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Modeling Aggressive Maneuvers on Loose Surfaces: The Cases of Trail-Braking and Pendulum-Turn

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Abstract— **In this work we initiate a mathematical analysis of rally racing techniques. We provide an empirical description of Trail-Braking (TB) and Pendulum-Turn (PT) cornering, two of the most common rally racing maneuvers. We introduce a low order vehicle model that can be efficiently used within an optimization scheme. The model incorporates the appropriate level of detail to reproduce modes of operation typical of those encountered in rally, off-road racing. We use a numerical scheme to study different trajectory optimization scenarios during cornering. We show that our modeling approach is capable of reproducing TB and PT as special cases of the minimum-time solution with additional constraints.**

I. INTRODUCTION

The state of the art in autonomous ground vehicles was perhaps demonstrated in the 2005 DARPA Grand Challenge, where several teams raced their vehicles autonomously over 131.2 miles of unpaved course in the Mojave desert within 10 hours. Technical reports of the teams participating in the final event can be found in [1]. The winning team was from Stanford University, which completed the course at an average speed of approximately 19 mph. It is envisioned that the next generation of autonomous ground vehicles will be able to travel autonomously such long distances faster than these moderate speeds, and perhaps as fast as human (expert) car drivers.

The problem of trajectory planning for high-speed ground vehicles is typically dealt with in the literature by means of numerical optimization [2], [3], [4], [5]. These results demonstrate that numerical techniques allow one to incorporate accurate, high order dynamical models, thus producing realistic results. On the other hand, these numerical optimization approaches are computationally intensive, and they cannot be readily applied in cases where the environment changes unpredictably. Analytical approaches have also been introduced in the literature [6], [7], [8], [9], [10] These analytical methodologies are computationally less intensive than numerical approaches. However, the assumptions used in the formulation of trajectory optimization problems aiming at analytic solutions tend to oversimplify the problem.

A new approach to real-time path planning of autonomous vehicles, which overcomes the limitations of both numerical and analytical optimization techniques has been developed in [11], [12] for aggressive autonomous operation of robotic helicopters. In these references the path optimization is solved by first producing (off-line) a library of maneuvers. By scheduling these maneuvers on the fly using a maneuver automaton one is able to perform real-time path optimization by numerically pasting together the prerecorded maneuvers from the maneuver library.

The scheme of [11], [12] is promising also for path planning of ground vehicles. For off-road aggressive driving scenarios the maneuver repertoire must be enriched with expert rally racing techniques. Unlike paved road racing or closed-circuit racing of high-performance vehicles (e.g., F1), to date there has been no concrete amount of work correlating driving techniques used by expert rally drivers with mathematical models. In this work, we use empirical information collected from our interaction with expert rally race drivers in order to develop a mathematical and computationally tractable framework that succinctly formalizes this empirical information.

In the following, we first present an empirical description of Trail-Braking (TB) and Pendulum-Turn (PT), two of the most common rally racing cornering maneuvers. A low order vehicle model is introduced that incorporates a sufficient level of fidelity to reproduce modes of operation in accordance to the empirical descriptions of rally maneuvers. Because of this low complexity, the model can be used efficiently within an optimization scheme. We propose such a numerical optimization scheme in order to study several cases of the minimum-time cornering problem. First, a smooth trajectory that maintains small vehicle slip angle is generated as the minimum-time solution in accordance to previous results in the literature [5]. Trail-Braking, which is characterized by higher vehicle slip angles, is generated as a special case of the minimum-time cornering problem. A simple parametrization of the input space of the minimum-time maneuver that applies to both the small slip angle solution and the Trail-Braking is presented, and is compared against the full space of allowable inputs. The Pendulum-Turn maneuver is generated as the minimum-time solution after varying the initial and final conditions of the minimum-time problem according to the empirical description of the maneuver. A simple parametrization of the input space is also presented for the Pendulum-Turn maneuver.

II. EMPIRICAL DESCRIPTION OF TRAIL-BRAKING AND PENDULUM-TURN MANEUVERS

Rally-racing provides an excellent platform for research and development of automotive systems for improving the safety and performance of passenger vehicles. As an example, we mention the development of All-Wheel-Drive (AWD) road vehicles after the successful introduction of the Audi Quattro in rally racing in the 1980's. Despite these technological successes, the techniques and driving style of expert rally race drivers have not yet been fully analyzed using a

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rigorous mathematical framework, at least in a way such that it can help researchers develop control systems which can operate a vehicle during extreme or abnormal driving conditions. This knowledge remains empirical and exclusive to few expert rally race drivers. In this section we present empirical information on Trail-Braking and Pendulum-Turn, two of the most commonly used rally racing maneuvers [13].

Trail-Braking is one of the techniques used by rally drivers to negotiate single corners at high speeds. Typically, the average driver negotiates a corner by first braking to regulate the speed, then releasing the brakes and steering the vehicle along the corner, and finally accelerating after the exit of the corner. Trail-Braking is used when the approach speed to the corner is high and the braking must continue even after the steering of the vehicle has started.

Consider for example a 90 deg left corner as in Fig. 1. Approaching the corner at high speed from the outer edge of the road, Trail-Braking begins by braking the vehicle without steering. The driver adjusts the brake pressure such that the maximum available friction is generated by the tire. This means that the maximum available deceleration is generated; subsequently, no friction is available for steering. As the vehicle approaches the corner, the driver starts steering. In Trail-Braking this is done by progressively releasing the brake (in order to allow cornering forces at the tires) and simultaneously–and progressively–increasing the steering angle. As the vehicle decelerates, the weight of the vehicle transfers from the rear to the front axle and thus, the front tires generate higher friction than the rear ones. The vehicle rotates about the vertical axis, counterclockwise. As the vehicle reaches the apex of the corner and its attitude is aligned with the exit of the corner, the driver accelerates and counter-steers (steers towards the opposite side of the corner) to stop the rotation of the vehicle and start the acceleration towards the exit of the corner. Acceleration causes weight transfer from the front axle to the rear. As a result, the rear tires generate more friction, resisting the counterclockwise rotation of the vehicle. Overall, TB involves high vehicle slip angles and yaw rates. This allows the vehicle to reach a controllable, straight line driving state as quickly as possible, and allow the driver to react to unexpected road condition changes ahead of the corner, which are typical in off-road rally racing.

Using the previous guidelines, we reproduced a TB maneuver for a FWD vehicle on a low μ surface ($\mu = 0.5$) using CarSim, a high fidelity vehicle dynamics simulation software [15]. The commands are manually provided in real time (Fig. 2(a)) and are shown in Fig. 2(b). The resulting trajectory is shown in Fig. 3.

The Pendulum-Turn is another high speed cornering maneuver. It is typically used when the vehicle approaches the corner at high speed coming from the inner edge of the road, for S-turns, or for connecting successive sharp corners. If there is not enough time for the driver to place the vehicle at the outer edge of the road and use TB, the pendulum turn is then appropriate. Consider again a 90deg left corner (Fig. 4). The PT technique starts with the driver releasing the throttle. This action results in load transfer from the rear to the front axle. The driver then steers the vehicle towards the outer edge of the road, opposite to the direction of the

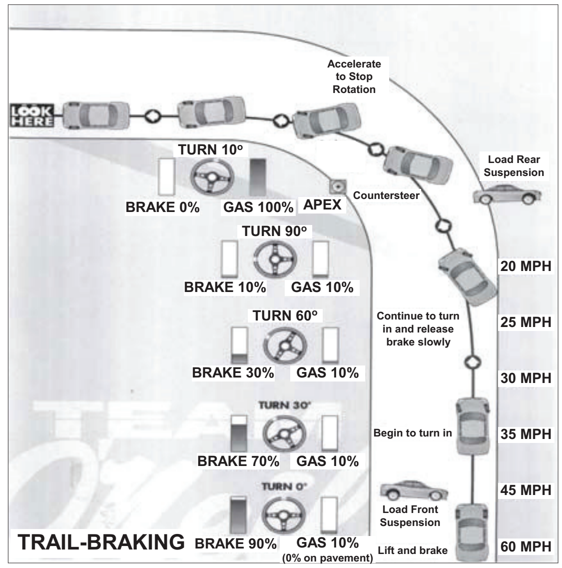
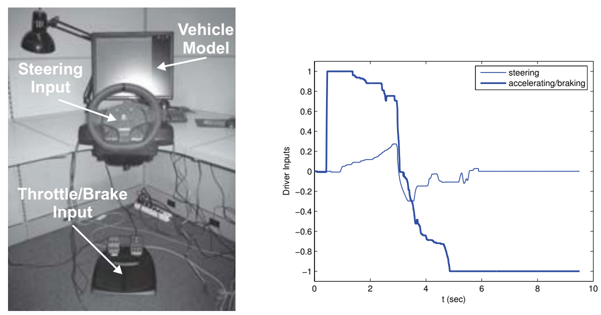


Fig. 1. Empirical description of the Trail-Braking maneuver; from [14].



(a) (b)

Fig. 2. (a) Steering wheel and throttle/brake pedals for real time manual inputs to the CarSim simulation; (b) Steering, throttle and braking commands used for Trail-Braking.

corner. As steering begins, the driver applies the brakes to decrease further the vertical load on the rear axle, which results in a high rate clockwise rotation of the vehicle. As the vehicle builds distance from the inner edge of the road, the driver turns the steering wheel to the left aiming now to the direction of the corner. As the load on the rear wheel has been reduced the change in the direction of the vehicle's rotation from clockwise to counterclockwise is fast (the rear wheels always damp the vehicle's yaw motion). As the vehicle aligns with the exit of the corner the driver releases the brakes and applies the throttle to start accelerating. Throttle and countersteer are used as in the TB case to stop the counterclockwise rotation of the vehicle at the exit of the corner. The attitude of the vehicle at the end of this maneuver can be fine-tuned by throttle and brake commands as necessary: throttle to reduce oversteer and brake to reduce understeer. This fine tuning is achieved with simultaneous application of throttle and brakes using both feet, which is called left-foot-braking (LFB) [14].

Once again, using the previous guidelines, we reproduced a PT maneuver for a FWD vehicle on a $\mu = 0.5$ surface using CarSim. The resulting trajectory is shown in Fig. 5.

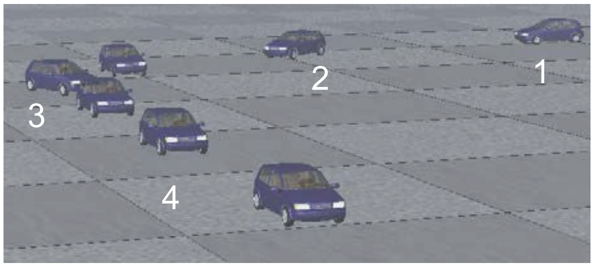


Fig. 3. Trail-Braking maneuver: 1. Hard braking with no steering, 2. Progressive increase of steering command and decrease of brake pressure, 3. Counter-steer and begin acceleration, 4. Hard acceleration.

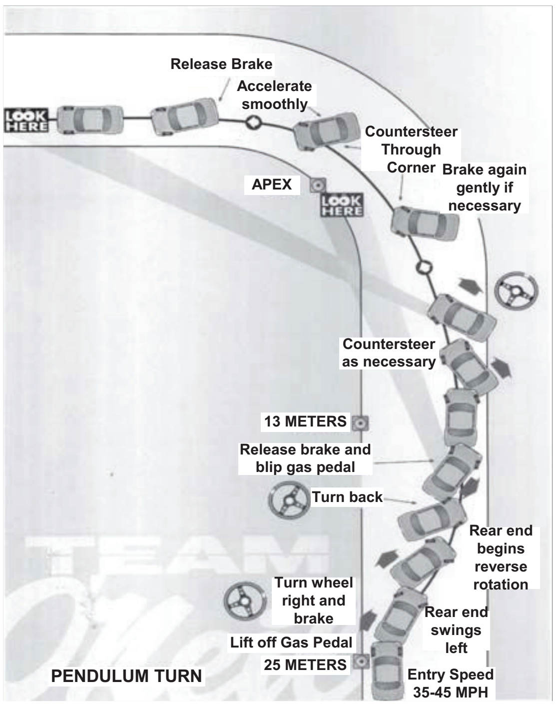


Fig. 4. Empirical description of the Pendulum-Turn technique; from [14].

III. Vehicle Modeling

In this section we introduce a vehicle model suitable for studying maneuvers such as TB and PT. The vehicle model has low order, such that it can be efficiently incorporated in a numerical optimization scheme. Nonetheless, we require that the model is capable of reproducing vehicle responses as those described in Figs. 1 and 4.

Rally drivers routinely take advantage of the load transfer from front to rear axles and vice versa in order to control the yaw motion of the vehicle. As a matter of fact, most of vehicle steering on loose surfaces at high speeds is done by using primarily load transfer as the control input rather than the steering wheel [13]. Since load transfer is so important for vehicle control on loose surfaces we choose a half-car model (Fig. 6) that incorporates longitudinal load transfer as a static map of the longitudinal acceleration. This has the benefit of avoiding the additional equations arising from the suspension dynamics [16].

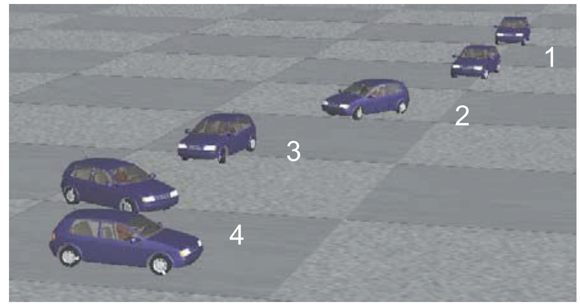


Fig. 5. Pendulum-Turn maneuver: 1. Stop accelerating, 2. Steer towards the opposite of the corner and progressively apply brakes, 3. Release brakes and steer towards the direction of the corner, 4. Accelerate and counter-steer.

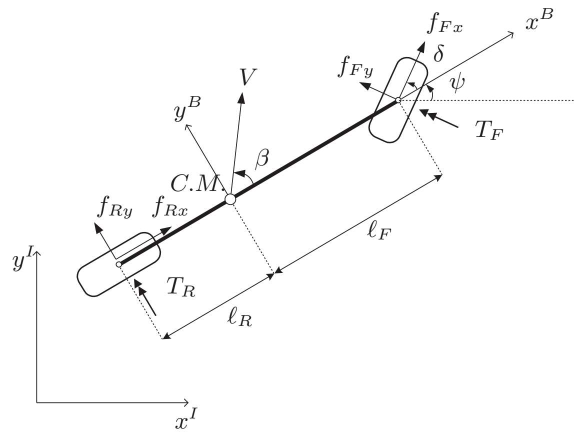


Fig. 6. Half-car vehicle model.

A. Equations of Motion

The equations of motion of the half-car model (Fig. 6) are given as follows.

$$
\begin{aligned}
m\ddot{x} &= f_{Fx}\cos(\psi+\delta) - f_{Fy}\sin(\psi+\delta) \\
&+ f_{Rx}\cos\psi - f_{Ry}\sin\psi \qquad (1)\\
m\ddot{y} &= f_{Fx}\sin(\psi+\delta) + f_{Fy}\cos(\psi+\delta) \\
&+ f_{Rx}\sin\psi + f_{Ry}\cos\psi \qquad (2)\\
I_z\ddot{\psi} &= (f_{Fy}\cos\delta + f_{Fx}\sin\delta)\,\ell_F - f_{Ry}\ell_R \qquad (3)\\
I_i\dot{\omega}_i &= T_i - f_{ix}r\ , \quad i = F, R. \qquad (4)
\end{aligned}
$$

In the above equations m is the vehicle's mass, I_z is the polar moment of inertia of the vehicle, I_i ($i = F, R$) are the moments of inertia of the front and rear wheels about the axis of rotation, r is the radius of each wheel, x and y are the cartesian coordinates of the center of mass in the inertial frame of reference, ψ is the yaw angle of the vehicle, ω_i ($i = F, R$) is the angular rate of the front and rear wheel respectively. By f_{ij} ($i = F, R$ and $j = x, y$) we denote the longitudinal and lateral friction forces at the front and rear wheels, respectively. In this model the inputs are the driving/braking torques T_F and T_R at the front and rear wheels, and δ is the steering angle of the front wheels.

B. Tire Forces

Assuming linear dependence of the friction forces on the normal load at each wheel, one obtains

$$f_{ij} = f_{iz}\mu_{ij}, \quad i = F, R, \; j = x, y, \tag{5}$$

where f_{iz} is the normal load at each of the front and rear axles, and μ_{ij} is the longitudinal and lateral friction coefficients of the front and rear tires. The friction coefficients μ_{ij} can be calculated using, for instance, Pacejka's Magic Formula [17].

In this work we neglect the suspension dynamics so as not to increase the order of the vehicle model. The normal loads at the front and rear axle are given by [16]

$$f_{Fz} = \frac{\ell_R mg - hmg\mu_{Rx}}{L + h\left(\mu_{Fx}\cos\delta - \mu_{Fy}\sin\delta - \mu_{Rx}\right)} \tag{6}$$

$$f_{Rz} = mg - f_{Fz}, \tag{7}$$

where $L = \ell_F + \ell_R$ and h is the vertical distance of the center of mass of the vehicle from the ground.

C. Control Inputs

Neglecting engine, transmission, brake and steering system dynamics, we use the following maps to calculate the inputs, T_F, T_R and δ from non-dimensional command signals $u_T \in [-1, 1]$ (throttle and brake command) and $u_\delta \in [-1, 1]$ (steering command) as follows:

$$\delta = C_\delta u_\delta,$$
$$T_i = \begin{cases} -\mathrm{sign}(\omega_i) C_{\mathrm{ibrk}} u_T & \text{for} \quad u_T \geq 0, \\ -C_{\mathrm{iacc}} u_T & \text{for} \quad u_T < 0, \end{cases} \tag{8}$$

with $i = F, R$. The constants C_δ, C_{Facc}, C_{Racc}, C_{Fbrk} and C_{Rbrk} determine the performance of the steering, engine and brake system. In this work we assume a Front Wheel Drive (FWD) vehicle, hence $C_{\mathrm{Racc}} = 0$.

The vehicle parameters used in this work are shown in Table I. The parameters B, C and D refer to Pacejka's Magic Formula tire friction model [17], under the assumption of uniform tire characteristics along the longitudinal and lateral directions.

TABLE I

Vehicle Parameters.

Parameter	Value
m (kgr)	1450
I_z (kgr m^2)	2740
ℓ_F (m)	1.1
ℓ_R (m)	1.6
r (m)	0.3
$I_{F,R}$ (kgr m^2)	1.8
h (m)	0.4
C_δ (deg)	60
C_{Facc} (Nm)	1000
C_{Racc} (Nm)	0
C_{Fbrk} (Nm)	700
C_{Rbrk} (Nm)	700
B	7
C	1.6
D	0.52

IV. Minimum-Time Cornering

A. Baseline Solution

In this section we calculate the solution to the minimum-time cornering problem along a 90 deg corner, as shown in Fig. 7. The minimum-time cornering problem was addressed in [5], where numerical optimization was used to compare minimum-time versus maximum exit speed cornering. In [5] independence of the T_F and T_R inputs was assumed. In this section we calculate the optimal u_T and u_δ profiles for minimum-time cornering on a low μ surface ($\mu = 0.5$). The optimal control problem is solved numerically using EZOPT, a direct optimization software available by Analytical Mechanics Associates Inc. It uses collocation to transcribe an optimal control problem to a nonlinear programming problem. It provides a gateway to NPSOL, a nonlinear optimization program (for details see [18]). The optimization algorithm involves discretization of the independent variable (time). The control inputs are approximated with constant functions for each time interval. The user is required to provide the continuous system dynamics, the cost to be optimized, state constraints, boundary conditions and an initial guess for the optimal control inputs and states time history.

The index to be minimized is the final time t_f. The road limits constitute the state constraint of the problem. The boundary conditions consist of fixed initial position, orientation and velocity of the vehicle, fixed final position and orientation, and free final speed. In particular, we use

$$x_0 = 18 \text{ m}, \; y_0 = -45 \text{ m}, \; \dot{x}_0 = 70 \text{ km/h}, \; \dot{y}_0 = 0,$$
$$\psi_0 = \pi/2, \; \dot{\psi}_0 = 0, \; \psi_f = \pi, \; \dot{\psi}_f = 0,$$
$$x_f = -45 \text{ m}, \; y_f = 15 \text{ m}, \; \dot{y}_f = 0. \tag{9}$$

The minimum-time trajectory, the optimal velocity profile, the vehicle slip angle β and the optimal control inputs are shown in Figs. 7, 8, 9 and 11, respectively (dashed lines). The minimum-time solution consists of a smooth trajectory that is tangent to the inner boundary of the road and a velocity profile that is decreasing before the apex of the corner and increasing after the apex, in accordance to the results in [5]. The control inputs of Fig. 11 agree with the guidelines of Section II on TB as far as the sequence of control actions is concerned: hard braking ($0 \leq t < 1$), followed by progressive increase of the steering input and simultaneous decrease of the braking input ($1 \leq t < 3.5$), followed by progressive decrease of the steering input, counter-steering and beginning of acceleration ($3.5 \leq t < 7.5$), followed by hard acceleration ($7.5 \leq t \leq t_f$). The vehicle slip angle β remains small throughout the minimum-time trajectory, indicating almost neutral steering.

We can efficiently approximate the minimum-time control inputs by simple mathematical functions (e.g., constants and ramps). To this end, consider the steering and throttle/brake input profiles (u_δ and u_T) of Fig. 10. These input profiles are constructed using constant and linear functions, and are characterized by a set of 18 parameters (t_{si}, c_{si}), $i = 1, 2, 3, 4$ and (t_{bj}, c_{bj}), $j = 1, .., 5$. The number of parameters can be further reduced by such assumptions as $c_{s1} = c_{s4} = 0$ and $c_{b3} = c_{b4}$. Notice that the input profiles of Fig. 10

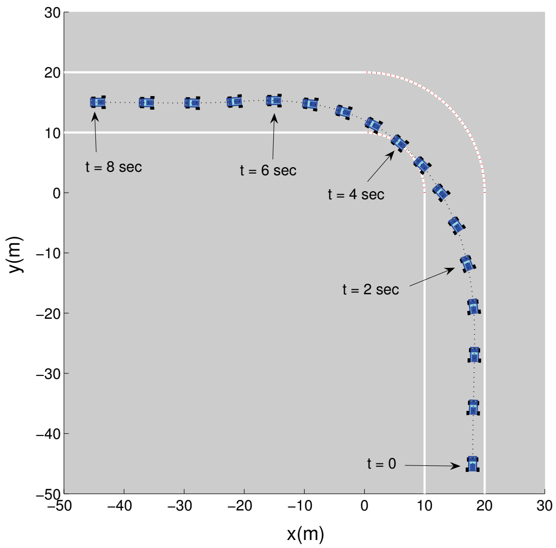


Fig. 7. Baseline minimum-time cornering: vehicle trajectory.

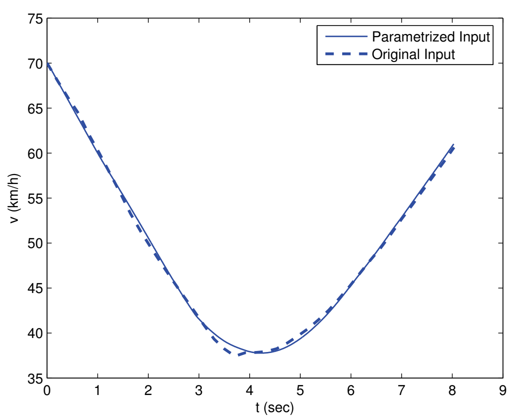


Fig. 8. Baseline minimum-time cornering: velocity profile.

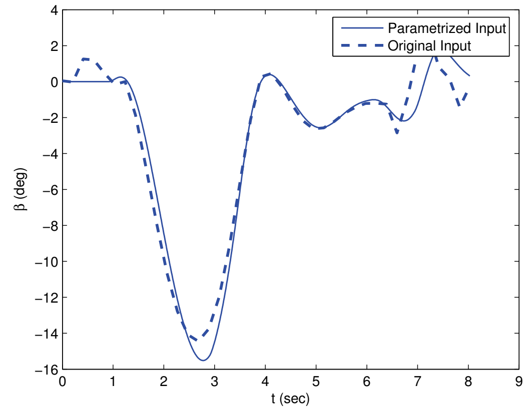


Fig. 9. Baseline minimum-time cornering: vehicle slip angle.

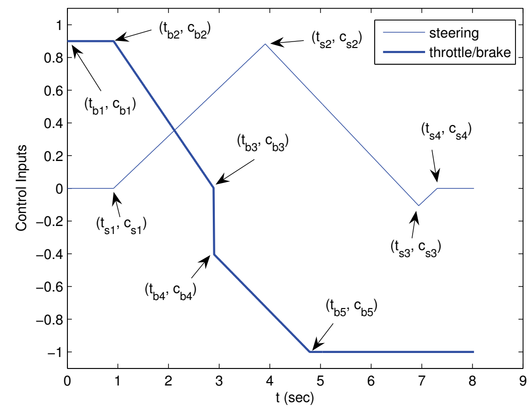


Fig. 10. Steering and throttle/brake command parametrization.

qualitatively agree with the sequence of control actions for the TB maneuver, described in Section II, namely, hard braking, followed by increase of steering and progressive decrease of braking, followed by progressive acceleration with progressive decrease in steering and counter-steering, followed by hard acceleration.

Using the same state constraints and boundary conditions, we resolve the minimum time problem using (t_{si}, c_{si}) and (t_{bj}, c_{bj}) as the free parameters. This time we use the Nelder-Mead minimization algorithm in MATLAB. The minimum-time velocity profile, vehicle slip angle and control inputs are shown in Figs. 8, 9 and 11 (solid lines). The resulting vehicle response using the previous parametrization of the control inputs is essentially the same as the one calculated by optimizing the control inputs at each time step. The calculated minimum time ($t_f = 8.03$ sec) is the same using both methods, within a tolerance of 0.01 sec.

B. Trail-Braking

In this section we consider an alternative optimization scenario than the one presented in Section IV-A, motivated by high-speed rally driving. Unlike road racing and track driving of high-performance (e.g., F1) vehicles, off-road rally-racing involves the additional challenges of an unpredictably changing environment and the lack of detailed information on the condition of the road. Rally drivers aim at bringing their vehicle in a controllable straight line driving state that will allow them to react to emergencies and unexpected changes in the environment, immediately after each corner.

Consider again the 90 deg corner of Section IV-A. We solve the minimum-time problem using the same boundary conditions as before, except for x_f, which is now set to $x_f = -15$ m. That is, we enforce the condition that the vehicle reaches straight line driving ($\dot{y}_f = 0$, $\psi_f = \pi$ and $\dot{\psi}_f = 0$) at 30 m ahead of the previous final state.

The minimum-time trajectory, the optimal control inputs, the optimal velocity profile and the vehicle slip angle are shown in Figs. 12, 13, 14 and 15 respectively (dashed lines). Once again, we observe the same control sequence as in Sections II (on the TB maneuver) and IV-A. This time, however, we also observe high vehicle slip angles, which are necessary for the vehicle to reach the straight line

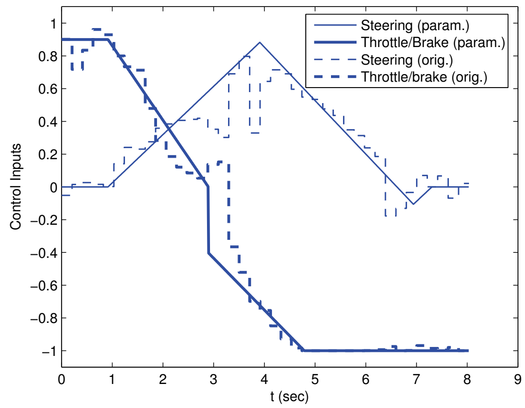


Fig. 11. Baseline minimum-time cornering: control inputs.

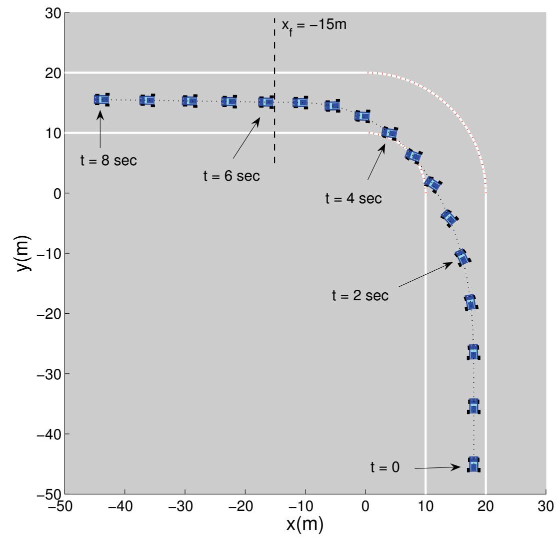


Fig. 12. Trail-Braking: vehicle trajectory.

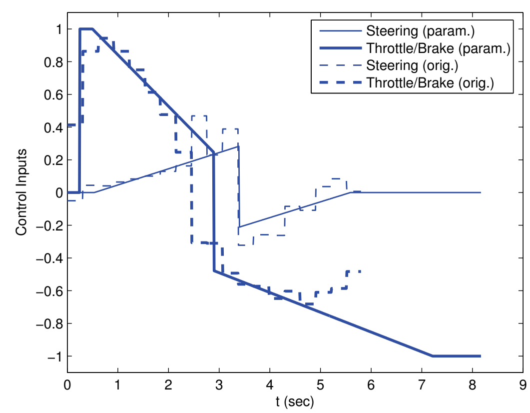


Fig. 13. Trail-Braking: control inputs.

driving condition in such a short distance after the corner. This minimum-time solution fits the description of a TB maneuver. The time till the vehicle reaches the final position (x_f = -15 m) is 5.9 sec.

In the following, we re-solve the minimum time problem using (t_{si}, c_{si}) and (t_{bj}, c_{bj}) as the parameters. The parametrized control inputs, the velocity profile and the vehicle slip angle and normal load of the front and rear axles are shown in Figs. 13, 14 and 15 respectively (solid lines). The normal load of the front and rear axles are shown in Fig. 16.

We observe that the TB maneuver results in a slightly smaller velocity at the $x = 45$ m mark (Fig. 14) than the baseline (small-β) minimum-time maneuver of Section IV-A. The time to reach $x = 45$ m is also higher for the TB case by 0.14 sec. In the TB maneuver however the vehicle yaw dynamics have been completely stabilized by $t = 6.5$ sec, as opposed to the small-β minimum-time maneuver, where the yaw dynamics are stabilized at $t = 8$ sec (Fig. 15).

Comparing the control inputs of the baseline minimum-time cornering and TB cornering (Figs. 11 and 13), we observe that qualitatively are very similar. In fact, the same input parametrization applies to both maneuvers. Quantitatively, however, they defer significantly. We observe that TB involves smaller steering commands and a larger change in u_T at $t = 2.6$ sec, when switching from braking to acceleration occurs. This sudden change in u_T results in a more pronounced normal load transfer as it can be verified in Fig. 16. This is consistent with the discussion at the beginning of Section III.

C. Pendulum-Turn

The Pendulum-Turn is a rally driving technique applied when connecting sharp corners. It is often applied when the vehicle is initially placed on the inside of the road with respect to the corner. In this section we solve the minimum-time problem along the same 90 deg corner as before, replacing the following in (9):

$$x_0 = 12 \text{ m}, \; \dot{x}_0 = 50 \text{ km/h}, \; x_f = 0 \text{ m}. \qquad (10)$$

The PT cannot be performed at the same high initial speed as the TB because it requires an almost immediate steering command, in contrast to the TB which is initiated by hard deceleration before any steering command is applied. For the same reason, PT is not applied for high center of gravity vehicles or on paved roads. We have thus chosen a lower initial speed in (10). The PT maneuver is used to quickly change the orientation of the vehicle. Thus, we require that the vehicle reaches a straight line driving condition immediately after the exit of the corner.

The minimum-time trajectory, the optimal control inputs, the optimal velocity profile and the vehicle slip angle are shown in Figs. 17, 18, 19 and 20, respectively (dashed lines). We observe the same control sequence as in the empirical description of the PT in Section II: reducing acceleration and turning towards the opposite direction of the corner, followed by progressive increase in braking, followed by steering towards the direction of the corner and reducing the brake pressure, followed by acceleration and counter-steer.

Similar to Sections IV-A and IV-B we can efficiently approximate the PT control inputs by simple functions

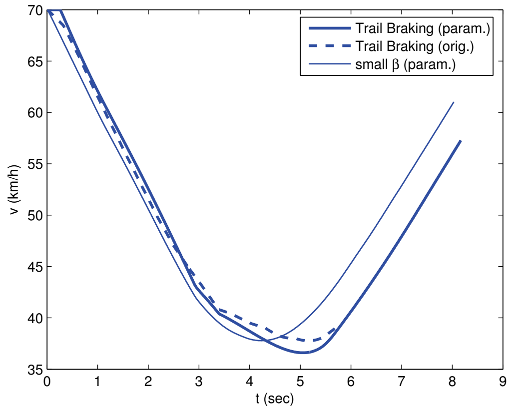


Fig. 14. Trail-Braking: velocity profile.

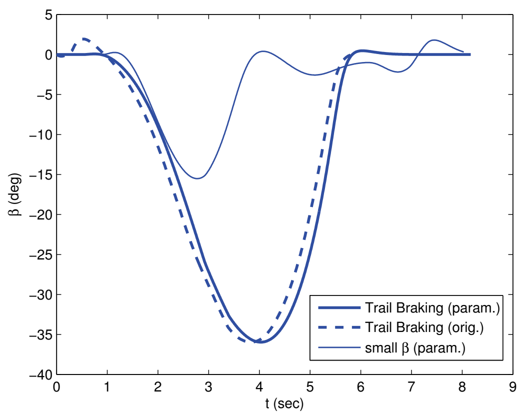


Fig. 15. Trail-Braking: vehicle slip angle.

(constants and ramps) as in Fig. 18 (solid lines). Using the same state constraints and boundary conditions as before, we re-solve the minimum-time problem using this simple parameterization of the control inputs. The optimal velocity profile and the vehicle slip angle are shown in Figs. 19 and 20, respectively (solid lines). We observe that the proposed parameterization of the control inputs generates essentially the same trajectory. The normal load of the front and rear axles are shown in Fig. 21.

V. CONCLUSIONS

In this paper we have initiated a mathematical analysis of rally racing techniques. We have concentrated our efforts on two specific techniques for high speed cornering used extensively by rally drivers, namely Trail-Braking and the Pendulum-Turn. We have introduced a simple dynamical model, which can be efficiently incorporated in a numerical optimization scheme. We have verified this model on the minimum-time cornering problem. When this problem is solved over a short horizon, the solution results in maneuvers reminiscent of Trail-Braking. We have also proposed a parametrization of the control input space that is shown to be appropriate for both standard (close to neutral) steering and Trail-Braking during cornering. Similar results are provided

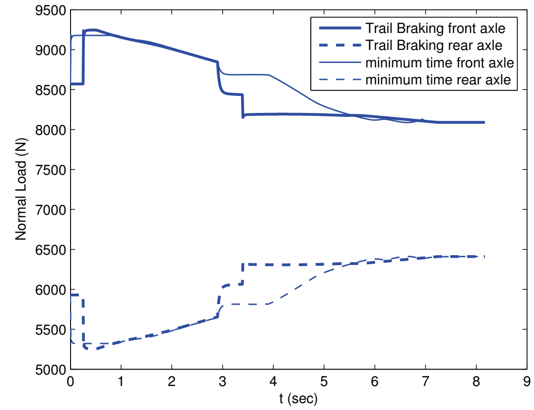


Fig. 16. Normal load on front and rear axles; comparison between Trail-Braking and the baseline minimum-time maneuver.

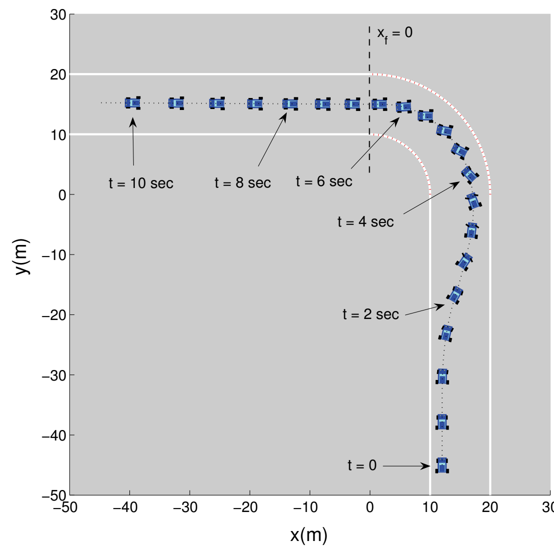


Fig. 17. Pendulum-Turn: vehicle trajectory.

for the case of the Pendulum-Turn, which is a technique used to negotiate very sharp corners or for connecting successive corners at high speed.

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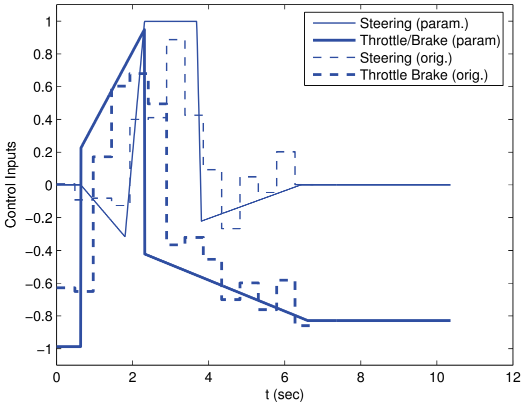

Fig. 18. Pendulum-Turn: control inputs.

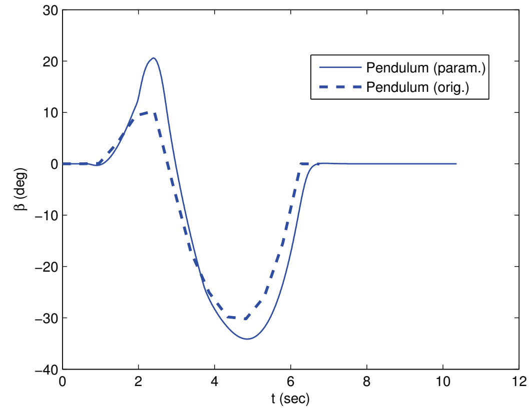

Fig. 20. Pendulum-Turn: vehicle slip angle.

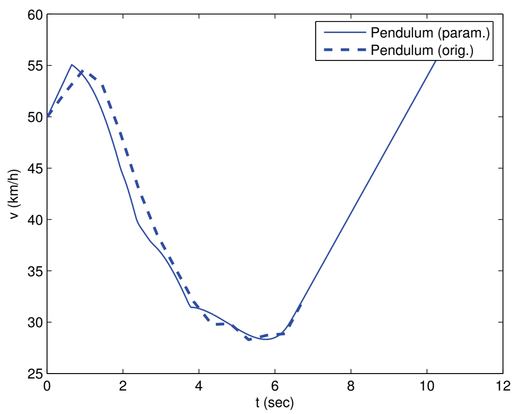

Fig. 19. Pendulum-Turn: velocity profile.

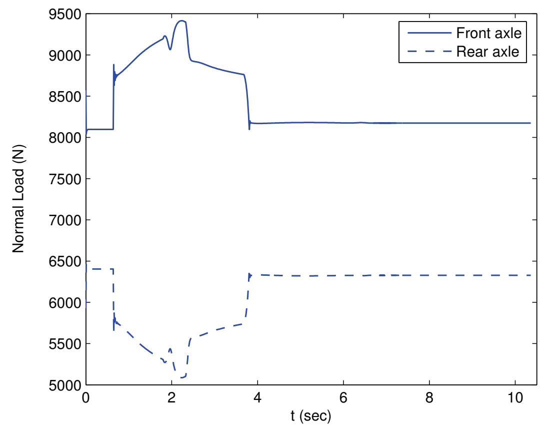

Fig. 21. Pendulum-Turn: normal load.

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