1.

$$\begin{array}{c|cc} 1/2 & 1/2 & 0 \\ c & a & 0 \\ \hline & 1/2 & 1/2 \end{array} \qquad \begin{array}{c|c} \bar{c} & A \\ \hline & \bar{b}^T \end{array}$$

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$c = a \ldots ?$

a) Write down all formulas for computation

$$hY_1' = hf(t_n + 1/2 h, Y_1)$$
$$hY_2' = hf(t_n + ch, Y_2)$$

Given below

$$Y_1 = Y_n + 1/2\, hY_1'$$

$$Y_2 = Y_n + 1/2\, hY_1' + ahY_2'$$

This yields next step

$$Y_{n+1} = Y_n + \frac{1}{2}hY_1' + \frac{1}{2}hY_2'$$

b) Determine parameters for A-stability
Applying linear test function $y' = \lambda y = f(y)$

$$hY_1' = h\lambda Y_1 = h\lambda(Y_n + 1/2\, hY_1') \Leftrightarrow hY_1' - h^2\lambda \frac{1}{2} Y_1' = h\lambda Y_n$$

$$hY_1'(1 - \tfrac{1}{2}h\lambda) = h\lambda Y_n \overset{h\lambda \neq 2}{\Longleftrightarrow} hY_1' = \frac{h\lambda}{1 - \frac{1}{2}h\lambda} Y_n$$

$$hY_2' = h\lambda Y_2 = h\lambda(Y_n + \tfrac{1}{2}hY_1' + ahY_2')$$

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$$hY_2' = h\lambda\left(Y_n + \frac{1}{2}\frac{h\lambda}{1-\frac{1}{2}h\lambda}Y_n + ahY_2'\right)$$

$$hY_2' - h^2\lambda a Y_2' = h\lambda\left(1 + \frac{1}{2}\frac{h\lambda}{1-\frac{1}{2}h\lambda}\right)Y_n$$

$$hY_2'(1-h\lambda a) = h\lambda\left(1+\frac{1}{2}\frac{h\lambda}{1-\frac{1}{2}h\lambda}\right)Y_n$$

$$hY_2' = \frac{h\lambda\left(1+\frac{1}{2}\frac{h\lambda}{1-\frac{1}{2}h\lambda}\right)Y_n}{1-h\lambda a}$$

We have

$$Y_{n+1} = Y_n + \frac{1}{2}hY_1' + \frac{1}{2}hY_2'$$

$$= Y_n + \left(\frac{1}{2}\frac{h\lambda}{1-\frac{1}{2}h\lambda} + \frac{1}{2}\frac{h\lambda\left(1+\frac{1}{2}\frac{h\lambda}{1-\frac{1}{2}h\lambda}\right)}{1-h\lambda a}\right)Y_n$$

stability function $R(h\lambda) = 1 + \frac{1}{2}h\lambda\left(\frac{1}{1-\frac{1}{2}h\lambda} + \frac{1+\frac{h\lambda}{2-h\lambda}}{1-h\lambda a}\right)$

Poles: $h\lambda = 2$, $h\lambda = \frac{1}{a}$. $h\lambda = \frac{1}{2}$

For A-stability all poles in $\mathbb{C}^+$ $\Rightarrow$ $a > 0$,
and $|R(i\omega)| \leq 1 \;\forall\; \omega \in \mathbb{R}$.

$$R(z) = 1 + \frac{1}{2}\frac{z}{1-\frac{1}{2}z} + \frac{1}{2}\frac{z\left(1+\frac{1}{2}\frac{z}{1-\frac{1}{2}z}\right)}{1-az}$$

A.G.

$$1+\frac{1}{2}\frac{Z}{1-\frac{1}{2}Z}+\frac{1}{2}\frac{Z\left(1+\frac{1}{2}\frac{Z}{1-\frac{1}{2}Z}\right)}{1-aZ}$$

$$=1+\frac{Z}{1-Z}+\frac{Z+Z^2\left(\frac{1}{2-Z}\right)}{2-2aZ}$$

$$\left|1+\frac{Z}{1-Z}+\frac{Z+Z^2\left(\frac{1}{2-Z}\right)}{2-2aZ}\right|$$

$$\left|1+\frac{iw}{1-iw}+\frac{iw-w^2\left(\frac{1}{2-iw}\right)}{2-i2aw}\right|\leq 1$$

$$\left|1+\frac{iw(1+iw)}{1+w^2}+\frac{iw-w^2\left(\frac{2+iw}{4+w^2}\right)(2+iaw)}{4+4a^2w^2}\right|\leq 1$$

$$\left|1+\frac{iw}{1+w^2}-\frac{w^2}{1+w^2}+\frac{iw-\frac{w^2}{4+w^2}\left(4+2iaw+2iw-aw^2\right)}{4+4a^2w^2}\right|\leq 1$$

$$\left|1+\frac{iw}{1+w^2}-\frac{w^2}{1+w^2}+\frac{iw-\frac{i2aw^3}{4+w^2}-\frac{i2w^3}{4+w^2}+\frac{aw^4}{4+w^2}-\frac{4w^2}{4+w^2}}{4+4a^2w^2}\right|\leq 1$$

$$\left|1-\frac{w^2}{1+w^2}+\frac{aw^4}{(4+w^2)(4+4a^2w^2)}-\frac{4w^2}{(4+w^2)(4+4a^2w^2)}+i\left(\frac{w}{1+w^2}+\frac{w}{4+4a^2w^2}-\frac{2w^3(a-1)}{(4+4a^2w^2)(4+w^2)}\right)\right|$$

This is not turning out very nice so I have probably miscalculated somewhere, but from the inequality I draw the conclusion that this is not smaller than 1 for all w, independent of selection of a. It is therefore not A-stable.

2. AG.

$$Y_{n+2} - Y_{n+1} = h\left(\beta_2 f(Y_{n+2}) + \beta_1 f(Y_{n+1}) + \beta_0 f(Y_n)\right)$$

a) Determine the parameters $\beta_0\, \beta_1\, \beta_2$ so that it is of maximal order.

Applying polynomial $Y = t^m$, $Y' = f(Y) = m t^{m-1}$ at $t=0$. Yields:

$m=0$ $Y=1$ $Y'=0$ LHS $=0$, RHS $= 0$ ☺

$m=1$ $Y=t$ $Y'=1$ LHS $= h$, RHS $= h(\beta_2+\beta_1+\beta_0)$

$m=2$ $Y=t^2$ $Y'=2t$ LHS $= 4h^2-h^2$, RHS $= h(\beta_2 \cdot 2 \cdot 2h + \beta_1 \cdot 2 \cdot h)$

$m=3$ $Y=t^3$ $Y'=3t^2$

$$\text{LHS} \overset{?}{=} (2h)^3 - h^3 = 7h^3$$

$$\text{RHS} = h(\beta_2 \cdot 3(2h)^2 + \beta_1 \cdot 3 \cdot h^2) = h^3(12\beta_2 + 3\beta_1)$$

These three equations yield $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & 3 \\ 12 & 3 & 0 & 7 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & 3 \\ 0 & -1 & 0 & -\frac{2}{3} \end{array}\right)$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 0 & 0 & 5/3 \\ 0 & 1 & 0 & 2/3 \end{array}\right) \Rightarrow \begin{cases} \beta_0 = \frac{5}{12} \\ \beta_1 = \frac{8}{12} \\ \beta_2 = -\frac{1}{12} \end{cases}$$

For $m=4$:

$$Y = t^4 \quad Y' = 4t^3 \quad \text{LHS} = (2h)^4 - h^4 \overset{?}{=} h\left(-\frac{1}{12} \cdot 4 \cdot (2h)^3 + \beta_1 \cdot 4 \cdot h^3\right) = \text{RHS}$$

$$\Rightarrow \text{LHS} = 15h^4 \quad \text{RHS} = \left(-\frac{1}{12} \cdot 4 \cdot 8 + 4\right) h^4$$

$$= \left(-\frac{32}{12} + \frac{48}{12}\right) h^4 = \underline{\frac{16}{12} h^4 \neq \text{LHS}}$$

<u>Order 3!</u>

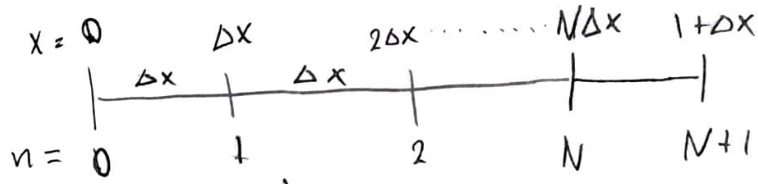
b) Since it is of order 3, Dahlquist's second barrier theorems says it cannot be A-stable since it has an order higher than 2. ($3 > 2$)

But it can be useful despite lacking A-stability ☺.

3.

$$y'' + \alpha y' + f(y) = 0 \qquad y(0) = \beta \quad y'(1) = \gamma$$

Introducing grid:



$$1 = N\Delta x \qquad \Delta x = \frac{1}{N}$$

Since we have no explicit value for $x = 1$ this has to be handled. The "fictional" gridpoint at $N+1$ gives this as with the derivative approximation for $x = 1$ gives $\frac{u_{N+1} - u_{N-1}}{2\Delta x} = \gamma \iff u_{N+1} = u_{N-1} + \gamma \cdot 2\Delta x$.

$$y'' \approx \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2}$$

$$\alpha y' \approx \alpha \frac{u_{n+1} - u_{n-1}}{2\Delta x}$$

$$\begin{cases} \dfrac{u_2 - 2u_1 + \beta}{\Delta x^2} + \alpha \dfrac{u_2 - \beta}{2\Delta x} + f(u_1) = 0 & n = 1 \\[2ex] \dfrac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2} + \alpha \dfrac{u_{n+1} - u_{n-1}}{2\Delta x} + f(u_n) = 0 & 1 < n < N \\[2ex] \dfrac{2\gamma\Delta x - 2u_N + 2u_{N-1}}{\Delta x^2} + \alpha\gamma + f(u_N) = 0 & n = N \end{cases}$$

$$F = \left\{ \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2} + \alpha \frac{u_{n+1} - u_{n-1}}{2\Delta x} + f(u_n) \right\}_{n=1}^{N} = 0$$

where $\begin{cases} u_0 = \beta \\ u_{N+1} = u_{N-1} + 2\Delta x\gamma \end{cases}$

A. 6

b)

Jacobian $\{F'_u\}$.

Last raw : $\overbrace{\left[0 \ldots \ldots 0 ; \frac{2}{\Delta x^2} ; \underbrace{\frac{-2}{\Delta x^2} + \frac{\partial f(u_N)}{\partial u_N}} \right]}^{\text{Length } N}$

$n = 1, \ldots N-2, N-1, N$

Answer

4. A.G.

$$\dot{u}(t,x) = \Delta u(t,x) \qquad u(0,x) = g(x) \qquad x \in [0,1]$$

Semidiscretization:

$$\dot{U}(t) = T_{\Delta x} U(t)$$

Show that solutions to both equations depend continuously on the data. For any $T \geq 0$ $\exists$ $C(T), D(T)$ s.t $\forall$ $t \in [0,T]$

$$\|u(t,\cdot)\| \leq C(T)\|u(0,\cdot)\|, \quad \|V(T)\| \leq D(T)\|V(0)\|$$

For the matrix-vector case we have

$$\frac{dU}{dt} = T_{\Delta x} U \iff \frac{d}{dt}\|U\| \leq \mu[T_{\Delta x}]\|U\|$$

This gives that the norm of the solution $\|u\|$ obides

$$\overbrace{\frac{d}{dt}\|U\| \leq \mu[T_{\Delta x}]\|U\|}^{\text{ODE.}} \iff \|U\| \leq e^{\mu[T_{\Delta x}]t} \cdot K$$

$$\|V(0)\| = \|g(x)\| \iff K = \|g(x)\|$$

We thus have

$$\|U\| \leq e^{\mu[T_{\Delta x}]t} \|g(x)\| .$$

with Sobolev's Lemma twice we have that

$$\mu[T_{\Delta x}] \approx \mu[\tfrac{d^2}{dx^2}] = \sup_{x \neq 0} \frac{Re\langle u, \frac{d^2}{dx^2} u\rangle}{\|u\|} = \sup_{x \neq 0} Re \frac{-\langle u', u'\rangle}{\|u\|} = \frac{-\|u'\|_2^2}{\|u\|_2^2}$$

As $\|u'\|_2^2 \geq \pi^2\|u\|_2^2$ we have $\mu[T_{\Delta x}] \leq -\pi^2$

$$-\|u'\|_2^2 \leq -\pi^2\|u\|_2^2$$

With a negative logarithmic norm we have

A. G.

$$\|U\| \le e^{\mu[T_{\Delta x}]t} \|g(x)\| = e^{-\pi^2 t} \underbrace{\|g(x)\|}_{=\|U(0)\|}$$

This means that the norm of the solution is strictly decreasing with time, as would be expected with heat diffusion. The norm of the solution $\|U(t)\|$ is therefore always bounded by

$$\|U(t)\| \le \text{constant} \|U(0)\| \qquad (\max\{e^{-\pi^2 t}\} = 1).$$

We may also use the eigenvalues of the operator as $T_{\Delta x} U = \lambda[T_{\Delta x}]$ gives

$$\dot{U}(t) = T_{\Delta x} U$$

$$\dot{U}(t) = \lambda[T_{\Delta x}] U \iff U = U(0) e^{\lambda[T_{\Delta x}]}$$

As with the given formula for eigenvalues of $T_{\Delta x}$ the eigenvalues are $\lambda[T_{\Delta x}] < 0$ the solution is decreasing with time.
Same holds for the analytic operator $\frac{d^2}{dx^2}$ with homogenous Dirichlet, as it yields $\lambda_k = -k\pi x$ and also they are strictly decreasing.
The norm of these strictly decreasing solutions are therefore bound by the norm of the initial value.

5.

a) $u_t = u_{xx} + u_x$

Periodic boundary conditions $u(0,x) = g(x)$

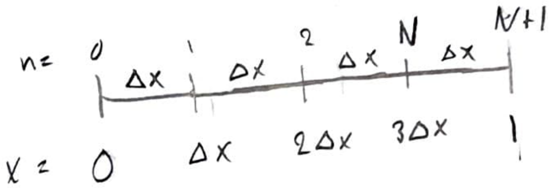
$$u_{xx} \approx \frac{u_{n+1} - 2u_n + u_{n-1}}{\Delta x^2}$$

$$u_x \approx \frac{u_{n+1} - u_{n-1}}{2\Delta x}$$

- with spacial boundaries $x=0,\ x=1$ we have

$$u(t,0) = u(t,1)$$

- [Since the value $u(1) = u(0)$ we have N inner grid points as $u(N+1) = u(0)$] →



$(N+1)\Delta x = 1 \iff \Delta x = \frac{1}{N+1}$

We have a discretization in space as

$$L = \frac{d^2}{dx^2} + \frac{d}{dx} \approx T_{\Delta x} + S_{\Delta x} \quad \text{where}$$

- $$T_{\Delta x} = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & 0 & \cdots \\ 0 & 1 & & & \\ \vdots & 0 & & & -1 \\ 1 & \vdots & & 1 & 0 \end{pmatrix} \quad S_{\Delta x} = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 1 & & & & \\ \vdots & 0 & & & & \\ 0 & & & & & -1 \\ -1 & 0 & \cdots & & 1 & 0 \end{pmatrix}$$

For periodic

for periodicity

- Periodic!

Periodicity.

For initial condition the first spacial function vector $U^0(x)$

$$U^0 = g(x) = \begin{pmatrix} g(x_0) = g(x_N) \\ g(\Delta x) \\ \vdots \\ g(x_N) = g(0) \end{pmatrix}$$

The most straight forward time-stepping method A. G.

b) would be the explicit Euler method but as this has a CFL-condition that restricts the time step for stability I would instead use the corresponding trapezoidal rule for PDE's; the Crank-Nicolson method. This is unconditionally stable but as it is implicit it uses more computational power per time step.

No CFL-condition.

We have

$$u^{j+1} = u^{j} + \frac{\Delta t}{2}\left(\left(T_{\Delta x} + S_{\Delta x}\right)u^{j} + \left(T_{\Delta x} + S_{\Delta x}\right)u^{j+1}\right)$$

That is

$$\left(I - \frac{\Delta t}{2}\left(T_{\Delta x} + S_{\Delta x}\right)\right)u^{j+1} = \left(I + \frac{\Delta t}{2}\left(T_{\Delta x} + S_{\Delta x}\right)\right)u^{j}$$

$$\Rightarrow \quad u^{j+1} = \left(I - \frac{\Delta t}{2}\left(T_{\Delta x} + S_{\Delta x}\right)\right)^{-1}\left(I + \frac{\Delta t}{2}\left(T_{\Delta x} + S_{\Delta x}\right)\right)u^{j}.$$

With $u^{0} = \{g(x_i)\}_{i=0}^{N}$.

6. $\dot{U}(t) = f(U(t)) \quad (1)$ AG.

a) $\|U(t)\| = C \iff \frac{d}{dt}\|U(t)\| = 0 \quad (2)$

Prove that $\langle v, f(v)\rangle = 0$.

$$\frac{d}{dt}\|U(t)\| = \frac{d}{dt}\langle U, U\rangle = \langle \frac{dU}{dt}, U\rangle + \langle U, \frac{dU}{dt}\rangle$$

$$= 2\langle U, \frac{dU}{dt}\rangle = 2\langle U, \dot{U}\rangle \overset{(1)}{=} 2\langle U, f(U)\rangle.$$

$$\left\{\begin{array}{l} \frac{d}{dt}\|U(t)\| = 0 \quad (2) \\ \frac{d}{dt}\|U(t)\| = 2\langle U, f(U)\rangle \end{array}\right. \Rightarrow \langle U, f(U)\rangle = 0 \;\not\downarrow$$

b) Explicit Euler: we want $\|V\| = C \Rightarrow$ we want $\langle U, f(U)\rangle = 0$

$$U_{n+1} - U_n = \Delta t\, f(U_n)$$

Inner products are distributive $\rightarrow$ $\langle U_n, U_{n+1}\rangle - \langle U_n, U_n\rangle = \Delta t \underbrace{\langle U_n, f(U_n)\rangle}_{=0}$

$$\Rightarrow \langle U_n, U_{n+1}\rangle = \langle u_n, u_n\rangle = \|U_n\|.$$

This implies that the norm of the solution $\|u_n\|$ is $= \langle u_n, u_{n+1}\rangle$, but unfortunately $\langle u_n, u_{n+1}\rangle \not\equiv \|u_{n+1}\|$ so we cannot deduce whether $\|U\|$ is constant or not.

c) Implicit Euler: A.G.

$$U_{n+1} - U_n = \Delta t\, f(U_{n+1})$$

$$\langle U_{n+1}, U_{n+1}\rangle - \langle U_{n+1}, U_n\rangle = \Delta t\, \overbrace{\langle U_{n+1}, f(U_{n+1})\rangle}^{=0}$$

$$\Rightarrow \|U_{n+1}\| = \langle U_{n+1}, U_n\rangle \neq \|U_n\|$$

Not constant!

d) Implicit midpoint method

$$U_{n+1} - U_n = \Delta t\, f\left(\frac{U_n + U_{n+1}}{2}\right)$$

Using the vector $\frac{U_n + U_{n+1}}{2}$:

$$\frac{1}{2}\langle U_n + U_{n+1}, U_{n+1}\rangle - \frac{1}{2}\langle U_n + U_{n+1}, U_n\rangle = \Delta t\, \overbrace{\left\langle \frac{U_n + U_{n+1}}{2}, f\left(\frac{U_n + U_{n+1}}{2}\right)\right\rangle}^{=0}$$

$$\Rightarrow \langle U_n + U_{n+1}, U_{n+1}\rangle = \langle U_n + U_{n+1}, U_n\rangle$$

$$\Rightarrow \langle U_n, U_{n+1}\rangle + \langle U_{n+1}, U_{n+1}\rangle = \langle U_n, U_n\rangle + \langle U_{n+1}, U_n\rangle$$

$$\Rightarrow \cancel{\langle U_{n+1}, U_n\rangle} + \|U_{n+1}\| = \|U_n\| + \cancel{\langle U_{n+1}, U_n\rangle}$$

$$\Rightarrow \|U_{n+1}\| = \|U_n\|.$$

As this holds for all n, we can start in $n=0$ and deduce that $\|U_n\| = \|U_0\| = \text{constant}$ for all $n \Leftrightarrow \forall t$.

Implicit midpoint preserves norm of solution!