Non-linear $\mathcal{O}$bservers Course

Arvind Balachandran

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Contents

Chapter 1

Observers for Linear Systems

1.1   Le 1.1

Model:

$$A = \begin{pmatrix} \mu_1 & 1 & 0 \\ 0 & \mu_1 & 1 \\ 0 & 0 & \mu_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ C A \\ C A^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mu_2 \\ 0 & 0 & \mu_2^2 \end{pmatrix}$$

The null space of the observability matrix

$$\mathcal{N}(\mathcal{O}) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Considering the following:

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix} = \begin{pmatrix} \lambda - \mu_1 & -1 & 0 \\ 0 & \lambda - \mu_1 & -1 \\ 0 & 0 & \lambda - \mu_2 \\ 0 & 0 & 1 \end{pmatrix}$$

Substituting $\lambda = \mu_1, \ \mu_2$,

$$\Lambda_1 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & \mu_1 - \mu_2 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \Lambda_2 = \begin{pmatrix} \mu_2 - \mu_1 & -1 & 0 \\ 0 & \mu_2 - \mu_1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

now,

$$\mathcal{N}(\Lambda_1) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}, \qquad\qquad \mathcal{N}(\Lambda_2) = \{\emptyset\}.$$

it is clear that $\mathcal{N}\begin{pmatrix} \lambda I - A \\ C \end{pmatrix}$ is not the same as $\mathcal{N}(\mathcal{O})$, but rather a subset.

1.2   Le 1.2

$$\dot{\theta}_1 = 0 \quad \dot{\theta}_2 = 0 \qquad\qquad z(t) = \begin{pmatrix} 1 - y(t) & y(t) \end{pmatrix} \theta$$

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad C = \begin{pmatrix} 1 - y(t) & y(t) \end{pmatrix}$$

The observability criterion is

$$\Sigma_0 = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

$$\Sigma_0 = \int_0^\infty e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} t} \begin{pmatrix} 1 - y(t) \\ y(t) \end{pmatrix} \begin{pmatrix} 1 - y(t) & y(t) \end{pmatrix} e^{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} t} dt$$

$$\Sigma_0 = \int_0^\infty I \begin{pmatrix} 1 - y(t) \\ y(t) \end{pmatrix} \begin{pmatrix} 1 - y(t) & y(t) \end{pmatrix} I dt$$

$$\Sigma_0 = \int_0^\infty \begin{pmatrix} (1 - y(t))^2 & y(t)\,(1 - y(t)) \\ y(t)\,(1 - y(t)) & y^2(t) \end{pmatrix} dt$$

$$\Sigma_0 = \begin{pmatrix} \int_0^\infty \left(1 + y^2(t) - 2y(t)\right) dt & \int_0^\infty \left(y(t) - y^2(t)\right) dt \\ \int_0^\infty \left(y(t) - y^2(t)\right) dt & \int_0^\infty y^2(t) dt \end{pmatrix}$$

Let,

$$Y_1 = \int_0^\infty y(t) dt \qquad\qquad \&,\ Y_2 = \int_0^\infty y^2(t) dt$$

Therefore,

$$\Sigma_0 = \begin{pmatrix} t + Y_2 - 2Y_1 & Y_1 - Y_2 \\ Y_1 - Y_2 & Y_2 \end{pmatrix}.$$

For the system to be observable $\det(\Sigma_0) \neq 0$, i.e.,

$$\det(\Sigma_0) \neq 0 \implies Y_2 t + Y_2^2 - 2Y_1 Y_2 + (Y_1 - Y_2)^2 \neq 0$$

$$Y_2 t + \cancel{Y_2^2} - \cancel{2Y_1 Y_2} - Y_1^2 - \cancel{Y_2^2} + \cancel{2Y_1 Y_2} \neq 0$$

$$Y_2 t - Y_1^2 \neq 0$$

$$t \int_0^\infty y^2(t) dt - \left(\int_0^\infty y(t) dt \right)^2 \neq 0$$

If $y(t)$ is a constant then $\det(\Sigma_0) = 0$.

1.3 Le 1.3

Show that the observability gramian Σ_o satisfies the Lyapunov equation

$$A^T \Sigma_o + \Sigma_o A + C^T C = 0,$$

or,

$$A^T \Sigma_o + \Sigma_o A = -C^T C.$$

Multiply e^{At} to the right and $e^{A^T t}$ to the left,

$$e^{A^T t} A^T \Sigma_o e^{At} + e^{A^T t} \Sigma_o A e^{At} = -e^{A^T t} C^T C e^{At}.$$

Since,

$$\frac{d}{dt} e^{At} = A e^{At}$$

as e^{At} and A are commutative,

$$\left(\frac{d}{dt} e^{A^T t} \right) \Sigma_o e^{At} + e^{A^T t} \Sigma_o \left(\frac{d}{dt} e^{At} \right) = -e^{A^T t} C^T C e^{At}.$$

Using the product rule,

$$\frac{d}{dt} \left(e^{A^T t} \Sigma_o e^{At} \right) = -e^{A^T t} C^T C e^{At}.$$

integrating both sides with limits from 0 to ∞

$$\int_0^\infty \frac{d}{dt} \left(e^{A^T t} \Sigma_o e^{At} \right) dt = \int_0^\infty -e^{A^T t} C^T C e^{At} dt.$$

$$\implies \left[e^{A^T t} \Sigma_o e^{At} \right]_0^\infty = \int_0^\infty -e^{A^T t} C^T C e^{At} dt.$$

for a linear system with the state equation $\dot{X} = A X$ solves to $X = e^{At}$ and

$$\lim_{t \to \infty} X = \lim_{t \to \infty} e^{At} = 0$$

if the eigenvalues are on the left-hand side of the complex plane. Using the stability requirement

$$\left[e^{A^T t} \Sigma_o e^{At} \right]_0^\infty = \int_0^\infty -e^{A^T t} C^T C e^{At} dt.$$

$$\text{becomes, } \Sigma_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt.$$

1.4   Le 1.4

Compute the observability gramian as a function of ϵ and interpret the result.

$$\dot{x} = \begin{pmatrix} -2 & -1 \\ \epsilon & -1 \end{pmatrix} x$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

the gramian is given as

$$\Sigma_o = \int_0^\infty \exp\left(A^T t\right) C^T C \exp\left(A t\right) dt = \int_0^\infty \mathcal{X} dt \tag{1.1}$$

or the solution to the Lyapunov equation,

$$A^T \Sigma_o + \Sigma_o A + C^T C = 0,$$

substituing A and C,

$$\begin{pmatrix} -2 & -1 \\ \epsilon & -1 \end{pmatrix}^T \Sigma_o + \Sigma_o \begin{pmatrix} -2 & -1 \\ \epsilon & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 1 \end{pmatrix} = 0,$$

$$\begin{pmatrix} -2 & -1 \\ \epsilon & -1 \end{pmatrix}^T \Sigma_o + \Sigma_o \begin{pmatrix} -2 & -1 \\ \epsilon & -1 \end{pmatrix} = -\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

Let,

$$\Sigma_o = \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{pmatrix},$$

then,

$$\begin{pmatrix} -2 & \epsilon \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{pmatrix} + \begin{pmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{pmatrix} \begin{pmatrix} -2 & -1 \\ \epsilon & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, = 0$$

$$\implies \begin{pmatrix} \epsilon\sigma_2 - 2\sigma_1 & \epsilon\sigma_3 - 2\sigma_2 \\ -\sigma_1 - \sigma_2 & -\sigma_2 - \sigma_3 \end{pmatrix} + \begin{pmatrix} \epsilon\sigma_2 - 2\sigma_1 & -\sigma_1 - \sigma_2 \\ \epsilon\sigma_3 - 2\sigma_2 & -\sigma_2 - \sigma_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, = 0$$

$$\implies \begin{pmatrix} 2\epsilon\sigma_2 - 4\sigma_1 & \epsilon\sigma_3 - 3\sigma_2 - \sigma_1 \\ \epsilon\sigma_3 - 3\sigma_2 - \sigma_1 & 2\sigma_2 - 2\sigma_3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, = 0$$

$$\implies \begin{pmatrix} 2\epsilon\sigma_2 - 4\sigma_1 + 1 & \epsilon\sigma_3 - 3\sigma_2 - \sigma_1 + 1 \\ \epsilon\sigma_3 - 3\sigma_2 - \sigma_1 + 1 & 2\sigma_2 - 2\sigma_3 + 1 \end{pmatrix} = 0,$$

$$\implies 2\epsilon\sigma_2 - 4\sigma_1 + 1 = 0$$
$$\epsilon\sigma_3 - 3\sigma_2 - \sigma_1 + 1 = 0$$
$$2\sigma_2 - 2\sigma_3 + 1 = 0.$$

$$\Sigma_o = \frac{1}{6(\epsilon + 2)} \begin{pmatrix} \epsilon^2 + 3\epsilon + 3 & 2\epsilon + 3 \\ 2\epsilon + 3 & \epsilon + 3 \end{pmatrix}$$

For Σ_o to have full-rank, $\det(\Sigma_o) \neq 0$, i.e.,

$$\frac{1}{6(\epsilon + 2)} \neq 0, \qquad\qquad \text{i.e., } \epsilon^2 + \epsilon \neq 0$$

For $\epsilon = -2$, Σ_o is signular.

For $\epsilon = 0$, rank(Σ_o) $= 1$.

1.5 Le 1.5

Considering partitioned state equations

$$\begin{pmatrix} \dot{x}_r \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A_r & a_r \\ a_n & a_{nn} \end{pmatrix} \begin{pmatrix} x_r \\ x_n \end{pmatrix} + \begin{pmatrix} b_r \\ b_u \end{pmatrix} u(t) \qquad\qquad y = x_n$$

This gives,

$$\dot{x}_r = A_r\, x_r + a_r\, x_n + b_r\, u(t) \qquad\qquad \dot{x}_n = a_n\, x_r + a_{nn}\, x_n + b_n\, u(t)$$

Considering

$$y_r = \dot{y} - a_{nn}\, y - b_n\, u(t).$$

Then,

$$\dot{x}_r = A_r\, x_r + a_r\, x_n + b_r\, u(t) \qquad\qquad y_r = a_n\, x_r$$

The observer is

$$\dot{\hat{x}}_r = A_r\, \hat{x}_r + a_r\, \hat{x}_n + b_r\, u(t) + K\ (y_r - a_n\, \hat{x}_r)$$

Substituting $x_n = \hat{x}_n = y$,

$$\dot{x}_r = A_r\, x_r + a_r\, y + b_r\, u(t)$$
$$\dot{\hat{x}}_r = A_r\, \hat{x}_r + a_r\, y + b_r\, u(t) + K\ (a_n\, x_r - a_n\, \hat{x}_r)$$

The error signal,

$$\begin{aligned}
\dot{\tilde{x}}_r &= \dot{x}_r - \dot{\hat{x}}_r \\
&= A_r\, x_r + \cancel{a_r\, y} + \cancel{b_r\, u(t)} - A_r\, \hat{x}_r - \cancel{a_r\, y} - \cancel{b_r\, u(t)} - K\ (a_n\, \hat{x}_r - a_n\, \hat{x}_r) \\
&= A_r\, x_r - A_r\, \hat{x}_r - K\ (a_n\, x_r - a_n\, \hat{x}_r) \\
&= (A_r - K\, a_n)\ \tilde{x}_r
\end{aligned}$$

The poles can be selected arbitrarily in the error dynamics of the reduced observer.

1.6   Le 1.6

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} X,$$
$$Y = \begin{pmatrix} 1 & 0 \end{pmatrix} X$$

Also,

$$\alpha_{\mathcal{O}}(s) = \det\left(SI - A + K\,C\right) = \det\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}\right]$$
$$= \det\left[\begin{pmatrix} s + k_1 & -1 \\ k_2 & s \end{pmatrix}\right]$$
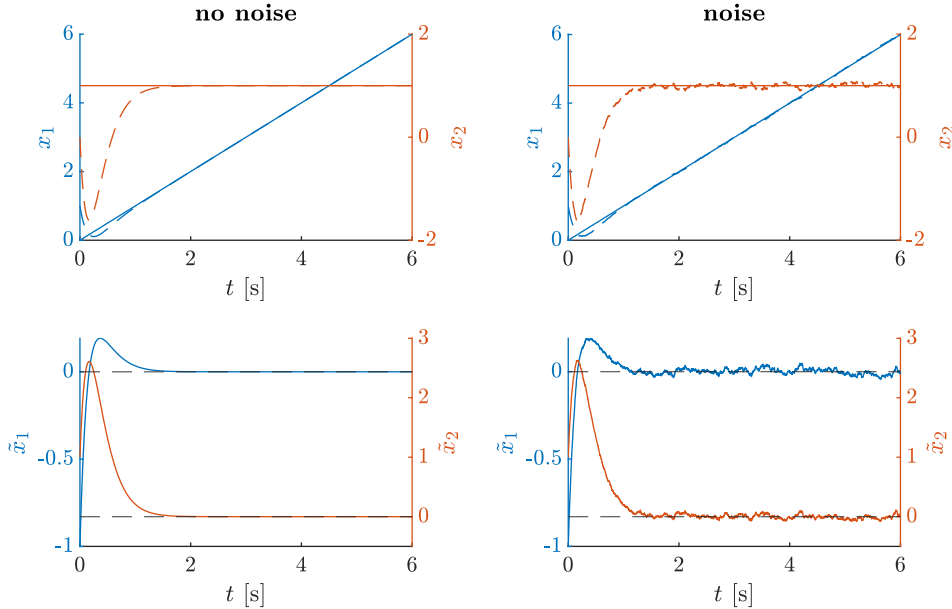$$= s^2 + k_1\,s + k_2$$

The closed-loop observer poles are at ω_α, i.e.,

$$\alpha_{\mathcal{O}}(s) = (s + \omega_\alpha)\,(s + \omega_\alpha) = s^2 + 2\,\omega_\alpha\,s + \omega_\alpha^2$$

Therefore,

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 2\omega_\alpha \\ \omega_\alpha^2 \end{pmatrix}$$

The figure below gives the simulation of the full-order observer with the poles at $\omega_\alpha = 5\,\mathrm{rad/s}$.


the blue and orange colors represent the states x_1 and x_2 respectively, the true state is shown by a solid line, and the estimated state is shown by a dashed line.

The reduced state observer is given as

$$\dot{w} = (0 - l)\,w + (0 + 0 - 0 - l\,1\,l)\,y, \qquad\qquad \hat{x}_1 = y, \quad \hat{x}_2 = w + l\,y,$$
$$\dot{w} = -l\,w - l^2\,y, \qquad\qquad\qquad\qquad\qquad \hat{x}_1 = y, \quad \hat{x}_2 = w + l\,y,$$

The figure below gives the simulation of the full-order observer with the poles at $l = 5\,\mathrm{rad/s}$. (placing the poles similar to the full-order observer)
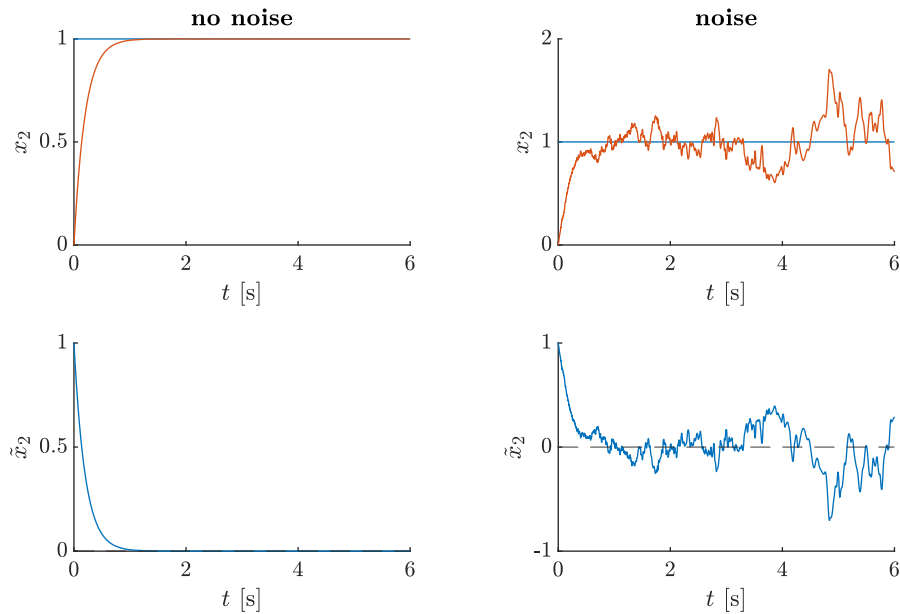
With the reduced order observer, the noise directly affects the observed 'state' since $x \propto y$ but in the full order observer, the noise is filtered since $\dot{x} \propto y$.

Code

```
% states
A = [0 1; 0 0];
C = [1, 0];
system = @(t,y) A*y;

% simulation
y0 = [0; 1];
tspan = linspace(0,6,5000);
[t_true,y_true] = ode45(@(t,y) system(t,y),tspan,y0);
% adding noise
y_noise = y_true(:,1) + (rand(size(t_true)) - 0.5)*0.5;

% simulation of full-order observer
y0 = [1; 0];
y_val = @(t) interp1(t_true, y_true(:,1), t, 'spline');
[t_fo,y_fo] = ode45(@(t,y) full_order(y,y_val(t),A,C), tspan, y0);
y_temp = interp1(t_true, y_true, t_fo, 'spline');
e_fo = y_temp - y_fo;

% simulation of full-order observer with noise
y_valN = @(t) interp1(t_true, y_noise(:,1), t, 'spline');
[t_foN,y_foN] = ode45(@(t,y) full_order(y,y_valN(t),A,C), tspan, y0);
y_temp = interp1(t_true, y_true, t_foN, 'spline');
e_foN = y_temp - y_foN;

% simulation of reduced-order observer
y0 = 0;
[t_ro,w] = ode45(@(t,y) reduced_order(y,y_val(t)),tspan,y0);
x2hat = w + 5 * y_val(t_ro);
y_temp = interp1(t_true, y_true(:,2), t_ro, 'spline');
e_ro = y_temp - x2hat;
```

```matlab
% simulation of reduced-order observer (with noise)
[t_roN,w] = ode45(@(t,y) reduced_order(y,y_valN(t)),tspan,y0);
x2hatN = w + 5 * y_val(t_roN);
y_temp = interp1(t_true, y_true(:,2), t_roN, 'spline');
e_roN = y_temp - x2hatN;

%% functions
function xdot = full_order(x,y,A,C)
    K = [2*5; 5^2]; % feedback gain
    xdot = A * x + K*(y - C * x);
end

function wdot = reduced_order(w,y)
    L = 5; % feedback gain
    wdot = -L * w - L^2 * y;
end
```

1.7 Kailath 2.3-1a

If

$$\mathcal{O}_1 T = \mathcal{O}_2,$$

then

$$c_1 T = c_2, \qquad\qquad c_1 A_1 T = c_2 A_2 = c_1 T A_2.$$

Multiplying T^{-1} on both sides to the right

$$c_1 A_1 T T^{-1} = c_1 T A_2 T^{-1} \qquad\qquad \implies c_1 A_1 = c_1 T A_2 T^{-1}.$$

Multiplying c_1^{-1} on both sides to the left

$$c_1^{-1} c_1 A_1 = c_1^{-1} c_1 T A_2 T^{-1} \qquad\qquad \implies A_1 = T A_2 T^{-1}.$$

This is not possible because c_1 is not invertible.

The non-observable states cannot be identified but if we measure all states then A_1 and A_2 will have similar dynamics.

1.8 Kilath 2.3.3

Realizing linear system to state-space

$$y = e\,\frac{s+1}{s(s+3)}$$

$$y = \left(u - y\frac{k}{s+a}\right)\frac{s+1}{s(s+3)}$$

$$y = u\,\frac{s+1}{s(s+3)} - y\,\frac{k\,(s+1)}{s(s+3)(s+a)}$$

$$y + y\,\frac{k\,(s+1)}{s(s+3)(s+a)} = u\,\frac{s+1}{s(s+3)}$$

$$y\,\frac{s(s+3)(s+a) + k\,(s+1)}{s(s+3)(s+a)} = u\,\frac{s+1}{s(s+3)}$$

$$y\,\frac{s(s+3)(s+a) + k\,(s+1)}{(s+a)} = u\,(s+1)$$

$$y\,s(s+3)(s+a) + y\,k\,(s+1) = u\,(s+1)(s+a)$$

$$y\,s\left(s^2 + s\,(3+a) + 3a\right) + y\,k\,(s+1) = u\left(s^2 + s\,(1+a) + a\right)$$

$$y\left(s^3 + s^2\,(3+a) + 3a\,s\right) + y\,k\,s + y\,k = u\left(s^2 + s\,(1+a) + a\right)$$

$$y\left(s^3 + s^2\,(3+a) + (3a+k)\,s + k\right) = u\left(s^2 + s\,(1+a) + a\right)$$

$$\frac{y}{u} = \frac{s^2 + s\,(1+a) + a}{s^3 + s^2\,(3+a) + (3a+k)\,s + k}$$

$$\frac{y}{u} = \frac{0\,s^3 + s^2 + s\,(1+a) + a}{s^3 + s^2\,(3+a) + (3a+k)\,s + k}$$

Controllable canonical state space model

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & -(3a+k) & -(3+a) \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} a & (1+a) & 1 \end{pmatrix}$$

Observable canonical state space model (*check something is wrong*)

$$A = \begin{pmatrix} -(3+a) & 1 & 0 \\ -(3a+k) & 0 & 1 \\ -k & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ (1+a) \\ a \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Controllability and observability

$$\mathcal{O}x = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -(3+a) & 1 & 0 \\ (a+3)^2 - k - 3a & -(a+3) & 1 \end{pmatrix}$$

Full-rank $\forall \, k$ and a.

1.9   Kilath 2.3-15

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \qquad\qquad C = \begin{pmatrix} \gamma_1 & \dots & \gamma_n \end{pmatrix}.v$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \dots & \gamma_n \\ \lambda_1\gamma_1 & \dots & \lambda_n\gamma_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1}\gamma_1 & \dots & \lambda_n^{n-1}\gamma_n \end{pmatrix} = \begin{pmatrix} \gamma_1 & \dots & \gamma_n \end{pmatrix} \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix}.$$

The rank$(\mathcal{O}) = n$ when λ_i is unique, $\lambda_i \neq 1$, and $\gamma_i \neq 0$.

1.10   Kilath 2.3-16 a-b

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad\qquad C = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \lambda\gamma_1 & \gamma_1 + \lambda\gamma_2 & \gamma_2 + \lambda\gamma_3 \\ \lambda^2\gamma_1 & 2\gamma_1\lambda + \lambda^2\gamma_2 & 2\gamma_2\lambda + \lambda^2\gamma_3 \end{pmatrix}$$

The rank$(\mathcal{O}) = n$ when $\gamma_1 \neq 0$.

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} \qquad\qquad C = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}.$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \lambda\gamma_1 & \gamma_1 + \lambda\gamma_2 & \mu\gamma_3 \\ \lambda^2\gamma_1 & 2\gamma_1\lambda + \lambda^2\gamma_2 & \mu^2\gamma_3 \end{pmatrix}$$

The rank$(\mathcal{O}) = n$ when $\gamma_1 \neq 0$ and $\mu \neq 0$.

1.11 Kilath 2.4-2

$$A = \begin{pmatrix} -0.5 & 1 & 0 \\ -1 & -0.5 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

The observability matrix is

$$\mathcal{O} = \begin{pmatrix} C \\ C\,A \\ C\,A^2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & -0.5 & 0 \end{pmatrix}$$

The system is completely observable.

1.12 Kilath 2.4-4

The equation is

$$y(t) + a_1\,y(t-1) + \cdots + a_n\,y(t-n) = b_1\,u(t-1) + \cdots + b_n\,u(t-n)$$
$$\implies y(t) = -a_1\,y(t-1) - \cdots - a_n\,y(t-n) + b_1\,u(t-1) + \cdots + b_n\,u(t-n)$$

In the discretized form, the state vector is

$$\theta(t) = \begin{pmatrix} y(t) & y(t-1) & \cdots & y(t-n+1) & u(t-1) & u(t-2) & \cdots & u(t-n+1) \end{pmatrix}^{\mathrm{T}}$$
$$\theta(t-1) = \begin{pmatrix} y(t-1) & y(t-2) & \cdots & y(t-n) & u(t-2) & u(t-3) & \cdots & u(t-n) \end{pmatrix}^{\mathrm{T}}$$
i.e., $\theta(t) = A\,\theta(t-1) + B\,u(t-1)$,

where

$$A = \begin{pmatrix}
-a_1 & \cdots & -a_n & b_2 & \cdots & b_n \\
1 & \ddots & 0 & 0 & \ddots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & 0 & 0 & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}, \qquad
B = \begin{pmatrix}
b_1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0
\end{pmatrix}$$

1.13 Kilath 2.4-5

Cascaded connections

$$H_1(s)\,H_2(s) = \frac{1}{s+1}\,\frac{s+1}{s+1} \qquad\qquad H_1(s)\,H_2(s) = \frac{1}{s+1}$$

1.14   Kilath 4.1.7

The system is modeled as

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = -\omega_0^2\, x(t), \qquad\qquad y(t) = v(t), \qquad\qquad x(t_0) = 0, \quad v(t_0) = 10.$$

$$A = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

The closed-loop observer poles are at ω_0, i.e.,

$$\alpha_{\mathcal{O}}(s) = (s + \omega_0)\,(s + \omega_0) = s^2 + 2\,\omega_0\,s + \omega_0^2$$

Also,

$$
\begin{aligned}
\alpha_{\mathcal{O}}(s) = \det\left(SI - A + K\,C\right) &= \det\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\right] \\
&= \det\left[\begin{pmatrix} s & -1 \\ \omega_0^2 & s \end{pmatrix} + \begin{pmatrix} 0 & k_1 \\ 0 & k_2 \end{pmatrix}\right] = \det\left[\begin{pmatrix} s & k_1 - 1 \\ \omega_0^2 & s + k_2 \end{pmatrix}\right] \\
&= s\,(s + k_2) - \omega_0^2\,(k_1 - 1) = s^2 + s\,k_2 + \omega_0^2\,(1 - k_1)
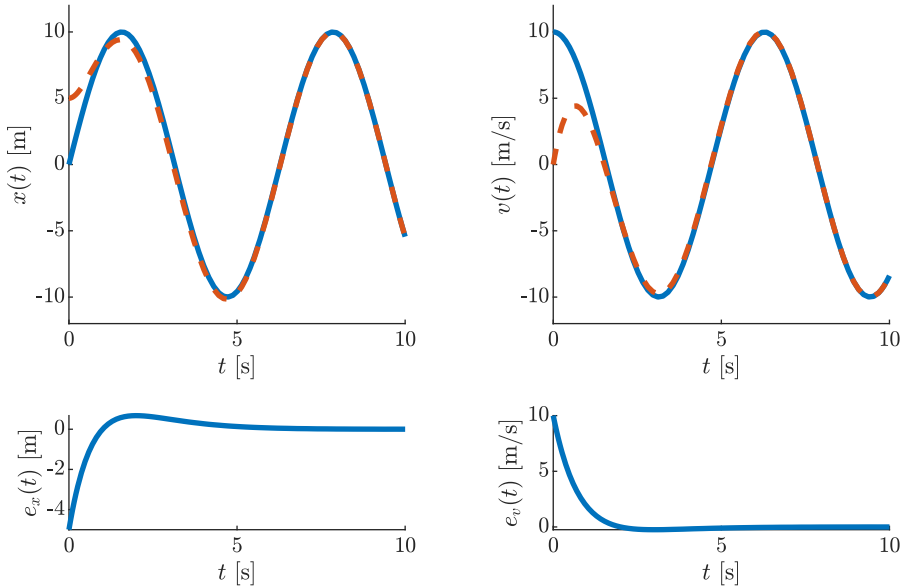\end{aligned}
$$

Therefore,

$$K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\omega_0 \end{pmatrix}$$

The solution to the ODE is

$$x(t) = 10\,\sin(\omega_o\, t), \qquad\qquad v(t) = 10\,\cos(\omega_o\, t).$$

The simulation results for $\omega_0 = 10\,\mathrm{rad/s}$ are presented in the figure:

The solid line is the true state and the dashed line is the estimate from the observer.

The error dynamics are

$$\dot{e} = (A - K\,C)\,e = \left(\begin{pmatrix} 0 & 1 \\ -\omega_0 & 0 \end{pmatrix} - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}\right) e = \begin{pmatrix} 0 & 1 - k_1 \\ -\omega_0^2 & -k_2 \end{pmatrix} e,$$

$$\dot{e}_x = (1 - k_1)\,e_v \quad \dot{e}_v = -\omega_0^2\,e_x - k_2\,e_v \qquad \Longrightarrow \qquad \dot{e}_x = e_v \quad \dot{e}_v = -\omega_0^2\,e_x - 2\,\omega_0\,e_v.$$

Therefore, the error will converge with a time constant of $1/\omega_0$.

Code

```
clear;

omega = 1;
v0 = 10;

% system true states
x = @(t) 10*sin(omega*t);
v = @(t) 10*cos(omega*t);

A = [0 1; -omega^2 0];
C = [0 1];

y0 = [5; 0];
tspan = [0 10];
[t,y] = ode45(@(t,y) observ(y,v(t),C,A,omega),tspan, y0);
```

functions

```
function xdot = observ(x,y,C,A,omega)

K = [0; 2*omega];

xdot = A*x + K*(y - C*x);

end
```

1.15   Kailath 4.1-8

The model:

$$\ddot{x} - 2\omega\dot{y} - 9\omega^2 x = 0 \qquad\qquad\qquad \ddot{y} + 2\omega\dot{x} + 4\omega^2 y = u.$$

Let,

$$\dot{x} = v_x, \quad \dot{y} = v_y \qquad\qquad \Longrightarrow \ddot{x} = \dot{v}_x, \quad \ddot{y} = \dot{v}_y.$$

Therefore,

$$\dot{v}_x - 2\omega v_y - 9\omega^2 x = 0 \qquad\qquad\qquad \dot{v}_y + 2\omega v_x + 4\omega^2 y = u,$$
$$\dot{v}_x = 2\omega v_y + 9\omega^2 x \qquad\qquad\qquad \dot{v}_y = u - 2\omega v_x - 4\omega^2 y.$$

The state matrix, $X = \begin{pmatrix} x & v_x & y & v_y \end{pmatrix}^T$. The system can be re-written as

$$\dot{X} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 9\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & -4\omega^2 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} X.$$

The closed-loop observer poles

$$\alpha_{\mathcal{O}}(s) = \det\left(SI - A + KC\right)$$

$$= \det\left(\begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 9\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & -4\omega^2 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} s & -1 & k_1 & 0 \\ -9\omega^2 & s & k_2 & -2\omega \\ 0 & 0 & k_3 + s & -1 \\ 0 & 2\omega & 4\omega^2 + k_4 & s \end{pmatrix}$$

$$= s^4 + k_3 s^3 + \left(k_4 - \omega^2\right) s^2 + (-5k_3\omega^2 - 2k_2\omega)s - 36\omega^4 - 18k_1\omega^3 - 9k_4\omega^2$$

The closed-loop observer poles are $s = -2\omega$, $s = -3\omega$, $s = -3\omega \pm j3\omega$

$$\alpha_{\mathcal{O}}(s) = (s + 2\omega)\,(s + 3\omega_0)\,(s + 3\omega - j3\omega)\,(s + 3\omega + j3\omega)$$
$$= s^4 + 11\omega s^3 + 54\omega^2 s^2 + 126\omega^3 s + 108\omega^4$$

Now,

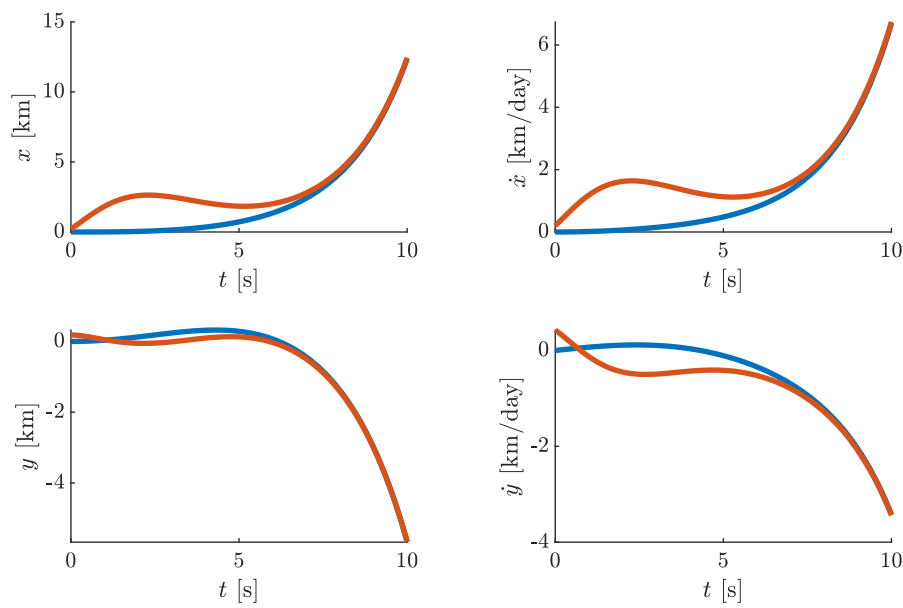$$\begin{aligned} k_3 &= 11\omega \\ k_4 - \omega^2 &= 54\omega^2 \\ 5k_3\omega^2 + 2k_2\omega &= -126\omega^3 \\ 36\omega^4 + 18k_1\omega^3 + 9k_4\omega^2 &= -108\omega^4 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} k_1 &= -\tfrac{71}{2}\omega \\ k_2 &= -\tfrac{181}{2}\omega^2 \\ k_3 &= 11\omega \\ k_4 &= 55\omega^2. \end{aligned}$$
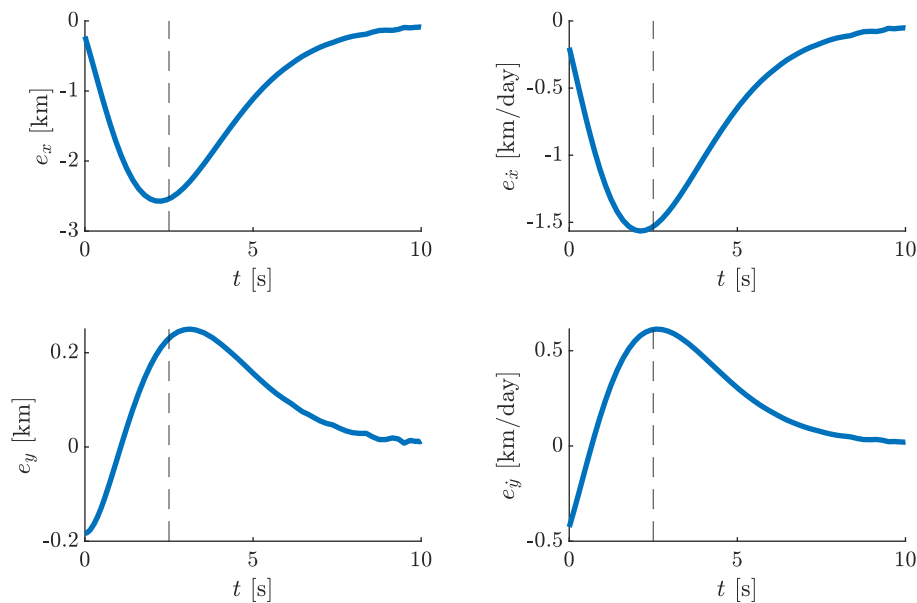
The observer is

$$\dot{\hat{X}} = A\hat{X} + Bu + K\left(y - C\hat{X}\right)$$

The simulation results are shown in the figure below, the blue curve is the 'true' state and rthe orange is the estimated from the observer.

The error between the estimated states and the 'true' states is shown

From the figure, it is clear that after 2.5 days the error starts to decay.

Code

Algebra

```
syms s w
e1 = collect((s + 2*w) * (s + 3*w) * ...
        ... (s + 3*w + 1j*3*w) * (s + 3*w - 1j*3*w),s);
c1 = coeffs(e1,s); c1(end) = [];
S = s*eye(4);
A = [0, 1, 0, 0;...
     9*w^2, 0, 0, 2*w;...
     0, 0, 0, 1;...
```

```
      0, -2*w, -4*w^2, 0];
K = sym('k',[4 1]); C = [0 0 1 0];
e2 = collect(det(S - A + K*C),s);
c2 = coeffs(e2,s); c2(end) = [];
eqns = c1 == c2;
Sol = solve(eqns,K);
```

system simulation (true-states)

```
omega = 2*pi/29.3; % orbital frequency [rad/s]
u = 1000 / (300 * omega^2) ; % input (F/(m*w^2)) [m/rad^2]
% state space representation [x; xdot; y; ydot]
A = [0, 1, 0, 0;...
     9*omega^2, 0, 0, 2*omega;...
     0, 0, 0, 1;...
     0, -2*omega, -4*omega^2, 0];
B = [0; 0; 0; 1]; C = [0, 0, 1, 0];
bryson_sattelite = @(t,x) A*x + B*u;

% simulation
y0 = [0; 0; 0; 0]; % initial states
tspan = [0 10]; % [days]
[t, y] = ode45(@(t,y) bryson_sattelite(t,y),tspan,y0);
```

observer

```
% state observer
K = [-71/2*omega; -181/2*omega^2; 11*omega; 55*omega^2];
mesh = @(x) interp1(t,y(:,3),x);
obs_bryson_sattelite = @(t,x) A*x + B*u + K*(mesh(t) - C*x);

% simulation
y0 = rand([4 1]) * 500; % initial states
[to, yo] = ode45(@(t,y) obs_bryson_sattelite(t,y),tspan,y0);

% error
for ii = 1:width(y)
    yq = interp1(t,y(:,ii),to);
    eq(:,ii) = yq - yo(:,ii);
end
```

1.16   Kailath 4.1-9

In discrete time systems, the error at time-step N is

$$e_N = (A - K\,C)\,e_{N-1}$$

The error $e_N = 0$, at time-step N ig K is the eigen-values of $(A - K\,C)$.

1.17   Kailath 4.3-4

The state-space model of the inertial navigator is

$$
\begin{pmatrix} \dot{v} \\ \dot{\varphi} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ \varphi \\ \varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}
$$

(a) Determining the solutions.

$$
X(s) = 0 \quad \det(sI - A) = 0
$$

$$
\det \begin{pmatrix} s & 1 & 0 \\ -1 & s & -1 \\ 0 & 0 & s \end{pmatrix} = 0
$$

$$
s\left(s^2 + 1\right) = 0.
$$

The open-loop eigen values are $\lambda_1 = 0$, $\lambda_{2,3} = \pm j$

(b) observer design

$$
y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ \varphi \\ \varepsilon \end{pmatrix}
$$

The observer is

$$
\dot{\hat{X}} = A\hat{X} + Bu + K\left(y - C\hat{X}\right)
$$

$$
\begin{pmatrix} \dot{\hat{v}} \\ \dot{\hat{\varphi}} \\ \dot{\hat{\varepsilon}} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\varphi} \\ \hat{\varepsilon} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix} + K\left(y - \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{v} \\ \hat{\varphi} \\ \hat{\varepsilon} \end{pmatrix}\right)
$$

The closed-loop observer poles

$$
\alpha_{\mathcal{O}}(s) = \det\left(SI - A + KC\right)
$$

$$
= \det\left(\begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}\right)
$$

$$
= \det\begin{pmatrix} k_1 + s & 1 & 0 \\ k_2 - 1 & s - 1 & 0 \\ k_3 & 0 & s \end{pmatrix}
$$

$$
= s^3 + k_1 s^2 + (1 - k_2) s - k_3
$$

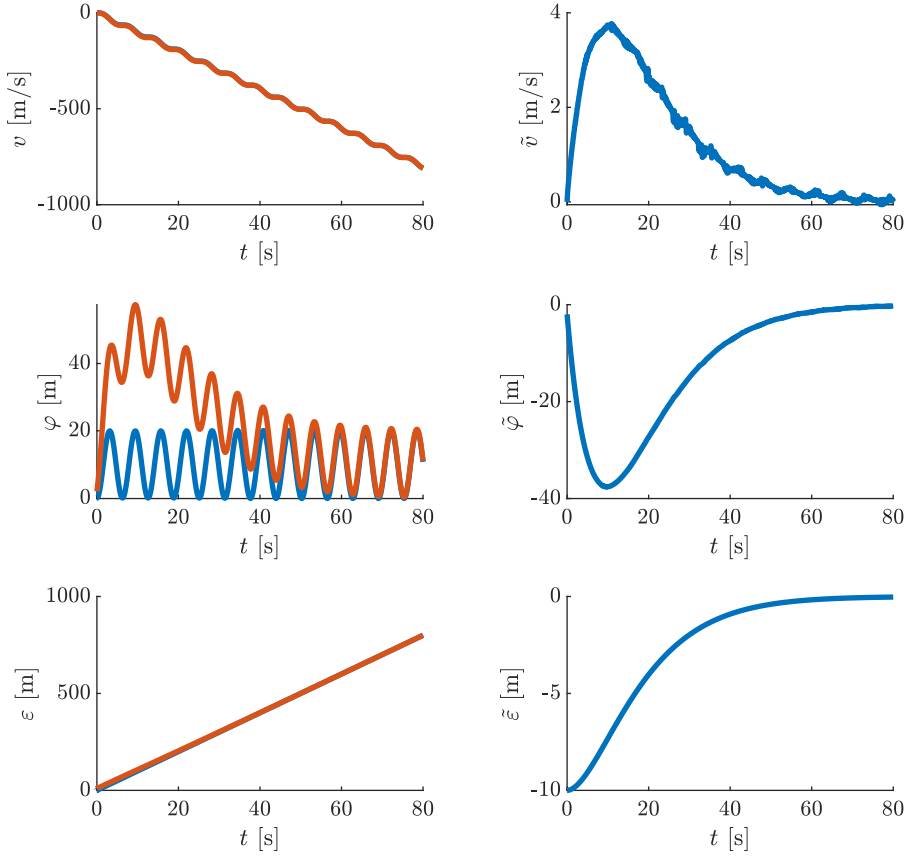The closed-loop observer poles are $s = -10$, $s = -0.1$, $s = -0.1$

$$
\alpha_{\mathcal{O}}(s) = (s + 10)(s + 0.1)(s + 0.1)
$$

$$
= s^3 + 10.2 s^2 + 2.01 s + 0.1
$$

The gains are

$$
K = \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix}^T = \begin{pmatrix} 10.2 & -1.01 & -0.1 \end{pmatrix}^T
$$

The simulation results of the observer is shown in the next page

The estimated states from the observer are shown in orange and the 'true' states are shown in blue.

(c) The *second order* observer. Original state equations

$$\begin{pmatrix} \dot{v} \\ \dot{\varphi} \\ \dot{\varepsilon} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ \varphi \\ \varepsilon \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}, \qquad \text{or} \quad \begin{pmatrix} \dot{\varepsilon} \\ \dot{\varphi} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \varepsilon \\ \varphi \\ v \end{pmatrix} + \begin{pmatrix} w \\ 0 \\ 0 \end{pmatrix}$$

Considering partitioned state equations

$$\begin{pmatrix} \dot{x}_r \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A_r & a_r \\ a_n & a_{nn} \end{pmatrix} \begin{pmatrix} x_r \\ x_n \end{pmatrix} + \begin{pmatrix} b_r \\ b_n \end{pmatrix} u(t) \qquad\qquad y = x_n$$

$$\begin{pmatrix} \begin{pmatrix} \dot{\hat{\varepsilon}} \\ \dot{\hat{\varphi}} \end{pmatrix} \\ \dot{\hat{v}} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \end{pmatrix} & 0 \end{pmatrix} \begin{pmatrix} \begin{pmatrix} \hat{\varepsilon} \\ \hat{\varphi} \end{pmatrix} \\ \hat{v} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} w \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} \qquad\qquad y = v$$

Change in partitioning affects the pole placement of the reduced order observer, with $\left(\begin{pmatrix} \varphi & \varepsilon \end{pmatrix} \quad v \right)^T$, only one pole (of the reduced order observer) can be placed arbitrarily.

The closed-loop poles

$$\alpha_{\mathcal{O}}(s) = \det\left(sI - A_r + L\, a_n\right)$$

$$= \det\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \begin{pmatrix} 0 & -1 \end{pmatrix} \right)$$

$$= \det\left(\begin{pmatrix} s-1 & -l_1 \\ -1 & s-l_2 \end{pmatrix} \right) = s^2 - l_2\, s - l_1$$

The closed-loop observer poles are $s = -0.1$, $s = -0.1$

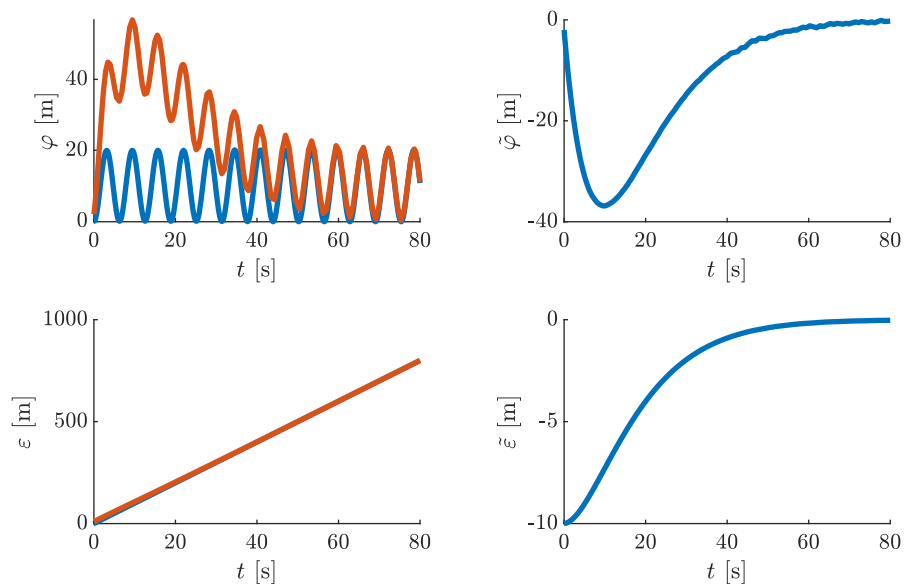$$\alpha_{\mathcal{O}}(s) = (s + 0.1)\,(s + 0.1) = s^2 + 0.2\,s + 0.01$$

The gains are

$$L = \begin{pmatrix} l_1 & l_2 \end{pmatrix}^T = \begin{pmatrix} -0.01 & -0.2 \end{pmatrix}^T$$

The observer is designed as,

$$\dot{\theta} = (A_r - L\,a_n)\,\theta + (a_r - L\,a_{nn} + A_r\,L - L\,a_n\,L)\,y + (b_r - b_n)\,u$$
$$\hat{x}_r = \theta + L\,y$$
$$\hat{x}_n = x_n = y$$

The simulation results of the observer are shown below, where the estimated states from the observer are shown in orange, and the 'true' states are shown in blue.

The reduced-order and the full-order observer are similar in performance.

Code

simulation (true system)

```
clear;
u = 10;
A = [0 -1 0; 1 0 1; 0 0 0];
B = [0; 0; 1]; C = [1 0 0];
int_nav = @(t,x) A*x + B*u;

% simulation
y0 = [0; 0; 0]; % initial states
tspan = [0 80]; % [s]
[t, y] = ode45(@(t,y) int_nav(t,y),tspan,y0);
```

Observer simulation

```matlab
% state observer
K = [10.2; -1.01; -0.1];
mesh = @(x) interp1(t,y(:,1),x,'spline');
obs_int_nav = @(t,x) A*x + B*u + K*(mesh(t) - C*x);

% simulation
y0 = [0; 2; 10]; % initial states
[to, yo] = ode45(@(t,y) obs_int_nav(t,y),tspan,y0);

% error
for ii = 1:width(y)
    yq = interp1(t,y(:,ii),to);
    eq(:,ii) = yq - yo(:,ii);
end
```

Reduced order observer simulation

```matlab
% A-matrix partition
Ar = [0 0; 1 0];
cr = [0 -1];
br = [0; 1];
ann = 0;
% B_matrix
gr = [1; 0];
gn = 0;
% observer gain
L = [-0.01; -0.2];
% state observer
red_obs_int_nav = @(t,z) (Ar - L*cr)*z ...
                  + (br -L*ann + Ar*L - L*cr*L)*mesh(t) ...
                  + (gr - L*gn)*u;

% simulation
y0 = [10; 2]; % initial states
[tr, zo] = ode45(@(t,y) red_obs_int_nav(t,y),tspan,y0);

% states
for ii = 1:length(tr)
    xr(ii,:) = zo(ii,:) + (L*mesh(tr(ii)))';
end

% error
yq = interp1(t,y(:,3),tr);
eq2(:,1) = yq - xr(:,1);
yq = interp1(t,y(:,2),tr);
eq2(:,2) = yq - xr(:,2);
```

Chapter 2

Observers for DAEs and non-Linear Systems

2.1 Le 2.1

Consider the following DAE:

$$E\,\dot{x} = A\,x + B\,u, \qquad\qquad y = C\,x$$

There exists invertible matrices P and T, where $x = T\,w$ and multiplying the model equations with P from the left and T, T^{-1} to the right of E gives

$$P\,E\,T\,T^{-1}\dot{x} = P\,A\,T\,T^{-1}\,x + P\,B\,T\,T^{-1}\,u.$$
$$P\,E\,T\,\dot{w} = P\,A\,T\,w + P\,B\,T\,T^{-1}\,u,$$

where

$$P\,E\,T = \begin{pmatrix} I & 0 \\ 0 & E_2 \end{pmatrix} \quad P\,A\,T = \begin{pmatrix} A_1 & 0 \\ 0 & I \end{pmatrix}, \quad \text{where } E_2 \text{ is nilpotent.}$$

Assuming that the ODE part, $\dot{w}_1 = A_1\,w_1 + B_1\,u$ is observable, i.e.,

$$\begin{pmatrix} \lambda\,I - A_1 \\ C \end{pmatrix} \quad \text{has full column rank } \forall\lambda, \text{ i.e.,} \quad \begin{pmatrix} \lambda\,I - A_1 \\ 0 \\ C \end{pmatrix} \text{ also has full column rank } \forall\lambda.$$

Considering the following:

$$\begin{pmatrix} \lambda\,E - A \\ C \end{pmatrix} \equiv \begin{pmatrix} \lambda\,P\,E\,T - P\,A\,T \\ C \end{pmatrix} = \begin{pmatrix} \lambda\begin{pmatrix} I & 0 \\ 0 & E_2 \end{pmatrix} - \begin{pmatrix} A_1 & 0 \\ 0 & E_2 \end{pmatrix} \\ \begin{pmatrix} C_1 & C_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \lambda\,I - A_1 & 0 \\ 0 & \lambda\,E_2 - I \\ C_1 & C_2 \end{pmatrix}$$

Any square matrix has a Jordan normal form if the field of coefficients is extended to one containing all the eigenvalues of the matrix [*from Wikipedia*]. The Jordan form is given as

$$\begin{pmatrix} \lambda_1 & 1 \\ & \lambda_1 & 1 \\ & & \lambda_1 \\ & & & \lambda_2 & 1 \\ & & & & \lambda_2 \\ & & & & & \lambda_3 \\ & & & & & & \ddots \\ & & & & & & & \lambda_n \end{pmatrix}, \text{for a nilpotent matrix, } \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$$

Thus, $\lambda E_2 - I$ has always full-column rank.

Therefore, if $\begin{pmatrix} \lambda I - A_1 & C \end{pmatrix}^T$ has full-column rank then $\begin{pmatrix} \lambda E - A & C \end{pmatrix}^T$ has full-column rank $\forall \lambda$.

If $\begin{pmatrix} \lambda E - A & C \end{pmatrix}^T$ has full-column rank then its minor $\begin{pmatrix} \lambda I - A_1 & C \end{pmatrix}^T$ also has full-column rank $\forall \lambda$.

2.2   Le 2.2

(a) Writing the model equations in the form

$$E\,\dot{x} = A\,x + B\,u$$

The model equations are

$$
\begin{array}{llll}
u - v_1 = R_1\,i_1 & v_3 - v_4 = R_2\,i_2 & v_1 - v_4 = R_3\,i_3 & C\,\dot{v}_1 - C\,\dot{v}_3 = i_2 \\
v_1 - v_2 = 0 & v_2 = 0 & i_1 = i_2 + i_3 & \\
v_1 + R_1\,i_1 = u & v_3 - v_4 - R_2\,i_2 = 0 & v_1 - v_4 - R_3\,i_3 = 0 & C\,\dot{v}_1 - C\,\dot{v}_3 - i_2 = 0 \\
v_1 - v_2 = 0 & v_2 = 0 & i_1 - i_2 - i_3 = 0 &
\end{array}
$$

Considering $x = \begin{pmatrix} i_1 & i_2 & i_3 & v_1 & v_2 & v_3 & v_4 \end{pmatrix}^T$,

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & C_c & 0 & -C_c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\dot{i}_1 \\ \dot{i}_2 \\ \dot{i}_3 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \\ \dot{v}_4
\end{pmatrix}
=
\begin{pmatrix}
-R_1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & -R_2 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & -R_3 & 1 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
i_1 \\ i_2 \\ i_3 \\ v_1 \\ v_2 \\ v_3 \\ v_4
\end{pmatrix}
$$

$$
+ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} u
$$

(b) Measurement signals:

Choosing C such that $\begin{pmatrix} \lambda\,E - A & C \end{pmatrix}^T$ has full-rank $\forall\,\lambda$.

$\lambda\,E - A$ is 0, when $\det\left(\lambda\,E - A\right) = 0$, i.e., $\lambda = \frac{-1}{C_c\,(R_2 + R_3)}$

The matrix $\begin{pmatrix} \lambda\,E - A & C \end{pmatrix}^T$ <u>does not</u> have full-rank with the measurement signals $\{i_1, v_1, v_2\}$ $\forall\,\lambda$.

(c) Building an observer

Simplifying the ODE,

$$
C\,\dot{v}_1 - C\,\dot{v}_3 - i_2 = 0 \qquad\qquad \implies C\,\dot{v}_3 = \frac{v_3 - v_4}{R_2}.
$$

The measurement equation:

$$y = v_4.$$

The observer design is

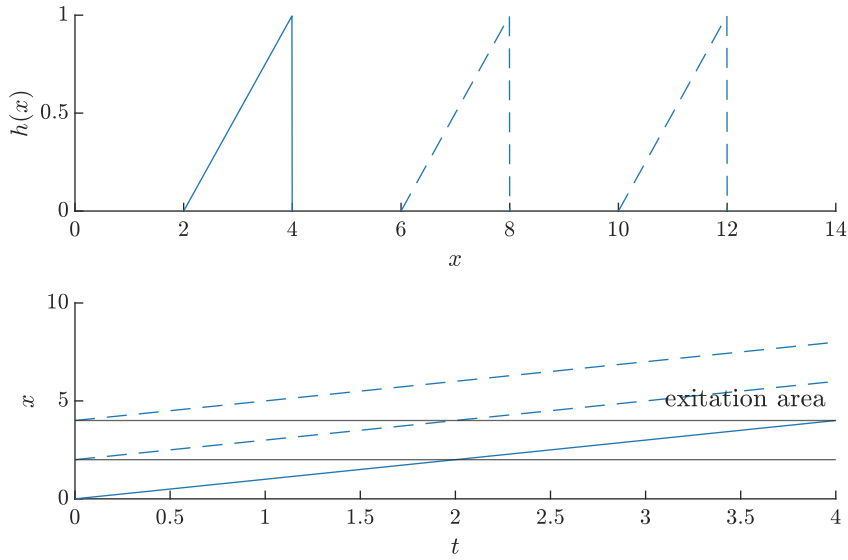$$\dot{\hat{v}}_3 = \frac{\hat{v}_3 - v_4}{R_2\,C_c} + K\,(v_4 - 0)$$

The poles for the observer is

$$
\begin{aligned}
\alpha_{\mathcal{O}}(s) &= \det\left(s\,I - A + K\,C\right) \\
&= s - \frac{1}{R_2\,C_c} + k_1 \times 0.
\end{aligned}
$$

2.3   Le 2.3

The system is defined as

$$\dot{x} = 1 \hspace{6cm} y = h(x)$$

(a) $\mathcal{M} \subset (0, 4)$ **Weakly observable**.

> At $x < 2$, $h(x) = 0$ and thus not locally observable. However, $\forall t >= 2$, $h(x) = k\,x$ and becomes observable.

(b) $\mathcal{M} = (2, 4)$ **Locally observable**.

> x is uniquly isolable for all $\mathcal{U} \subset \mathcal{M}$.

(c) $\mathcal{M} = \mathcal{R}^+ \,'\cdots'$ **Weakly observable**.

> $2 \leq t \leq 4$, $h(x) = k\,x$ and becomes observable.

Note:

> if $h(x)$ is a sawtooth waveform, then the system is **locally weakly observable**.

2.4   Le 2.4

Considering the ODE

$$\dot{x} = 1 \qquad\qquad\qquad y = x^3$$

Let,

$$F(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^q h(x) \end{pmatrix} = \begin{pmatrix} x^3 \\ 3x^2 \\ 6x \\ \vdots \end{pmatrix}$$

The system is indeed locally weakly observable. 1

Considering the ODE

$$\dot{x} = 1 \qquad\qquad\qquad y = x\,e^{-1/x^2}$$

$$F(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^q h(x) \end{pmatrix} = \begin{pmatrix} x\,e^{-1/x^2} \\ \vdots \end{pmatrix}$$

Since $x\,e^{-1/x^2}\big|_{x=0}$ is $\div 0$. Therefore the locally weakly observable at $x = 0$ is defined as

$$\lim_{x\to 0} \frac{\partial}{\partial x} F(x) = \lim_{x\to 0} \frac{\partial}{\partial x}\left(h(x)\right)$$

$$= \lim_{x\to 0} \begin{pmatrix} x\,\lambda + \frac{2\,\lambda}{x^2} \\ \frac{4\,\lambda}{x^5} - \frac{2\,\lambda}{x^3} \\ \frac{6\,\lambda}{x^4} - \frac{24\,\lambda}{x^6} + \frac{8\,\lambda}{x^8} \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

where $\lambda = e^{-1/x^2}$

In this case, even as $q \to \infty$, the matrix is not full-rank as $x \to 0$. However, the system is locally weakly observable.

> Even if q is ∞, it does not imply that the system is not observable.

Code

```
syms x f(x) h(x)
f(x) = 1; h(x) = x*exp(-1/x^2);
F(x) = [h(x)*f(x);...
    diff(h(x),x)*f(x); ...
    diff(diff(h(x),x),x)*f(x);...
    diff(diff(diff(h(x),x),x),x)*f(x)];
dF(x) = diff(F(x),x); val = limit(dF(x),x,0); rank_Fx = rank(val)
```

2.5   Le 2.5

Model, let:

$$A = f_x(x)\Big|_{x=x^0} \quad C = h_x(x)\Big|_{x=x^0},$$

where $f_x(x) = u$ and $h_x(x) = x^2$.

At $x^0 = 0$, the observability matrix of $(C,\ A)$ is

$$\mathcal{O}x = \left(h_x(x)|_{x=0}\right) = \left(2\,x\right)_{x=0} = 0.$$

It is not full-rank and from the linearization.   However, the system is *locally weakly observable*.

Nothing can be concluded about the system's observability if C and A are not observable.

2.6   Le 2.6

The model of the pendulum is

$$\dot{x} = f(x, u) = \begin{pmatrix} x_2 \\ -x_2 - \sin(x_1) + u \end{pmatrix}, \quad y = x_1$$

The observer is

$$\dot{X} = f(\hat{x}, u) + K(y - \hat{x}_1) = \begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_2 - \sin(\hat{x}_1) + u \end{pmatrix} + K(y - \hat{x}_1),$$

$$\dot{\hat{x}}_1 = \hat{x}_2 + k_1(y - \hat{x}_1), \quad \dot{\hat{x}}_2 = -\hat{x}_2 - \sin(\hat{x}_1) + u + k_2(y - \hat{x}_1).$$

The error signals are

$$\dot{e} = \dot{X} - \dot{\hat{X}} \quad \text{i.e.,} \quad \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 - \dot{\hat{x}}_1 \\ \dot{x}_2 - \dot{\hat{x}}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \dot{x}_1 - \dot{\hat{x}}_1 \\ \dot{x}_2 - \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} x_2 - \hat{x}_2 - k_1(x_1 - \hat{x}_1) \\ -x_2 - \sin(x_1) + \not{u} + \hat{x}_2 + \sin(\hat{x}_1) - \not{u} - k_2(y - \hat{x}_1) \end{pmatrix}$$

$$= \begin{pmatrix} e_2 - k_1 e_1 \\ -e_2 - \sin(x_1) + \sin(\hat{x}_1) - k_2 e_1 \end{pmatrix}$$

The Lyapunov equation is

$$V(e) = e_1^2 + \beta e_2^2, \ \beta > 0$$
$$\dot{V}(e) = 2 e_1 \dot{e}_1 + 2 \beta e_2 \dot{e}_2$$
$$= 2 e_1 (e_2 - k_1 e_1) + 2 \beta e_2 (-e_2 - \sin(x_1) + \sin(\hat{x}_1) - k_2 e_1)$$
$$= 2 e_1 e_2 - 2 k_1 e_1^2 - 2 \beta e_2^2 - 2 \beta k_2 e_1 e_2 - 2 \beta e_2 (\sin(x_1) - \sin(\hat{x}_1))$$

Considering the inequality

$$0 \le \frac{\sin(x) - \sin(y)}{x - y} \le 1, \implies 0 \le \sin(x) - \sin(y) \le x - y \qquad \text{if } -\pi/2 \le x, \ y \le \pi/2$$

The maximum error $x_1 - \hat{x}_1$ will not exceed π, (the pendulum is hanging on a wall)

Therefore for the worst error $x_1 - \hat{x}_1$, $\dot{V}(e)$ is

$$\dot{V}(e) = 2 e_1 e_2 - 2 k_1 e_1^2 - 2 \beta e_2^2 - 2 \beta k_2 e_1 e_2 - 2 \beta e_2 e_1$$
$$= 2 e_1 e_2 (1 - k_2 - \beta) - 2 k_1 e_1^2 - 2 \beta e_2^2$$

Choosing $k_2 = 1$ gives

$$\dot{V}(e) = 2 e_1 e_2 \left(1 - k_2^{0} - \beta \right) - 2 k_1 e_1^2 - 2 \beta e_2^2 = -2 \beta e_1 e_2 - 2 k_1 e_1^2 - 2 \beta e_2^2$$
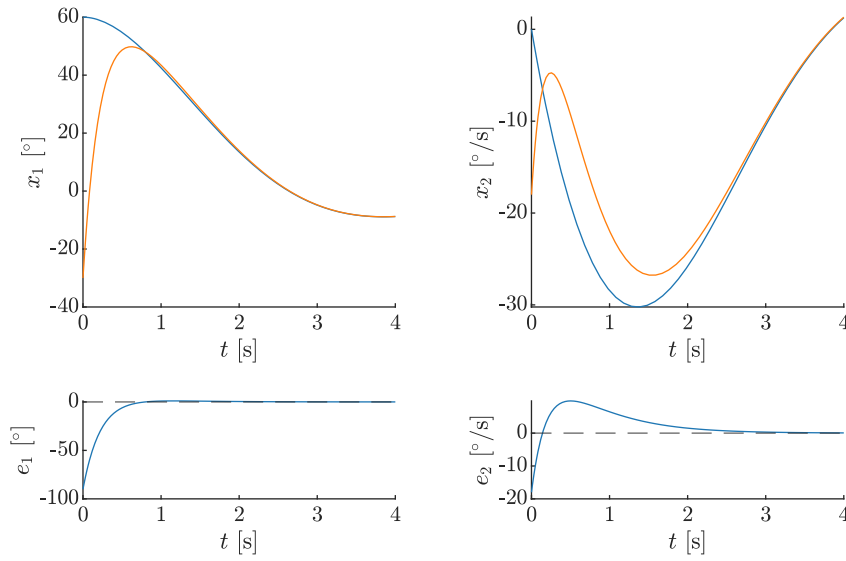
Choosing $k_1 = \beta$ gives

$$\dot{V}(e) = -2 \beta \left(e_1 e_2 + e_1^2 + e_2^2 \right) \quad \implies \quad \dot{V}(e) < 0 \ \forall \ \beta > 0.$$

The observer is

$$\dot{X} = \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_2 - \sin(\hat{x}_1) + u \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} (y - \hat{x}_1) \qquad\qquad K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} \beta \\ 1 \end{pmatrix}$$

The simulation results are shown in the following figure

The blue line is the true state and the orange line is estimated.

Code

main system

```
u = @(t) 0;
fx = @(t,x) [ x(2) ;...
             -x(2) - sin(x(1)) + u(t)];

tspan = [0 4];
y0 = [pi/3; 0];
[t,y] = ode45(@(t,y) fx(t,y),tspan,y0);
```

observer design

```
y_val = @(tt) interp1(t,y(:,1),tt,'spline');
beta = 5;
K = [beta; 1];
fo = @(t,x) fx(t,x) + K * (y_val(t) - x(1));

y0 = [-pi/6; -pi/10];
[to,yo] = ode45(@(t,y) fo(t,y),tspan,y0);
```

error signals

```
yoo = interp1(t,y,to,'spline');
err = (yo - yoo);
```

2.7 Le 2.7

Consider the system,

$$\dot{x} = f(x) + g(x)\,u \qquad\qquad y = h(x)$$

differentiating $y(t)$ one time

$$\dot{y}(t) = \frac{\partial h(x)}{\partial x}\,\dot{x} = \frac{\partial h(x)}{\partial x}\,(f(x) + g(x)\,u) = L_f\,h(x) + L_g\,h(x)\,u.$$

Similarly,

$$L_f^2\,h(x) = \frac{\partial L_f\,h(x)}{\partial x}\,f(x), \qquad\qquad L_g\,L_f\,h(x) = \frac{\partial L_f\,h(x)}{\partial x}\,g(x)$$

The system is of relative degree equal to the number of states n, then

$$L_g\,L_f^k\,h(x) = 0, \forall\,x\;\in\;\mathcal{N}(x^0)\;\forall\,k < n-1 \qquad\qquad L_g\,L_f^{n-1}\,h(x) \neq 0.$$

Considering the observability criteria,

$$\mathcal{O} = \frac{\partial}{\partial x}\begin{pmatrix} h(x) \\ L_f\,h(x) \\ \vdots \\ L_f^{n-1}h(x) \end{pmatrix} = \begin{pmatrix} L_f\,h(x) \\ L_f^2\,h(x) \\ \vdots \\ L_f^n h(x) \end{pmatrix}.$$

The rank of $\mathcal{O}$ is equal to n, the number of states if the relative degree equal to the number of states.

2.8   Le 2.8

Practice

The system:

$$\dot{X} = \begin{pmatrix} \left(\frac{x_2}{x_1}\right)^2 \\ \left(\frac{x_2}{x_1}\right)^3 - x_2 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} u \qquad\qquad y = x_1\, x_2$$

let, $z = \Phi(x) = \begin{pmatrix} x_1 \\ \frac{x_2}{x_1} \end{pmatrix}$, then

$$\dot{z} = \frac{\partial \Phi\,(x)}{\partial x}\, f\,(x,u)\Big|_{x=\Phi^{-1}(z)} = \begin{pmatrix} 1 & 0 \\ -\frac{x_2}{x_1^2} & \frac{1}{x_1} \end{pmatrix} f\,(x,u)\Big|_{x=\Phi^{-1}(z)}$$

$$= \begin{pmatrix} \left(\frac{x_2}{x_1}\right)^2 \\ -\frac{x_2}{x_1} \end{pmatrix} + \begin{pmatrix} x_1 \\ 0 \end{pmatrix} u\Big|_{x=\Phi^{-1}(z)}$$

it is also defined that

$$\Phi^{-1}(x) = \begin{pmatrix} z_1 \\ z_2\, z_1 \end{pmatrix}$$

Therefore,

$$\dot{z} = \begin{pmatrix} z_2^2 \\ -z_2 \end{pmatrix} + \begin{pmatrix} z_1 \\ 0 \end{pmatrix} u \quad y = z_1^2\, z_2,$$

$$\dot{z}_1 = z_2^2 + z_1\, u \quad \dot{z}_2 = -z_2 \quad y = z_1^2\, z_2, \implies \dot{z}_1 = \frac{1}{z_1^4} z_2^2\, z_1^4 + z_1\, u \quad \dot{z}_2 = -z_2 \quad y = z_1^2\, z_2,$$

Excercise

The system:

$$\dot{X} = \begin{pmatrix} x_1^2\, x_2^2 + u \\ 1 - x_1\, x_2^3 - x_2 - \frac{x_2}{x_1} u \end{pmatrix} \quad y = x_1\, x_2$$

let,

$$z = \Phi(x) = \begin{pmatrix} x_2 x_1 \\ x_1 \end{pmatrix}, \qquad\qquad \implies x = \Phi^{-1}(x) = \begin{pmatrix} z_2 \\ \frac{z_1}{z_2} \end{pmatrix},$$

Then

$$\dot{z} = \frac{\partial \Phi\,(x)}{\partial x}\, f\,(x,u)\Big|_{x=\Phi^{-1}(z)} = \begin{pmatrix} x_2 & x_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1^2\, x_2^2 + u \\ 1 - x_1\, x_2^3 - x_2 - \frac{x_2}{x_1} u \end{pmatrix}\Big|_{x=\Phi^{-1}(z)}$$

$$= \begin{pmatrix} x_1\,(1 - x_2) \\ x_1^2\, x_2^2 + u \end{pmatrix}\Big|_{x=\Phi^{-1}(z)} = \begin{pmatrix} -z_1 + z_2 \\ z_1^2 + u \end{pmatrix}$$

Now,

$$y = h(x)\big|_{x=\Phi^{-1}(z)} = x_1\, x_2\big|_{x=\Phi^{-1}(z)} = z_1.$$

Therefore,

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ y^2 \end{pmatrix} \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

For a linear system, the observability is

$$\mathcal{O}z = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ has full-column rank}$$

If Z is observable then X is also observable since $X = \Phi^{-1}(Z)$, i.e., an algebraic expression.

The observer can be designed as

$$\dot{\hat{Z}} = A\hat{Z} + f(u, y) + K\left(y - C\hat{Z}\right)$$

The closed-loop observer poles

$$\begin{aligned}
\alpha_{\mathcal{O}}(s) &= \det\left(SI - A + KC\right) \\
&= \det\left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\right) \\
&= \det\begin{pmatrix} k_1 + s + 1 & -1 \\ k_2 & s \end{pmatrix} \\
&= s^2 + s\,(k_1 + 1) + k_2
\end{aligned}$$

Chapter 3

Design techniques, part-1

Implementation of EKF

An EKF was implemented according to the lecture slides. However, the gains are rather challenging to determine.

However, the covariance matrices (Q, R), and the initial covariance matrix (P_0)

$$Q = \begin{pmatrix} 0.05 & & \\ & 0.05 & \\ & & 0.05 \end{pmatrix}, \qquad R = \begin{pmatrix} 0.08 & \\ & 0.2 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 0.01 & & \\ & 0.1 & \\ & & 0.01 \end{pmatrix},$$

> **How did I guess?**
> Q is assumed as the variance in the model (not covariances, since it is a diagonal matrix) and since, in this case, G (noise 'component' to the model), is non-existent, that is why low values of Q are assumed.
> R is assumed as the variance in the measurement. High values are chosen because I want to trust the measurements less and more focus on the model.
> P_0 is the initial co-variance matrix (how much we trust the initial guess), and low values are chosen for top-right and bottom-left since the initial guesses are quite good for x and θ, but the initial guess for y is bad.

A least-square optimization is performed to determine the matrices Q, R, and P_0. The matrices are

$$Q = \begin{pmatrix} 0.1914 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 1.79 \times 10^5 & \\ & 0.87 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 145.66 & & \\ & 1.43 \times 10^4 & \\ & & 0 \end{pmatrix}$$

The results of the optimization seem to agree with my idea.

An array square-root algorithm was implemented and optimization resulted in the following co-variance matrices:

$$Q = \begin{pmatrix} 0.059 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 608.11 & \\ & 0.35 \end{pmatrix}, \qquad P_0 = \begin{pmatrix} 56.96 & & \\ & 1.4 \times 10^5 & \\ & & 0 \end{pmatrix}$$

The results from EKF for robot tracking is
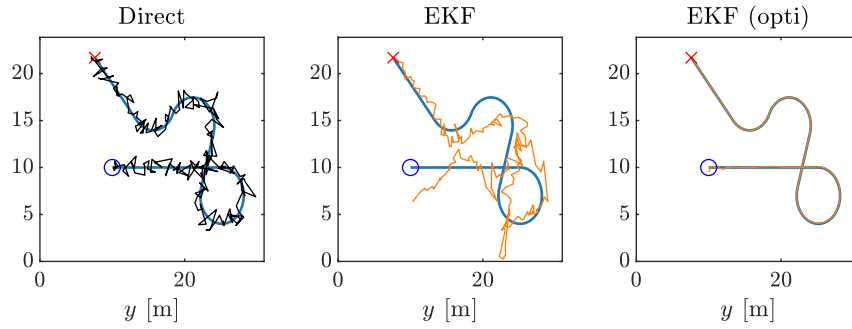
The results from EKF for robot tracking with array SR implementation is
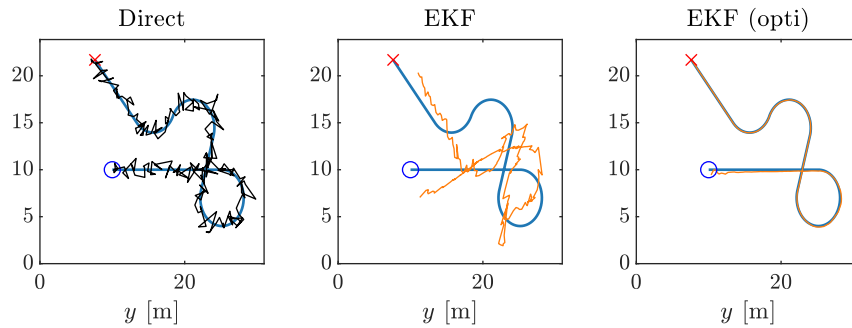
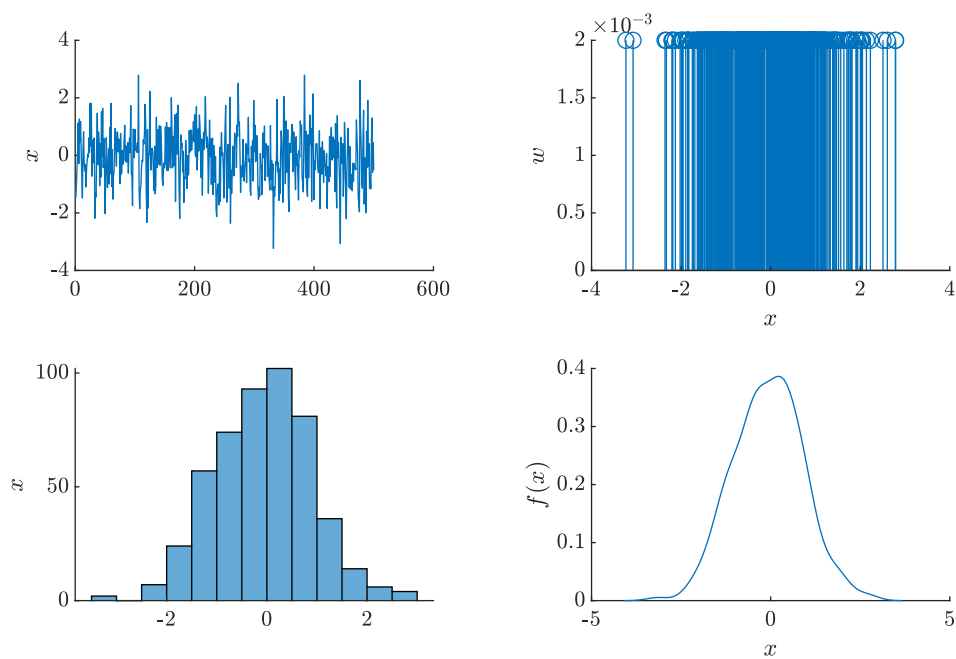Both the mothods have similar performance.

The code is available at 🐙.

Chapter 4

Design techniques, part-2

The code for this assignment is available at [GitHub].

4.1 Le 3b-1

The Sample generated by 500 particles to represent the distribution with equal weight is as follows:

The particles from the exercise were used to compute new particles such that the new particles represent the distribution with the following relation:

$$y = \sin(2\pi x^2),$$

representing the distribution is shown in the figure below:
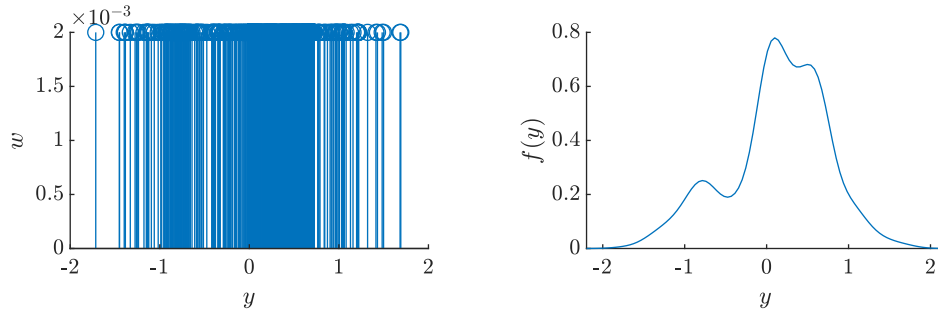
4.2   Le 3b.2

Computing particles and weights before the measurement update step 2. The figure shows the
particles and a kernel density estimate of the distribution
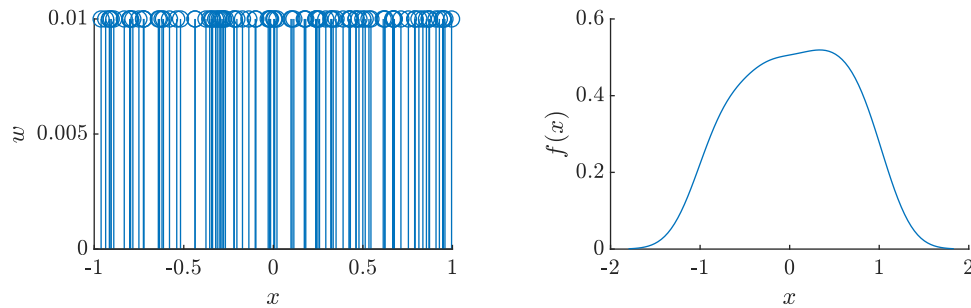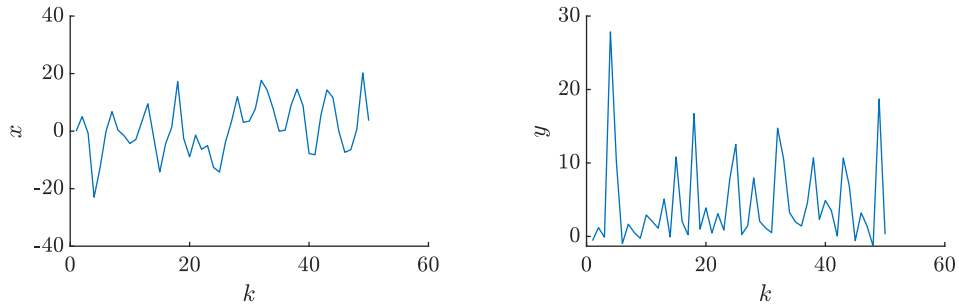

Computing particles and weights after the measurement update step 2.
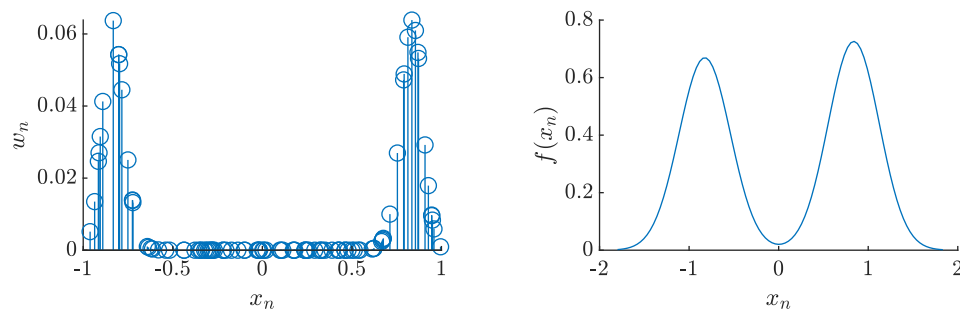
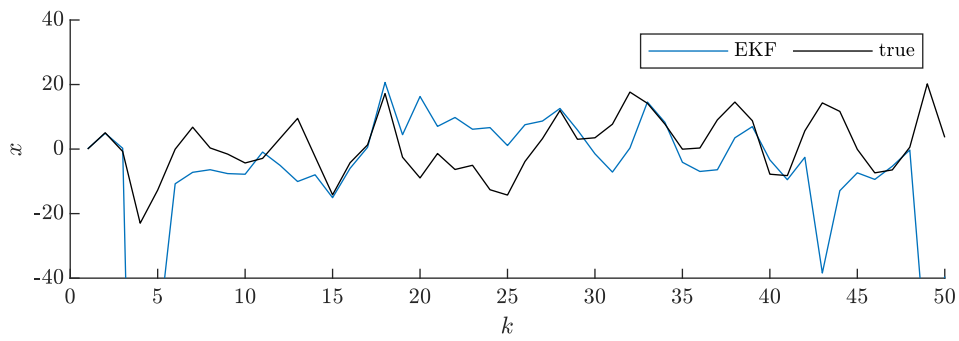Finally, resampling and the number of effective particles are **22.66**.
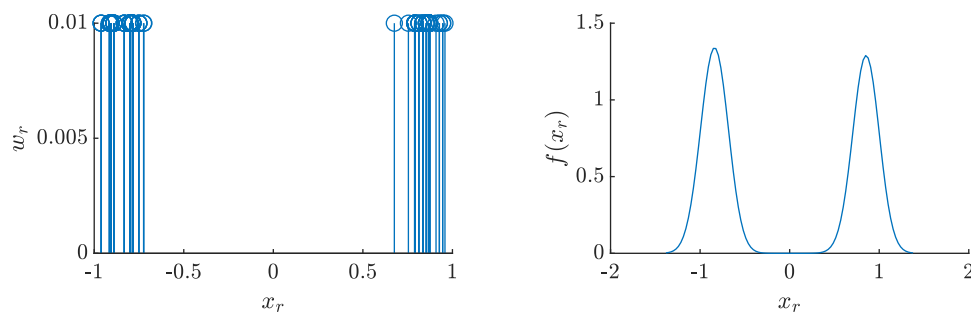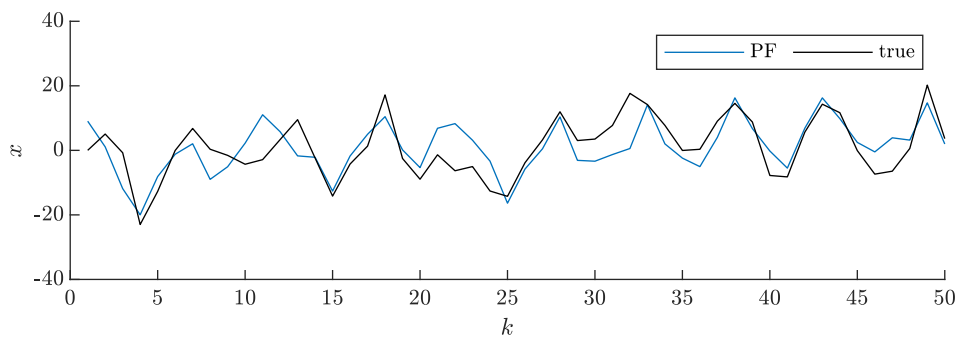
4.3   Le 3b.3

Simulated system results are

The EKF estimation is

The particle filter estimation is

Chapter 5

Moving Horizon Estimation

The code for this assignment is available at ⛓.

The moving horizon estimation (MHE) problem presented in the paper: Rao, Christopher V., James B. Rawlings, and David Q. Mayne. "*Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations*", IEEE transactions on automatic control 48.2 (2003): 246-258. (https://doi.org/10.1109/TAC.2002.808470).

The problem is to estimate the state of a nonlinear discrete-time system subject to constraints on the state and input. The problem is solved using a moving horizon estimation (MHE) approach. The MHE problem is formulated as a nonlinear programming problem (NLP) and solved using an IPOPT algorithm. The performance of the MHE algorithm is compared with that of a Kalman filter (KF) using a numerical example presented in the article (equation (8)). The results show that the MHE algorithm provides better estimates of the state and input than the KF as seen in the figure below:
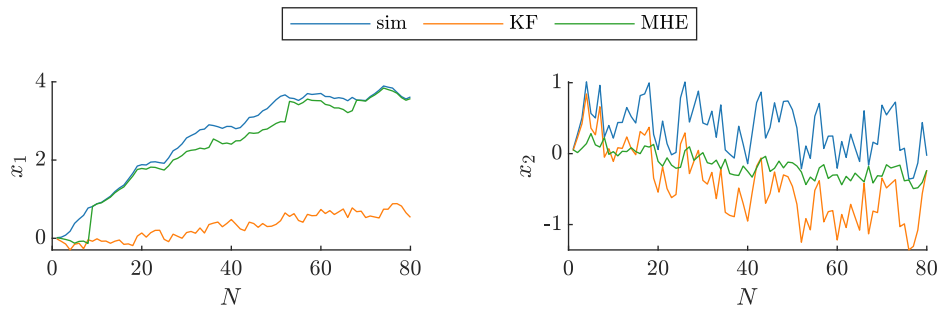
The cumulative squared errors for the two states using the two methods are as follows:

Method	$\tilde{x}_1$	$\tilde{x}_2$
KF	482.16	54.99
MHE	7.98	27.72

Comparing the performance of MHE and EKF, a model with non-linear perturbation (given by (9) in the article) is used as an example and the results are as follows:
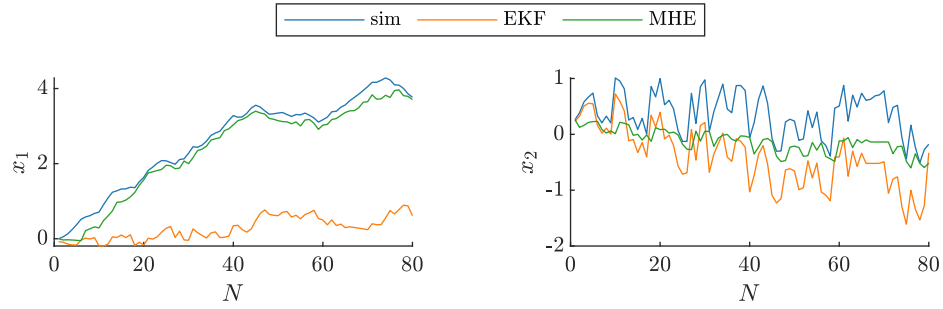
The cumulative squared errors for the two states using the two methods are as follows:

Method	$\tilde{x}_1$	$\tilde{x}_2$
EKF	187.08	61.39
MHE	17.75	39.13