Simulation of a Permanent Magnet Synchronous Machine

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The aim of the project is to simulate an electric machine and show that the numerical solution gives the same solution as the differential equation.

1 PMSM Model

This section presents the model of a PMSM electric machine.

Instantaneous three-phase voltages (v_a, v_b, and v_c) and currents (i_a, i_b, and i_c) can be written as follows:

$$v_a(t) = \hat{v}_a \cos\left(\omega_e\, t\right), \qquad v_b(t) = \hat{v}_b \cos\left(\omega_e\, t + \frac{2\,\pi}{3}\right), \qquad v_c(t) = \hat{v}_c \cos\left(\omega_e\, t - \frac{2\,\pi}{3}\right),$$
$$i_a(t) = \hat{i}_a \cos\left(\omega_e\, t - \phi\right), \quad i_b(t) = \hat{i}_b \cos\left(\omega_e\, t + \frac{2\,\pi}{3} - \phi\right), \quad i_c(t) = \hat{i}_c \cos\left(\omega_e\, t - \frac{2\,\pi}{3} - \phi\right).$$

Assuming a balanced 3-phase network, i.e:

$$v_a + v_b + v_c = 0, \qquad\qquad i_a + i_b + i_c = 0$$

Transformation to an equivalent two-phase system, i.e, Clarke transformation, is presented as follows:

$$\begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} = \frac{2\,K}{3} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} v_a \\ v_b \\ v_c \end{pmatrix}, \qquad \begin{pmatrix} i_\alpha \\ i_\beta \end{pmatrix} = \frac{2\,K}{3} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} i_a \\ i_b \\ i_c \end{pmatrix},$$

where K is a constant (1, peak-scaling and $\sqrt{3/2}$ for power-scaling).

In-order to simplify the analysis, the reference frame of a three-phase vectors are rotated. i.e:

$$\begin{pmatrix} v_d \\ v_q \end{pmatrix} = \begin{pmatrix} \cos(\omega_e\, t) & -\sin(\omega_e\, t) \\ \sin(\omega_e\, t) & \cos(\omega_e\, t) \end{pmatrix} \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} \qquad \begin{pmatrix} i_d \\ i_q \end{pmatrix} = \begin{pmatrix} \cos(\omega_e\, t) & -\sin(\omega_e\, t) \\ \sin(\omega_e\, t) & \cos(\omega_e\, t) \end{pmatrix} \begin{pmatrix} i_\alpha \\ i_\beta \end{pmatrix}$$

The PMSM can thus be modeled as follows:

$$\begin{aligned} L_d\,\frac{di_d}{dt} &= v_d - R_s\, i_d + L_q\, i_q\, \omega_m, \\ L_q\,\frac{di_q}{dt} &= v_q - R_s\, i_q - \left(L_d\, i_d + \Psi\right)\, \omega_m, \\ J\,\frac{d\omega_m}{dt} &= \frac{3\, n_p}{2\, K^2}\, \left(\left(L_d - L_q\right)\, i_d\, i_q + i_q\, \Psi\right) - T_l, \end{aligned}$$

where ω_m is PMSM rotor rotating magnetic field's angular velocity, R_s is the stater resistance per phase, L_d and L_q are d and q axis inductances, respectively, Ψ is the flux constant of the permanent magnet, n_p is the number of pole pairs, and T_l is the load torque.

The system equations are given below:

$$v_a = \hat{v}_a \, \cos(\omega_e \, t), \qquad \text{(input)}, \tag{1}$$

$$v_b = \hat{v}_b \, \cos(\omega_e \, t + \frac{2\,\pi}{3}), \qquad \text{(input)}, \tag{2}$$

$$v_c = \hat{v}_c \, \cos(\omega_e \, t - \frac{2\,\pi}{3}), \qquad \text{(input)}, \tag{3}$$

$$v_\alpha = \frac{2\,K}{3} \left(v_a - \frac{1}{2}\, v_b - \frac{1}{2}\, v_c \right), \tag{4}$$

$$v_\beta = \frac{2\,K}{3} \left(\frac{\sqrt{3}}{2}\, v_b - \frac{\sqrt{3}}{2}\, v_c \right), \tag{5}$$

$$v_d = \cos(\omega_e \, t) \, v_\alpha - \sin(\omega_e \, t) \, v_\beta, \tag{6}$$

$$v_q = \sin(\omega_e \, t) \, v_\alpha + \cos(\omega_e \, t) \, v_\beta, \tag{7}$$

$$L_d \frac{di_d}{dt} = v_d - R_s \, i_d + L_q \, i_q \, \omega_m, \tag{8}$$

$$L_q \frac{di_q}{dt} = v_q - R_s \, i_q - (L_d \, i_d + \Psi) \, \omega_m, \tag{9}$$

$$J \frac{d\omega_m}{dt} = \frac{3\,n_p}{2\,K^2} \left((L_d - L_q) \; i_d \, i_q + i_q \, \Psi \right) - T_l, \tag{10}$$

$$i_s = i_d + j \, i_q, \qquad \text{(output)}, \tag{11}$$

$$T_e = \frac{3\,n_p}{2\,K^2} \left((L_d - L_q) \; i_d \, i_q + i_q \, \Psi \right), \qquad \text{(output)}, \tag{12}$$

$$N_r = \frac{\omega_m}{n_p} \, \frac{30}{\pi}, \qquad \text{(output)}. \tag{13}$$

The above gives the ODE with three dynamic variables, i_d, i_q, ω_m.

The main aim of the project is to show that the numerical solution gives the same solution as the differential equation. However, we cannot determine the solution analytically. Therefore, we need to try perform the several checks. The following sections cover these in detail.

Two different single step numerical methods are chosen: a fixed-step and a variables step length methods. *Euler forward* method is used as the fixed step solver. The *Dormand-Prince 4(5)* method is employed as the variable step length solver.

The table Table 1 below provides the simulation parameters:

Parameter Name	Parameter symbol	value
Peak input voltage	$\hat{v}_a, \; \hat{v}_b, \; \hat{v}_c$	800 V
Load torque	T_l	0
Motor inertia	J	1.09 kgm^2
Number of pole pairs	n_p	20
Rated speed (Nominal speed)	$N_{r(nom)}$	1000 rpm
Electrical frequency	f_1	166.67 Hz
Stator inductance (direct axis)	L_d	48.35 μH
Stator inductance (quadrature axis)	L_d	48.35 μH
Stator resistance	R_s	0.052
Flux constant	Ψ	0.38
Clarke Constant	K	1

Table 1: Simulation parameters for the PMSM

2 Fixed step solver

2.1 Test Equation

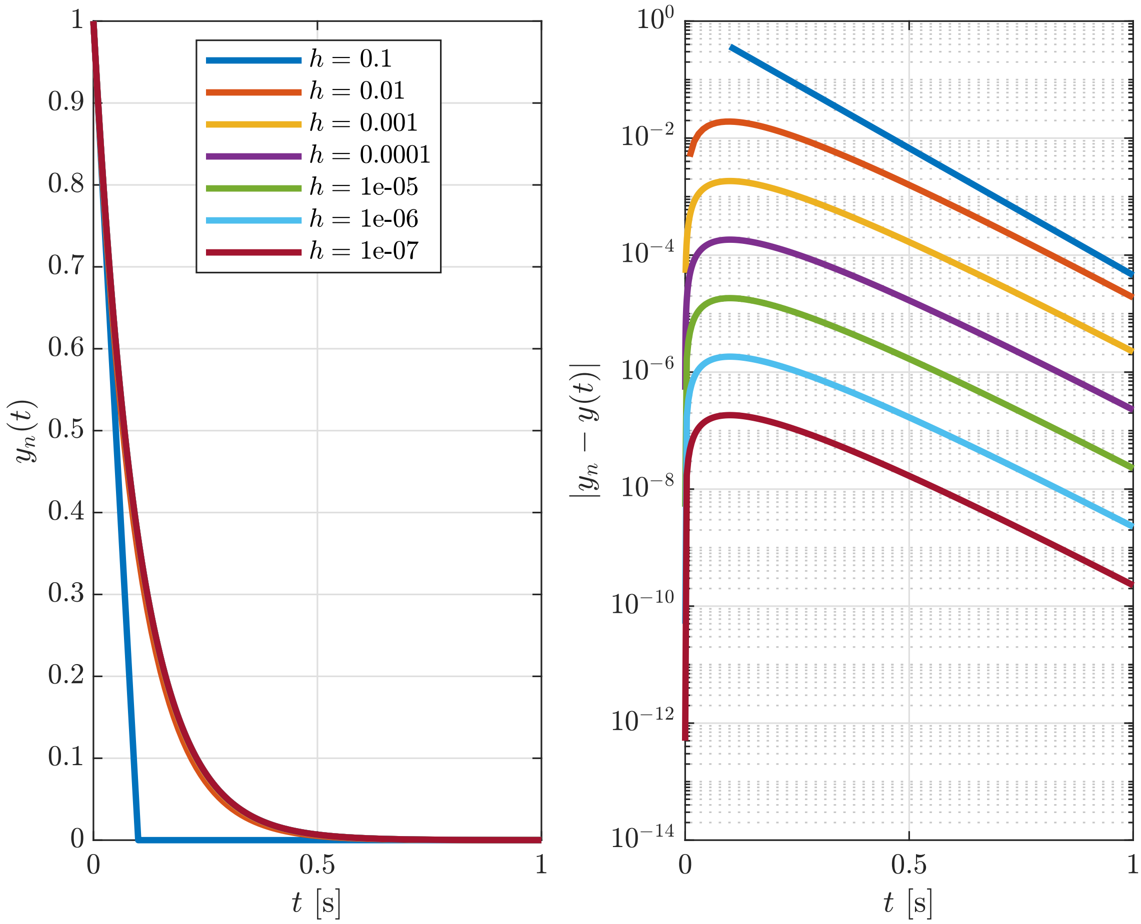

Figure 1: Simulation of the test equation using the Euler forward method with $\lambda = 1$.

The consistency of the Euler forward method is checked by using a test equation. The test equation is given as follows:

Test equation: $f(t, y) = -\lambda\, y, \quad y(0) = 1.$ **Analytical solution:** $y = e^{-\lambda\, t}.$

Using the test equation, the solver's (or numerical method's) *convergence* is checked.

2.1.1 Convergence

The solver arrives at a solution that is close to the exact solution within some pre-specified error tolerance or other convergence criterion. This is the significance of convergence.

By definition, the solver is said to be *convergent of order p* if the global error e_n, satisfies

$$e_n = O(h^p)$$

The simulation results of the test equation are presented in Figure 1. From the figure, the order of the system is calculated and presented in the table below:
From the table and the figure, it is clear that, as $h \to 0$, $|y_n - y(t)| \to 0$. Therefore, according to the fundamental convergence theorem, the solver is *convergent of order p*.

2.2 Motor Equation

Figure 2 presents the simulation results of a PMSM using the *forward Euler* method. The figure shows the states (dynamic variables) i_d, i_q and ω_m as a function of time. From the figure it is clear

h	$\lvert y_n - y(t) \rvert$	p
0.1	1.58e-3	
0.01	1.67e-4	0.95
0.001	1.68e-5	0.99
0.0001	1.68e-6	1
1e-5	1.69e-7	1
1e-6	1.69e-8	1
1e-7	1.69e-9	1

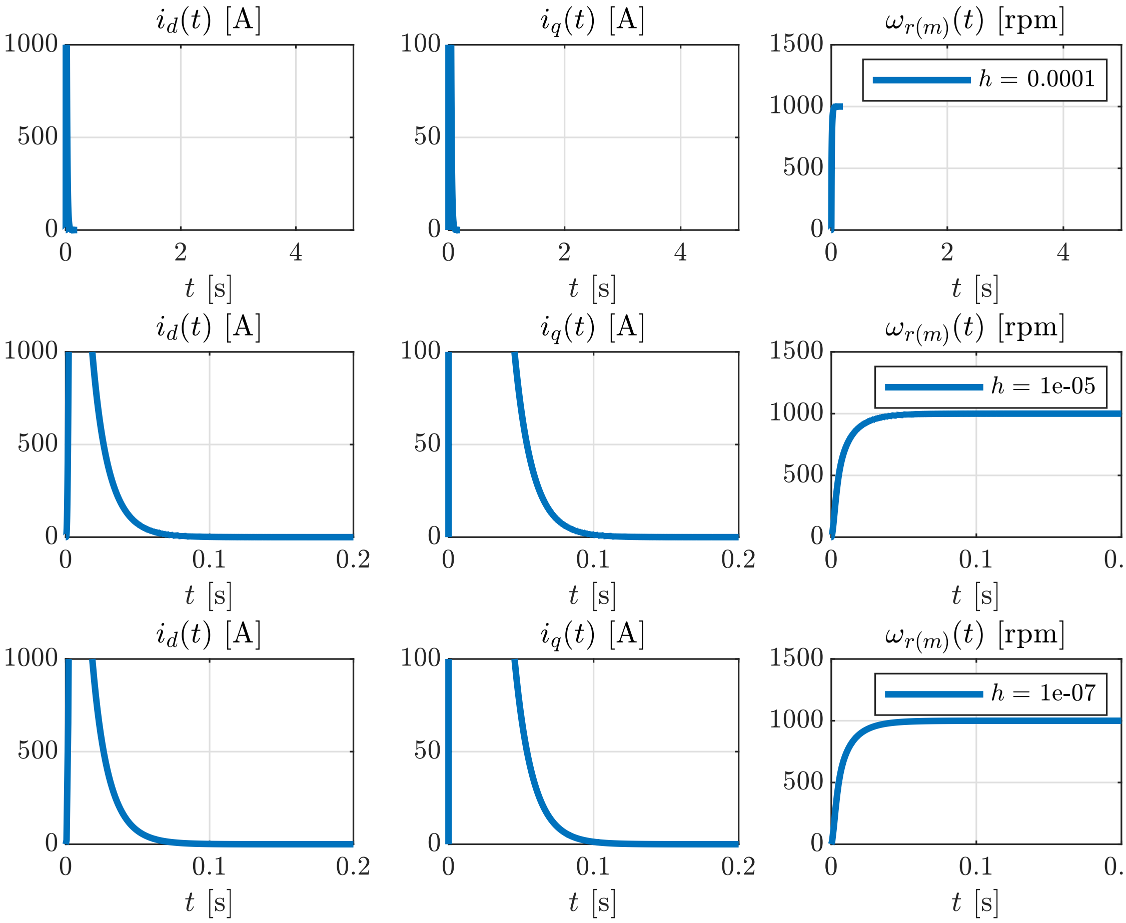

Figure 2: Simulation of the PMSM using Euler forward method with $v_d = v_q = 800\,\mathrm{V}$ at three different step sizes, $h = 0.001$, $h = 1\mathrm{e}-5$, and $h = 1\mathrm{e}-7$.

that as $h \to 0$. all the states converges to a constant value. This implies that the ODE system is stable.

The *convergence* of the numerical method was checked in the previous section. Therefore, it can be concluded that the simulated results with $h \leq 1\mathrm{e}-4$ gives the a solution that is very close to the true solution to the ode system.

3 Variable step solver

Similar to the previous method, the *convergence* of the *Dormand and Prince 4(5)* solver is investigated using a test equation.

3.1 Test Equation

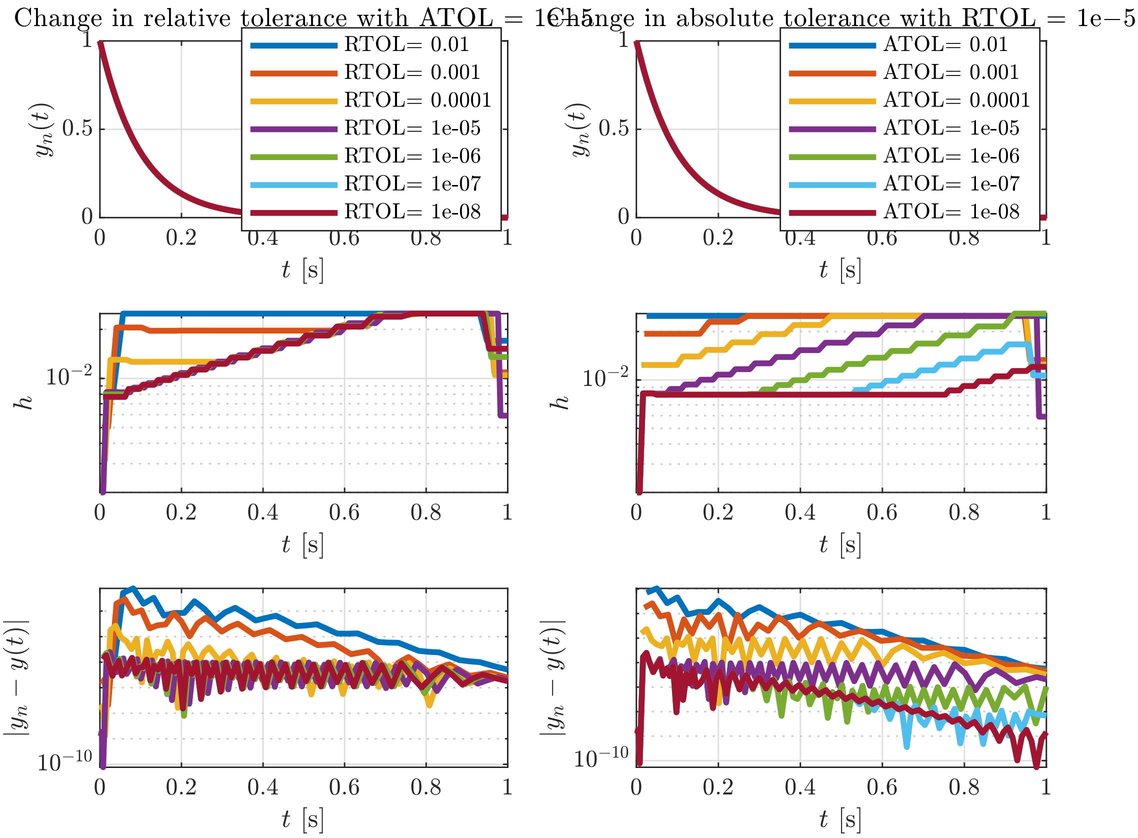

Figure 3: Simulation of the test equation using the *Dormand and Prince 4(5)* method with $\lambda = 1$.

The consistency of the *Dormand-Prince 4(5)* method is checked by using the same test equation that was used for the *Euler forward* method. The test equation is given as follows:

$$\textbf{Test equation: } f(t, y) = -\lambda\, y, \quad y(0) = 1. \qquad \textbf{Analytical solution: } y = e^{-\lambda\, t}.$$

Contrary to the fixed step method, the step length h is varied throughout the simulation depending on the specified relative (RTOL) and absolute (ATOL) tolerances.

The simulation results of the test equation are presented in Figure 3. From the figure, it is clear that by varying only the relative tolerance (RTOL), there is no significant difference in the error signal ($e(t_N) = |y_N - y(t_N)|$) at time $t = t_N$. This is because h at $t = t_N$ for different RTOL are similar. However, varying the absolute tolerance (ATOL), it is clear that as $h \to 0$, $|y_n - y(t)| \to 0$. At time $t = t_{N-1}$ there is a sudden change in h possibly be due to numerical rounding errors. Therefore time t_{N-1} was considered for consistency check. According to the fundamental convergence theorem, the solver is convergent. Furthermore, from the figure it can be concluded RTOL controls the rate of change of h and ATOL controls the absolute value of h

The average step size for convergence in the *Dormand and Prince 4(5)* method have larger h than the *Euler forward* method. At the same time, the error in *Dormand and Prince 4(5)* is lower than the error in the *Euler forward*. This indicates the *Dormand and Prince 4(5)* is a higher order method than the *Euler forward*.

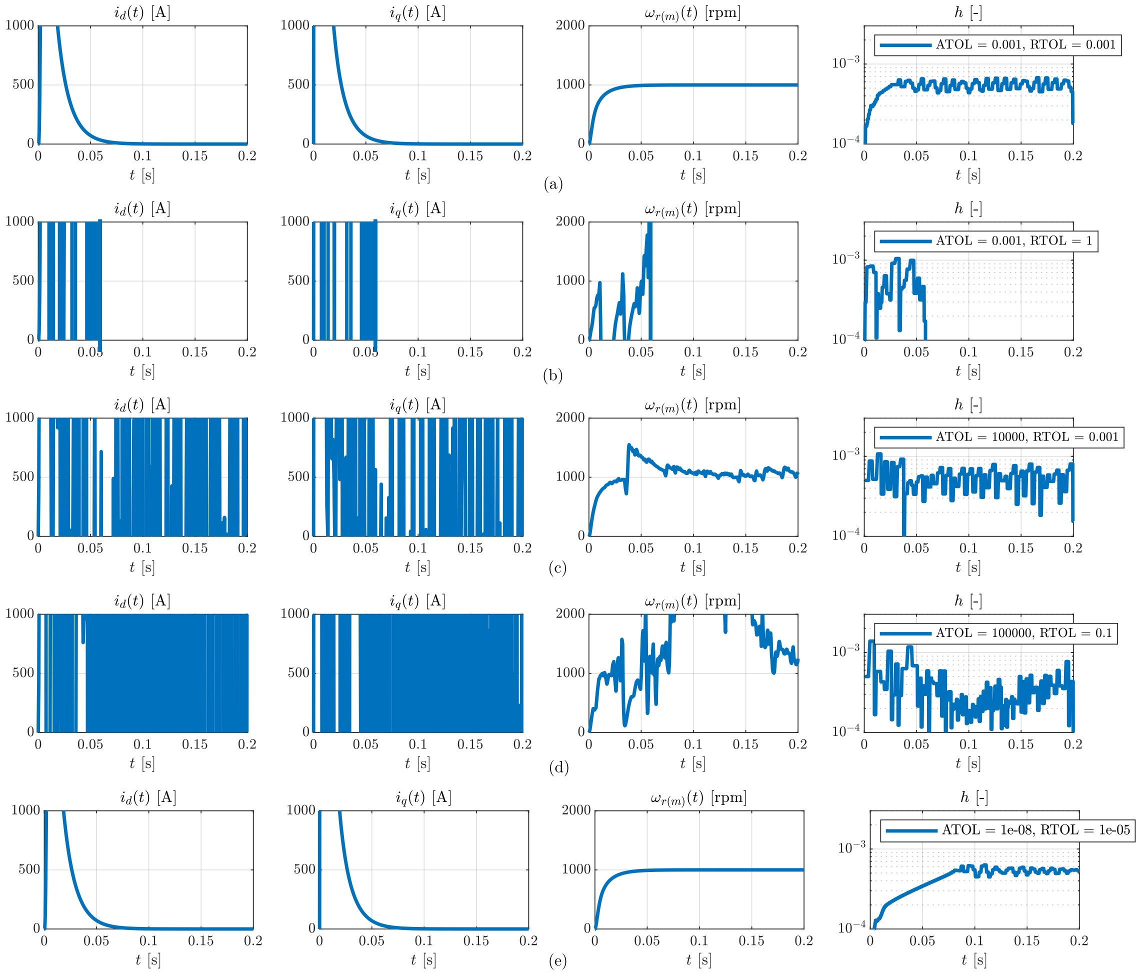

Figure 4: Simulation of the PMSM using Euler forward method with $v_d = v_q = 800\,\mathrm{V}$ at different absolute (ATOL) and relative tolerances (RTOL). (a) ATOL = 0.001 RTOL = 0.001, (b) ATOL = 0.001 RTOL = 1, (c) ATOL = 1e4 RTOL = 0.001, (d) ATOL = 1e5 RTOL = 0.1, and (e) ATOL = 1e-8 RTOL = 1e-5

3.2 Motor equation

Figure 4 presents the simulation results of a PMSM using the *Dormand and Prince 4(5)* method. The figure shows the states (dynamic variables) i_d, i_q and ω_m as a function of time at different ATOL and RTOL. Figure 4(a) presents the simulation results considering ATOL = RTOL = 0.001 and it is clear the dynamic variables i_d, i_q, and ω_m, converge implying the stability of both the ODE and the solver. Furthermore, the average set size is about 0.5e-3 s and a deviation of 0.1e-3 s. Moreover, at $t < 0.5\,\mathrm{s}$ the step size increases gradually from lower than 1e-4 s to about 0.5e-3 s. Figure 4(b) presents simulation results with ATOL = 0.001 and RTOL = 1. From the figure it is clear that the simulation does not converter after $t \approx 0.05\,\mathrm{s}$ and the solutions is ∞. h at t_1 (1^{st} step) is lower than 1e-4. However, just after a few steps, h increases to 0.9e-3 as a consequence, the solver is does not converge. Figure 4(c) and Figure 4(d) presents simulation results with ATOL = 1e4 and RTOL = 0.001, and ATOL = 1e5 and RTOL = 0.1, respectively. From the figure it is clear that in both cases h varies between 1e-4 and 1e-3 s throughout the simulation and as a result, the simulation is although bounded, is not stable, ex: the drastic change in ω_m at $t = 0.035\,\mathrm{s}$ indicates the instability of the solver. Figure 4(e) presents simulation results with ATOL = 1e-8 and RTOL = 1e-5 and from the figure it is clear that the simulation converges. Furthermore, h increases gradually to 0.5e-5 s at $t = 0.75\,\mathrm{s}$ due to very low RTOL.

4 Conclusion

A permanent magnet synchronous machine (PMSM) was modeled and simulated using a fixed-step method and a variable step-method

4.1 Fixed-step method

A simple *Euler forward* numerical method was selected and the convergence of the solver was verified using a test equation. The PMSM was simulated between $0 \leq t \leq 2\,\mathrm{s}$ for different step-lengths h. For $h \leq 1 \times 10^{-4}\,\mathrm{s}$ the solver is convergent. Using the definition of convergence, it is concluded that as $h \to 0$, the numerical solution is close to the true solution of the ODE.

4.2 Variable-step method

A *Dormand-Prince 4(5)* numerical method was selected and the convergence of the solver was verified using a test equation. The PMSM was simulated between $0 \leq t \leq 2\,\mathrm{s}$ for different absolute (ATOL) and relative (RTOL) tolerances. For ATOL ≤ 10 and RTOL ≤ 0.1 the solver is both stable and the solution converges. Using the definition of convergence, it is concluded that as $h \to 0$, i.e reducing ATOL and RTOL $\to 0$, the numerical solution is close to the true solution of the ODE.