Non-linear $\mathcal{O}$bservers Course

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2024-02-01

Chapter 1

Observers for Linear Systems

1.1 Le 1.1

Model:

$$A = \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 1 \\ 0 & 0 & \mu_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ C\,A \\ C\,A^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mu_2 \\ 0 & 0 & \mu_2^2 \end{pmatrix}$$

The null space of the observability matrix

$$\mathcal{N}\left(\mathcal{O}\right) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Considering the following:

$$\begin{pmatrix} \lambda\,I - A \\ C \end{pmatrix} = \begin{pmatrix} \lambda - \mu_1 & -1 & 0 \\ 0 & \lambda - \mu_1 & -1 \\ 0 & 0 & \lambda - \mu_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Substituting $\lambda = \mu_1,\ \mu_2,$

$$\Lambda_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & \mu_1 - \mu_2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} \mu_2 - \mu_1 & -1 & 0 \\ 0 & \mu_2 - \mu_1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

now,

$$\mathcal{N}\left(\Lambda_1\right) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad \mathcal{N}\left(\Lambda_2\right) = \left\{ \emptyset \right\}.$$

it is clear that $\mathcal{N}\begin{pmatrix} \lambda\,I - A \\ C \end{pmatrix}$ is not the same as $\mathcal{N}\left(\mathcal{O}\right)$, but rather a subset.

1.2 Le 1.2

$$\dot{\theta}_1 = 0 \quad \dot{\theta}_2 = 0 \qquad\qquad z(t) = \begin{pmatrix} 1 - y(t) & y(t) \end{pmatrix} \theta$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad C = \begin{pmatrix} 1 - y(t) & y(t) \end{pmatrix}$$

The observability criterion is

$$\begin{aligned}
\Sigma_0 &= \int_0^\infty e^{A^T t}\, C^T\, C\, e^{At} dt \\
\Sigma_0 &= \int_0^\infty e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} t} \begin{pmatrix} 1 - y(t) \\ y(t) \end{pmatrix} \begin{pmatrix} 1 - y(t) & y(t) \end{pmatrix} e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} t} dt \\
\Sigma_0 &= \int_0^\infty I \begin{pmatrix} 1 - y(t) \\ y(t) \end{pmatrix} \begin{pmatrix} 1 - y(t) & y(t) \end{pmatrix} I dt \\
\Sigma_0 &= \int_0^\infty \begin{pmatrix} (1 - y(t))^2 & y(t)\ (1 - y(t)) \\ y(t)\ (1 - y(t)) & y^2(t) \end{pmatrix} dt \\
\Sigma_0 &= \begin{pmatrix} \int_0^\infty \left(1 + y^2(t) - 2y(t)\right) dt & \int_0^\infty \left(y(t) - y^2(t)\right) dt \\ \int_0^\infty \left(y(t) - y^2(t)\right) dt & \int_0^\infty y^2(t)\, dt \end{pmatrix}
\end{aligned}$$

Let,

$$Y_1 = \int_0^\infty y(t) dt \qquad\qquad \&,\ Y_2 = \int_0^\infty y^2(t) dt$$

Therefore,

$$\Sigma_0 = \begin{pmatrix} t + Y_2 - 2Y_1 & Y_1 - Y_2 \\ Y_1 - Y_2 & Y_2 \end{pmatrix}.$$

For the system to be observable $\det(\Sigma_0) \neq 0$, i.e.,

$$\begin{aligned}
\det(\Sigma_0) \neq 0 \implies Y_2 t + Y_2^2 - 2Y_1 Y_2 + (Y_1 - Y_2)^2 &\neq 0 \\
Y_2 t + \cancel{Y_2^2} - \cancel{2Y_1 Y_2} - Y_1^2 - \cancel{Y_2^2} + \cancel{2Y_1 Y_2} &\neq 0 \\
Y_2 t - Y_1^2 &\neq 0 \\
t \int_0^\infty y^2(t) dt - \left(\int_0^\infty y(t) dt \right)^2 &\neq 0
\end{aligned}$$

If $y(t)$ is a constant then $\det(\Sigma_0) = 0$.

1.3 Le 1.3

Show that the observability gramian Σ_o satisfies the Lyapunov equation

$$A^T\,\Sigma_o + \Sigma_o\,A + C^T\,C = 0,$$

or,

$$A^T\,\Sigma_o + \Sigma_o\,A = -C^T\,C.$$

Multiply e^{At} to the right and $e^{A^T t}$ to the left,

$$e^{A^T t}\,A^T\,\Sigma_o\,e^{At} + e^{A^T t}\,\Sigma_o\,A\,e^{At} = -e^{A^T t}\,C^T\,C\,e^{At}.$$

Since,

$$\frac{d}{dt}e^{At} = A\,e^{At}$$

as e^{At} and A are commutative,

$$\left(\frac{d}{dt}e^{A^T t}\right)\Sigma_o\,e^{At} + e^{A^T t}\,\Sigma_o\left(\frac{d}{dt}e^{At}\right) = -e^{A^T t}\,C^T\,C\,e^{At}.$$

Using the product rule,

$$\frac{d}{dt}\left(e^{A^T t}\,\Sigma_o\,e^{At}\right) = -e^{A^T t}\,C^T\,C\,e^{At}.$$

integrating both sides with limits from 0 to ∞

$$\begin{aligned}\int_0^\infty \frac{d}{dt}\left(e^{A^T t}\,\Sigma_o\,e^{At}\right)dt &= \int_0^\infty -e^{A^T t}\,C^T\,C\,e^{At}dt.\\ \Longrightarrow\ \left[e^{A^T t}\,\Sigma_o\,e^{At}\right]_0^\infty &= \int_0^\infty -e^{A^T t}\,C^T\,C\,e^{At}dt.\end{aligned}$$

for a linear system with the state equation $\dot{X} = A\,X$ solves to $X = e^{At}$ and

$$\lim_{t\to\infty} X = \lim_{t\to\infty} e^{At} = 0$$

if the eigenvalues are on the left-hand side of the complex plane. Using the stability requirement

$$\begin{aligned}\left[e^{A^T t}\,\Sigma_o\,e^{At}\right]_0^\infty &= \int_0^\infty -e^{A^T t}\,C^T\,C\,e^{At}dt.\\ \text{becomes, } \Sigma_o &= \int_0^\infty e^{A^T t}\,C^T\,C\,e^{At}dt.\end{aligned}$$

1.4 Le 1.4

Compute the observability gramian as a function of ϵ and interpret the result.

$$\begin{aligned}\dot{x} &= \begin{pmatrix}-2 & -1\\ \epsilon & -1\end{pmatrix} x\\ y &= \begin{pmatrix}1 & 1\end{pmatrix} x\end{aligned}$$

the gramian is given as

$$\Sigma_o = \int_0^\infty \exp\left(A^T t\right)\, C^T\, C\, \exp\left(At\right) dt = \int_0^\infty \mathcal{X} dt \tag{1.1}$$

or the solution to the Lyapunov equation,

$$A^T\,\Sigma_o + \Sigma_o\, A + C^T\, C = 0,$$

substituing A and C,

$$\begin{aligned}\begin{pmatrix}-2 & -1\\ \epsilon & -1\end{pmatrix}^T \Sigma_o + \Sigma_o \begin{pmatrix}-2 & -1\\ \epsilon & -1\end{pmatrix} + \begin{pmatrix}1 & 1\end{pmatrix}^T \begin{pmatrix}1 & 1\end{pmatrix} = 0,\\ \begin{pmatrix}-2 & -1\\ \epsilon & -1\end{pmatrix}^T \Sigma_o + \Sigma_o \begin{pmatrix}-2 & -1\\ \epsilon & -1\end{pmatrix} = -\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},\end{aligned}$$

Let,

$$\Sigma_o = \begin{pmatrix}\sigma_1 & \sigma_2\\ \sigma_2 & \sigma_3\end{pmatrix},$$

then,

$$\begin{aligned}\begin{pmatrix}-2 & \epsilon\\ -1 & -1\end{pmatrix}\begin{pmatrix}\sigma_1 & \sigma_2\\ \sigma_2 & \sigma_3\end{pmatrix} + \begin{pmatrix}\sigma_1 & \sigma_2\\ \sigma_2 & \sigma_3\end{pmatrix}\begin{pmatrix}-2 & -1\\ \epsilon & -1\end{pmatrix} + \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, &= 0\\ \implies \begin{pmatrix}\epsilon\,\sigma_2 - 2\sigma_1 & \epsilon\,\sigma_3 - 2\sigma_2\\ -\sigma_1 - \sigma_2 & -\sigma_2 - \sigma_3\end{pmatrix} + \begin{pmatrix}\epsilon\,\sigma_2 - 2\sigma_1 & -\sigma_1 - \sigma_2\\ \epsilon\,\sigma_3 - 2\sigma_2 & -\sigma_2 - \sigma_3\end{pmatrix} + \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, &= 0\\ \implies \begin{pmatrix}2\epsilon\,\sigma_2 - 4\sigma_1 & \epsilon\,\sigma_3 - 3\sigma_2 - \sigma_1\\ \epsilon\,\sigma_3 - 3\sigma_2 - \sigma_1 & 2\sigma_2 - 2\sigma_3\end{pmatrix} + \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, &= 0\\ \implies \begin{pmatrix}2\epsilon\,\sigma_2 - 4\sigma_1 + 1 & \epsilon\,\sigma_3 - 3\sigma_2 - \sigma_1 + 1\\ \epsilon\,\sigma_3 - 3\sigma_2 - \sigma_1 + 1 & 2\sigma_2 - 2\sigma_3 + 1\end{pmatrix} &= 0,\\ \implies 2\epsilon\,\sigma_2 - 4\sigma_1 + 1 &= 0\\ \epsilon\,\sigma_3 - 3\sigma_2 - \sigma_1 + 1 &= 0\\ 2\sigma_2 - 2\sigma_3 + 1 &= 0.\end{aligned}$$

$$\Sigma_o = \frac{1}{6\,(\epsilon+2)}\begin{pmatrix}\epsilon^2 + 3\,\epsilon + 3 & 2\,\epsilon + 3\\ 2\,\epsilon + 3 & \epsilon + 3\end{pmatrix}$$

For Σ_o to have full-rank, $\det(\Sigma_o) \neq 0$, i.e.,

$$\frac{1}{6\,(\epsilon+2)} \neq 0, \qquad \text{i.e., } \epsilon^2 + \epsilon \neq 0$$

For $\epsilon = -2$, Σ_o is signular.

For $\epsilon = 0$, $\text{rank}(\Sigma_o) = 1$.

1.5 Le 1.5

Considering partitioned state equations

$$\begin{pmatrix} \dot{x}_r \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A_r & a_r \\ a_n & a_{nn} \end{pmatrix} \begin{pmatrix} x_r \\ x_n \end{pmatrix} + \begin{pmatrix} b_r \\ b_u \end{pmatrix} u(t) \qquad\qquad y = x_n$$

This gives,

$$\dot{x}_r = A_r\,x_r + a_r\,x_n + b_r\,u(t) \qquad\qquad \dot{x}_n = a_n\,x_r + a_{nn}\,x_n + b_n\,u(t)$$

Considering

$$y_r = \dot{y} - a_{nn}\,y - b_n\,u(t).$$

Then,

$$\dot{x}_r = A_r\,x_r + a_r\,x_n + b_r\,u(t) \qquad\qquad y_r = a_n\,x_r$$

The observer is

$$\hat{x}_r = A_r\,\hat{x}_r + a_r\,\hat{x}_n + b_r\,u(t) + K\,\left(y_r - a_n\,\hat{x}_r\right)$$

Substituting $x_n = \hat{x}_n = y$,

$$\begin{aligned} \dot{x}_r &= A_r\,x_r + a_r\,y + b_r\,u(t) \\ \dot{\hat{x}}_r &= A_r\,\hat{x}_r + a_r\,y + b_r\,u(t) + K\,\left(a_n\,x_r - a_n\,\hat{x}_r\right) \end{aligned}$$

The error signal,

$$\begin{aligned} \dot{\tilde{x}}_r &= \dot{x}_r - \dot{\hat{x}}_r \\ &= A_r\,x_r + \cancel{a_r\,y} + \cancel{b_r\,u(t)} - A_r\,\hat{x}_r - \cancel{a_r\,y} - \cancel{b_r\,u(t)} - K\,\left(a_n\,\hat{x}_r - a_n\,\hat{x}_r\right) \\ &= A_r\,x_r - A_r\,\hat{x}_r - K\,\left(a_n\,x_r - a_n\,\hat{x}_r\right) \\ &= \left(A_r - K\,a_n\right)\,\tilde{x}_r \end{aligned}$$

The poles can be selected arbitrarily in the error dynamics of the reduced observer.

1.6 Le 1.6

$$
\begin{aligned}
X &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X, \\
Y &= \begin{pmatrix} 1 & 0 \end{pmatrix} X
\end{aligned}
$$

Also,

$$
\begin{aligned}
\alpha_{\mathcal{O}}(s) = \det\left(SI - A + K\,C\right) &= \det\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}\right] \\
&= \det\left[\begin{pmatrix} s + k_1 & -1 \\ k_2 & s \end{pmatrix}\right] \\
&= s^2 + k_1\,s + k_2
\end{aligned}
$$

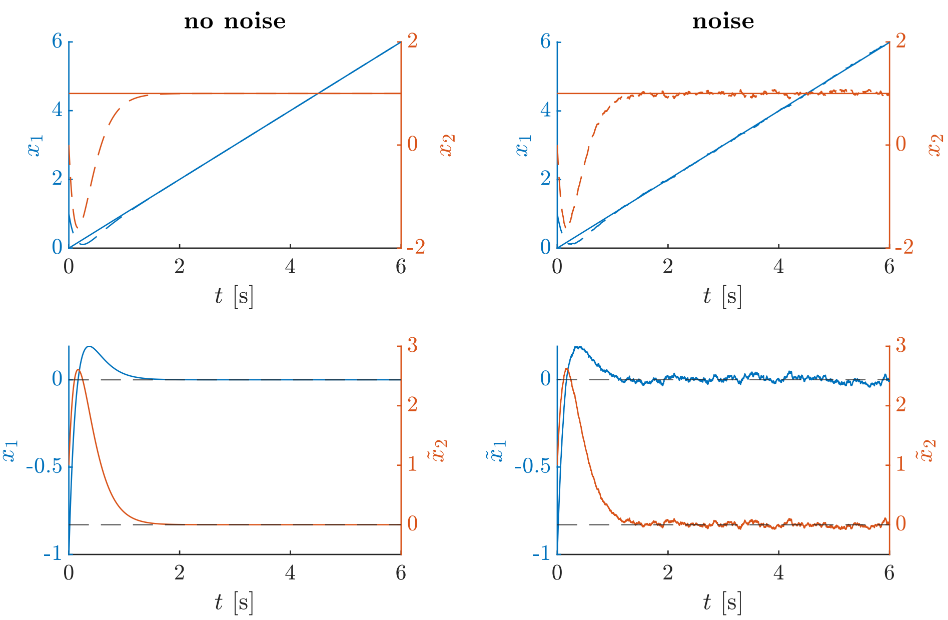
The closed-loop observer poles are at ω_α, i.e.,

$$
\alpha_{\mathcal{O}}(s) = (s + \omega_\alpha)\ (s + \omega_\alpha) = s^2 + 2\,\omega_\alpha\,s + \omega_\alpha^2
$$

Therefore,

$$
K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 2\omega_\alpha \\ \omega_\alpha^2 \end{pmatrix}
$$

The figure below gives the simulation of the full-order observer with the poles at $\omega_\alpha = 5\,\mathrm{rad/s}$.

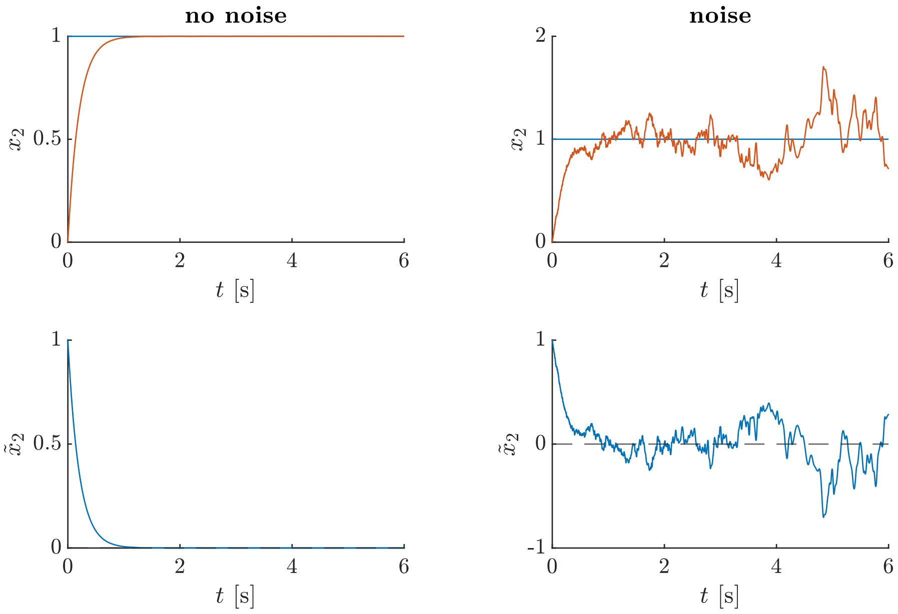


the blue and orange colors represent the states x_1 and x_2 respectively, the true state is shown by a solid line, and the estimated state is shown by a dashed line.

The reduced state observer is given as

$$
\begin{aligned}
\dot{w} &= (0 - l)\ w + (0 + 0 - 0 - l\,1\,l)\ y, \qquad & \hat{x}_1 = y, \quad \hat{x}_2 = w + l\,y, \\
\dot{w} &= -l\,w - l^2\,y, \qquad & \hat{x}_1 = y, \quad \hat{x}_2 = w + l\,y,
\end{aligned}
$$

The figure below gives the simulation of the full-order observer with the poles at $l = 5\,\mathrm{rad/s}$. (placing the poles similar to the full-order observer)



With the reduced order observer, the noise directly affects the observed 'state' since $x \propto y$ but in the full order observer, the noise is filtered since $\dot{x} \propto y$.

Code

```
% states
A = [0 1; 0 0];
C = [1, 0];
system = @(t,y) A*y;

% simulation
y0 = [0; 1];
tspan = linspace(0,6,5000);
[t_true,y_true] = ode45(@(t,y) system(t,y),tspan,y0);
% adding noise
y_noise = y_true(:,1) + (rand(size(t_true)) - 0.5)*0.5;

% simulation of full-order observer
y0 = [1; 0];
y_val = @(t) interp1(t_true, y_true(:,1), t, 'spline');
[t_fo,y_fo] = ode45(@(t,y) full_order(y,y_val(t),A,C), tspan, y0);
y_temp = interp1(t_true, y_true, t_fo, 'spline');
e_fo = y_temp - y_fo;

% simulation of full-order observer with noise
y_valN = @(t) interp1(t_true, y_noise(:,1), t, 'spline');
[t_foN,y_foN] = ode45(@(t,y) full_order(y,y_valN(t),A,C), tspan, y0);
y_temp = interp1(t_true, y_true, t_foN, 'spline');
e_foN = y_temp - y_foN;

% simulation of reduced-order observer
y0 = 0;
[t_ro,w] = ode45(@(t,y) reduced_order(y,y_val(t)),tspan,y0);
x2hat = w + 5 * y_val(t_ro);
y_temp = interp1(t_true, y_true(:,2), t_ro, 'spline');
e_ro = y_temp - x2hat;
```

```
% simulation of reduced-order observer (with noise)
[t_roN,w] = ode45(@(t,y) reduced_order(y,y_valN(t)),tspan,y0);
x2hatN = w + 5 * y_val(t_roN);
y_temp = interp1(t_true, y_true(:,2), t_roN, 'spline');
e_roN = y_temp - x2hatN;

%% functions
function xdot = full_order(x,y,A,C)
    K = [2*5; 5^2]; % feedback gain
    xdot = A * x + K*(y - C * x);
end

function wdot = reduced_order(w,y)
    L = 5; % feedback gain
    wdot = -L * w - L^2 * y;
end
```

1.7 Kailath 2.3-1a

If

$$\mathcal{O}_1\, T = \mathcal{O}_2,$$

then

$$c_1\, T = c_2, \qquad\qquad c_1\, A_1\, T = c_2\, A_2 = c_1\, T\, A_2.$$

Multiplying T^{-1} on both sides to the right

$$c_1\, A_1\, T\, T^{-1} = c_1\, T\, A_2\, T^{-1} \qquad\qquad \Longrightarrow\ c_1\, A_1 = c_1\, T\, A_2\, T^{-1}.$$

Multiplying c_1^{-1} on both sides to the left

$$c_1^{-1} c_1\, A_1 = c_1^{-1} c_1\, T\, A_2\, T^{-1} \qquad\qquad \Longrightarrow\ A_1 = T\, A_2\, T^{-1}.$$

This is not possible because c_1 is not invertible.

The non-observable states cannot be identified but if we measure all states then A_1 and A_2 will have similar dynamics.

1.8 Kilath 2.3.3

Realizing linear system to state-space

$$\begin{aligned}
y &= e\, \frac{s+1}{s\,(s+3)}\\
y &= \left(u - y\frac{k}{s+a}\right)\, \frac{s+1}{s\,(s+3)}\\
y &= u\, \frac{s+1}{s\,(s+3)} - y\, \frac{k\ (s+1)}{s\,(s+3)\,(s+a)}\\
y + y\, \frac{k\ (s+1)}{s\,(s+3)\,(s+a)} &= u\, \frac{s+1}{s\,(s+3)}\\
\frac{y\, s\,(s+3)\,(s+a) + y\, k\ (s+1)}{s\,(s+3)\,(s+a)} &= u\, \frac{s+1}{s\,(s+3)}\\
\frac{y\, s\,(s+3)\,(s+a) + y\, k\ (s+1)}{(s+a)} &= u\ (s+1)\\
y\, s\,(s+3)\,(s+a) + y\, k\ (s+1) &= u\ (s+1)\ (s+a)\\
y\, s\left(s^2 + s\,(3+a) + 3a\right) + y\, k\ (s+1) &= u\ \left(s^2 + s\,(1+a) + a\right)\\
y\left(s^3 + s^2\,(3+a) + 3a\, s\right) + y\, k\, s + y\, k &= u\ \left(s^2 + s\,(1+a) + a\right)\\
y\left(s^3 + s^2\,(3+a) + (3a+k)\ s + k\right) &= u\ \left(s^2 + s\,(1+a) + a\right)\\
\frac{y}{u} &= \frac{s^2 + s\,(1+a) + a}{s^3 + s^2\,(3+a) + (3a+k)\ s + k}\\
\frac{y}{u} &= \frac{0\, s^3 + s^2 + s\,(1+a) + a}{s^3 + s^2\,(3+a) + (3a+k)\ s + k}
\end{aligned}$$

Controllable canonical state space model

$$A = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ -k & -(3a+k) & -(3+a) \end{pmatrix}, \qquad B = \begin{pmatrix} 0\\ 0\\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} a & (1+a) & 1 \end{pmatrix}$$

Observable canonical state space model (*check something is wrong*)

$$A = \begin{pmatrix} -(3+a) & 1 & 0\\ -(3a+k) & 0 & 1\\ -k & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1\\ (1+a)\\ a \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Controllability and observability

$$\mathcal{O}x = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -(3+a) & 1 & 0 \\ (a+3)^2 - k - 3a & -(a+3) & 1 \end{pmatrix}$$

Observability on k.

1.9 Kilath 2.3-15

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \qquad C = \begin{pmatrix} \gamma_1 & \dots & \gamma_n \end{pmatrix}.v$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ C\,A \\ \vdots \\ C\,A^2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \dots & \gamma_n \\ \lambda_1\gamma_1 & \dots & \lambda_n\gamma_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1}\gamma_1 & \dots & \lambda_n^{n-1}\gamma_n \end{pmatrix} = \begin{pmatrix} \gamma_1 & \dots & \gamma_n \end{pmatrix} \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix}.$$

The rank$(\mathcal{O}) = n$ when λ_i is unique, $\lambda_i \neq 1$, and $\gamma_i \neq 0$.

1.10 Kilath 2.3-16 a-b

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad C = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ C\,A \\ C\,A^2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \lambda\,\gamma_1 & \gamma_1 + \lambda\,\gamma_2 & \gamma_2 + \lambda\,\gamma_3 \\ \lambda^2\,\gamma_1 & 2\gamma_1\,\lambda + \lambda^2\,\gamma_2 & 2\gamma_2\,\lambda + \lambda^2\,\gamma_3 \end{pmatrix}$$

The rank$(\mathcal{O}) = n$ when $\gamma_1 \neq 0$.

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad C = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ C\,A \\ C\,A^2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \lambda\,\gamma_1 & \gamma_1 + \lambda\,\gamma_2 & \mu\,\gamma_3 \\ \lambda^2\,\gamma_1 & 2\gamma_1\,\lambda + \lambda^2\,\gamma_2 & \mu^2\,\gamma_3 \end{pmatrix}$$

The rank$(\mathcal{O}) = n$ when $\gamma_1 \neq 0$ and $\mu \neq 0$.

1.11 Kilath 4.3.7

The system is modeled as

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = -\omega_0^2\, x(t), \qquad y(t) = v(t), \qquad x(t_0) = 0, \quad v(t_0) = 10.$$

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

The closed-loop observer poles are at ω_0, i.e.,

$$\alpha_{\mathcal{O}}(s) = (s+\omega_0)\ (s+\omega_0) = s^2 + 2\,\omega_0\, s + \omega_0^2$$

Also,

$$\begin{aligned} \alpha_{\mathcal{O}}(s) = \det\left(SI - A + K\,C\right) &= \det\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}\right] \\ &= \det\left[\begin{pmatrix} s & -1 \\ \omega_0^2 & s \end{pmatrix} + \begin{pmatrix} 0 & k_1 \\ 0 & k_2 \end{pmatrix}\right] = \det\left[\begin{pmatrix} s & k_1 - 1 \\ \omega_0^2 & s + k_2 \end{pmatrix}\right] \\ &= s\ (s + k_2) - \omega_0^2\ (k_1 - 1) = s^2 + s\, k_2 + \omega_0^2\ (1 - k_1) \end{aligned}$$

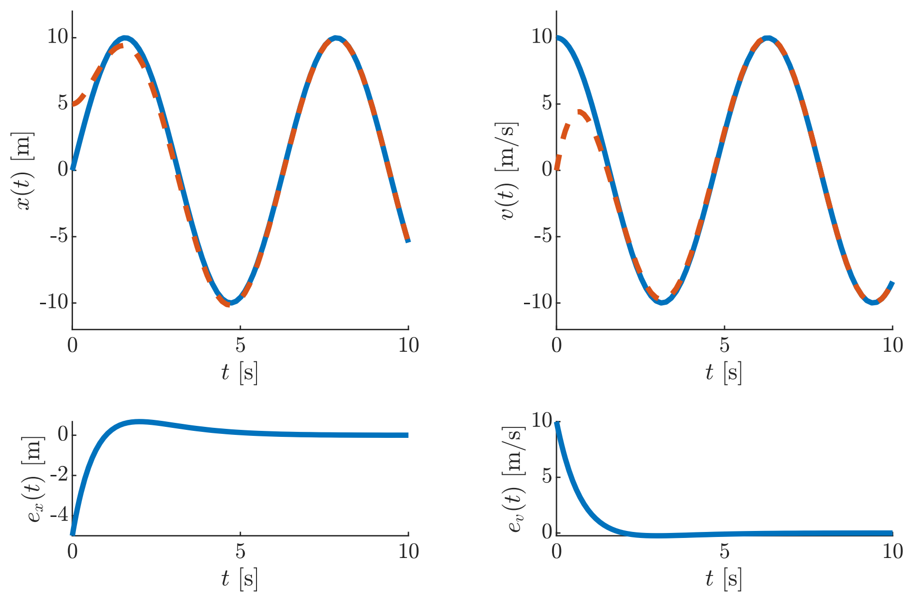
Therefore,

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\omega_0 \end{pmatrix}$$

The solution to the ODE is

$$x(t) = 10\,\sin(\omega_o\, t), \qquad v(t) = 10\,\cos(\omega_o\, t).$$

The simulation results for $\omega_0 = 10\,\mathrm{rad/s}$ are presented in the figure:



The solid line is the true state and the dashed line is the estimate from the observer.

The error dynamics are

$$\dot{e} = (A - K\,C)\ e = \left(\begin{pmatrix} 0 & 1 \\ -\omega_0 & 0 \end{pmatrix} - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix}\right)\ e = \begin{pmatrix} 0 & 1 - k_1 \\ -\omega_0^2 & -k_2 \end{pmatrix}\ e,$$

$$\dot{e}_x = (1 - k_1)\, e_v \quad \dot{e}_v = -\omega_0^2\, e_x - k_2\, e_v \qquad \Longrightarrow \dot{e}_x = e_v \quad \dot{e}_v = -\omega_0^2\, e_x - 2\,\omega_0\, e_v.$$

Therefore, the error will converge with a time constant of $1/\omega_0$.

Code

```
clear;

omega = 1;
v0 = 10;

% system true states
x = @(t) 10*sin(omega*t);
v = @(t) 10*cos(omega*t);

A = [0 1; -omega^2 0];
C = [0 1];

y0 = [5; 0];
tspan = [0 10];
[t,y] = ode45(@(t,y) observ(y,v(t),C,A,omega),tspan, y0);
```

functions

```
function xdot = observ(x,y,C,A,omega)

K = [0; 2*omega];

xdot = A*x + K*(y - C*x);

end
```

1.12 Kailath 4.1-8

The model:

$$\ddot{x} - 2\omega\dot{y} - 9\omega^2 x = 0 \qquad\qquad \ddot{y} + 2\omega\dot{x} + 4\omega^2 y = u.$$

Let,

$$\dot{x} = v_x, \quad \dot{y} = v_y \qquad\qquad \Longrightarrow \ddot{x} = \dot{v}_x, \quad \ddot{y} = \dot{v}_y.$$

Therefore,

$$\begin{aligned} \dot{v}_x - 2\omega v_y - 9\omega^2 x = 0 & \qquad \dot{v}_y + 2\omega v_x + 4\omega^2 y = u, \\ \dot{v}_x = 2\omega v_y + 9\omega^2 x & \qquad \dot{v}_y = u - 2\omega v_x - 4\omega^2 y. \end{aligned}$$

The state matrix, $X = \begin{pmatrix} x & v_x & y & v_y \end{pmatrix}^T$. The system can be re-written as

$$\dot{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 9\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & -4\omega^2 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} X.$$

The closed-loop observer poles

$$\begin{aligned} \alpha_{\mathcal{O}}(s) &= \det\left(SI - A + K\,C\right) \\ &= \det\left(\begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 9\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & -4\omega^2 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} s & -1 & k_1 & 0 \\ -9\omega^2 & s & k_2 & -2\omega \\ 0 & 0 & k_3 + s & -1 \\ 0 & 2\omega & 4\omega^2 + k_4 & s \end{pmatrix} \\ &= s^4 + k_3 s^3 + \left(k_4 - \omega^2\right) s^2 + (-5k_3\omega^2 - 2k_2\omega)s - 36\omega^4 - 18k_1\omega^3 - 9k_4\omega^2 \end{aligned}$$

The closed-loop observer poles are $s = -2\omega$, $s = -3\omega$, $s = -3\omega \pm j3\omega$

$$\begin{aligned} \alpha_{\mathcal{O}}(s) &= (s + 2\omega)\,(s + 3\omega_0)\,(s + 3\omega - j3\omega)\,(s + 3\omega + j3\omega) \\ &= s^4 + 11\omega s^3 + 54\omega^2 s^2 + 126\omega^3 s + 108\omega^4 \end{aligned}$$

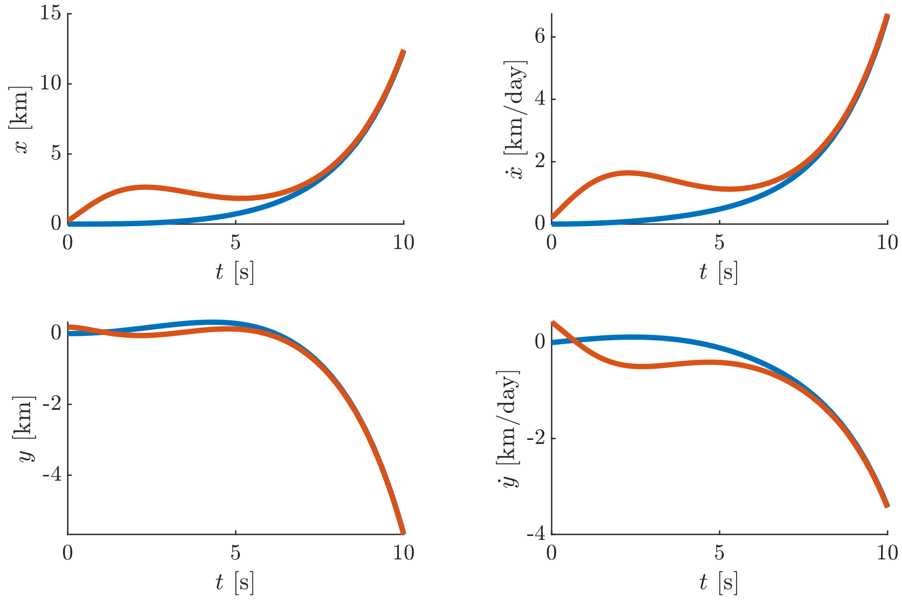
Now,

$$\begin{array}{rcl} k_3 &=& 11\omega \\ k_4 - \omega^2 &=& 54\omega^2 \\ 5k_3\omega^2 + 2k_2\omega &=& -126\omega^3 \\ 36\omega^4 + 18k_1\omega^3 + 9k_4\omega^2 &=& -108\omega^4 \end{array} \quad \Longrightarrow \quad \begin{array}{rcl} k_1 &=& -\frac{71}{2}\omega \\ k_2 &=& -\frac{181}{2}\omega^2 \\ k_3 &=& 11\omega \\ k_4 &=& 55\omega^2. \end{array}$$

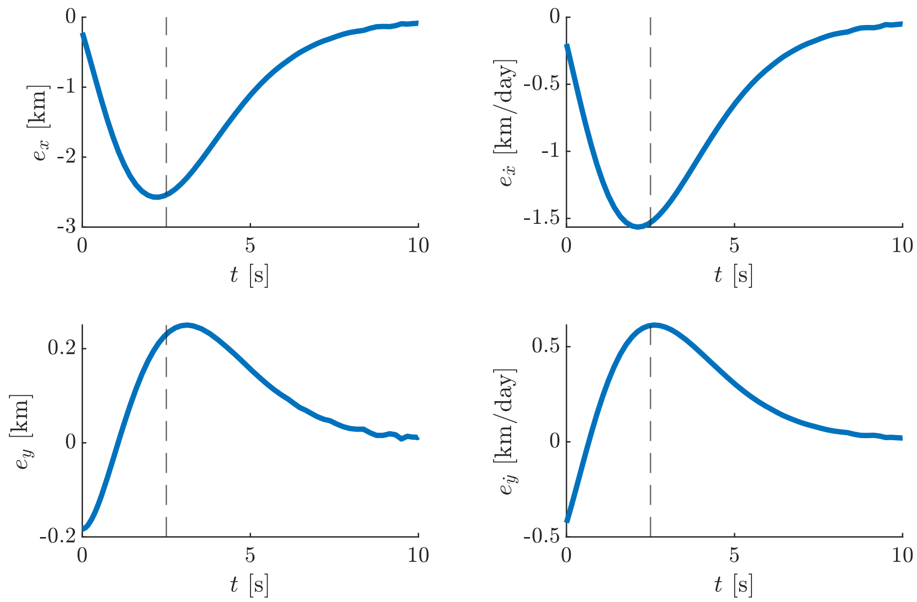
The observer is

$$\dot{\hat{X}} = A\,\hat{X} + B\,u + K\,\left(y - C\,\hat{X}\right)$$

The simulation results are shown in the figure below, the blue curve is the 'true' state and rthe orange is the estimated from the observer.



The error between the estimated states and the 'true' states is shown



From the figure, it is clear that after 2.5 days the error starts to decay.

Code

Algebra

```
syms s w
e1 = collect((s + 2*w) * (s + 3*w) * ...
        ... (s + 3*w + 1j*3*w) * (s + 3*w - 1j*3*w),s);
c1 = coeffs(e1,s); c1(end) = [];
S = s*eye(4);
A = [0, 1, 0, 0;...
     9*w^2, 0, 0, 2*w;...
     0, 0, 0, 1;...
```

```
     0, -2*w, -4*w^2, 0];
K = sym('k',[4 1]); C = [0 0 1 0];
e2 = collect(det(S - A + K*C),s);
c2 = coeffs(e2,s); c2(end) = [];
eqns = c1 == c2;
Sol = solve(eqns,K);
```

system simulation (true-states)

```
omega = 2*pi/29.3; % orbital frequency [rad/s]
u = 1000 / (300 * omega^2) ; % input (F/(m*w^2)) [m/rad^2]
% state space representation [x; xdot; y; ydot]
A = [0, 1, 0, 0;...
     9*omega^2, 0, 0, 2*omega;...
     0, 0, 0, 1;...
     0, -2*omega, -4*omega^2, 0];
B = [0; 0; 0; 1]; C = [0, 0, 1, 0];
bryson_sattelite = @(t,x) A*x + B*u;

% simulation
y0 = [0; 0; 0; 0]; % initial states
tspan = [0 10]; % [days]
[t, y] = ode45(@(t,y) bryson_sattelite(t,y),tspan,y0);
```

observer

```
% state observer
K = [-71/2*omega; -181/2*omega^2; 11*omega; 55*omega^2];
mesh = @(x) interp1(t,y(:,3),x);
obs_bryson_sattelite = @(t,x) A*x + B*u + K*(mesh(t) - C*x);

% simulation
y0 = rand([4 1]) * 500; % initial states
[to, yo] = ode45(@(t,y) obs_bryson_sattelite(t,y),tspan,y0);

% error
for ii = 1:width(y)
    yq = interp1(t,y(:,ii),to);
    eq(:,ii) = yq - yo(:,ii);
end
```

1.13 Kailath 4.1-9

In descrete time systems, the error at time-step N is

$$e_N = (A - K\,C)\; e_{N-1}$$

The error $e_N = 0$, at time-step N for the eigen-values of $(A - K\,C)$ at

1.14 Kailath 4.3-4

The state-space model of the inertial navigator is

$$\begin{pmatrix} \dot{v} \\ \dot{\varphi} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ \varphi \\ \varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}$$

(a) Determining the solutions.

$$\begin{aligned} X(s) = 0 \quad \det\left(sI - A\right) &= 0 \\ \det \begin{pmatrix} s & 1 & 0 \\ -1 & s & -1 \\ 0 & 0 & s \end{pmatrix} &= 0 \\ s\,\left(s^2 + 1\right) &= 0. \end{aligned}$$

The open-loop eigen values are $\lambda_1 = 0$, $\lambda_{2,3} = \pm j$

(b) observer design

$$y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ \varphi \\ \varepsilon \end{pmatrix}$$

The observer is

$$\begin{aligned} \dot{\hat{X}} &= A\,\hat{X} + B\,u + K\,\left(y - C\,\hat{X}\right) \\ \begin{pmatrix} \dot{\hat{v}} \\ \dot{\hat{\varphi}} \\ \dot{\hat{\varepsilon}} \end{pmatrix} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\varphi} \\ \hat{\varepsilon} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix} + K \left(y - \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\varphi} \\ \hat{\varepsilon} \end{pmatrix} \right) \end{aligned}$$

The closed-loop observer poles

$$\begin{aligned} \alpha_{\mathcal{O}}(s) &= \det\left(SI - A + K\,C\right) \\ &= \det \left(\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \right) \\ &= \det \begin{pmatrix} k_1 + s & 1 & 0 \\ k_2 - 1 & s - 1 & \\ k_3 & 0 & s \end{pmatrix} \\ &= s^3 + k_1\,s^2 + \left(1 - k_2\right)\,s - k_3 \end{aligned}$$

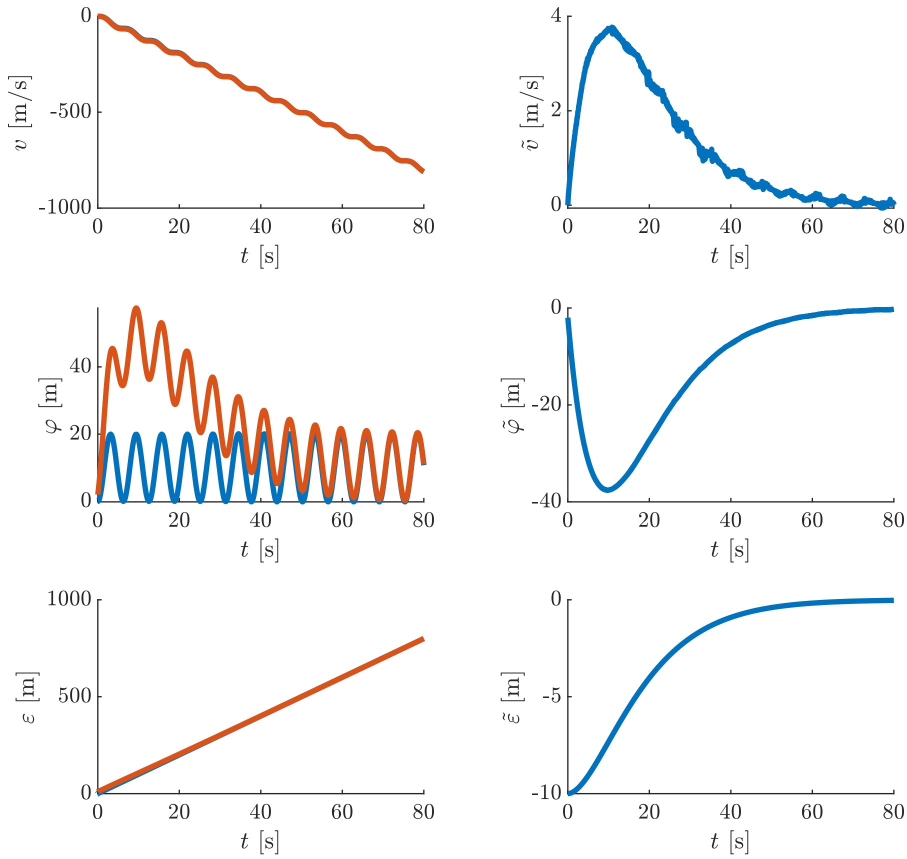
The closed-loop observer poles are $s = -10$, $s = -0.1$, $s = -0.1$

$$\begin{aligned} \alpha_{\mathcal{O}}(s) &= (s + 10)\,(s + 0.1)\,(s + 0.1) \\ &= s^3 + 10.2\,s^2 + 2.01\,s + 0.1 \end{aligned}$$

The gains are

$$K = \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix}^T = \begin{pmatrix} 10.2 & -1.01 & -0.1 \end{pmatrix}^T$$

The simulation results of the observer is shown in the next page



The estimated states from the observer are shown in orange and the 'true' states are shown in blue.

(c) The *second order* observer. Original state equations

$$\begin{pmatrix}\dot v\\ \dot\varphi\\ \dot\varepsilon\end{pmatrix}=\begin{pmatrix}0&-1&0\\1&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}v\\ \varphi\\ \varepsilon\end{pmatrix}+\begin{pmatrix}0\\0\\w\end{pmatrix},\qquad \text{or}\quad \begin{pmatrix}\dot\varepsilon\\ \dot\varphi\\ \dot v\end{pmatrix}=\begin{pmatrix}0&0&0\\1&0&1\\0&-1&0\end{pmatrix}\begin{pmatrix}\varepsilon\\ \varphi\\ v\end{pmatrix}+\begin{pmatrix}w\\0\\0\end{pmatrix}$$

Considering partitioned state equations

$$\begin{pmatrix}\dot x_r\\ \dot x_n\end{pmatrix}=\begin{pmatrix}A_r & a_r\\ a_n & a_{nn}\end{pmatrix}\begin{pmatrix}x_r\\ x_n\end{pmatrix}+\begin{pmatrix}b_r\\ b_n\end{pmatrix}u(t)\qquad\qquad y=x_n$$

$$\begin{pmatrix}\begin{pmatrix}\dot{\hat\varepsilon}\\ \dot{\hat\varphi}\end{pmatrix}\\ \dot{\hat v}\end{pmatrix}=\begin{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix}&\begin{pmatrix}0\\1\end{pmatrix}\\ \begin{pmatrix}0&-1\end{pmatrix}&0\end{pmatrix}\begin{pmatrix}\begin{pmatrix}\hat\varepsilon\\ \hat\varphi\end{pmatrix}\\ \hat v\end{pmatrix}+\begin{pmatrix}\begin{pmatrix}w\\0\end{pmatrix}\\0\end{pmatrix}\qquad\qquad y=v$$

Change in partitioning affects the pole placement of the reduced order observer, with $\begin{pmatrix}\begin{pmatrix}\varphi & \varepsilon\end{pmatrix} & v\end{pmatrix}^T$, only one pole (of the reduced order observer) can be placed arbitrarily.

The closed-loop poles

$$\begin{aligned}\alpha_{\mathcal{O}}(s)&=\det\left(sI-A_r+L\,a_n\right)\\&=\det\left(\begin{pmatrix}s&0\\0&s\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}+\begin{pmatrix}l_1\\l_2\end{pmatrix}\begin{pmatrix}0&-1\end{pmatrix}\right)\\&=\det\left(\begin{pmatrix}s-1&-l_1\\-1&s-l_2\end{pmatrix}\right)=s^2-l_2\,s-l_1\end{aligned}$$

The closed-loop observer poles are $s = -0.1$, $s = -0.1$

$$\alpha_{\mathcal{O}}(s) = (s + 0.1)\ (s + 0.1) = s^2 + 0.2\, s + 0.01$$

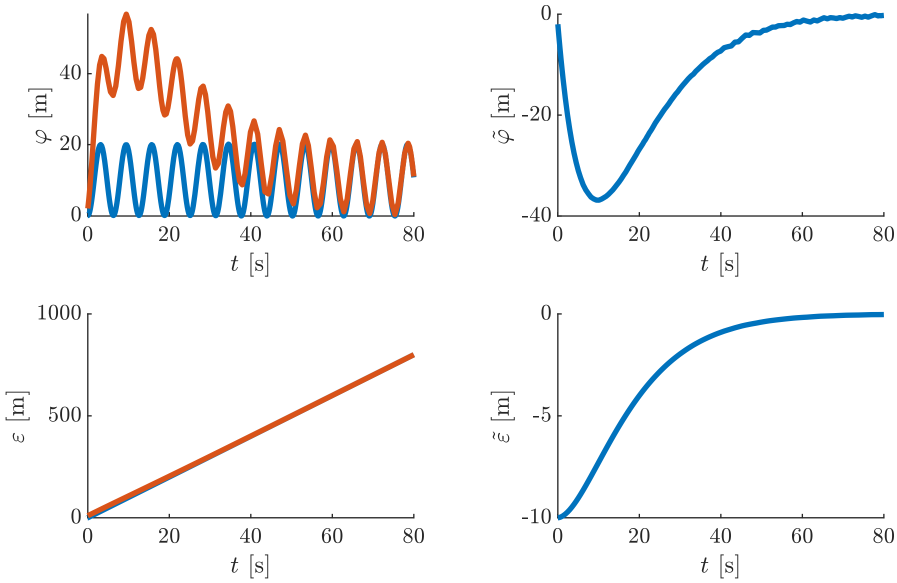
The gains are

$$L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^T = \begin{pmatrix} -0.01 & -0.2 \end{pmatrix}^T$$

The observer is designed as,

$$\begin{aligned}
\dot{\hat{\theta}} &= (A_r - L\, a_n)\ \theta + (a_r - L\, a_{nn} + A_r\, L - L\, a_n\, L)\ y + (b_r - b_n)\, u \\
\hat{x}_r &= \theta + L\, y \\
\hat{x}_n &= x_n = y
\end{aligned}$$

The simulation results of the observer are shown below, where the estimated states from the observer are shown in orange, and the 'true' states are shown in blue.



$\boxed{\text{The reduced-order and the full-order observer are similar in performance.}}$

Code

simulation (true system)

```
clear;
u = 10;
A = [0 -1 0; 1 0 1; 0 0 0];
B = [0; 0; 1]; C = [1 0 0];
int_nav = @(t,x) A*x + B*u;

% simulation
y0 = [0; 0; 0]; % initial states
tspan = [0 80]; % [s]
[t, y] = ode45(@(t,y) int_nav(t,y),tspan,y0);
```

Observer simulation

```
% state observer
K = [10.2; -1.01; -0.1];
mesh = @(x) interp1(t,y(:,1),x,'spline');
obs_int_nav = @(t,x) A*x + B*u + K*(mesh(t) - C*x);

% simulation
y0 = [0; 2; 10]; % initial states
[to, yo] = ode45(@(t,y) obs_int_nav(t,y),tspan,y0);

% error
for ii = 1:width(y)
    yq = interp1(t,y(:,ii),to);
    eq(:,ii) = yq - yo(:,ii);
end
```

Reduced order observer simulation

```
% A-matrix partition
Ar = [0 0; 1 0];
cr = [0 -1];
br = [0; 1];
ann = 0;
% B_matrix
gr = [1; 0];
gn = 0;
% observer gain
L = [-0.01; -0.2];
% state observer
red_obs_int_nav = @(t,z) (Ar - L*cr)*z ...
                  + (br -L*ann + Ar*L - L*cr*L)*mesh(t) ...
                  + (gr - L*gn)*u;

% simulation
y0 = [10; 2]; % initial states
[tr, zo] = ode45(@(t,y) red_obs_int_nav(t,y),tspan,y0);

% states
for ii = 1:length(tr)
    xr(ii,:) = zo(ii,:) + (L*mesh(tr(ii)))';
end

% error
yq = interp1(t,y(:,3),tr);
eq2(:,1) = yq - xr(:,1);
yq = interp1(t,y(:,2),tr);
eq2(:,2) = yq - xr(:,2);
```

Chapter 2

Observers for DAEs and non-Linear Systems

2.1 Le 2.1

Consider the following DAE:

$$E\,\dot{x} = A\,x + B\,u, \qquad\qquad y = C\,x$$

There exists invertible matrices P and T, where $x = T\,w$ and multiplying the model equations with P from the left and T, T^{-1} to the right of E gives

$$\begin{gathered} P\,E\,T\,T^{-1}\dot{x} = P\,A\,T\,T^{-1}\,x + P\,B\,T\,T^{-1}\,u. \\ P\,E\,T\,\dot{w} = P\,A\,T\,w + P\,B\,T\,T^{-1}\,u, \end{gathered}$$

where

$$P\,E\,T = \begin{pmatrix} I & 0 \\ 0 & E_2 \end{pmatrix} \quad P\,A\,T = \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix}, \quad \text{where } E_2 \text{ is nilpotent.}$$

Assuming that the ODE part, $\dot{w}_1 = A_1\,w_1 + B_1\,u$ is observable, i.e.,

$$\begin{pmatrix} \lambda\,I - A_1 \\ C \end{pmatrix} \quad \text{has full column rank } \forall\lambda, \text{ i.e.,} \begin{pmatrix} \lambda\,I - A_1 \\ 0 \\ C \end{pmatrix} \text{ also has full column rank } \forall\lambda.$$

Considering the following:

$$\begin{pmatrix} \lambda\,E - A \\ C \end{pmatrix} \equiv \begin{pmatrix} \lambda\,P\,E\,T - P\,A\,T \\ C \end{pmatrix} = \begin{pmatrix} \lambda\begin{pmatrix} I & 0 \\ 0 & E_2 \end{pmatrix} - \begin{pmatrix} A_1 & 0 \\ 0 & E_2 \end{pmatrix} \\ \begin{pmatrix} C_1 & C_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \lambda\,I - A_1 & 0 \\ 0 & \lambda\,E_2 - I \\ C_1 & C_2 \end{pmatrix}$$

Any square matrix has a Jordan normal form if the field of coefficients is extended to one containing all the eigenvalues of the matrix [*from Wikipedia*]. The Jordan form is given as

$$\begin{pmatrix} \lambda_1 & 1 & & & & & & \\ & \lambda_1 & 1 & & & & & \\ & & \lambda_1 & & & & & \\ & & & \lambda_2 & 1 & & & \\ & & & & \lambda_2 & & & \\ & & & & & \lambda_3 & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_n \end{pmatrix}, \text{for a nilpotent matrix}, \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

Thus, $\lambda\, E_2 - I$ has always full-column rank.

Therefore, if $\begin{pmatrix}\lambda\, I - A_1 & C\end{pmatrix}^T$ has full-column rank then $\begin{pmatrix}\lambda\, E - A & C\end{pmatrix}^T$ has full-column rank $\forall\ \lambda$.

If $\begin{pmatrix}\lambda\, E - A & C\end{pmatrix}^T$ has full-column rank then its minor $\begin{pmatrix}\lambda\, I - A_1 & C\end{pmatrix}^T$ also has full-column rank $\forall\ \lambda$.

2.2 Le 2.2

(a) Writing the model equations in the form

$$E\,\dot{x} = A\,x + B\,u$$

The model equations are

$$\begin{array}{cccc} u - v_1 = R_1\,i_1 & v_3 - v_4 = R_2\,i_2 & v_1 - v_4 = R_3\,i_3 & C\,\dot{v}_1 - C\,\dot{v}_3 = i_2 \\ v_1 - v_2 = 0 & v_2 = 0 & i_1 = i_2 + i_3 & \\ v_1 + R_1\,i_1 = u & v_3 - v_4 - R_2\,i_2 = 0 & v_1 - v_4 - R_3\,i_3 = 0 & C\,\dot{v}_1 - C\,\dot{v}_3 - i_2 = 0 \\ v_1 - v_2 = 0 & v_2 = 0 & i_1 - i_2 - i_3 = 0 & \end{array}$$

Considering $x = \begin{pmatrix} i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & v_4 \end{pmatrix}^T$,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_c & 0 & -C_c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_3 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4 \end{pmatrix} = \begin{pmatrix} -R_1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -R_2 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -R_3 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u$$

(b) Measurement signals:

Choosing C such that $\begin{pmatrix} \lambda\,E - A & C \end{pmatrix}^T$ has full-rank $\forall\ \lambda$.

$\lambda\,E - A$ is 0, when $\det\left(\lambda\,E - A\right) = 0$, i.e., $\lambda = \frac{-1}{C_c\,(R_2+R_3)}$

The matrix $\begin{pmatrix} \lambda\,E - A & C \end{pmatrix}^T$ <u>does not</u> have full-rank with the measurement signals $\left\{i_1, v_1, v_2\right\}$ $\forall\ \lambda$.

(c) Building an observer

Simplifying the ODE,

$$C\,\dot{v}_1 - C\,\dot{v}_3 - i_2 = 0 \qquad \Longrightarrow\ C\,\dot{v}_3 = \frac{v_3 - v_4}{R_2}.$$

The measurement equation:

$$y = v_4.$$

The observer design is

$$\dot{\hat{v}}_3 = \frac{\hat{v}_3 - v_4}{R_2\,C_c} + K\ (v_4 - 0)$$

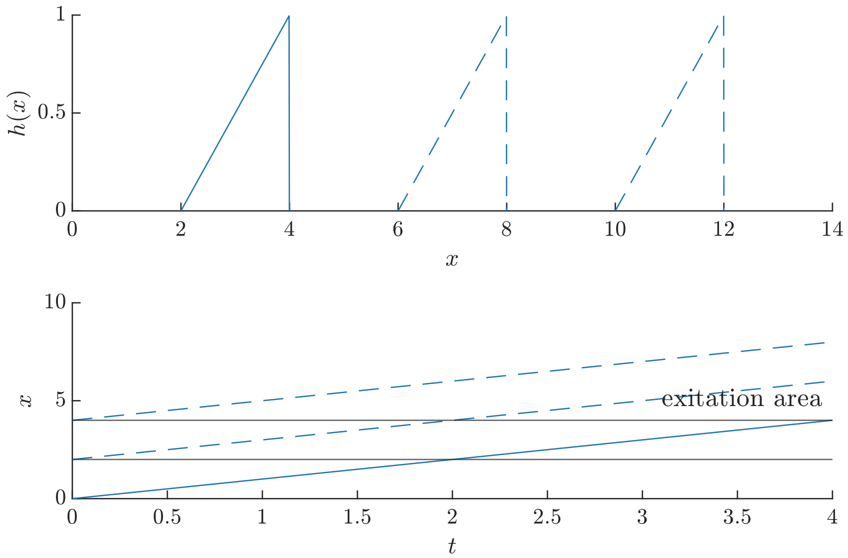
The poles for the observer is

$$\begin{aligned} \alpha_{\mathcal{O}}(s) &= \det\left(s\,I - A + K\,C\right) \\ &= s - \frac{1}{R_2\,C_c} + k_1 \times 0. \end{aligned}$$

2.3 Le 2.3

The system is defined as

$$\dot{x} = 1 \qquad\qquad y = h(x)$$



(a) $\mathcal{M} \subset (0, 4)$ **Weakly observable**.

At $x < 2$, $h(x) = 0$ and thus not locally observable. However, $\forall t >= 2$, $h(x) = k\,x$ and becomes observable.

(b) $\mathcal{M} = (2, 4)$ **Locally observable**.

x is uniquly isolable for all $\mathcal{U} \subset \mathcal{M}$.

(c) $\mathcal{M} = \mathcal{R}^{+}\,'\ldots'$ **Weakly observable**.

$2 \leq t \leq 4$, $h(x) = k\,x$ and becomes observable.

Note:

if $h(x)$ is a sawtooth waveform, then the system is **locally weakly observable**.

2.4 Le 2.4

Considering the ODE

$$\dot{x} = 1 \qquad\qquad y = x^3$$

Let,

$$F(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^q h(x) \end{pmatrix} = \begin{pmatrix} x^3 \\ 3x^2 \\ 6x \\ \vdots \end{pmatrix}$$

The system is indeed locally weakly observable. 1

Considering the ODE

$$\dot{x} = 1 \qquad\qquad y = x\, e^{-1/x^2}$$

$$F(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^q h(x) \end{pmatrix} = \begin{pmatrix} x\, e^{-1/x^2} \\ \vdots \end{pmatrix}$$

Since $x\, e^{-1/x^2}|_{x=0}$ is $\div 0$. Therefore the locally weakly observable at $x = 0$ is defined as

$$\begin{aligned} \lim_{x\to 0} \frac{\partial}{\partial x} F(x) &= \lim_{x\to 0} \frac{\partial}{\partial x} \big(h(x)\big) \\ &= \lim_{x\to 0} \begin{pmatrix} x\,\lambda + \frac{2\,\lambda}{x^2} \\ \frac{4\,\lambda}{x^5} - \frac{2\,\lambda}{x^3} \\ \frac{6\,\lambda}{x^4} - \frac{24\,\lambda}{x^6} + \frac{8\,\lambda}{x^8} \\ \vdots \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \end{aligned}$$

where $\lambda = e^{-1/x^2}$

In this case, even as $q \to \infty$, the matrix is not full-rank as $x \to 0$. However, the system is locally weakly observable.

$\boxed{\text{Even if } q \text{ is } \infty\text{, it does not imply that the system is not observable.}}$

Code

```
syms x f(x) h(x)
f(x) = 1; h(x) = x*exp(-1/x^2);
F(x) = [h(x)*f(x);...
    diff(h(x),x)*f(x); ...
    diff(diff(h(x),x),x)*f(x);...
    diff(diff(diff(h(x),x),x),x)*f(x)];
dF(x) = diff(F(x),x); val = limit(dF(x),x,0); rank_Fx = rank(val)
```

2.5 Le 2.5

Model, let:

$$A = f_x(x)\Big|_{x=x^0} \quad C = h_x(x)\Big|_{x=x^0},$$

where $f_x(x) = u$ and $h_x(x) = x^2$.

At $x^0 = 0$, the observability matrix of $(C,\ A)$ is

$$\mathcal{O}x = \big(h_x(x)|_{x=0}\big) = \big(2\,x\big)_{x=0} = 0.$$

It is not full-rank and from the linearization. However, the system is *locally weakly observable.*

$\boxed{\text{Nothing can be concluded about the system's observability if } C \text{ and } A \text{ are not observable.}}$

2.6 Le 2.6

The model of the pendulum is

$$\dot{x} = f(x,u) = \begin{pmatrix} x_2 \\ -x_2 - \sin(x_1) + u \end{pmatrix}, \quad y = x_1$$

The observer is

$$\begin{aligned}
&\dot{\hat{X}} = f(\hat{x},u) + K\,(y - \hat{x}_1) = \begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_2 - \sin(\hat{x}_1) + u \end{pmatrix} + K\ (y - \hat{x}_1)\,, \\
&\dot{\hat{x}}_1 = \hat{x}_2 + k_1\ (y - \hat{x}_1)\,, \quad \dot{\hat{x}}_2 = -\hat{x}_2 - \sin(\hat{x}_1) + u + k_2\ (y - \hat{x}_1)\,.
\end{aligned}$$

The error signals are

$$\begin{aligned}
\dot{e} &= \dot{X} - \dot{\hat{X}} \quad \text{i.e.,} \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 - \dot{\hat{x}}_1 \\ \dot{x}_2 - \dot{\hat{x}}_2 \end{pmatrix} \\
&= \begin{pmatrix} \dot{x}_1 - \dot{\hat{x}}_1 \\ \dot{x}_2 - \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} x_2 - \hat{x}_2 - k_1\ (x_1 - \hat{x}_1) \\ -x_2 - \sin(x_1) + \cancel{u} + \hat{x}_2 + \sin(\hat{x}_1) - \cancel{u} - k_2\ (y - \hat{x}_1) \end{pmatrix} \\
&= \begin{pmatrix} e_2 - k_1\, e_1 \\ -e_2 - \sin(x_1) + \sin(\hat{x}_1) - k_2\, e_1 \end{pmatrix}
\end{aligned}$$

The Lyapunov equation is

$$\begin{aligned}
V(e) &= e_1^2 + \beta\, e_2^2,\ \beta > 0 \\
\dot{V}(e) &= 2\, e_1\, \dot{e}_1 + 2\,\beta\, e_2\, \dot{e}_2 \\
&= 2\, e_1\ (e_2 - k_1\, e_1) + 2\,\beta\, e_2\ (-e_2 - \sin(x_1) + \sin(\hat{x}_1) - k_2\, e_1) \\
&= 2\, e_1\, e_2 - 2\, k_1\, e_1^2 - 2\,\beta\, e_2^2 - 2\,\beta\, k_2\, e_1\, e_2 - 2\,\beta\, e_2\ (\sin(x_1) - \sin(\hat{x}_1))
\end{aligned}$$

Considering the inequality

$$0 \le \frac{\sin(x) - \sin(y)}{x - y} \le 1, \implies 0 \le \sin(x) - \sin(y) \le x - y \qquad \text{if } -\pi/2 \le x,\ y \le \pi/2$$

The maximum error $x_1 - \hat{x}_1$ will not exceed π, (the pendulum is hanging on a wall)

Therefore for the worst error $x_1 - \hat{x}_1$, $\dot{V}(e)$ is

$$\begin{aligned}
\dot{V}(e) &= 2\, e_1\, e_2 - 2\, k_1\, e_1^2 - 2\,\beta\, e_2^2 - 2\,\beta\, k_2\, e_1\, e_2 - 2\,\beta\, e_2\, e_1 \\
&= 2\, e_1\, e_2\, (1 - k_2 - \beta) - 2\, k_1\, e_1^2 - 2\,\beta\, e_2^2
\end{aligned}$$

Choosing $k_2 = 1$ gives

$$\dot{V}(e) = 2\, e_1\, e_2\, \left(\cancelto{0}{1 - k_2} - \beta\right) - 2\, k_1\, e_1^2 - 2\,\beta\, e_2^2 = -2\,\beta\, e_1\, e_2 - 2\, k_1\, e_1^2 - 2\,\beta\, e_2^2$$

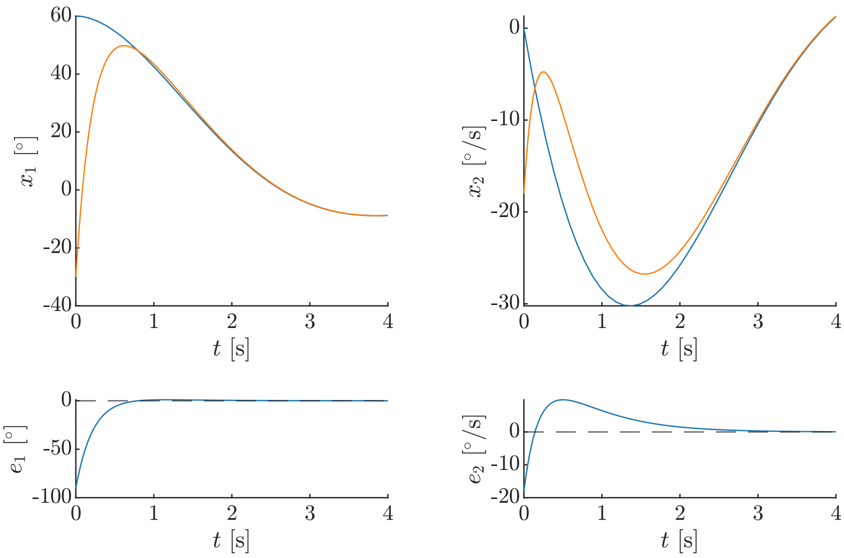
Choosing $k_1 = \beta$ gives

$$\dot{V}(e) = -2\,\beta\ \left(e_1\, e_2 + e_1^2 + e_2^2\right) \quad \implies \dot{V}(e) < 0\ \forall\ \beta > 0.$$

The observer is

$$\dot{\hat{X}} = \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_2 - \sin(\hat{x}_1) + u \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\ (y - \hat{x}_1) \qquad K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

The simulation results are shown in the following figure



The blue line is the true state and the orange line is estimated.

Code

main system

```
u = @(t) 0;
fx = @(t,x) [ x(2) ;...
            -x(2) - sin(x(1)) + u(t)];

tspan = [0 4];
y0 = [pi/3; 0];
[t,y] = ode45(@(t,y) fx(t,y),tspan,y0);
```

observer design

```
y_val = @(tt) interp1(t,y(:,1),tt,'spline');
beta = 5;
K = [beta; 1];
fo = @(t,x) fx(t,x) + K * (y_val(t) - x(1));

y0 = [-pi/6; -pi/10];
[to,yo] = ode45(@(t,y) fo(t,y),tspan,y0);
```

error signals

```
yoo = interp1(t,y,to,'spline');
err = (yo - yoo);
```

2.7 Le 2.8

Practice

The system:

$$\dot{X} = \begin{pmatrix} \left(\frac{x_2}{x_1}\right)^2 \\ \left(\frac{x_2}{x_1}\right)^3 - x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} u \qquad\qquad y = x_1\, x_2$$

let, $z = \Phi(x) = \begin{pmatrix} x_1 \\ \frac{x_2}{x_1} \end{pmatrix}$, then

$$\begin{aligned} \dot{z} &= \frac{\partial \Phi\,(x)}{\partial x}\, f\,(x,u)\bigg|_{x=\Phi^{-1}(z)} = \begin{pmatrix} 1 & 0 \\ -\frac{x_2}{x_1^2} & \frac{1}{x_1} \end{pmatrix} f\,(x,u)\bigg|_{x=\Phi^{-1}(z)} \\ &= \left(\begin{pmatrix} \left(\frac{x_2}{x_1}\right)^2 \\ -\frac{x_2}{x_1} \end{pmatrix} + \begin{pmatrix} x_1 \\ 0 \end{pmatrix} u \right|_{x=\Phi^{-1}(z)} \end{aligned}$$

it is also defined that

$$\Phi^{-1}(x) = \begin{pmatrix} z_1 \\ z_2\, z_1 \end{pmatrix}$$

Therefore,

$$\begin{aligned} &\dot{z} = \begin{pmatrix} z_2^2 \\ -z_2 \end{pmatrix} + \begin{pmatrix} z_1 \\ 0 \end{pmatrix} u \quad y = z_1^2\, z_2, \\ &\dot{z}_1 = z_2^2 + z_1\, u \quad \dot{z}_2 = -z_2 \quad y = z_1^2\, z_2, \implies \dot{z}_1 = \frac{1}{z_1^4} z_2^2\, z_1^4 + z_1\, u \quad \dot{z}_2 = -z_2 \quad y = z_1^2\, z_2, \end{aligned}$$

Excercise

The system:

$$\dot{X} = \begin{pmatrix} x_1^2\, x_2^2 + u \\ 1 - x_1\, x_2^3 - x_2 - \frac{x_2}{x_1}\, u \end{pmatrix} \quad y = x_1\, x_2$$

let,

$$z = \Phi(x) = \begin{pmatrix} x_2 x_1 \\ x_1 \end{pmatrix}, \qquad\qquad \implies x = \Phi^{-1}(x) = \begin{pmatrix} z_2 \\ \frac{z_1}{z_2} \end{pmatrix},$$

Then

$$\begin{aligned} \dot{z} &= \frac{\partial \Phi\,(x)}{\partial x}\, f\,(x,u)\bigg|_{x=\Phi^{-1}(z)} = \begin{pmatrix} x_2 & x_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1^2\, x_2^2 + u \\ 1 - x_1\, x_2^3 - x_2 - \frac{x_2}{x_1}\, u \end{pmatrix}\bigg|_{x=\Phi^{-1}(z)} \\ &= \begin{pmatrix} x_1\,(1 - x_2) \\ x_1^2\, x_2^2 + u \end{pmatrix}\bigg|_{x=\Phi^{-1}(z)} = \begin{pmatrix} -z_1 + z_2 \\ z_1^2 + u \end{pmatrix} \end{aligned}$$

Now,

$$y = h(x)\big|_{x=\Phi^{-1}(z)} = x_1\, x_2\big|_{x=\Phi^{-1}(z)} = z_1.$$

Therefore,

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

For a linear system, the observability is

$$\mathcal{O}z = \begin{pmatrix} C \\ C\,A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{has full-column rank}$$

If Z is observable then X is also observable since $X = \Phi^{-1}(Z)$, i.e., an algebraic expression.

The observer can be designed as

$$\dot{\hat{Z}} = A\,\hat{Z} + f(u, y) + K\,\left(y - C\,\hat{Z}\right)$$

The closed-loop observer poles

$$\begin{aligned}
\alpha_{\mathcal{O}}(s) &= \det\left(SI - A + K\,C\right) \\
&= \det\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}\right) \\
&= \det\begin{pmatrix} k_1 + s + 1 & -1 \\ k_2 & s \end{pmatrix} \\
&= s^2 + s\,\left(k_1 + 1\right) + k_2
\end{aligned}$$

Chapter 3

Design techniques, part-1

Implementation of EKF

An EKF was implemented according to the lecture slides. However, the gains are rather challenging to determine.

However, the covariance matrices (Q, R), and the initial covariance matrix (P_0)

$$Q = \begin{pmatrix} 0.05 & & \\ & 0.05 & \\ & & 0.05 \end{pmatrix}, \qquad R = \begin{pmatrix} 0.08 & \\ & 0.2 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 0.01 & & \\ & 0.1 & \\ & & 0.01 \end{pmatrix},$$

How did I guess?
Q is assumed as the variance in the model (not covariances, since it is a diagonal matrix) and since, in this case, G (noise 'component' to the model), is non-existent, that is why low values of Q are assumed.
R is assumed as the variance in the measurement. High values are chosen because the gain is high.
P_0 is the initial co-variance matrix (how much we trust the initial guess), and low values are chosen for top-right and bottom-left since the initial guesses are quite good for x and θ, but the initial guess for y is bad.

A least-square optimization is performed to determine the matrices Q, R, and P_0. The matrices are

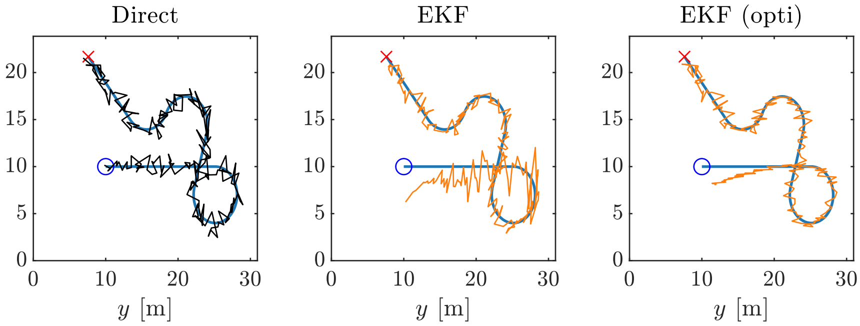
$$Q = \begin{pmatrix} 8978 & & \\ & 0 & \\ & & 0.012 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & \\ & 0.57 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 1.33 & & \\ & 4.42 & \\ & & 0 \end{pmatrix}$$

The results of the optimization do not seem to agree with my guess. No idea why.

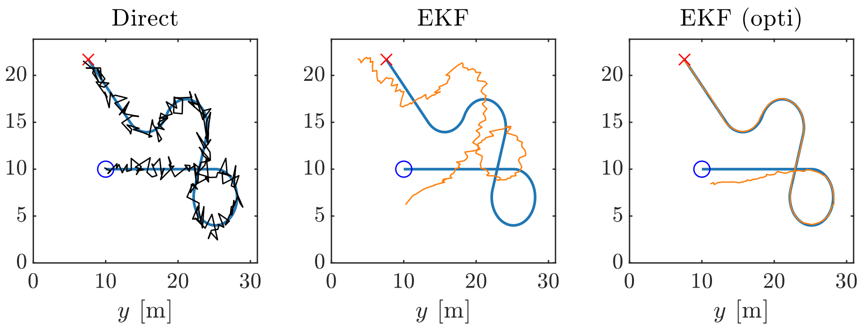
An array square-root algorithm was implemented and optimization resulted in the following co-variance matrices:

$$Q = \begin{pmatrix} 0.077 & & \\ & 0.0032 & \\ & & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1.511 & \\ & 0 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 5.51 & & \\ & 31.96 & \\ & & 0.0011 \end{pmatrix}$$

The results from EKF for robot tracking is



The results from EKF for robot tracking with array SR implementation is



Cannot say why the SR algorithm performs much better than EKF.

Chapter 4

Design techniques, part-2

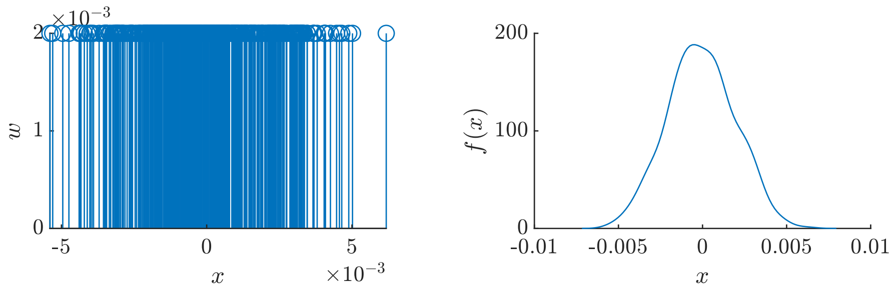
4.1 Le 3b-1

(a)

The code is as follows:

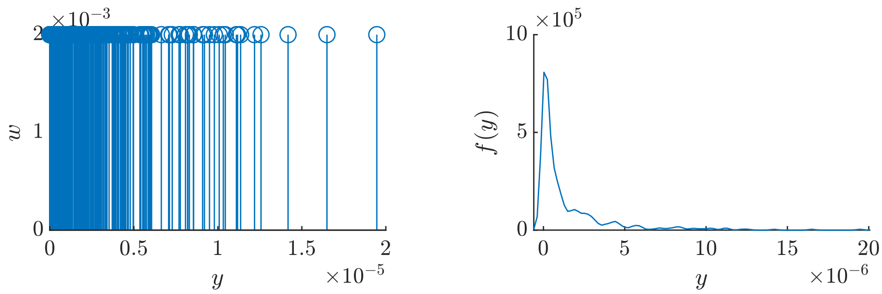
```
N = 500;
w = 1/N + zeros(1,N);
for ii = 1:N
    x(ii) = normrnd(0,1) * w(ii);
end
[fx,xk]=ksdensity(x);
```

The data sample is shown in the following figure:



(b)

The data sample is shown in the following figure:



The width of a text is: 14.99786cm (5.90666in)
The width of a column is: 14.99786cm (5.90666in)
The height of a column is: 24.69649cm (9.72632in)