Lund University
Centre for Mathematical Sciences
Mathematical Statistics

FMS051/MASM17

Take home exam 2022

The take home exam should be solved **individually** and be presented in written form (handwritten is ok as long as it is easily readable). All solutions should be clear, concise, and well motivated. Please start a new problem on a new page. **Write your name on all pages**. If you do not solve all problems, please indicate this as well, so that we know and do not spend time searching for any "missing" solutions. The take-home exam is voluntary and you only need to complete it if you aim for a higher grade.

You may not discuss the exam with anyone other than the teaching staff; we expect you to honor this. Please state on your exam that you have not collaborated with anyone when solving it and **sign** with your name.

The exam is due on January 13th, at 13.15. Please submit your exam via Canvas before this. If you have problems with the interpretation of any part, please get in touch!

We hope you have enjoyed the course and have found it rewarding. Please take the time to complete the course evaluation. This helps us to make the course better and future students will love you for it. All forms of feedback on the course is much appreciated.

Good luck!

Problem 1

Consider a (real-valued) stationary process x_t with mean m_x. Show that optimal predictor $\hat{x}_{t+k|t}$ of the form $ax_t + b$ is obtained by choosing $a = \rho_x(k)$ and $b = m_x - m_x\rho_x(k)$, where $\rho_x(k)$ denotes the autocorrelation of x_t.
(*7 marks*)

Problem 2

Let $\{x_t\}$ denote a (real-valued) sequence of independent normal random variables, each with zero mean and variance σ_x^2. Determine the values of the constant c such that

$$y_t = \sin(ct)x_t + \cos(ct)x_{t-1}$$

is a weakly stationary process.
(*7 marks*)

Problem 3

Let

$$x_t + a_1 x_{t-1} + a_2 x_{t-2} = e_t$$

where e_t is a zero-mean (real-valued) white process with variance σ_e^2.

a) Derive the 1- and 2-step predictors for x_t, i.e., form $\hat{x}_{t+1|t}$ and $\hat{x}_{t+2|t}$, and compute their prediction error variances.

b) Construct new predictors based on linear extrapolation from x_{t-1} and x_t and compute the prediction error variances for these predictors.

c) Determine the a_1 and a_2 values, if any, for which the linear extrapolation estimators are optimal.

(*8 marks*)

Problem 4

We have during the course introduced the RLS estimate for the (real-valued) linear process

$$y_t = \mathbf{x}_t^T \theta + e_t$$

As most estimators minimizing a squared cost function, the RLS estimate is sensitive to the occurrence of outliers. In an attempt to construct less sensitive algorithms, there has been significant interest in developing other, more robust, minimization criteria. One example of such a robust criteria is the so-called Huber criteria, which is quadratic for small residuals, and then linear for larger ones, i.e., the cost function at time t is

$$f(t,\theta) = \sum_{\ell=1}^{t} \rho\left(\sigma_e^{-1}(y_\ell - \mathbf{x}_\ell^T \theta)\right)\sigma_e^2$$

where σ_e^2 denotes the variance of e_t and

$$\rho(u) = \begin{cases} \frac{u^2}{2} & \text{if } |u| \le c \\ c|u| - \frac{c^2}{2} & \text{if } |u| > c \end{cases}$$

for some constant c. For simplicity, we will here assume that σ_e^2 is known (otherwise, one needs to find also a robust estimate of this). Derive a recursive algorithm minimizing the Huber criteria by making the quadratic approximation

$$f(t,\theta) \approx f(t,\hat{\theta}_{t-1}) + (\theta - \hat{\theta}_{t-1})^T f'(t,\hat{\theta}_{t-1}) + \frac{1}{2}(\theta - \hat{\theta}_{t-1})^T f''(t,\hat{\theta}_{t-1})(\theta - \hat{\theta}_{t-1})$$

How does this "robust" algorithm differ from the RLS algorithm?
(*8 marks*)