right] \tag{4}$$

Now suppose the sequence of outputs $r_1, r_2, ..., r_k$ is observed. The joint pdf of $r_1, r_2, ..., r_k$ may be expressed as a product of K marginal pdfs, i.e.,

$$p(r_1, r_2, ..., r_k|s_m) = p(r_1|s_1)p(r_2|s_2)...p(r_k|s_m) \tag{5}$$

$$p(r_1, r_2, ..., r_k|s_m) = \prod_{i=1}^{k} p(r_i|s_i) \tag{6}$$

$$p(r_1, r_2, ..., r_k|s_m) = \prod_{i=1}^{k} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left[\frac{-(r_i-s_i)^2}{2\sigma^2}\right] \tag{7}$$

$$p(r_1, r_2, ..., r_k|s_m) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^k exp\left[-\sum_{k=1}^{k}\frac{(r_k - s_k)^2}{2\sigma^2}\right] \tag{8}$$

By taking the log of Equation 8, the final expression is achieved which is given as follows:

$$= kln\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}\sum_{k=1}^{k}(r_k - s_k)^2 \tag{9}$$

By neglecting the terms that are constant in the above Equation 9, the ML sequence detector selects that sequence that minimizes the euclidean distance metric given as follows:

$$D(r, s_m) = \sum_{k=1}^{k}(r_k - s_k)^2 \tag{10}$$

III. Performance of BER with varying SNR

The received signal is compared with the ML threshold of 0 since the input signals are equally probable. If the received signal is above 0, it is decoded as +1, otherwise it is decoded as -1. The simulation records the Probability of Error with SNR beginning from -500 dB to +50 dB with increments of 5. In the Figure 1, it can be seen that the performance of AWGN channel improves as the SNR increases. The unnecessary details from -500 dB to -30 dB and from 10 dB to 50 dB are omitted in the figure. The probability of error was close to 0.5 when the SNR was -500 dB and this was consistent until the SNR reached -30 dB. The probability of error then exponentially reduces with increase in SNR until it crosses 10 dB mark. Onwards from 10 dB SNR, the probability of error is very much close to 0.

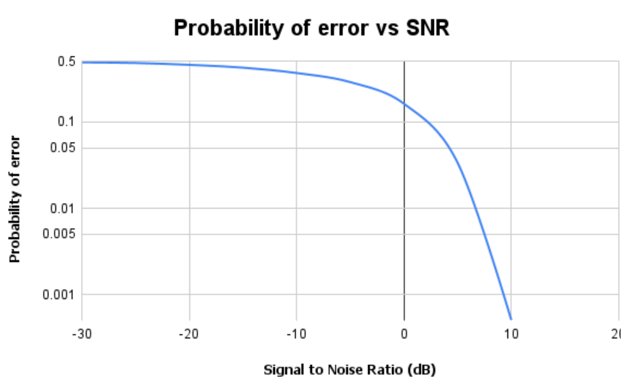

Fig. 1. Performance of BER with varying SNR

IV. Comparison of Monte Carlo simulation with the theoretical expression

In this section, the performance of Monte Carlo simulation is compared with the analytical expression as stated in the Equation 1 which is otherwise known as Q-function. The math library in C programming language cannot compute the Q-function directly but it can compute the complimentary error function value. The relationship between Q-function and complimentary error function is as follows:

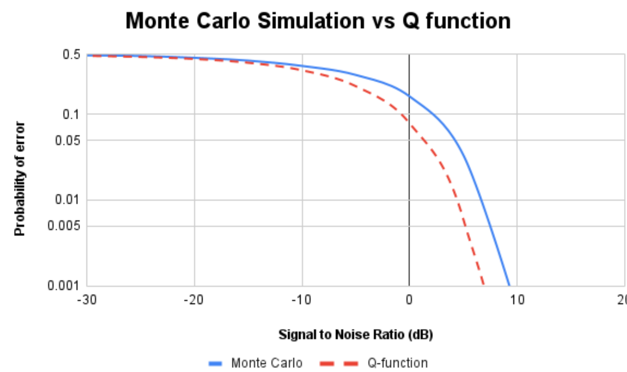

Fig. 2. Performance comparison of Monte Carlo with Q function

$$Q(x) = \frac{1}{2}erfc\left(\frac{x}{\sqrt{2}}\right) \tag{11}$$

It can be observed from the Figure 2, that the Monte Carlo simulation has a little deviation from the Q-function when the SNR increases onwards from approximately -15 dB. This little deviation is intuitively expected as there are some randomness generated when dealing with simulations. Q-function represents ideal case in this scenario as it is a theoretical expression. Overall, Monte Carlo simulation follows the similar trends with that of Q-function.

V. Conclusion

8000 bipolar bits are transmitted by source to the destination using AWGN channel and the simulation is run 1000 times. Performance of the channel in terms of BER is evaluated with varying SNR using Monte Carlo simulation written in C programming language. Finally the performance of Monte Carlo simulation is compared with Q-function with varying SNR.

VI. Derivation of Probability of Error

The probability of error is likelihood of misrepresenting the bit due to a noise which in this case is AWGN. Considering the bipolar transmission, we can have $x_1(t) = A + n(t)$ for transmission of bit 1 and $x_2(t) = -A + n(t)$ for bit -1. Now its known that the power spectral density of AWGN noise is $N_o/2$ as it contains all the frequency components [1].We can then write $x_1(t) \sim N(A, \frac{N_o}{2T})$ and $x_2(t) \sim N(-A, \frac{N_o}{2T})$ where T is a period. The likelihood of misinterpretation of bit is denoted as

$$p_e = p(-1|1)p_1 + p(1|-1)p_{-1} \tag{12}$$

where p_1 is the probability of transmitting bit 1 and p_{-1} is the probability of transmitting bit -1. Each of the terms on Right Hand Side of Equation 3 can be written as follows:

$$p(1|-1)p_1 = 0.5\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}}exp\left(\frac{-(y+1)^2}{2\sigma^2}\right)dy \tag{13}$$

$$p(-1|1)p_{-1} = 0.5 \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(\frac{-(y-1)^2}{2\sigma^2}\right) dy \quad (14)$$

where y is the received signal at the decoder. By using the average energy of the signal $E_b = A^2T$ and combining the Equations 13, 14, the expression can be simplified as:

$$p_e = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2E/N_o}}^{\infty} exp(-x^2/2)dx \quad (15)$$

The integral of the Gaussian function, i.e. CDF, is computed by Q-function [1] and the final expression is reduced as follows:

$$p_e = Q\left(\sqrt{\frac{2E_b}{N_o}}\right) = 0.5erfc\left(\sqrt{\frac{E_b}{N_o}}\right) \quad (16)$$

References

[1] J. G. Proakis and M. Salehi, *Digital communications*. McGraw-hill New York, 2001, vol. 4.