American University of Armenia
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Partial Differential Equations

F01 - American Options Pricing

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Abstract

This paper discusses American Option Pricing problem and its solution using numerical methods. We first give an introduction to Options and other related concepts, then more detailed explanation of European and American Options. Later we concentrate on the American Option Pricing, introduce the Obstacle problem, its numerical solution, then use that solution while solving the American Option pricing problem. At the end we present the Python code of the solution, consider some applications, provide explanations and examples.

Keywords: partial differential equations, option, European option pricing, American option pricing. obstacle problem, free boundary problem, numerical methods

Chapter 1

Introduction

For simplicity let us first take a look at what is European call option. It is a contract which has conditions so that at the expiry date, which is some point in the future, the option owner can purchase the underlying asset for an exercise or strike price that is some prescribed amount.

This contract is not an obligation, instead, it is a right. Yet, the writer, which is the other party to the contract, is under an obligation to sell the asset if the holder of it chooses to purchase it. And as the option gives power to the holder for having right without any obligation, it has value and it must be paid for as the contract is being opened.

Now let's cover some terminology used in finance so that further discussions will be clear and no questions will arise.

- **Gearing**

It is the effect of option prices responding in an exaggerated way to the underlying asset price changes. The value of the call option on some specific day depends on that day's share price. What relates to call option price and exercise price dependence, the lower the exercise price, the higher the option price as less need to be paid on exercise.

- **Volatility**

  It is a statistical measure of return dispersion for a given market index. The larger the volatility, the more likely a profitable outcome at the expiry date. And as the volatility increases, the call option value is increasing. It’s worth to note that the option is paid at the moment of opening the contract, while the payoff comes later.

- **Call vs Put option**

  The call option is an option to buy a specified quantity of a security at a specified price until its expiration. The put option is an option to sell a specified quantity of a security at a specified price until its expiration.

  The owner of the call option desires to see rising asset price as greater profit will be guaranteed when there is a higher asset price at expiry. Meantime, the owner of the put option desires to see falling asset price as here it works in the opposite way.

- **Arbitrage**

  In term of mathematics, arbitrage resembles no free lunch theorem, which in financial terms will be no opportunity gives an immediate risk-free profit. So for a greater return, one needs to take a greater risk. So the work of person engaging in arbitrage includes seeking out the irregularities and mispricings and deriving benefit from it.

- **Risk**

  The commonly described types of risks are specific and non-specific(market/systematic risk) ones. It is very important to have a clear understanding of these two types and be able to distinguish between them based on their behavior within a large portfolio. Having a portfolio with a large number of assets from different sectors of the market, it is possible to diversify specific type risks, while it is impossible to diversify non-specific ones. It is believed that taking a specific risk will not be rewarded, while that of a non-specific will be greatly rewarded by a huge return.

The simple definition of risk is the variance of return, which does not take into account return distribution. So it gives the same weight to both the possibility of greater and less return than the expected one. There are other more complicated definitions of risk, which give different weights to different returns based on various criterias.

Chapter 1

Theoretical Background

1.1 General Theory

For understanding the basic option theory, let us introduce the list of parameters on which the value of the option is dependent. We will also give the notations of each parameter as we are going to use them on further discussions.

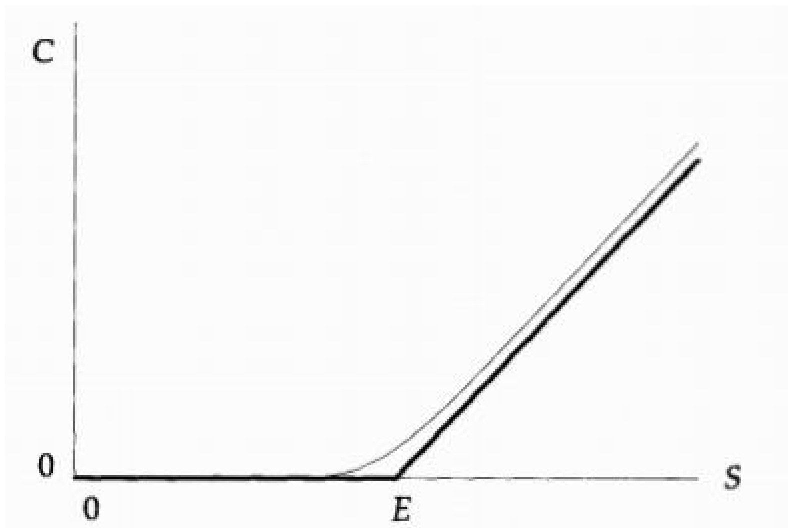
We denote the value of the option as **V**. It is dependent on the current value of the underlying asset **S** and the time **t**(i.e. V = V(S, t)). Whenever we want to emphasize the importance of the distinct option that is put or call, we will use P(S, t) or C(S, t), respectively. Despite these two parameters, the option value is dependent on the volatility of the underlying asset(σ **)**, the exercise price(**E**), the expiry(**T**), the interest rate(**r**).

Having the current value of the underlying asset(**S**) bigger than the exercise price(**E**) at expiry(**T**), it is reasonable to exercise the call option. So we are handing over **E** amount for obtaining an asset worth of **S**. Thus the profit of the transaction is **S - E**. Else if we have **S** less than **E** at **T**, it would be unreasonable to exercise the option as we would make a “profit” of **E - S**, which is a negative value, so it is a loss not a gain. Thus the value of the call option(**C(S, t)**) at expiry(**T**) can be calculated using

$$C(S,\ T)\ =\ max(S\ -\ E,\ 0)$$

As the expiry date is approaching, we are expecting the value of the call option to approach the value of $max(S - E, 0)$. The difference of the option value before and at the expiry is called **time value**, and the value at expiry is called **intrinsic value**.

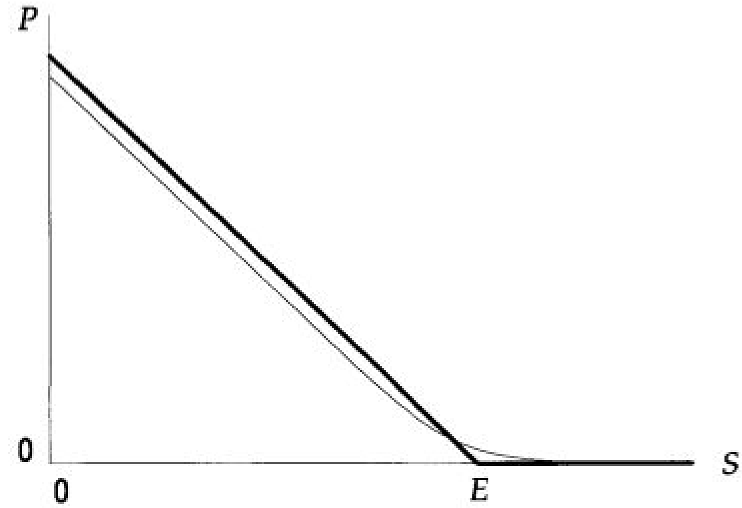
The plot of the call option payoff(**C(S, T)**) and the option value(**C(S, t)**) before the expiry(**T**) is given as a function of **S**.



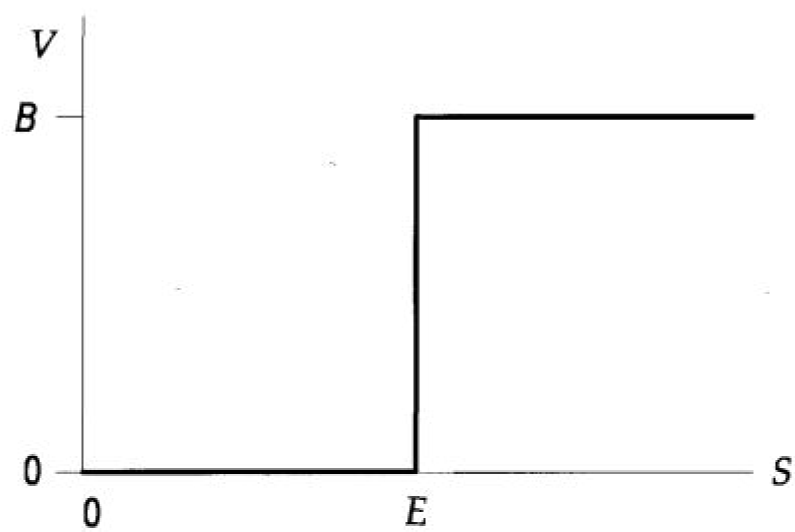
Each option has its payoff at expiry. If **S** is greater than **E** at expiry, the put option is worthless, yet if **S** is less than **E** at expiry it has a value of **S - E**. So the put option payoff at expiry is given with the following equation

$$P(S, T) = max(E - S, 0)$$

The plot of the put option payoff(P**(S, T)**) and the option value(**P(S, t)**) before the expiry(**T**) is given as a function of **S**.



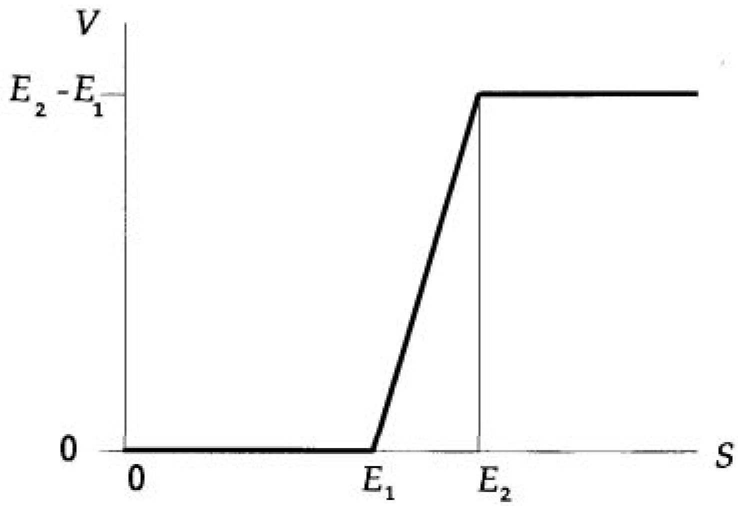
The put and call options are the most basic structures for payoff, however, there are more general payoffs. Such options are called **binaries** or **digitals**. Example of general payoff is $BH(S - E)$ (Heaviside function, which has value of 1 if its argument is positive, otherwise it is zero), this option is called **cash-or-nothing call**. See the payoff diagram below.



There is another one called **bullish vertical spread** option, which has payoff of

$$max(S - E_1, 0) - max(S - E_2, 0)$$

where $E_2 > E_1$. See the payoff diagram below.



There is a relationship between the underlying asset and its price called **put-call parity**. It is a risk elimination example, which is done by completing one transaction in the asset and each of the options. The relationship is given below.

$$S + P - C = Ee^{-r(T-t)}$$

Assumptions made to remember throughout the article.

1. The price of the asset follows the log-normal random walk. That is the movements of the stock prices are independent of one another, and also the size and direction are random(except the fact that stock prices are increasing over time).
2. The interest rate(**r**) and the asset volatility(σ) are known functions depending on time(life of the option).
3. There are not any transaction costs related to hedging a portfolio.
4. The underlying asset does not pay any dividends during the life of the option.
5. Arbitrages are not possible, that is all risk-free portfolios have the same return.

6. Underlying asset trading can happen continuously.
7. Assets are separable and short-selling is allowed.

1.2 The Black-Scholes equation

The Black-Scholes partial differential equation is given below.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

We aim to solve this PDE and get the option price. Sometimes, numerical methods can be used as the exact value is not always possible to be found. Also the partial differential equation usually has many solutions, yet the value of the option should be unique. So for making an appropriate decision, we must give some boundary conditions, which tend to specify the solution behaviour at some part of the domain it is defined.

Our PDE is a parabolic equation as its highest derivative w.r.t. S is a second derivative and that of t is a first derivative. If the signs of these second and first derivatives are the same, when they are on the same side of the parabolic equation, the equation is called **backward parabolic** one, otherwise it is called **forward parabolic** one. In our case, the PDE is backward parabolic equation. So now as we have decided the type of our equation to be parabolic, it is time to make statements about boundary conditions(two conditions in S and one in t). Example of such conditions are

$$V(S,\ t) = V_a(t) \qquad on \qquad S = a$$

$$V(S,\ t) = V_b(t) \qquad on \qquad S = b$$

where V_a and V_b are given functions of t.

If the equation is backward parabolic one, we must add another condition as well.

$$V(S,\ t)\ =\ V_T\,(S)\quad on\quad t\ =\ T$$

where V_T is a given function.

If we solve backward parabolic equation, we solve the problem for V in the region $t\ <\ T$, that is, backwards in time. Otherwise, if we aim to solve a forward one, we set an initial condition of $t\ =\ t_1$, and solve in region $t\ >\ t_1$, now in the forward direction. It is worth to mention that for changing from backward to a froward one, we simply need to change variables $t'\ =\ -t$. So mathematically, both forward and backward types are the same, however, it is important to remember about the right direction.

1.3 European Option

The final condition for European option is applied at t = T, and comes from the arbitrage argument we have described before, and the payoff is given by

$$C(S,\ t)\ =\ max(S\ -\ E,\ 0)$$

The asset price boundary conditions are applied at 0 asset price, that is when $S\ =\ 0$ and as $S\ \to\infty$. If $S\ =\ 0$ at expiry the payoff is 0. Therefore, even if there is a long time to expiry, the call option is worthless on $S\ =\ 0$.

$$C(0,\ t)\ =\ 0\qquad on\qquad S\ =\ 0$$

The price of the asset increases boundlessly and thus it becomes more likely that the option will be exercised and the exercise price magnitude becomes less and less. So as $S\ \to\infty$ the option value becomes the assets value.

$$C(S, t) \sim S \quad as \quad S \to \infty$$

For a put option, the final condition is the payoff of

$$P(S, t) = max(E - S, 0)$$

If $S = 0$, then it must remain 0. In this case, the final put payoff is known with E certainty. And thus, for getting $P(0, t)$, we just need to calculate the current value of an amount E that has been received at time T. Having the assumption of constant interest rates, we can find the boundary condition at $S = 0$

$$P(0, t) = Ee^{-r(T - t)}$$

For a time-dependent interest rate, we have

$$P(0, t) = Ee^{-\int_t^T r(r)dr}$$

The option is unlikely to be exercised as $S \to \infty$.

$$P(S, t) \to 0 \quad as \quad S \to \infty$$

Now let us talk about the exact solution of the European call option problem having as constants the interest rate and the volatility. So when r and σ are constant, the exact and explicit solution for the European call is

$$C(S, t) = S\,N(d_1) - E\,e^{-r(T - t)}\,N(d_2)$$

where the function N() is the cumulative distribution function for a standardised normal random variable

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy$$

where d_1 and d_2 are as follows

$$d_1 = \frac{log(S/E) + (r + \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = \frac{log(S/E) + (r - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$$

For a put option, the solution is

$$P(S, t) = E\, e^{-r(T - t)} N(-d_2) - S\, N(-d_1)$$

It is important to note that these satisfy the put-call parity, that is

$$S + P - C = E\, e^{-r(T - t)}$$

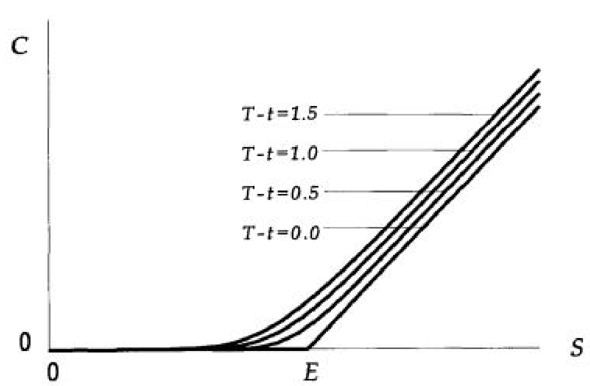
The delta for a European call and put are as follow,

$$\Delta = \frac{\partial C}{\partial S} = N(d_1)$$

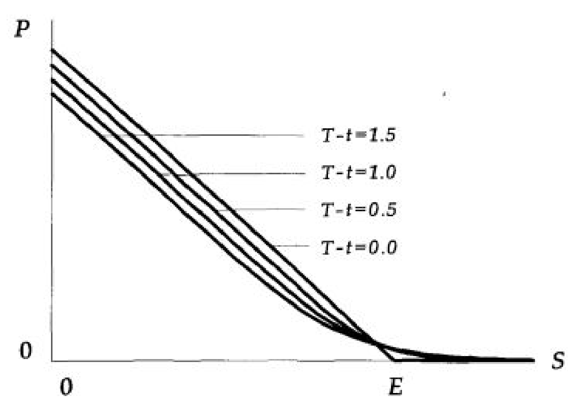
$$\Delta = \frac{\partial P}{\partial S} = N(d_1) - 1$$

The plots of the European call value $C(S, t)$ as a function of S and the European put option $P(S, t)$ as a function of S for various values of expiry time are given below

$$r = 0.1,\ \sigma = 0.2,\ E = 1,\ T - t = 0,\ 0.5,\ 1.0,\ 1.5$$

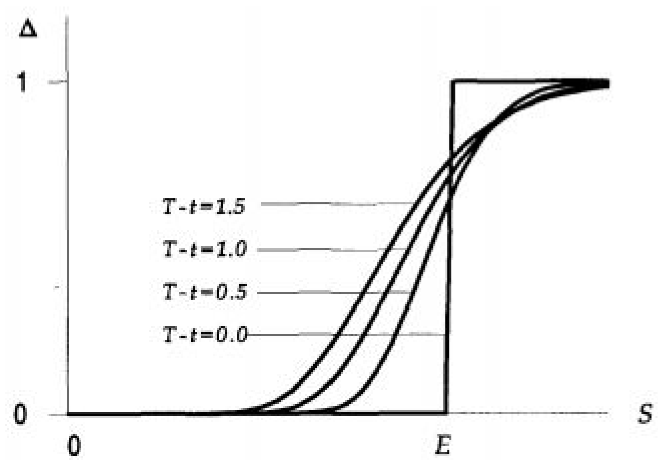


Call



Put

And here we can see the graph of the European call delta as a function of S, for several expiry times. The delta is between 0 and 1, and approaches a step function as $t \to T$.



1.4 Option on dividend-paying asset

Many assets pay out dividends. And as the price of the option is affected by these payments, we should modify Black-Scholes analysis. We are going to modify it for a continuous and constant dividend, which we will need later when discussing American options.

Let us suppose that at time dt, the underlying asset pays out dividend of $D_o S\, dt$, where D_0 is a constant. This payment is independent of time(except through the dependence on S). The ratio of the dividend payment and the asset price is called the **dividend yield**. In our case, the dividend $D_o S\, dt$ is a constant continuous dividend yield.

When considering arbitrage, the asset price falls by the dividend payment amount i.e. the random walk for the asset price is modified to

$$dS = \sigma S\, dX + (\mu - D_0)\, S\, dt$$

As we get $D_0 S\,dt$ for every held asset and as we hold $-\Delta$ of the underlying, our portfolio changes by the dividend our assets recieve, that is by $-D_0 S\Delta\,dt$. So now we will have

$$d\Pi = dV - \Delta\,dS - D_0\,S\,dt$$

Thus, we get the modified version of the Black-Scholes equation as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\,S\,\frac{\partial V}{\partial S} - rV = 0$$

Now, let us consider the effect of a non-zero dividend yield on the boundary and final conditions. For the call option the final condition is the same as before, that is

$$C(S,\ t) = max(S - E,\ 0)$$

and the boundary condition at $S = 0$ also remains the same, i.e.

$$C(0,\ t) = 0$$

The only change to the boundary conditions is

$$C(S,\ t) \sim S\,e^{-D_0(T-t)} \qquad as \quad S \to \infty$$

This is due to the fact that the option becomes equivalent to the asset but without its dividend. So the value of the European call option with the addition of a constant dividend yield D_0 is

$$C(S,\ t) = e^{-D_0(T-t)}\,S\,N(d_{10}) - E\,e^{-r(T-t)}\,N(d_{20})$$

where d_{10} and d_{20} are as follow

$$d_{10} = \frac{log(S/E) + (r - D_0 + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_{20} = d_1 - \sigma\sqrt{T - t}$$

1.4 American Option

The American option allows exercising at any time during the life of an option, and thus it has a higher value.

Let us suppose that $P(S,\ t)\ <\ max(E\ -\ S,\ 0)$ and as the American option allows exercising at any time, it can be exercised at this moment. Here comes the opportunity of arbitrage. We can purchase the option for P, exercise it by selling the asset for E and then repurchase it for S. In this scenario, we are making a risk-free profit of $E\ -\ P\ -\ S$. It is obvious that this opportunity is not a long-lasting one and that it will last till the option value is pushed up by the arbitrageurs demand. Thus we have a constraint

$$V(S,\ t)\ \geq\ max(S\ -\ E,\ 0)$$

Considering the call option on a dividend-paying asset and that the American option must be more valuable than that of the European, it must satisfy

$$C(S,\ t)\ \geq max(S\ -\ E,\ 0)$$

The valuation of American options is called the free boundary problem. At every time t, there is an S value marking the boundary between two regions. First side is holding the option and the second side is exercising it. The boundary is denoted by $S_f\,(t)$. Here we do not know a priori where to apply the boundary conditions.

The problem of the American option valuation can be uniquely specified by some set of constraints:

1. The value of the option is greater than or equal to the payoff function.
2. The Black-Scholes equality is changed by inequality.

3. The option value is a continuous function of S.

4. The option delta is continuous.

The American option has an exercise boundary $S = S_f(t)$ and the option should be exercised if $S < S_f(t)$ and held otherwise. Having the assumption that $S_f(t) < E$, the payoff function, which is $max(S - E, 0)$, slope at the contact point is -1. So there are three possibilities for the delta of the option $\frac{\partial P}{\partial S}$, at $S = S_f(t)$.

1. $\frac{\partial P}{\partial S} < -1$

2. $\frac{\partial P}{\partial S} > -1$

3. $\frac{\partial P}{\partial S} = -1$

The increase in P is passed on by the PDE to all values of $S > S_f$ and by decreasing S_f we reach to the crossover point between first and second possibilities(these are proven to be incorrect), which maximizes the benefit to the holder and avoids arbitrage. This yields to the correct possibility that is $\frac{\partial P}{\partial S} = -1$ at $S = S_f(t)$.

The return from the portfolio can not be greater than that of the bank deposit. So for an American put, we will have

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r S \frac{\partial P}{\partial S} - r P \leq 0$$

When it is a good choice to hold the option, Black-Scholes equation is valid and the inequality constraint for the option price must be satisfied. Or else exercising the option is optimal and only having the above inequality hold and the equality of the option price constraint to be satisfied, the obstacle is the solution.

For vanilla options including a dividend yield, we have

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)\, S \frac{\partial V}{\partial S} - r\, V \le 0$$

Chapter 2

American Put Option as Free Boundary Problem

2.1 American Option

Referring to American Options we can talk about American put options and American call options. The aim of this particular project is to study the American put option. Hence this and the following chapters will concentrate mainly on that type of American options.

Now we have already showed that the Black - Scholes formula written for European put option will not generally work for American put. Hence we need to do some changes in order to get correct results for American put as well.

Recall that the American put option was different from European put option in a sense that it could be exercised at any time including the expiry date. This fact leads to a free boundary problem for the price of an American put option. The fact that the option can be exercised at any time makes as set its value to be always greater than or equal to the intrinsic price which is $max(S - E, 0)$ in order to not have a possibility of arbitrage (this past was discussed in the previous chapter as well). Now, if we did not have this restriction we would witness that in some cases Black - Scholes value became lower than the intrinsic value, hence there is a range of prices in case of which it would be better to exercise the option and range of prices in case which it would not. Let’s call the dividing price between these 2 sets of prices an optimal exercise price

$S_f(t)$. $S_f(t)$ depends on time remaining to expiry and some other parameters of the problem. Sf(t) is not known a priori that is why it is called a free boundary for the associated Black - Scholes partial differential equation. Logically, the problem of finding the price of the option will be called free boundary problem. Looking at American put option we can see that there is only one free boundary $S = S_f(t)$. If S is greater than $S_f(t)$ the option can be held, if it is less than $S_f(t)$ then it needs to be exercised. In general there can be more than one free boundaries.

Now let's discuss the American put option pricing in more details. Recall Black - Scholes formula for European put option:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r\,S\,\frac{\partial P}{\partial S} - r\,P = 0$$

It's payoff function is

$$P(S,\ T) = max(E - s,\ 0)$$

and the boundary conditions are

$$P(0,\ t) = E\,e^{-r(T - t)}$$

$$P(S,\ t) \rightarrow 0 \qquad as \qquad S \rightarrow \infty$$

As we already have mentioned, for some values of S European put option price may be less than its intrinsic price. Let's consider the case when $S = 0$. In this case the intrinsic value $max(E - S,\ 0)$ will be equal to E, while from boundary conditions we can see that

$$P(0,\ t) = E\,e^{-r(T - t)} \leq E$$

Hence, for $t < T$ the option price is less than it intrinsic value. In such cases there will be arbitrage possibilities (which is not acceptable). Thus for the case of American put option we must set the condition

$$P(S,t) \geq max(E - S,\ 0)$$

For American put option the Black-Scholes formula will be changed to the following inequality

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r\,S\,\frac{\partial P}{\partial S} - r\,P \leq 0$$

Summing up what we have discussed so far we see that there must be a free boundary as European put option formula does not satisfy the constraint of $P(S,t)$. Moreover if $P = E - S$ for some $S < E$, then it will probably not satisfy the Black-Scholes formula.

Anyway, it will satisfy the corresponding inequality. In our case when $P = E - S$ we face the case when the return from portfolio is less than the return for bank deposit, thus it is better to exercise the option. As we have already mentioned for any t we divide the S axis into 2 parts. In one the exercise is optimal

$$P = E - S$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r\,S\,\frac{\partial P}{\partial S} - r\,P \leq 0$$

In the second one, it is better to hold the option

$$P > E - S$$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + r\,S\,\frac{\partial P}{\partial S} - r\,P = 0$$

We have that $S_f(t)$ is the largest value of S at t for which we have $P(S,t) = max(E - S,\ 0)$.

Hence

$$P(S_f(t),\ t)\ =\ max(E\ -\ S_f\,(t),\ 0)$$

But we know that

$$P(S,\ t)\ >\ max(E\ -\ S,\ 0)\qquad if\,S\ >\ S_f\,(t)$$

This defines the free boundary $S_f\,(t)$.

Some analysis near the free boundary(not to be covered in this paper) show that

$$\frac{\partial P}{\partial S}(S_f\,(t),\ t)\ =\ -1$$

This result gives as 2 free boundary conditions.

$$P(S_f\,(t),\ t)\ =\ max(E\ - S_f(t),\ 0)$$

$$\frac{\partial P}{\partial S}(S_f(t),\ t)\ =\ -1$$

The first can help us to find the option price on the free boundary and the second one cal help to find the location of the free boundary.

2.2 American Options as Variational Inequalities and the Obstacle Problem

Continuing the discussion on the possible solutions on American option pricing problem it is important to mention that no explicit solutions are found to solve this problem. In this paper our aim will be to use efficient numerical methods and find approximation to the solution of the problem.

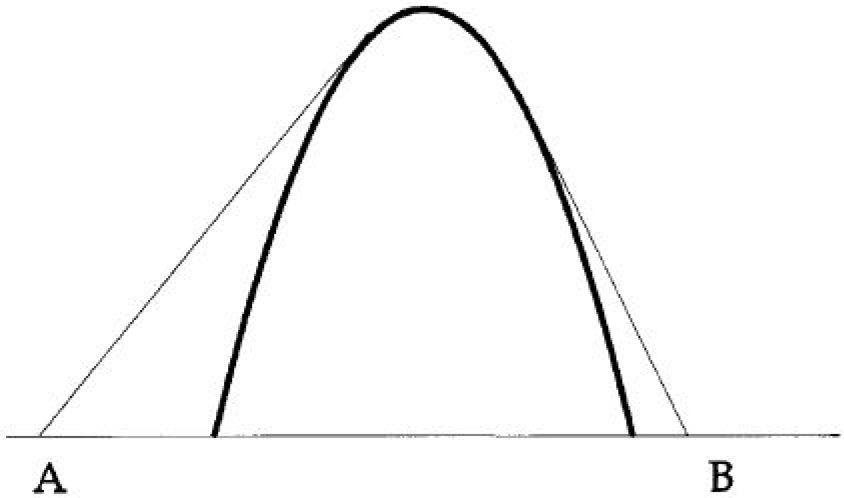
As we have already mentioned the American put option pricing problem can be introduced as a free boundary problem and basically our task will be to discuss and solve a free boundary problem. For that we concentrate on the canonical free boundary problem, **the obstacle problem**. From the first sight it might be not clear why we discuss this problem but there are some strong reasons to do so:

1. The obstacle problem is a quite easy free boundary problem.
2. It is independent of time.
3. Can be used to introduce linear complementarity problems and variational inequalities(will be discussed later),
4. The questions of the existence and uniqueness of the solutions are easy to discuss within its scope.

The logic behind the **variational inequalities** is to formulate the given free boundary problem to exclude its explicit dependence on the free boundary because we do not know its value a priori. By doing such change we can deal with the free boundary after solving the problem.

Now we will discuss the obstacle problem, which is a free boundary problem and then go to the more complex one, American options. These two free boundary problems hase linear complementarity as well as variational inequality formulations due to which we can have efficient numerical solution schemes with good accuracy.

To understand this problem let’s imagine a string which is held fixed at to its ends and should lie above a fixed obstacle with height



The end of the string are at -1 and +1. The displacement of the string is expressed by $u(x)$. $f(1) < 0,\ f(-1) < 0,\ f(x) > 0\ on\ 1 < x < 1$, hence there is definitely a contact region between the obstacle and the string. In addition we assume that $f' < 0$ which means that $f(x)$ is concave, hence there is only 1 contact region. The contact region consist of all points from A to B. These points are unknown a priori and will be found while solving the problem.

Looking at the plot it is easy to notice that when the obstacle meets the string $u = f$ and in other case the string is straight, hence $u'' = 0$. If we knew the values at the endpoints of contact-free regions we would easily find the straight parts. We know the value of f at +1 and -1, but it is unknown on A and B. Hence we need two other boundary conditions. Using the fact that u` and u should be continuous at A and B(based on force balance) we will obtain the following boundary conditions:

$$u(-1) = 0$$

$$u'' = 0 \qquad -1 < x < A$$

$$u(A) = f(A) \qquad u'(A) = f'(A)$$

$$u(x) = f(x) \qquad A < x < B$$

$$u(B) = f(B) \qquad u'(B) = f'(B)$$

$$u'' = 0 \qquad B < x < 1$$

$$u(1) = 0$$

Concluding our aim will be to find $u(x)$ having the presented boundary conditions. In almost all cases except when $f(x)$ is a very simple function points A and B should be found using some numerical methods.

2.3 The linear complementarity formulation

Another approach to a obstacle problem is to take the fact that either string is above the obstacle $u > f$ and it is straight $u'' = 0$, or that the string is in contact with the obstacle $u = f$ and $u'' = f'' < 0$ (concavity). Using these conditions we can write the obstacle problem as a complementarity problem:

$$u''(u - f) = 0$$

$$-u'' \geq 0$$

$$u - f \geq 0$$

with conditions

$$u(-1) = u(1) = 0 \qquad u,\ u' \text{ are continuous}$$

2.4 The variational inequality formulation

This formulation of the problem is given using the following facts. Let K be the set of all functions $v(x)$ for which

- $v(-1) = 0 \; and \; v(1) = 0$
- $v(x) \geq f(x) \;\; for \; -1 \leq x \leq 1$
- $v(x) \; is \; continuous$
- $v'(x) \; is \; piecewise \; continuous$

We call the functions from the set K as **test functions**. If $u(x)$ should be continuous the functions in K should be only pointwise continuous. Now let's apply the results from the previous part to this formulation.

For any v, we have $(v-f) \geq 0$ and as $-u'' \geq 0 \Rightarrow -u''(v-f) \geq 0$

From this we can also state that $\int_{-1}^{1} -u''(v-f)dx \geq 0$. We know that $\int_{-1}^{1} -u''(u-f)dx = 0$ Subtracting we obtain $\int_{-1}^{1} -u''(v-u)dx \geq 0$. Looking at the last example we can see that it does not include $f(x)$. Hence we do not have the explicit presence of the $f(x)$ function which was needed to obtain by the logic of variational inequalities. Let's obtain a better representation of this result by doing an integration by parts:

$$[-u'(v-u)]_{-1}^{1} + \int_{-1}^{1} u'(v-u)'dx \geq 0$$

Having $u = v$ and $x = \pm 1$ we obtain

$$\int_{-1}^{1} u'(v - u)'dx \geq 0$$

If u is the solution of our problem then the obtained formula is true for any v.

Concluding, the variational inequality formulation of the free boundary problem will be in the following way:

$$\text{Find } u \,\varepsilon\, K \text{ s.t } \int_{-1}^{1} u'(v - u)'dx \geq 0 \text{ for every } v \,\varepsilon\, K.$$

We can notice that the minimum value is obtained whenever $u = v$.

It is important to note that the solution found for variational inequality problem is a solution for complementarity and initial problem as well and vice versa. Another important fact is that the solution of variational inequality is unique. The last statement is useful when showing the existence and uniqueness of the solution in more complicated cases (such as American put option problem).

To conclude, some free boundary problems can be represented as variational inequalities. American options are one of these problems and this fact will help us tremendously while giving a numerical solution to American option problem.

To sum up why this kind of representation is helpful we can mention 2 main reasons:

1. The free boundary is not appearing explicitly, hence it can be found later, after solving the variational inequality problem.

2. We have only first order derivatives, hence we only need to approximate the first derivative of the unknown function which will result in better accuracy and efficiency.

2.5 A variational inequality for the American put

As we have already mentioned previously Black - Scholas formulation of the American put can be written as a variational inequality and linear complementarity problem.

Before going deep into that process let's express Black - Scholes differential equation as a diffusion equation by doing change of variables. Set:

$$S = E e^{x}$$

$$t = T - \tau / \tfrac{1}{2}\sigma^2$$

$$P(S, t) = E e^{-\frac{1}{2}(k_1 - 1)x - \frac{1}{4}(k_1 + 1)^2 \tau} u(x, \tau)$$

where $k_1 = r / \frac{1}{2}\sigma^2$.

After doing the change we will obtain

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

wherever $P(S, t) \geq max(E - S, 0)$

Alongside with this let's define

$$g(x, \tau) = e^{\frac{1}{4}(k_1 + 1)^2 \tau} max(e^{\frac{1}{2}(k_1 - 1)x} - e^{\frac{1}{2}(k_1 + 1)x}, 0)$$

The Initial condition(IC) for $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ will be $u(x, 0) = g(x, 0)$, $P(S, t) \geq max(E - S, 0)$ will be changed to $u(x, \tau) \geq g(x, \tau)$. Besides we have the following conditions:

$$u(x, \tau) \rightarrow 0 \quad as\ x \rightarrow \infty,\ u\ and\ \frac{\partial u}{\partial x}\ are\ continuous\ everywhere$$

After calculations we obtain

$$\frac{\partial g}{\partial \tau} - \frac{\partial^2 g}{\partial x^2} \geq 0 \quad for \quad x \neq 0$$

the return from portfolio is less than the riskless interest rate r.

We are going to solve this problem numerical methods and take some set of grid points. Hence we can consider the problem in a finite interval now as well.

From now on we consider the problem in $-x^- < x < x^+$ interval ($-x^-$ and x^+ are large).

The boundary conditions will be changes in the following way:

$$u(x^+, \tau) = 0, u(-x^-, \tau) = g(-x^-, \tau)$$

We do these changes by assuming that we can replace exact boundary conditions by approximations that for small values of S, $P = E - S$, while for its large values $P = 0$.

As we mentioned American put is very similar to the obstacle problem hence, locially, it has linear complementarity and variational inequality formulations too. The difference is that in case of American put the obstacle (transformed payoff function $g(x, tau)$) is time dependent. In addition as our PDE is parabolic the variational inequality formulation of this problem is called **parabolic variational inequality**.

As a linear complementarity problem we can write:

$$(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2})(u(x, \tau) - g(x, \tau)) = 0$$

$$(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}) \geq 0$$

$$u(x, \tau) - g(x, \tau) \geq 0$$

with IC as

$$u(x, 0) = g(x, 0)$$

and BC as

$$u(-x^{-}, \tau) = g(-x^{-}, \tau), \; u(x^{+}, \tau) = g(x^{+}, \tau) = 0$$

and the conditions

$$u(x, \tau) \; and \; \frac{\partial u}{\partial x}(x, \tau) \; are \; continuous$$

In this formulation as discussed previously we have two possibilities: it is optimal to exercise the option $u = g$ and it is better to hold it $u > g$.

Following the logic of the obstacle problem, we introduce the set of test functions M for American put problem. Other than that we need to convert the linear complementarity formulation of American put problem into (parabolic) variational inequality.

Now, the space of test functions M, which consists of functions $\phi(x, t)$ satisfies the following points(again very similar to the obstacle problem):

- $\phi(x, \tau) \; and \; \partial\phi/\partial\tau$ are both continuous and $\partial\phi/\partial x$ is piecewise continuous
- $\phi(x, \tau) \geq g(x, \tau)$ for all x for τ
- $\phi(x^{+}, \tau) = g(x^{+}, \tau) = 0 \; and \; \phi(-x^{-}, \tau) = g(-x^{-}, \tau)$
- $\phi(x, 0) = g(x, 0)$

Now we can guess that $u(x, t)$ is from M too. Besides for any ϕ belonging to M, $\phi \geq g$, hence

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(\phi(x, \tau) - g(x, \tau)) \geq 0$$

, consider the integral formula

$$\int_{-x^{-}}^{x^{+}} \left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(\phi(x, \tau) - g(x, \tau))dx \geq 0$$

Besides, we knew that

$$\int_{-x^-}^{x^+} \left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\ \tau)\ -\ g(x,\ \tau))dx\ \geq\ 0$$

Subtracting the previous two formulas we obtain

$$\int_{-x^-}^{x^+} \left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(\phi(x,\ \tau)\ -\ u(x,\ \tau))dx\ \geq\ 0$$

which is true for any $\phi(x,t)$. In addition similar to the obstacle problem the formula does not contain $g(x,t)$ explicitly. $g(x,t)$ enters into the formulation only via space of test functions M.

Now, we do integration by parts

$$\int_{-x^-}^{x^+} \frac{\partial u}{\partial \tau}(\phi - u) + \frac{\partial u}{\partial x}\left(\frac{\partial \phi}{\partial x} - \frac{\partial u}{\partial x}\right)dx\ - \left[\frac{\partial u}{\partial x}(\phi\ -\ u\right]_{-x^-}^{x^+} \geq\ 0$$

Since all test functions satisfy the same conditions we are left with the following variational inequality:

$$\int_{-x^-}^{x^+} \frac{\partial u}{\partial \tau}(\phi - u) + \frac{\partial u}{\partial x}\left(\frac{\partial \phi}{\partial x} - \frac{\partial u}{\partial x}\right)dx\ \geq\ 0$$

The global formulation of the problem is the following:

$$\int_{0}^{\frac{1}{2}\sigma^2 T} \int_{-x^-}^{x^+} \frac{\partial u}{\partial \tau}(\phi - u) + \frac{\partial u}{\partial x}\left(\frac{\partial \phi}{\partial x} - \frac{\partial u}{\partial x}\right)dx\ d\tau\ \geq\ 0$$

Again as we stated while discussing the obstacle problem the solutions of the variational inequality for American put is also solution for its linear complementarity formulation and the original problem. This implies that the solution will also solve the free boundary problem stated earlier.

To conclude, the advantage of the variational inequality for American put is that there is no explicit mention of the free boundary and by solving it we will find the optimal exercise boundary by the condition that defines it. Another important outcome of this representation is that it can be shown that the solution of variational inequality exists and is unique. Hence the solution of the American put has the same properties which would be hard to show if we had only the initial problem statement.

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Chapter 3

Numerical Solution

To find the numerical solution of the problem we start with the simple case. Let us take the Obstacle problem and solve it numerically. We have

$$u''(u - f) = 0$$

$$-u'' \geq 0$$

$$u - f \geq 0$$

The conditions are $u(-1) = u(1) = 0$ **,** $u,\ u'$ are continuous. To solve this problem numerically we use finite difference method. We divide the [-1, 1] range into 2N equal parts, where

$$-1 = -N\Delta x < (-n+1)\Delta x < \ldots < n\Delta x < \ldots < N\,\Delta x = 1$$

Then vor u'' we use the approximation

$$u'' = \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2} + O(\Delta x^2)$$

We also have that $u_n = u(n\Delta x),\ f_n = f(n\Delta x)$. Along with this information we take v_n as an approximation to u_n and get

$$-v_{n+1} + 2v_n - v_{n-1} \geq 0, \quad v_n \geq f_n, \quad (v_n - f_n)(v_{n+1} - 2v_n + v_{n-1}) = 0,$$

In addition to this, we obtain $v_{-N} = v_N = 0$.

The formula given above can also be given in the following matrix form:

$$\mathbf{B}\boldsymbol{v} \geq \mathbf{0}, \quad \boldsymbol{v} \geq \boldsymbol{f}, \quad (\boldsymbol{v} - \boldsymbol{f}) \cdot \mathbf{B}\boldsymbol{v} = 0,$$

were B and v are f have the following structure:

$$\mathbf{B} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix},$$

$$\boldsymbol{v} = \begin{pmatrix} v_{N-1} \\ \vdots \\ \vdots \\ \vdots \\ v_{-N+1} \end{pmatrix}, \quad \boldsymbol{f} = \begin{pmatrix} f_{N-1} \\ \vdots \\ \vdots \\ \vdots \\ f_{-N+1} \end{pmatrix}.$$

Using the SOR (Successive Over Relaxation) solution scheme and the modifications of the problem we have given the solution of Obstacle problem using Python.

Note: The paper is written based on "Option Pricing(mathematical models and computation)" book (authors: Paul Wilmott, Jeff Dewynne, Sam Howison).

Chapter 4

Python Code

Obstacle Problem: Numerical solution and Visualization

In [736]:
```
a = 1
N = 50
omega = 1.7 #if 0<omega<2 then the method will converge
err = 0
eps = 0.0000001
```

In [737]:
```
x = a*np.arange(-N,N+1)/N
dx = a/N
```

In [738]:
```
x
```

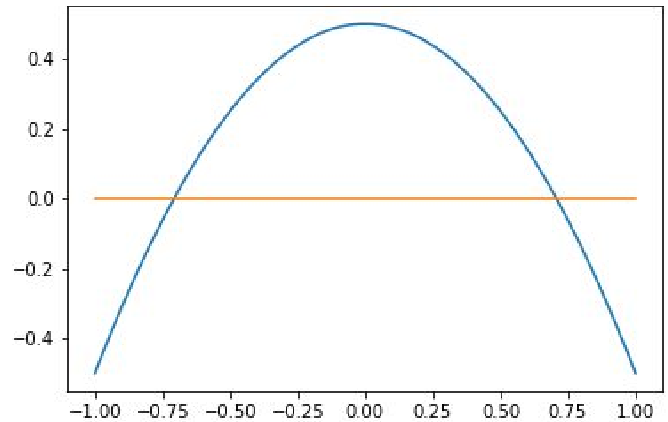
Out[738]:
```
array([-1.  , -0.98, -0.96, -0.94, -0.92, -0.9 , -0.88, -0.86, -0.84,
       -0.82, -0.8 , -0.78, -0.76, -0.74, -0.72, -0.7 , -0.68, -0.66,
       -0.64, -0.62, -0.6 , -0.58, -0.56, -0.54, -0.52, -0.5 , -0.48,
       -0.46, -0.44, -0.42, -0.4 , -0.38, -0.36, -0.34, -0.32, -0.3 ,
       -0.28, -0.26, -0.24, -0.22, -0.2 , -0.18, -0.16, -0.14, -0.12,
       -0.1 , -0.08, -0.06, -0.04, -0.02,  0.  ,  0.02,  0.04,  0.06,
        0.08,  0.1 ,  0.12,  0.14,  0.16,  0.18,  0.2 ,  0.22,  0.24,
        0.26,  0.28,  0.3 ,  0.32,  0.34,  0.36,  0.38,  0.4 ,  0.42,
        0.44,  0.46,  0.48,  0.5 ,  0.52,  0.54,  0.56,  0.58,  0.6 ,
        0.62,  0.64,  0.66,  0.68,  0.7 ,  0.72,  0.74,  0.76,  0.78,
        0.8 ,  0.82,  0.84,  0.86,  0.88,  0.9 ,  0.92,  0.94,  0.96,
        0.98,  1.  ])
```

In [739]:
```
f = 1/2 - (x)**2 #obstacle
```

In [740]:
```
zero = np.zeros(2*N+1)
```

In [741]:
```
plt.plot(x, f,x,zero)
```

Out[741]:
```
[<matplotlib.lines.Line2D at 0x1260155c0>,
 <matplotlib.lines.Line2D at 0x126015710>]
```



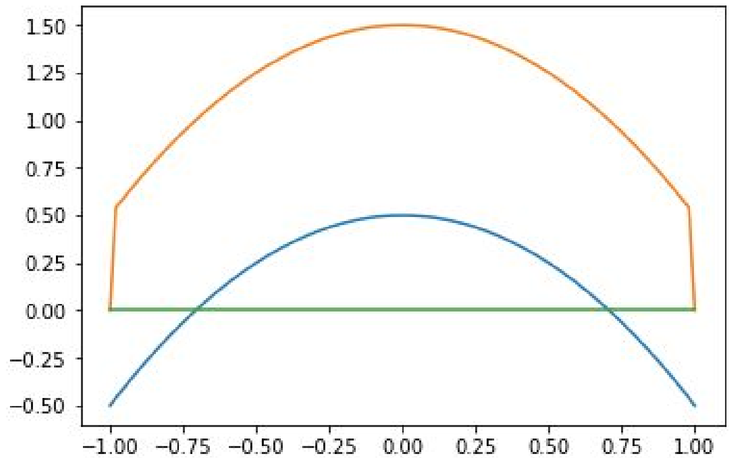
In [742]:
```
v = f+1
v[0] = 0
v[2*N] = 0
```

In [743]:
```
plt.plot(x, f, x, v, x, zero )
```

Out[743]:
```
[<matplotlib.lines.Line2D at 0x1260e90b8>,
 <matplotlib.lines.Line2D at 0x1260e9208>,
 <matplotlib.lines.Line2D at 0x1260e95c0>]
```

In [743]:
```
plt.plot(x, f, x, v, x, zero )
```

Out[743]:
```
[<matplotlib.lines.Line2D at 0x1260e90b8>,
 <matplotlib.lines.Line2D at 0x1260e9208>,
 <matplotlib.lines.Line2D at 0x1260e95c0>]
```

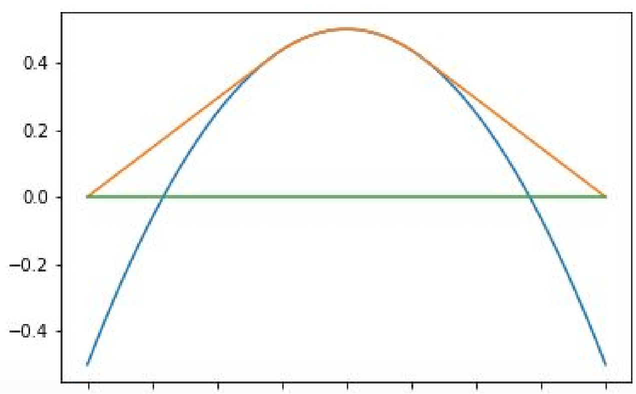


In [744]:
```
#Run for 1 time to initialize v0
for i in range(1,2*N):
        y = (v[i-1]+v[i+1])/2
        y = max(f[i], v[i]+omega*(y-v[i]))
        err += (v[i]-y)*(v[i]-y)
        v[i] = y
```

```
In [744]: #Run for 1 time to initialize v0
          for i in range(1,2*N):
                  y = (v[i-1]+v[i+1])/2
                  y = max(f[i], v[i]+omega*(y-v[i]))
                  err += (v[i]-y)*(v[i]-y)
                  v[i] = y
```

```
In [745]: #Run until err <= eps
          while err > eps*eps:
              for i in range(1,2*N):
                  err = 0
                  y = (v[i-1]+v[i+1])/2
                  y = max(f[i], v[i]+omega*(y-v[i]))
                  err += (v[i]-y)*(v[i]-y)
                  v[i] = y;
```

```
In [746]: plt.plot(x, f, x, v, x, zero )

Out[746]: [<matplotlib.lines.Line2D at 0x1262352b0>,
           <matplotlib.lines.Line2D at 0x126235400>,
           <matplotlib.lines.Line2D at 0x1262357b8>]
```



Using the numerical solution of the obstacle problem we, then implemented it on American put option problem and obtained the following solution.

American Option Pricing: Numerical Solution

Ex. American Put with interest rate r = 0.1, volatility sigma = 0.5, Excercise Price E = 10.

In [793]:
```
M = 10000
dt = 1/M
theta = 0
err1 = 0
r = 0.1
sigma = 0.5
```

In [805]:
```
k = r/((1/2)*sigma**2)
```

In [806]:
```
alpha #for convergency alpha < 0.5
```

Out[806]:
```
0.25
```

In [807]:
```
def payoff(x,t):
    return math.exp(((1/4)*(k+1)**2)*t) * max(math.exp((1/4)*(k+1)*x) - math.exp((1/2)*(k+1)*x), 0)
```

In [808]:
```
g = np.zeros(2*N+)
```

In [809]:
```
g
```

Out[809]:
```
array([0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
       0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
       0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
       0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
       0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.,
       0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0., 0.])
```

In [817]:
```
b = np.zeros(2*N+1)
```

In [818]:
```
v1 = np.zeros(2*N+1)
```

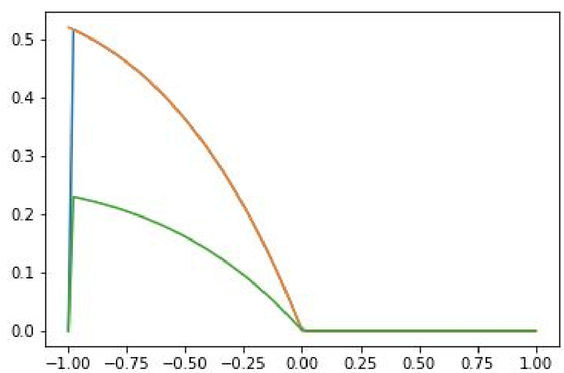
In [819]:
```
a = alpha*theta
```

In [820]:
```
tau = 0
alpha = dt/(dx)**2
```

In [821]:
```
#initial value
for i in range(0,2*N+1):
    v[i] = payoff(x[i], tau)
```

```
In [849]: for i in range(1, M+1):
              tau = i*dt
              for j in range(1,2*N-1):
                  g[j] = payoff(x[j], tau)
                  b[j] = v[j] + alpha*(1-theta)*(v[j+1]+v[j-1]-2*v[j]);

              g[0] = payoff(x[0], tau)
              g[2*N] = payoff(x[2*N], tau)
              b[1] += alpha*theta*g[0]
              b[2*N] += alpha*theta*g[2*N]

              for i in range(1,2*N):
                  y1 = (b[i]+a*(v1[i-1]+v1[i+1]))/(1+2*a)
                  y1 = max(g[i], v1[i]+omega*(y1-v1[i]))
                  err1 += (v1[i]-y1)*(v1[i]-y1)
                  v1[i] = y1;

              while err1 > eps*eps:
                  for i in range(1,2*N):
                      err1 = 0
                      y1 = (b[i]+a*(v1[i-1]+v1[i+1]))/(1+2*a)
                      y1 = max(g[i], v1[i]+omega*(y1-v1[i]))
                      err1 += (v1[i]-y1)*(v1[i]-y1)
                      v1[i] = y1;
```

```
In [853]: plt.plot(x, v1, x, g, x, b)

Out[853]: [<matplotlib.lines.Line2D at 0x126b24668>,
           <matplotlib.lines.Line2D at 0x126b249b0>,
           <matplotlib.lines.Line2D at 0x12690f160>]
```



GitHub URL

https://github.com/arusyak97/pde_project?fbclid=IwAR2Izun7I_EY24_T_-MOegijddxare5anSK7Od4mjtdAHxGoNwYYMUi5tNY

Reference

Wilmott, P., Dewynne, J., & Howison, S. (2000). *Option pricing: Mathematical models and computation*. Oxford: Oxford Financial Press.