Technical Report

Optimal Vehicle Maneuvers

Not a course in vehicle dynamics, of course!

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*I never procrastinate

1 Friction-limited particle with and without rate-limitation

Investigating the friction-limited particle and the friction-limited particle with rate-limited direction control with a comparison.

The optimization problems follow the following general equation structure:

$$\underset{u}{\text{Min}} \quad J \tag{1.1}$$

$$\text{subject to} \quad f_u(u) <= 0 \tag{1.2}$$

$$f_o(x, u) <= 0 \tag{1.3}$$

$$\dot{x} = f(x, u), \tag{1.4}$$

$$x_0, \ x_f, \tag{1.5}$$

where u is(are) the optimization variable(s), J is the cost function, $f_u(u) <= 0$ and $f_o(x, u) <= 0$ denote the constraints on the u and the states x, $\dot{x} = f(x, u)$ are the ODEs or dynamic constraints, and x_0, & x_f are the boundry conditions.

Opimization variables: the longitudinal and lateral forces u_x and u_y on the vehicle.

Cost function: to minimize time t.

Constraints: The forces on the vehicle are limited elliptically, i.e.,

$$u_x^2 + u_y^2 \leq (\mu m g)^2.$$

The force limit on the vehicle for the rate-limited direction control is given by

$$u_1^2 \leq (\mu m g)^2.$$

The obstacle is given by the following equation:

$$\left(\frac{x - X_a}{R_1}\right)^n + \left(\frac{y}{R_2}\right)^n \geq 1$$

Vehicle model: The friction-limited particle is given as follows:

$$\begin{aligned} \dot{x} &= v_x, \\ \dot{x} &= v_x, \\ m\,\dot{v}_x &= u_x, \\ m\,\dot{v}_x &= u_y. \end{aligned}$$

The friction-limited particle with rate-limited direction control is given as follows:

$$\begin{aligned} \dot{x} &= v_x, \\ \dot{x} &= v_x, \\ m\,\dot{v}_x &= u_1 \cos(\delta), \\ m\,\dot{v}_y &= u_1 \sin(\delta), \\ \dot{\delta} &= u_2. \end{aligned}$$

Miscellaneous constraints: In order to ensure that the solution is within the desired operating space, certain miscellaneous constraints are included such as,

$$\begin{aligned} x_0 &\leq x \leq x_f \\ y_0 &\leq y \leq y_f \\ y_{min} &\leq y \leq y_{max} \\ 0 &\leq v_x \end{aligned}$$

For the rate-limited direction control model, the steering angle and steering rate is also constrained, i.e.,

$$\begin{aligned} |\delta| &\leq \delta_{max}, \\ |\dot{\delta}| &\leq \dot{\delta}_{max}, \end{aligned}$$

The model, optimization, and obstacle parameters are presented in Tables 1.1a, b, and c, respectively.

-	value
m	500 kg
g	9.8 m/s^2
μ	0.8

(a) Model parameters

-	value
x_0	0 m
x_f	100 m
y_0 & y_f	1 m
v_x	40 km/h
v_y	0 km/h

(b) Initial parameters

-	value
X_a	50 m
R_1	2 m
R_2	1.5 m
n	6

(c) Obstacle parameters

Table 1.1: Model, optimization, and obstacle parameters.

The optimal control problem (OCP), was solved with direct multiple-shooting with 40 control intervals for the optimization using Matlab and CasADi. The ODE was solved using the fixed-step Runge-Kutta 4 integration method.

The results of the optimization are presented in

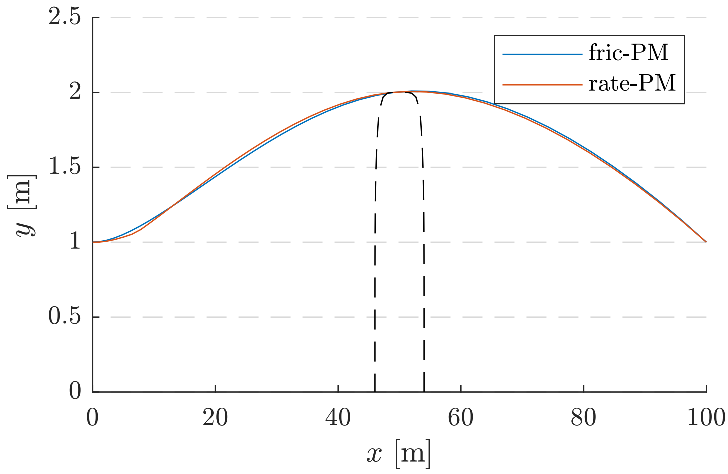

Figure 1.1: Obstacle avoidance trajectory for the friction- and rate-limited particle model.

-	min t	min $-v_f$
t	3.83 s	3.94 s
$v_x(t_f)$	147.59 km/h	146.03 km/h

Table 1.2: Results for friction-limited and rate-limited particle model.

From Table 1.2, it is clear that the friction-limited particle model (fric-PM) is slightly faster than the rate-limited particle model (rate-PM). This is because in rate-PM the rate of change of direction of the particle is limited and as a result, the ability of the vehicle to make a sharp turn is restricted and thus takes a longer time to complete the maneuver. This is visible in the control signals and state variables for the optimal trajectory shown in Figure 1.2.

Additional constraints and initial values for the firc-PM and rate-PM are presented in Table 1.3.

fric-PM		rate-PM	
y_{max}	5	δ_{max}	$\pi/2$
-	-	$\dot{\delta}_{max}$	$\pi/6$

(a) Constraints.

fric-PM		rate-PM	
v_x	40 km/h	v_x	40 km/h

(b) Initial conditions.

Table 1.3: Constraints for the fric-PM and rate-PM.

Some reflections:
The fric-PM can converge faster than the rate-PM in some cases. Since 'ipopt' is used to solve the optimization problem, fric-PM is more sensitive toward the initializations (initial guesses). Therefore, additional constraints may be necessary to improve convergence. Furthermore, the rate-PM does not have this problem and thus can have a lower computational time. However, the computational time can get long with 'bad' initialization conditions. The convergence of this model seems better than the fric-PM.
It is worth mentioning that the terms 'improved convergence' and 'better convergence' mean the ability of the solver to produce an 'optimal solution found' even with ridiculous guesses. However, one should take care not to fall into local minima pits.

1.1 Code

The source codes for this problem can be found at
https://github.com/arvba41/optimal_vehicle_maneuvers.

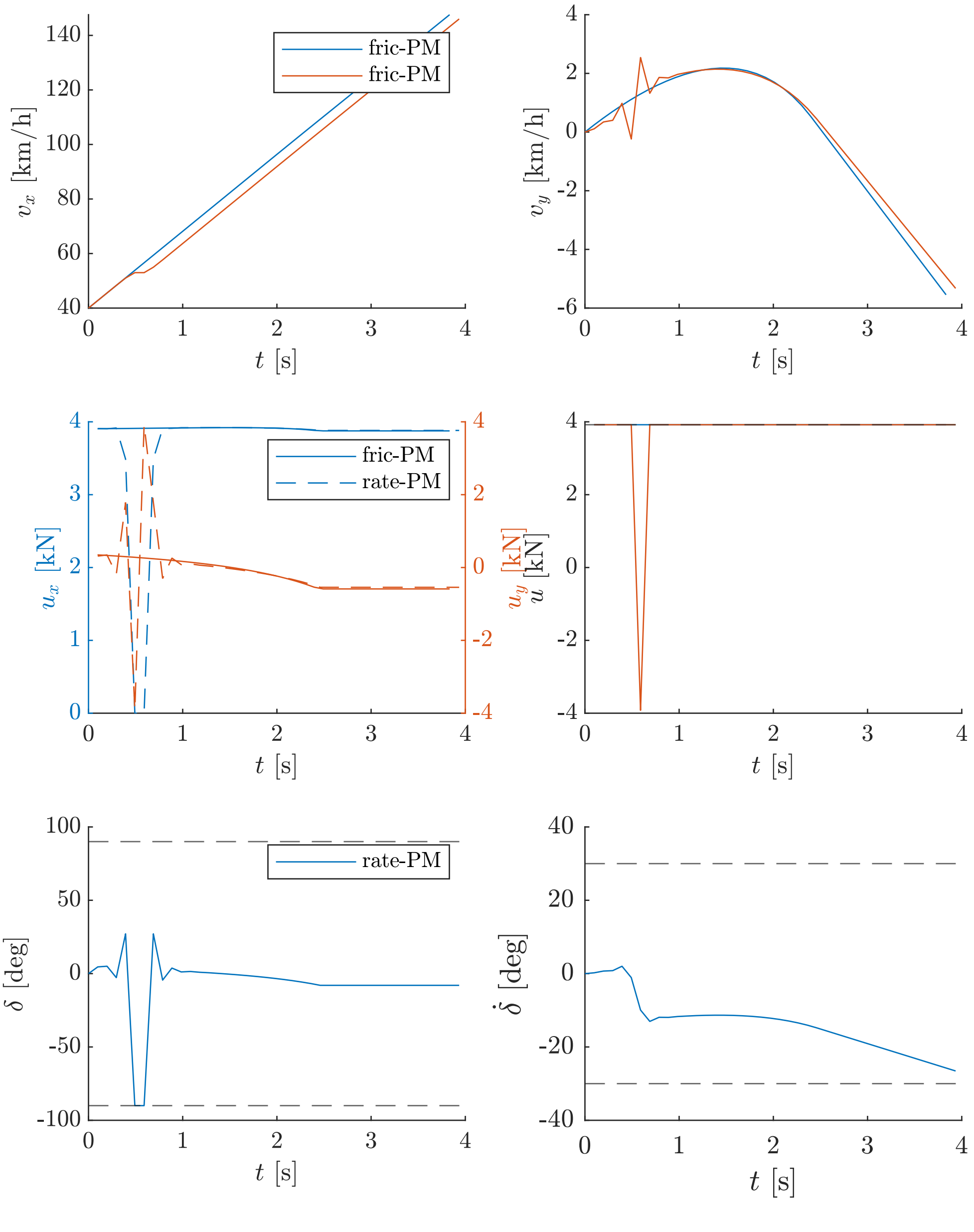

Figure 1.2: Detailed optimal trajectory states and control inputs for friction- and rate-limited particle models.

2 Maneuver with DT for hairpin 4$^{\text{th}}$ degree super ellipse, without LT or WT.

In addition to the double-track (DT) model equations state equations, the input forces are filtered using a low pass filter with a time constant τ_f to prevent NaN errors during the optimization.

$$\dot{F}_{x(f)} = \frac{1}{\tau_f}\left(F^*_{x(f)} - F_{x(f)}\right), \tag{2.1}$$

$$\dot{F}_{x(r)} = \frac{1}{\tau_f}\left(F^*_{x(r)} - F_{x(r)}\right), \tag{2.2}$$

where $F^*_{x(f)}$ and $F^*_{x(r)}$ are the inputs to the model.

The hairpin is modeled as two ellipses,

$$\frac{x}{R_1^i} + \frac{y}{R_2^i} \geq 1, \qquad \frac{x}{R_1^o} + \frac{y}{R_2^o} \leq 1. \tag{2.3}$$

A straightforward model of combined forces is based on the friction ellipses and the Magic Formula. The tire parameters are taken from Berntorp, Karl, et al. "Models and methodology for optimal trajectory generation in safety-critical road–vehicle manoeuvres." Vehicle System Dynamics 52.10 (2014): 1304-1332.

The nominal normal force F_z resting on the respective wheel in the steady state is given by

$$F_{z(1)} = \frac{1}{2}mg\frac{l_r}{l_f+l_r}, \quad F_{z(2)} = \frac{1}{2}mg\frac{l_r}{l_f+l_r}, \quad F_{z(3)} = \frac{1}{2}mg\frac{l_f}{l_f+l_r}, \quad F_{z(4)} = \frac{1}{2}mg\frac{l_f}{l_f+l_r}. \tag{2.4}$$

2.1 Constrains

$v_x > 0$ to avoid $\div$ by zero error while calculating the lateral slips, α.

$-\delta_{max} \leq \delta \leq \delta_{max}$ steering angle limit.

$-\dot{\delta}_{max} \leq \dot{\delta} \leq \dot{\delta}_{max}$ steering angle rate limit.

$-\epsilon\, D_{x(f)} \leq F^*_{x(f)} \leq \epsilon\, D_{x(f)}$, limiting the forces on the front wheel, and ϵ is a number close to 1 to avoid NaN errors.

$-\epsilon\, D_{x(r)} \leq F^*_{x(r)} \leq 0$, limiting the forces on the rear wheel.

$X_f - \beta \leq X_p(\textbf{end}) \leq X_f + \beta$, Allowing some error on the final X_P.

$Y_f - \beta \leq Y_p(\textbf{end}) \leq Y_f + \beta$, Allowing some error on the final Y_P.

The model is front-wheel driven but can brake on all four wheels.

2.2 Cost function

The const function J is defined as

$$J = \min\left(t + 0.5\beta\right), \tag{2.5}$$

The number 0.5 is arbitrarily chosen.

The vehicle parameters and constraints for the the optimal control problem are presented in Table 2.1.

parameter	value
m	2100 kg
l_f	1.3 m
l_r	s1.5 m
w	0.8 m
g	9.82 m/s^2
I_{zz}	3900 kgm^2

(a) Vehicle parameters

parameter	value
R_1^i	2 m
R_2^i	50 m
R_1^o	7 m
R_2^o	55 m

(b) Hairpin parameters

parameter	value
δ_{max}	30°
$\dot{\delta}_{max}$	45°/s
τ_f	0.1 s
ϵ	0.99

(c) Constrains

Table 2.1: The vehicle, hairpin, and constraints for the DT-optimal vehicle Maneuver problem.

The optimal trajectory for the DT model through the hairpin is presented in Figure 2.1.

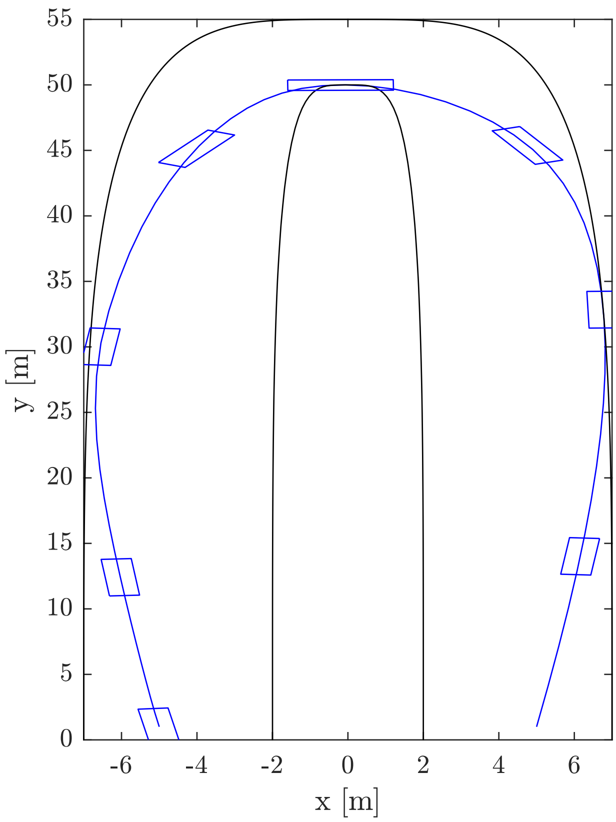

Figure 2.1: Optimal trajectory for a harpin maneuver with minimum time for DT-model.

The states and forces acting on the wheels for the DT model for the hairpin are presented in Figure 2.2.

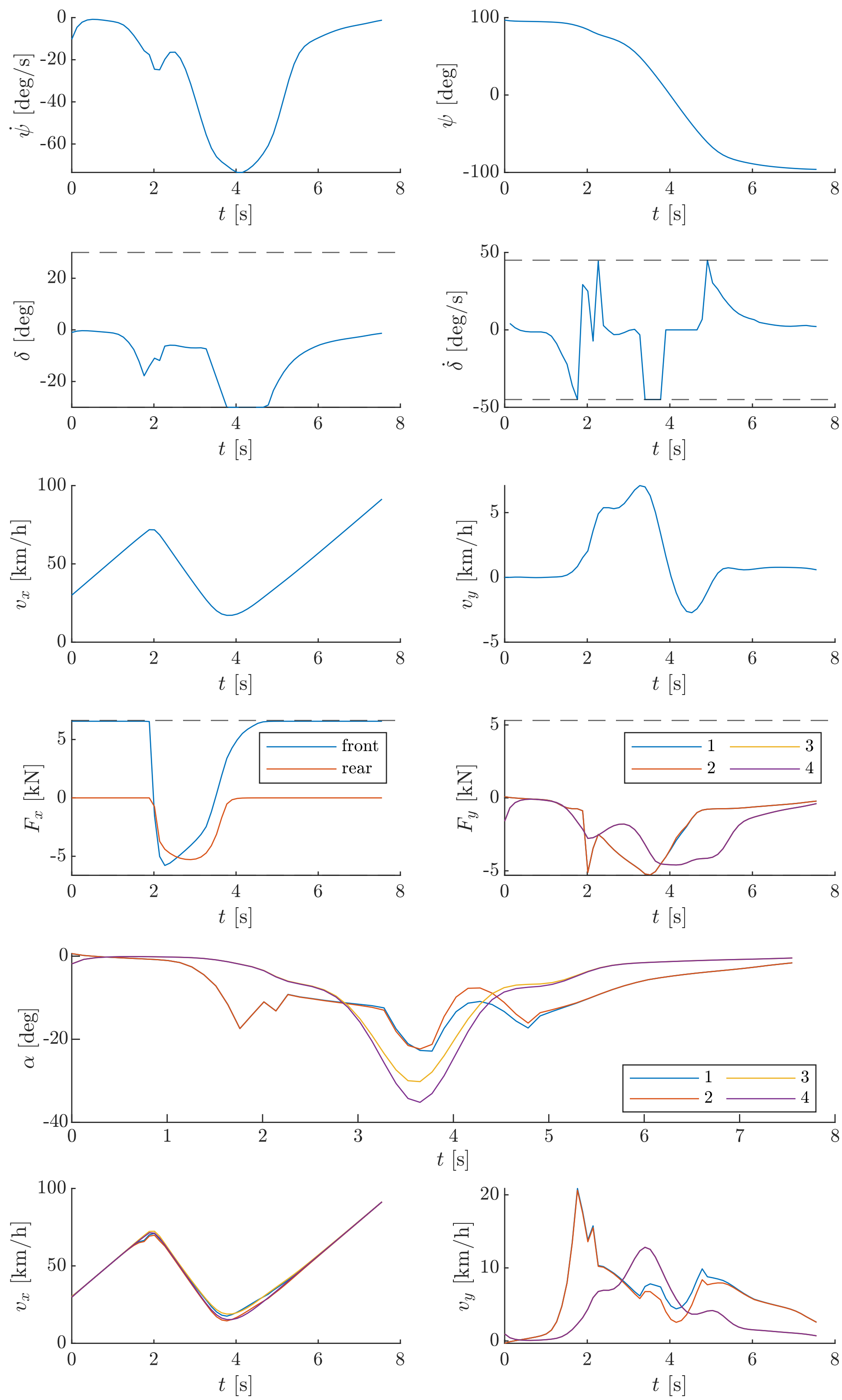

Figure 2.2: velocities, forces, and steering angels during the optimal trajectory for a harpin maneuver with minimum time for DT-model.

Some reflections:
The OCP for the DT model hairpin turn maneuver can have difficulty finding the optimal solution if the initial guesses are non-trivial. Therefore to improve convergence, the height of the hairpin was slowly increased, and the simulation results from the previous iterations were used as initializations for the next.

2.3 Code

The source codes for this problem can be found at
https://github.com/arvba41/optimal_vehicle_maneuvers.

3 Verification of brake or evade criteria

To verify the brake or evade criteria the following OCP was formulated:

$$\underset{u}{\text{Min}} \quad \mu \tag{3.1}$$

$$\text{subject to} \quad f_u(u) <= 0 \tag{3.2}$$

$$\dot{x} = f(x, u), \tag{3.3}$$

$$x_0, \ x_f. \tag{3.4}$$

3.1 Straight-line braking

For straight-line braking, the following constraints are set up:

$$f_u(u): \quad 0 \geq F_x \leq -F_{\max} \qquad F_y = 0, \tag{3.5}$$

$$\dot{x} = f(x, u): \quad \dot{x} = v_x, \qquad \dot{y} = v_y, \qquad \dot{v}_x = \frac{F_x}{m}, \qquad \dot{v}_y = \frac{F_y}{m}, \tag{3.6}$$

$$x_0, \ x_f: \quad x(t_o) = 0, \qquad y(t_o) = 0, \qquad v_x(t_o) = v_o, \qquad v_y(t_o) = 0, \tag{3.7}$$

$$x(t_f) = x_f, \qquad y(t_f) = 0, \qquad v_x(t_f) = 0, \qquad v_y(t_f) = 0, \tag{3.8}$$

where $F_{\max} = \mu m g$. The parameters of the vehicle are presented in Table 3.1a.

The numerical verification for the break or evade criteria for straight-line braking is presented in Table 3.1b.

Parameters	Value
m	2000 kg
g	9.81 m/s^2

(a) Vehicle PM parameters.

	x_f	Analytical μ	Simulation μ
Dry Asphalt	20.3 m	1	1.0043
Wet Asphalt	34 m	0.6	0.5996
Ice Asphalt	68 m	0.3	0.2998

(b) Numerical and analytical solutions for road friction for straight-line braking with $v_0 = 20$ m/s.

Table 3.1: Brake or evade for straight-line braking.

3.1.1 Code

The source code for this problem can be found at
https://github.com/arvba41/optimal_vehicle_maneuvers.

3.2 Evading

This section presents the numerical verification of evading criteria considering a PM.

3.2.1 Wet asphalt maximum obstacle height

To verify the largest obstacle that can be avoided without any braking on wet asphalt,the following OCP is formulated:

$$\begin{aligned}
\underset{u}{\text{Max}} \quad & y(t_f) && (3.9)\\
\text{subject to} \quad & F_x = 0 \qquad 0 \leq F_y \leq F_{\max}, && (3.10)\\
& \dot{x} = v_x, \qquad \dot{y} = v_y, \qquad \dot{v}_x = \frac{F_x}{m}, \qquad \dot{v}_y = \frac{F_y}{m}, && (3.11)\\
& x(t_o) = 0, \qquad y(t_o) = 0, \qquad v_x(t_o) = v_o, \qquad v_y(t_o) = 0, && (3.12)\\
& x(t_f) = x_f, && (3.13)
\end{aligned}$$

where $F_{\max} = \mu m g$, and the vehicle parameters are presented in Table 3.1a.

x_f	v_0	μ	Analytical $y(t_f)$	Simulation $y(t_f)$
34 m	20 m/s	0.6	8.5 m	8.5053 m

Table 3.2: Numerical and analytical solutions for maximum obstacle height that a vehicle can avoid.

Code

The source code for this problem can be found at
https://github.com/arvba41/optimal_vehicle_maneuvers.

3.2.2 Minimum required friction

To verify the minimum required friction to avoid an obstacle with a height of 1.7 m at a distance of 34 m, the following optimization problem was formulated:

$$\begin{aligned}
\underset{u}{\text{Min}} \quad & \mu && (3.14)\\
\text{subject to} \quad & F_x = 0 \qquad 0 \leq F_y \leq F_{\max}, && (3.15)\\
& \dot{x} = v_x, \qquad \dot{y} = v_y, \qquad \dot{v}_x = \frac{F_x}{m}, \qquad \dot{v}_y = \frac{F_y}{m}, && (3.16)\\
& x(t_o) = 0, \qquad y(t_o) = 0, \qquad v_x(t_o) = v_o, \qquad v_y(t_o) = 0, && (3.17)\\
& x(t_f) = x_f, \qquad y(t_f) = y_f, && (3.18)
\end{aligned}$$

where $F_{\max} = \mu m g$, and the vehicle parameters are presented in Table 3.1a.

x_f	y_f	v_0	Analytical $y(t_f)$	Simulation $y(t_f)$
34 m	1.7 m	20 m/s	0.12	0.1199

Table 3.3: Numerical and analytical solutions for the minimum required friction to avoid an obstacle.

Code

The source code for this problem can be found at
https://github.com/arvba41/optimal_vehicle_maneuvers.

3.2.3 Minimum distance to object

To verify the minimum distance to an object with a height of 1.7 m on wet asphalt, the following optimization problem was formulated:

$$\underset{u}{\text{Max}} \quad x(t_f) \tag{3.19}$$

$$\text{subject to} \quad F_x = 0 \qquad 0 \leq F_y \leq F_{\max}, \tag{3.20}$$

$$\dot{x} = v_x, \qquad \dot{y} = v_y, \qquad \dot{v}_x = \frac{F_x}{m}, \qquad \dot{v}_y = \frac{F_y}{m}, \tag{3.21}$$

$$x(t_o) = 0, \qquad y(t_o) = 0, \qquad v_x(t_o) = v_o, \qquad v_y(t_o) = 0, \tag{3.22}$$

$$y(t_f) = y_f, \tag{3.23}$$

where $F_{\max} = \mu m g$, and the vehicle parameters are presented in Table 3.1a.

μ	y_f	v_0	Analytical $x(t_f)$	Simulation $x(t_f)$
0.6	1.7 m	20 m/s	15.2 m	15.2006 m

Table 3.4: Numerical and analytical solutions for the maximum distance to an obstacle that can be avoided.

Code

The source code for this problem can be found at
https://github.com/arvba41/optimal_vehicle_maneuvers.

4 Numerical solution of optimal avoidance for PM

To determine the numerical solution of optimal avoidance for PM, the following OCP is formulated:

$$\min_{F_x,\ F_y} \quad \mu \tag{4.1}$$

$$\text{subject to} \quad F_x \le 0 \qquad 0 \le F_y \quad F_x^2 + F_y^2 \le F_{\max}^2, \tag{4.2}$$

$$\dot{x} = v_x, \qquad \dot{y} = v_y, \qquad \dot{v}_x = \frac{F_x}{m}, \qquad \dot{v}_y = \frac{F_y}{m}, \tag{4.3}$$

$$x(t_o) = 0, \qquad y(t_o) = 0, \qquad v_x(t_o) = v_o, \qquad v_y(t_o) = 0, \tag{4.4}$$

$$x(t_f) = \mathrm{A} + 25, \tag{4.5}$$

$$f_o(x, y) >= 1, \tag{4.6}$$

where

$$f_o(x, y) = \left(\frac{x - \mathrm{A}}{R_1}\right)^n + \left(\frac{y}{R_2}\right)^n \tag{4.7}$$

is the obstacle, $F_{\max} = \mu m g$, and the vehicle parameters are presented in Table 4.1a.

Vehicle	
m	2000 kg
g	9.81 m/s^2
Obstacle	
R_1	5
R_2	3
n	20

(a) Vehicle PM and obstacle parameters.

A	B	Analytical μ	Simulation μ
20.5047 m	2.9012 m	0.5546	0.5583

(b) Numerical and analytical solutions for minimum road friction for optimal obstacle with $v_0 = 20$ m/s.

Table 4.1: Brake or evade for straight-line braking.

4.1 Analytical solution

The analytical solution is determined by first calculating γ,

$$\gamma = \arctan\left(\frac{\mathrm{B}}{\mathrm{A}}\right), \tag{4.8}$$

where B and A are the height and distance of the obstacle and the trajectory intersection point, assuming that the vehicle is starting from (0,0). θ is calculated using the relation

$$\theta = \frac{\gamma + \arcsin\left(3\sin(\gamma)\right)}{2}. \tag{4.9}$$

Now, μ is calculated using

$$\mu = \frac{2\sin(2\gamma)\cos(\theta)}{\cos(\theta - \gamma)^2}\frac{v_o^2}{2g\mathrm{A}}. \tag{4.10}$$

The comparison between the optimal and analytically calculated μ is presented in Table 4.1b.

4.1.1 Code

The source codes for this problem can be found at
https://github.com/arvba41/optimal_vehicle_maneuvers.

The detailed results of the optimization problem are presented in Figure 4.1.

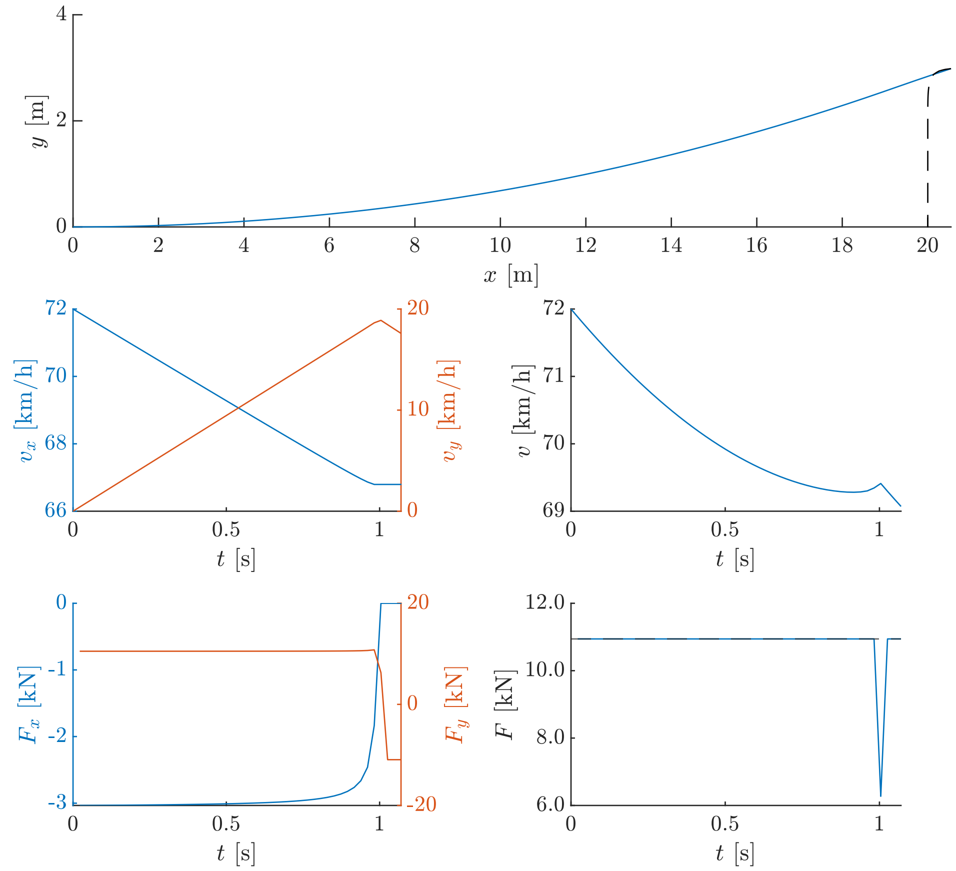

Figure 4.1: Numerical solution for the optimal obstacle avoidance.

5 Avoidance with criterion max $F_{c,y}$ as an exercise with both a scenario-centric and a force-centric perspective

To determine the optimal avoidance for PM, the OCP presented in Chapter 4 is used. The vehicle and the optimization properties are reused from Table 4.1a.

The angle of the obstacle (ψ_v) is the angle between the component of the global force (F_x) and the velocity vector, and is calculated using

$$\psi_v(t) = \frac{dy(t)}{dx(t)}. \tag{5.1}$$

The vehicle centric control force vectors ($F_{c,x}$ and $F_{c,y}$) are calculated using

$$\begin{bmatrix} F_{v,x}(t) \\ F_{v,y}(t) \end{bmatrix} = \begin{bmatrix} \cos(\psi_v(t)) & \sin(\psi_v(t)) \\ -\sin(\psi_v(t)) & \cos(\psi_v(t)) \end{bmatrix} \begin{bmatrix} F_x(t) \\ F_y(t) \end{bmatrix}. \tag{5.2}$$

ψ_v when the vehicle just touches the obstacle at time t_o is calculated using

$$\theta = \psi_v(t_o)|_{f_o(x,y)=1}. \tag{5.3}$$

The scenario centric control force vectors ($F_{c,x}$ and $F_{c,y}$) are calculated using

$$\begin{bmatrix} F_{c,x}(t) \\ F_{c,y}(t) \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} F_x(t) \\ F_y(t) \end{bmatrix}. \tag{5.4}$$

The optimization results for the force and scenario centric are presented in Figure 5.1. From the figure, it is clear that the control force magnitude for the y direction is constant for both the scenario-, vehicle-, and global-centric. However, the x is different for scenarios, which is rather intuitive.

Some reflections:
It is worth mentioning that, in Figures 4.1 and 5.1, the intersection occurs at $t = 1.07$s. However, the vehicle starts to decelerate in the y-direction at around 1 s, when $x = 20$ m. This is probably due to the order of sharpness of the obstacle. when the degree of the super ellipse (obstacle model) is reduced, the vehicle starts to decelerate earlier.

5.0.1 Code

The source codes for this problem can be found at
https://github.com/arvba41/optimal_vehicle_maneuvers.

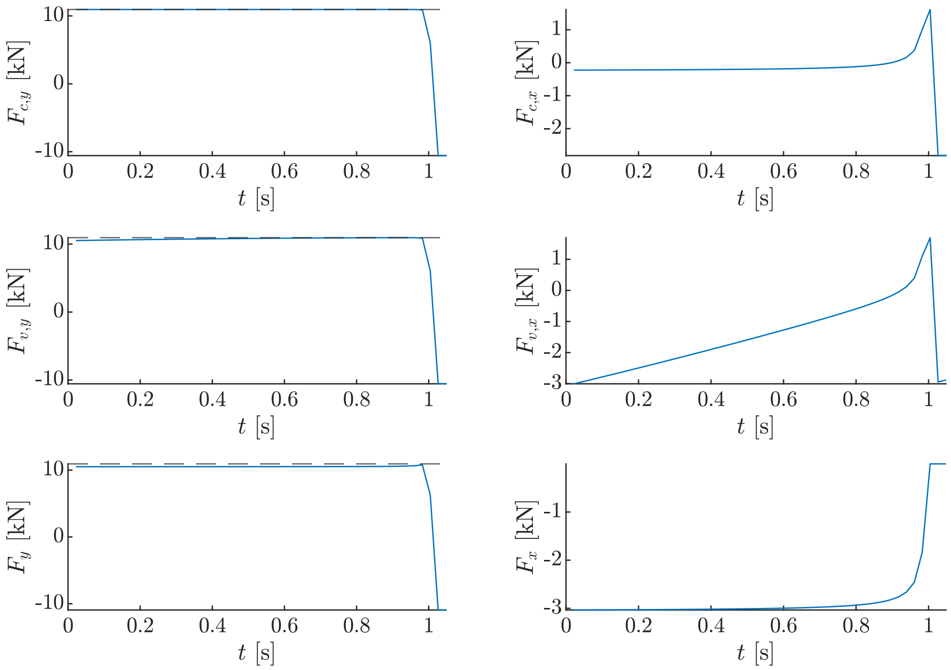

Figure 5.1: Criteria for max $F_{c,y}$ for both a scenario-centric and a force-centric perspective.

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