Home Assignment 1

MASM11 - Monte Carlo and empirical methods for stochastic inference

2023-02-07

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1. Random number generation

Let X be a random variable on $\mathbb{R}$ with density f_X and invertible distribution function F_X. Here f_X, F_X, and the inverse F_X^{-1} are assumed to be known. Let $I = (a, b)$ be an interval such that $P(X \in I) > 0$.

a) Conditional distribution

Find the conditional distribution function $F_{X|X\in I}(x) = P(X \leq x | X \in I)$ and density $f_{X|X\in I}(x)$ of X given that $X \in I$.

We may write the distribution function as

$$\begin{aligned} F_{X|X\in I}(x) &= P(X \leq x | X \in I) && (1) \\ &= \frac{P(X \leq x, X \in I)}{P(X \in I)} && (2) \\ &= \frac{P(X \leq x, a < X < b)}{P(a < X < b)} && (3) \end{aligned}$$

according to the laws of conditional distribution.

The numerator of this probability becomes zero for $x < a$ since it is not possible for X to be simultaneously smaller than x and larger than a. For $x > b$, however, X will always be smaller than x if it is in the interval I. This makes the conditional probability 1. For $a < x < b$, finally, the numerator becomes the probability that X is smaller than x (and thus also smaller than b) but larger than a, a probability reached by subtracting the distribution functions:

$$\begin{aligned} (3) &= \frac{P(a < X \leq x)}{P(a < X < b)} && (4) \\ &= \frac{P(X \leq x) - P(X \leq a)}{P(X \leq b) - P(X \leq a)} && (5) \\ &= \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} && (6) \end{aligned}$$

In summary,

$$F_{X|X\in I}(x) = \begin{cases} 0 & x < a \\ \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a < x < b \\ 1 & x > b \end{cases}$$

and we may prove that this fulfills the requirements of a distribution function. Since the distribution function F_X is invertible, we know that it is also strictly monotonically increasing. This, together with the inequality $a < b$, leads to the result $F_X(b) - F_X(a) > 0$, which further means that $\frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} \geq 0$ for $x > a$. The maximum of $\frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}$, with the restriction $x < b$, will be found as x approaches b, and the expression will approach $\frac{F_X(b) - F_X(a)}{F_X(b) - F_X(a)} = 1$. The function is monotonically non-decreasing between a minimum of zero and a maximum of one, making it a proper distribution function.

To find the density function $f_{X|X\in I(x)}$, we simply take the partial derivative on x of the distribution function:

$$f_{X|X\in I}(x) = \frac{\partial}{\partial x}F_{X|X\in I}(x) \tag{7}$$

$$= \begin{cases} \frac{\partial}{\partial x}0 = 0 & x < a \\ \frac{\partial}{\partial x}\frac{F_X(x)-F_X(a)}{F_X(b)-F_X(a)} = \frac{f_X(x)}{F_X(b)-F_X(a)} & a < x < b \\ \frac{\partial}{\partial x}1 = 0 & x > b \end{cases} \tag{8}$$

b) Inverse distribution function

Find the inverse $F^{-1}_{X|X\in I}$. How can this be used for simulating X conditionally on $X \in I$?

The inverse fulfills the condition that $F_X^{-1}(F_X(x)) = x$. Thus, the inverse takes as its input a variable which takes values between 0 and 1 (the limits of F_X) and gives x as its output. We may simulate a uniform random variable $U \sim \mathcal{U}(0,1)$ and use its realisations u as inputs to the inverse function in order to get simulated random realisations from the distribution of X. For method to work, u must be equal to $F_X(x)$. In this case, we have the equation

$$u = F_{X|X\in I}(x) \tag{9}$$

$$= \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)}, a < x < b \tag{10}$$

so for $a < x < b$, this solves to

$$F_X(x) - F_X(a) = (F_X(b) - F_X(a))u \tag{11}$$

$$F_X(x) = (F_X(b) - F_X(a))u + F_X(a) \tag{12}$$

$$F^{-1}_{X|X\in I}(u) = x = F_X^{-1}((F_X(b) - F_X(a))u + F_X(a)) \tag{13}$$

which cannot be further simplified without specifying the distribution of X.

In order to simulate N random realisations from X with the restriction that X be in the interval I, we can use this result and simulate N random realisations from the uniform random variable U, find the value of F_X for arguments a and b, and find the inverse distribution function. We would take the difference between $F_X(b)$ and $F_X(a)$, multiply it by each random realisation u, add $F_X(a)$ again, and set the sum as argument to the unconditional inverse distribution function of X. This will give N simulated conditional random realisations from $X|X \in I$.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
λ	10.6	9.7	9.2	8.0	7.8	8.1	7.8	8.1	9.1	9.9	10.6	10.6
k	2.0	2.0	2.0	1.9	1.9	1.9	1.9	1.9	2.0	1.9	2.0	2.0

Table 1: Parameter values for the Weibull distribution every month of the year.

2. Power production of a wind turbine

a) Standard and truncated Monte Carlo

Create an approximate 95% confidence interval for the expected amount of power generated by the wind turbine using draws from a Weibull distribution. Compare the width of the confidence interval using standard Monte Carlo and the truncated version considered in problem 1. Remember to properly adjust for the conditioning when computing the estimate using the truncated Weibull distribution.

Our first simulation was done using a standard Monte Carlo method, where we first simulated wind speed using the Weibull distribution. This means that the wind speed is assumed to follow the density function

$$f_X(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, x \geq 0$$

with corresponding distribution function

$$F_X(x) = 1 - e^{-(x/\lambda)^k}, x \geq 0$$

where the parameters k, λ varied for each month. The parameter values are summarized in table 1. Using these parameters, we simulated 1 000 samples of wind speed from each month and ran each of these simulated wind speed measures through the function $P(v) = \frac{1}{2}\rho\pi\frac{d^2}{4}v^4$ (accessed through the provided file $powercurve_V164.mat$). For each month, this gave the mean estimate, sample variance and 95% confidence interval bounds provided in table 2. The confidence intervals are made based on the assumption that the central limit theorem will make the sample mean approximately normal distributed, and is thus constructed according to

$$CI = \left[\hat{\mathbb{E}}(P(V)) - \lambda_{0.025}\frac{1}{\sqrt{N}}\mathbb{D}[\hat{\mathbb{E}}(P(V))] \quad \hat{\mathbb{E}}(P(V)) + \lambda_{0.025}\frac{1}{\sqrt{N}}\mathbb{D}[\hat{\mathbb{E}}(P(V))]\right] \tag{14}$$

where $\lambda_{0.025}$ is the 2.5th percentile of the standard normal distribution and $\mathbb{D}[\hat{\mathbb{E}}(P(V))]$ is the square root of the simulated sample variance. This method for constructing confidence intervals is used throughout the assignment, with the main difference being that the sample variance and thereby the interval width are reduced through various methods.

Next, we tried using a truncated version of the same simulation, where we only simulated wind of speeds larger than 3.5 m/s and smaller than 25 m/s, since only these windspeeds lead to non-zero power output. To create this simulation, we used our results from part 1 of the assignment, applying the inverse distribution function

$$F^{-1}_{X|X\in I}(u) = x = F^{-1}_X((F_X(b) - F_X(a))u + F_X(a))$$

Month	Jan	Feb	Mar	Apr	May	Jun
Estimate [MW]	4.7408	4.2043	3.9248	3.1239	2.7885	3.1249
Variance [10^{12}]	13.595	12.709	12.501	11.003	9.2107	10.674
CI lower [MW]	4.5490	4.0189	3.7409	2.9513	2.6306	2.9550
CI upper [MW]	4.9326	4.3897	4.1088	3.2964	2.9464	3.2948
Month	Jul	Aug	Sep	Oct	Nov	Dec
Estimate [MW]	2.7819	2.9027	3.5352	4.2870	4.7502	4.7063
Variance [10^{12}]	9.480	9.960	11.253	13.394	14.248	13.548
CI lower [MW]	2.6217	2.7386	3.3607	4.0966	4.5538	4.5148
CI upper [MW]	2.9420	3.0669	3.7097	4.4773	4.9465	4.8977

Table 2: Estimated mean power, estimate variance and 95% confidence interval for the standard Monte Carlo simulation.

and inserting the Weibull distribution function as $F_X(x)$ and making interval limits $a = 3.5$ and $b = 25$. We now calculate the inverse distribution function of the Weibull distribution:

$$F_X(x) = y = 1 - e^{-(x/\lambda)^k} \tag{15}$$
$$e^{-(x/\lambda)^k} = 1 - y \tag{16}$$
$$(x/\lambda)^k = -ln(1-y) \tag{17}$$
$$x/\lambda = \sqrt[k]{-ln(1-y)} \tag{18}$$
$$x = \lambda \sqrt[k]{-ln(1-y)} \tag{19}$$

where $y = (F_X(b) - F_X(a))u + F_X(a)$ and u is a sample from a $\mathcal{U}(0,1)$ random variable. Using this, we simulate random samples from a truncated Weibull distribution. Figure 1 shows that a histogram of a realisation of this distribution follows a very similar pattern to that of the original $W(\lambda, k)$ distribution. However, due to the cut-off at windspeeds lower than 3.5 m/s, the truncated distribution shows no frequencies below this speed. Instead, the frequencies between windspeeds 3.5 and 25 are slightly higher for the truncated distribution, especially for those windspeed intervals that are already high-frequency. Note that the comparison is not of the simulated power but of the windspeed, since this is the variable that follows a Weibull distribution directly.

Unfortunately, the cut-off of the lowest windspeeds leads to a large bias in the estimation of the mean power, with the truncated simulation giving estimates around 50 000 W lower than the standard Monte Carlo simulation. In order to correct this, we observe that the expected power output may be written as

$$E[P(V)] = E[P(V)|V \in I]P(V \in I) + E[P(V)|V \notin I]P(V \notin I) \tag{20}$$
$$= E[P(V)|V \in I](F_X(b) - F_X(a)) + 0 \cdot P(V \notin I) \tag{21}$$
$$= E[P(V)|V \in I](F_X(b) - F_X(a)) \tag{22}$$

where $E[P(V)|V \in I]$ is the expected value which we have just calculated. Multiplying the estimate by $(F_X(b) - F_X(a))$, we get an unbiased estimate with a slightly lower variance. The estimates for each month, together with their variance and the confidence interval limits, are found in table 3. Here, as well as in the

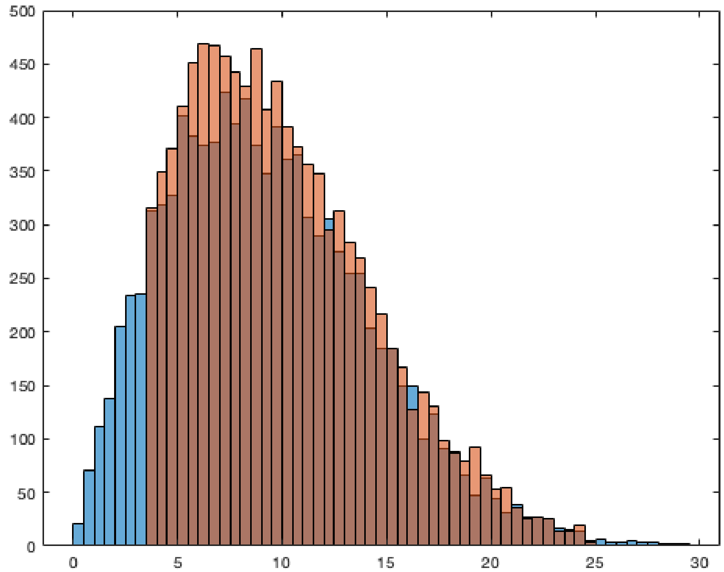

Figure 1: Histogram of a standard Monte Carlo simulation of windspeed (blue) and a truncated Monte Carlo simulation of windspeed (red). $\lambda = 10.6$, $k = 2.0$ (January), sample size 10 000.

corresponding tables for other methods, we also present the reduction in variance relative to the variance of the standard Monte Carlo estimator. The truncated simulation seems to give a variance reduction of on average 32.5%. This is of course good, but other methods should be able to reduce it further.

b) Variance reduction with control variate

Use the wind V as a control variate to decrease the variance and estimate a 95% confidence interval for the expected power.

In order to use a control variate to reduce the variance of an estimate we want to use an additional random variable, with known expected value. If this random variable is highly correlated with our target function, we can expect a large reduction of variance in the new estimate. Since the wind is modelled as draws from a Weibull distribution, we know expected value and variance and may calculate them explicitly. Since the output power is a direct function of the wind speed, we can also expect high correlation between the two variables. We implement this, as described in pages 13-15 in Lecture 4 slides, in the following way. First, draw N wind speeds $V \in W(\lambda, k)$ from a Weibull distribution and use these to calculate the turbine output power $P(V)$, as described in task a). Then, for the same distribution, calculate the closed form expected value and variance m, σ^2 as:

Month	Jan	Feb	Mar	Apr	May	Jun
Estimate [MW]	4.7404	4.2320	3.8104	2.9740	2.9836	3.1006
Variance [10^{12}]	9.5117	8.8902	8.3816	6.3431	6.3499	6.8161
CI lower [MW]	4.5800	4.0769	3.6598	2.8430	2.8528	2.9648
CI upper [MW]	4.9009	4.3871	3.9609	3.1050	3.1147	3.2364
Variance reduction	30.0	30.0	33.0	42.4	31.1	36.1
Month	Jul	Aug	Sep	Oct	Nov	Dec
Estimate [MW]	2.7539	2.9900	3.8016	4.2992	4.7099	4.5719
Variance [10^{12}]	6.1400	6.6899	8.1362	9.1260	9.5611	9.8529
CI lower [MW]	2.6250	2.8555	3.6533	4.1421	4.5490	4.4087
CI upper [MW]	2.8828	3.1246	3.9500	4.4563	4.8707	4.7351
Variance reduction [%]	35.2	32.8	27.7	31.9	32.9	27.3

Table 3: Estimated mean power, estimate variance and 95% confidence interval for the truncated Monte Carlo simulation, as well as variance reduction compared to the standard Monte Carlo simulation.

$$m = \mathbb{E}\{V\} = \lambda\Gamma(1 + 1/k) \tag{23}$$

$$\sigma^2 = \mathbb{V}\{V\} = \mathbb{E}\{V^2\} - \mathbb{E}^2\{V\} = \lambda^2\left(\Gamma\left(1 + \frac{2}{k}\right) - \left(\Gamma\left(1 + \frac{1}{k}\right)\right)^2\right) \tag{24}$$

with values for λ and k as found in table 1. We then use the two random variables $P(V)$ and V to form the composite variable $Z = P(V) + \beta(V - m)$. With β as $-\mathbb{C}(P(V), V)/\sigma^2$ one can show that this has the same expected value as $P(V)$ and reduced variance. We simulate Z many times and take the mean of these simulations as our expected value. The estimates, variance, confidence intervals and variance reduction are all found in table 4. Here we see a much better variance reduction than with the truncated sampling, averaging at 86%.

c) Variance reduction using importance sampling

Compare the above result to an approximate 95% confidence interval created by means of importance sampling based on some convenient instrumental distribution.

The idea for this method is to sample more from areas that are more important, i.e. where the integrand is large, to lower the variance. This means that we sample from an instrumental distribution $g(v)$ instead of the actual f, where g should have large values for wind speeds where $f(v)P(v)$ is large. We then compensate for the bias that occurs due to sampling from an incorrect distribution by multiplying the values with the importance weight function $\omega(v) = f(v)/g(v)$. As seen in figure 2, a plot of $f(v)P(v)$ shows that a Weibull distribution with parameters $\lambda = 14$ and $k = 3$ would be a good fit since its densities follow a similar curve as the target distributions. Since the turbine does not produce any power for wind speeds outside [3.5, 25] m/s we truncate the instrumental distribution at these boundaries. It should be noted that the best model

Month	Jan	Feb	Mar	Apr	May	Jun
Estimate [MW]	4.6912	4.1991	3.7921	3.0176	2.8491	3.0315
Variance [10^{12}]	2.3719	1.3535	1.4237	0.8951	0.8542	1.0907
CI lower [MW]	4.6111	4.1386	3.7300	2.9684	2.8010	2.9772
CI upper [MW]	4.7713	4.2596	3.8542	3.0669	2.8972	3.0858
Variance reduction	82.5	89.4	88.6	91.9	90.7	89.8
Month	Jul	Aug	Sep	Oct	Nov	Dec
Estimate [MW]	2.9052	3.0389	3.798	4.2531	4.6724	4.6070
Variance [10^{12}]	0.9694	1.0268	1.1604	3.1203	2.7301	3.7908
CI lower [MW]	2.8539	2.9862	3.742	4.1612	4.5864	4.5057
CI upper [MW]	2.9564	3.0916	3.8541	4.345	4.7583	4.7083
Variance reduction [%]	89.8	89.7	89.7	76.7	81.0	72.0

Table 4: Estimated mean power, estimate variance and 95% confidence interval for the control variate simulation, as well as variance reduction compared to the standard Monte Carlo simulation.

would be to fit a new instrumental distribution to each month, since they have different parameter values. However, this is slightly beyond the time budget for this project, and we use the same instrumental for all months (using decembers values). The resulting $P(v)\omega(v) = P(v)f(v)/g(v)$ is found in figure 3, and shows that the resulting integrand has been somewhat flattened. When a good instrumental distribution has been found, we sample from this and multiply with $f(v)/g(v)$ to compensate for the bias and calculate the average as our estimate of the expected value. Implementing this yields the values found in table 5. The variance reduction is large and comparable to that from the control variate method. The best results come from the months October, November, December and January. They have very similar (or same) parameters as the distribution we tried to fit the instrumental distribution to (December), which could have been expected.

d) Variance reduction using antithetic sampling

The power curve $P(v)$ is monotonously increasing over the interval $(3.5, 14)$ and constant over $(14, 25)$. Use this for reducing the variance of the estimator in (a) via antithetic sampling. Construct a new 95% confidence interval using the robustified estimator and compare it to the ones obtained in (a), (b) and (c).

Our final approach to reduce the variance was to introduce two antithetic variables which have the same mean, variance and distribution, but are negatively correlated. Running both of these variables through the same monotone function, the function values will also have a negative correlation. In this case, we let our first variable be $U \sim \mathcal{U}(0, 1)$ and its antithetic variable be $T(U) = 1 - u \sim \mathcal{U}(0, 1)$. Setting these as arguments for the inverse of the Weibull distribution function, and using the resulting wind speeds as inputs to the power function, we receive two power variables P and $\tilde{P}$ with negative correlation. Finally, we calculate $Q = \frac{P+\tilde{P}}{2}$

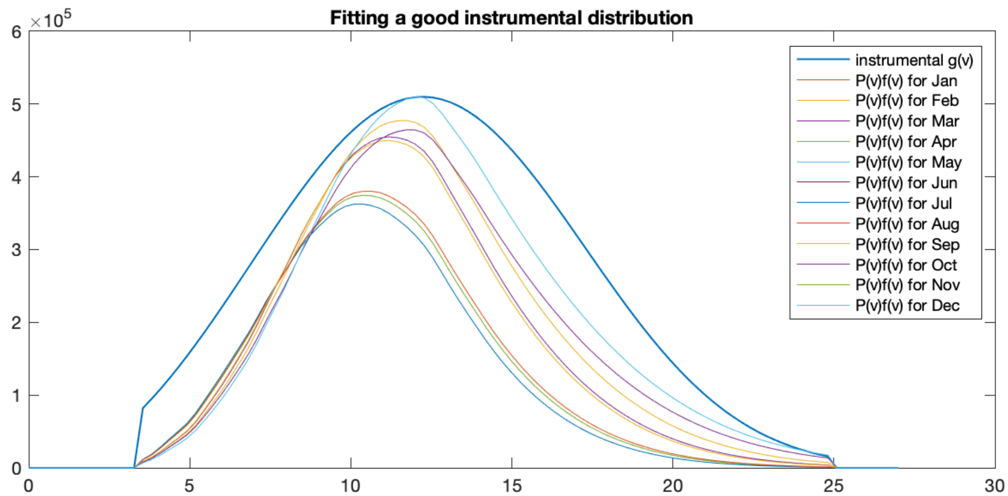

Figure 2: Fitting an instrumental density, the instrumental density function (blue) compared to the product of the original densities and the power function for the different months.

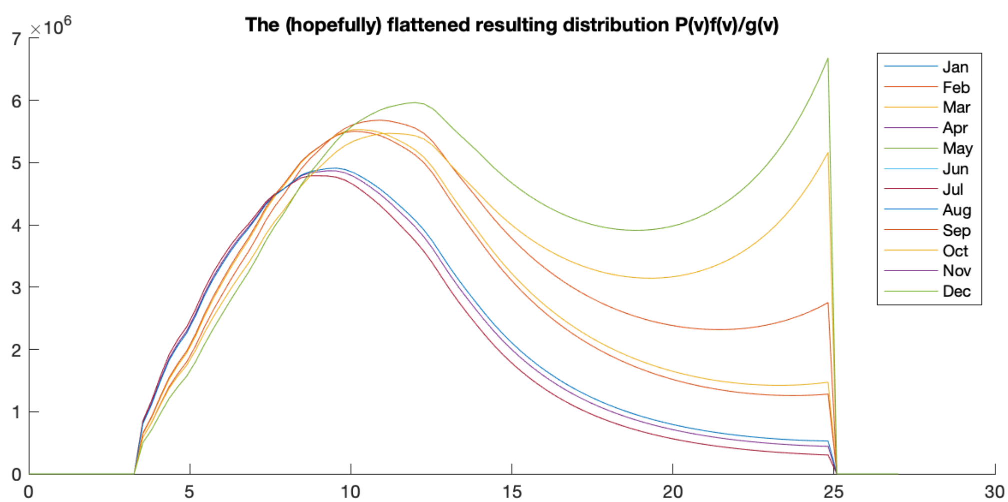

Figure 3: The resulting $\omega = P(v)f(v)/g(v)$

. As can be seen it is more flat than $P(v)f(v)$ which is the goal.

Month	Jan	Feb	Mar	Apr	May	Jun
Estimate [MW]	4.6739	4.1084	3.8496	3.0195	2.8176	2.9299
Variance [10^{12}]	1.089	1.598	1.898	2.1052	2.137	2.138
CI lower [MW]	4.695	4.0363	3.7577	2.9440	2.7416	2.8538
CI upper [MW]	4.768	4.1678	3.9010	3.0950	2.8936	3.0059
Variance reduction [%]	92.0	87.4	84.8	80.9	76.8	80.0
Month	Jul	Aug	Sep	Oct	Nov	Dec
Estimate [MW]	2.9249	3.1286	3.7056	4.1830	4.6872	4.5994
Variance [10^{12}]	2.137	2.075	1.987	1.228	1.292	1.311
CI lower [MW]	2.8488	3.0537	3.6323	4.1254	4.6280	4.5398
CI upper [MW]	3.0009	3.2035	3.7789	4.2406	4.7463	4.6589
Variance reduction [%]	77.5	79.2	82.3	90.8	90.9	90.3

Table 5: Estimated mean power, estimate variance and 95% confidence interval for the importance sampling simulation, as well as variance reduction compared to the standard Monte Carlo simulation. We can see that the month that we used to fit the instrumental distribution (December) has among the best reductions, as well as other months during the same season (October, November, January).

and use the mean of Q as our antithetic estimate of the expected power output. Theoretically,

$$\begin{aligned} E[Q] &= E\big[\frac{P+\tilde{P}}{2}\big] && (25)\\ &= \frac{1}{2}(E[P]+E[\tilde{P}]) && (26)\\ &= \frac{1}{2}(\tau+\tau) && (27)\\ &= \tau && (28) \end{aligned}$$

so that the estimate is unbiased, and

$$\begin{aligned} V[Q] &= V\big[\frac{P+\tilde{P}}{2}\big]\\ &= \frac{1}{4}(V[P]+V[\tilde{P}]+2C[P,\tilde{P}])\\ &= \frac{1}{4}(2V[P]+2C[P,\tilde{P}])\\ &= \frac{1}{2}(V[P]+C[P,\tilde{P}]) \end{aligned}$$

Then, since we in practice use a collection of N independent samples of Q, we have mean

$$\begin{aligned} E[Q_N] &= E\big[\frac{1}{N}\sum_{i=1}^{N} q_i\big] && (29)\\ &= \frac{1}{N}\sum_{i=1}^{N} E[Q] && (30)\\ &= \tau && (31) \end{aligned}$$

Month	Jan	Feb	Mar	Apr	May	Jun
Estimate [MW]	4.6670	4.1606	3.8226	3.0762	2.8328	3.0448
Variance [10^{12}]	0.2250	0.4011	0.5947	1.3206	1.3868	1.2012
CI lower [MW]	4.6424	4.1277	3.7825	3.0164	2.7715	2.9878
CI upper [MW]	4.6917	4.1936	3.8627	3.1359	2.8940	3.1018
Variance reduction [%]	98.3	96.8	95.2	88.0	84.9	88.7
Month	Jul	Aug	Sep	Oct	Nov	Dec
Estimate [MW]	2.8887	3.0729	3.7438	4.229	4.6483	4.6796
Variance [10^{12}]	1.3778	1.2337	0.6553	0.4244	0.2483	0.1602
CI lower [MW]	2.8276	3.0151	3.7016	4.1890	4.6224	4.6587
CI upper [MW]	2.9498	3.1306	3.7859	4.2568	4.6742	4.7004
Variance reduction [%]	85.5	87.6	94.2	96.8	98.3	98.8

Table 6: Estimated mean power, estimate variance and 95% confidence interval using antithetic sampling, as well as variance reduction compared to the standard Monte Carlo simulation.

and estimate variance

$$
\begin{aligned}
V[Q_N] &= V\big[\frac{1}{N}\sum_{i=1}^{N} q_i] && (32)\\
&= \frac{1}{N^2}\sum_{i=1}^{N} V[Q] && (33)\\
&= \frac{1}{N}\frac{1}{2}(V[P] + C[P,\tilde{P}]) && (34)
\end{aligned}
$$

However, in order to have a fair comparison with a standard Monte Carlo sampling we would need to halve the number of samples, since P and $\tilde{P}$ are only sampled $N/2$ times each if we take a total of N samples. Thus, the variance becomes

$$
\begin{aligned}
V[Q_N] &= \frac{1}{N/2}\frac{1}{2}(V[P] + C[P,\tilde{P}]) && (35)\\
&= \frac{1}{N}(V[P] + C[P,\tilde{P}]) && (36)
\end{aligned}
$$

which will still be smaller than $V[P_N] = \frac{1}{N}V[P]$ since $\tilde{P}$ is chosen such that $C[P,\tilde{P}] \leq 0$.

Implementing this in our simulation, we get the means and variances shown in Table 6. The variance reduction is the largest out of any methods, averaging around 93% and for some months reaching over 98%.

A comparison of the convergences of the different estimators are found in figure 4. We see that the fastest converging estimates are the antithetic and control variate methods, which is the same conclusion which we could draw from studying the variance reduction in the tables 3-6.

e) Probability of power output

Calculate/estimate the probability that the turbine delivers power, $\mathbb{P}(P(V) > 0)$.

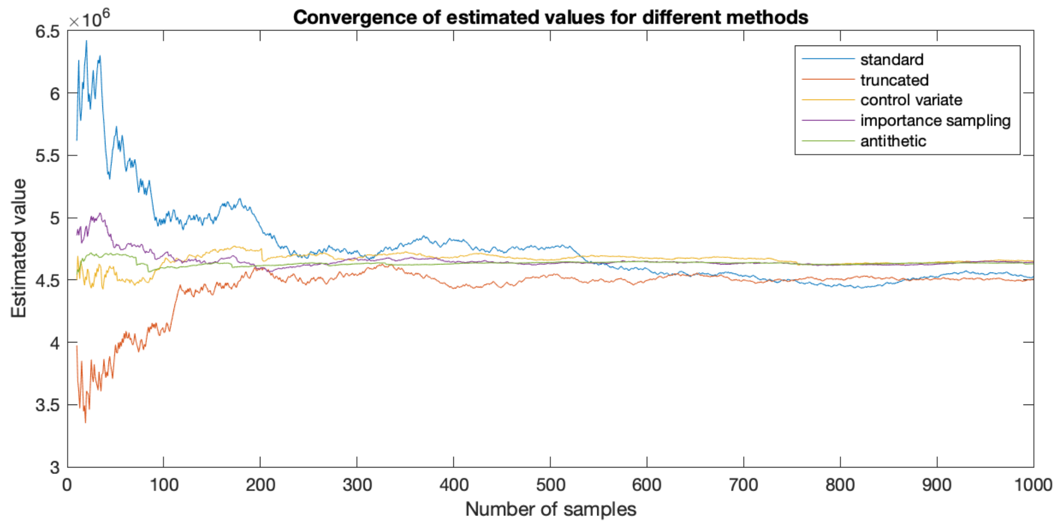

Figure 4: The estimated values after $N \in [1, 1000]$ samples for the five different methods

Month	Jan	Feb	Mar	Apr	May	Jun
Theoretical probability	0.8929	0.8741	0.8646	0.8121	0.8039	0.8160
Estimated probability	0.8950	0.8920	0.8610	0.8350	0.8180	0.8240
Month	Jul	Aug	Sep	Oct	Nov	Dec
Theoretical probability	0.8039	0.8160	0.8620	0.8675	0.8929	0.8929
Estimated probability	0.8100	0.8370	0.8670	0.8560	0.8850	0.8880

Table 7: Theoretical and estimated probabilities $\mathbb{P}(P(V) > 0)$ for each month.

The turbine will deliver power if the windspeed is between 3.5 and 25 m/s. The probability that this occurs can both be simulated and calculated explicitly. We first write the explicit solution (for January) using the Weibull distribution function:

$$\begin{aligned}
\mathbb{P}(P(V) > 0) &= \mathbb{P}(3.5 < V < 25) && (37)\\
&= F_V(25) - F_V(3.5) && (38)\\
&= (1 - e^{-(25/10.6)^2}) - (1 - e^{-(3.5/10.6)^2}) && (39)\\
&= e^{-(3.5/10.6)^2} - e^{-(25/10.6)^2} && (40)\\
&= 0.89287 && (41)
\end{aligned}$$

We simulate the probability in Matlab by making a standard Monte Carlo simulation ($N = 1000$) of the power output and creating a new index vector which has element zero when the output is zero, and one otherwise. The mean of this index vector was 0.8534, making it close to the theoretical value.

Corresponding theoretical and simulated values for each month are found in table 7. A plot of the estimated versus theoretical probability is found in figure 5, and show that there is a strong correlation.

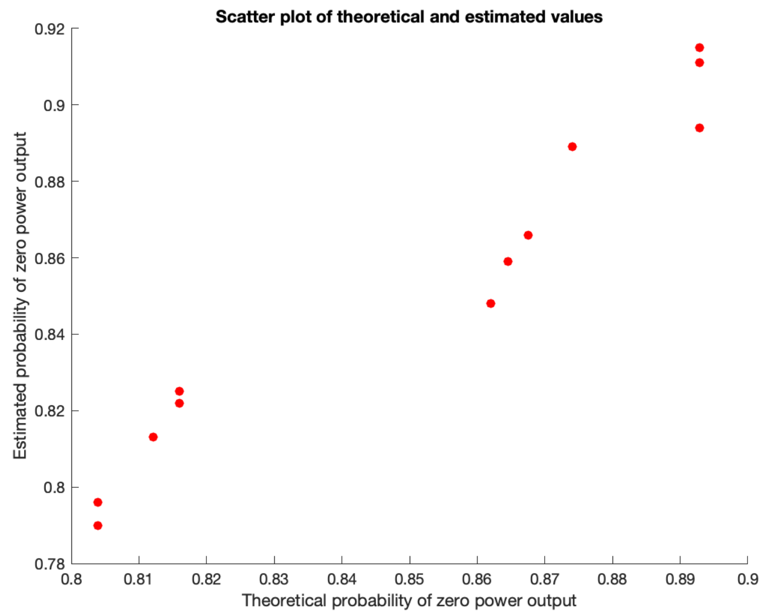

Figure 5: A scatter plot of the theoretical value of the probability of the turbine outputting power as well as the estimated value, using standard Monte Carlo simulation. There is a clear correlation between the two values.

f) Ratio of turbine output

Create an approximate 95% confidence interval for the average ratio of actual wind turbine output to total wind power (average power coefficient).

The total wind power may be calculated explicitly using the relation between wind and power

$$P_{tot}(V) = \frac{1}{2}\rho\pi\frac{d^2}{4}v^3$$

and the fact that the m:th moment of the Weibull(k, λ) distribution may be written as

$$\mathbb{E}[V] = \Gamma(1 + \frac{m}{k})\lambda^m$$

making the expected total wind power

$$\begin{aligned} \mathbb{E}P_{tot}(V) &= E\big[\frac{1}{2}\rho\pi\frac{d^2}{4}v^3\big] && (42)\\ &= \frac{1}{2}\rho\pi\frac{d^2}{4}E[v^3] && (43)\\ &= \frac{1}{2}\rho\pi\frac{d^2}{4}\Gamma(1 + \frac{3}{k})\lambda^3 && (44) \end{aligned}$$

With $\rho = 1.225$ and $d = 164$,

$$\begin{aligned} \mathbb{E}P_{tot}(V) &= \frac{1}{2}1.225\pi\frac{164^2}{4}\Gamma(1 + \frac{3}{k})\lambda^3 && (45)\\ &= 4118.45\pi\Gamma(1 + \frac{3}{k})\lambda^3 && (46) \end{aligned}$$

For January, this becomes

$$\mathbb{E}P_{tot}(V) = 4118.45\pi\Gamma(1+\frac{3}{2})10.6^3 \tag{47}$$
$$= 20485064.66 \tag{48}$$

The numerator of the ratio cannot be calculated explicitly but must be estimated. Since our best estimator for this numerator is found by antithetic sampling, we scale this estimator and its error margin by $\mathbb{E}P_{tot}(V)$ in order to get the confidence interval for the ratio. For January, the point estimate is

$$\left[\frac{\widehat{\mathbb{E}P(V)}}{\mathbb{E}P_{tot}(V)}\right] = \frac{\hat{\mathbb{E}}P(V)}{\mathbb{E}P_{tot}(V)} \tag{49}$$
$$= \frac{4.6670}{20.4851}10^6 \tag{50}$$
$$= 0.2278 \tag{51}$$

while the variance and standard deviation of the estimate become

$$\mathbb{V}\left[\frac{\widehat{\mathbb{E}P(V)}}{\mathbb{E}P_{tot}(V)}\right] = \frac{1}{\mathbb{E}^2 P_{tot}(V)}\mathbb{V}[\hat{\mathbb{E}}P(V)] \tag{52}$$
$$= \frac{1}{20485064.66^2}0.2250 \cdot 10^{12} \tag{53}$$
$$\mathbb{D}\left[\frac{\widehat{\mathbb{E}P(V)}}{\mathbb{E}P_{tot}(V)}\right] = \sqrt{\mathbb{V}\left[\frac{\widehat{\mathbb{E}P(V)}}{\mathbb{E}P_{tot}(V)}\right]} \tag{54}$$
$$= \frac{1}{20485064.66}4.7434 \cdot 10^5 \tag{55}$$
$$= 0.0232 \tag{56}$$

This means that we may set up the 95% confidence interval, with λ_α being the α:th percentile of the standard normal distribution

$$CI = \left[\left[\frac{\widehat{\mathbb{E}P(V)}}{\mathbb{E}P_{tot}(V)}\right] - \lambda_{0.025}\frac{1}{\sqrt{N}}\mathbb{D}\left[\frac{\widehat{\mathbb{E}P(V)}}{\mathbb{E}P_{tot}(V)}\right] \quad \left[\frac{\widehat{\mathbb{E}P(V)}}{\mathbb{E}P_{tot}(V)}\right] + \lambda_{0.025}\frac{1}{\sqrt{N}}\mathbb{D}\left[\frac{\widehat{\mathbb{E}P(V)}}{\mathbb{E}P_{tot}(V)}\right]\right] \tag{57}$$
$$= \left[0.2278 - 1.96\frac{1}{\sqrt{1000}}0.0232 \quad 0.2278 + 1.96\frac{1}{\sqrt{1000}}0.0232\right] \tag{58}$$
$$= \left[0.2278 - 0.0014 \quad 0.2278 + 0.0014\right] \tag{59}$$
$$= \left[0.2264 \quad 0.2292\right] \tag{60}$$

Repeating these calculations for all twelve months, we get the results in table 8. The ratio estimates and CI boundaries are rescaled from the corresponding values in table 6. The capacity varies throughout the year but seems to average at around 28%.

g) Capacity and availability

Two important characteristics of power plants are the capacity factor, or the ratio of the actual output over a time period and the maximum possible output during that time (9.5 MW times time span for the Vestas V164

Month	Jan	Feb	Mar	Apr	May	Jun
$\mathbb{E}P_{tot}(V)$ [MW]	20.4851	15.6977	13.3932	9.3230	8.6411	9.6770
$\mathbb{E}P(V)$ [MW]	4.6670	4.1606	3.8226	3.0762	2.8328	3.0448
Ratio estimate [%]	22.78	26.50	28.54	33.0	32.78	31.46
Lower CI of the ratio [%]	22.64	26.29	28.24	32.35	32.07	30.88
Upper CI of the ratio [%]	22.92	26.71	28.84	33.64	33.49	32.05
Month	Jul	Aug	Sep	Oct	Nov	Dec
$\mathbb{E}P_{tot}(V)$	8.6411	9.6770	12.9612	17.6681	20.4851	20.4851
$\mathbb{E}P(V)$	2.8887	3.0729	3.7438	4.229	4.6483	4.6796
Ratio estimate [%]	33.43	31.75	28.88	23.94	22.69	22.84
Upper CI of the ratio [%]	32.72	31.16	28.56	23.71	22.56	22.74
Lower CI of the ratio [%]	34.14	32.35	29.21	24.09	22.82	22.95

Table 8: Theoretical and estimated probabilities $\mathbb{P}(P(V) > 0)$ for each month.

9.5 MW); and the availability factor, or the amount of time that electricity is produced during a given period divided by the length of the period (you can here re-use the result from (a)-(d)). Wind turbines typically have a capacity factor of 20–40% and an availability greater than 90%. Does this seem like a good site to build a wind turbine? Look here at the averages over all months, i.e. first calculate the measures for each month and then take averages.

To find the capacity factor, we use our estimates of the power output from antithetic sampling (see table 6). Averaging over the 12 months, we estimate the output to be 3.7389 MW. Assuming equal time span and thus simply dividing by the maximum output of 9.5 MW, we estimate the capacity factor to be $\frac{3.7389}{9.5} \approx 0.3936$. Since the variance of the output is on average $7.6909 \cdot 10^{11}$ (again, see table 6), the standard deviation becomes $\sqrt{7.6909 \cdot 10^{11}} = 8.7698 \cdot 10^5$. Dividing this by 9.5 MW as a scaling factor, we receive a standard deviation of $\frac{8.7698 \cdot 10^5}{9.5 \cdot 10^6} \approx 0.0923$. This lets us make a 95% confidence interval in the same manner as in previous problems:

$$CI = \Big[0.3936 - 1.96 \cdot \tfrac{1}{\sqrt{1000}} 0.0923 \quad 0.3936 + 1.96 \cdot \tfrac{1}{\sqrt{1000}} 0.0923\Big] \tag{61}$$

$$= \Big[0.3936 - 0.0057 \quad 0.3936 + 0.0057\Big] \tag{62}$$

$$= \Big[0.3879 \quad 0.3993\Big] \tag{63}$$

We can thus see that the expected capacity is clearly in the upper part of the interval of typical capacities, almost 40% where 20-40% is typical.

The availability factor, on the other hand, is calculated explicitly for each month (see 2e) and thus has no uncertainty (assuming that the model for windspeed is correct). The average over all months is 84.99%, putting it slightly below the typical availability of over 90%.

Whether it is a good idea to build a turbine at this site is therefore slightly ambiguous. The capacity factor is high and fully acceptable, even above average, but the availability is lower than the typical range which might lead to the electricity supply being slightly too unreliable. It would have to be a judgement call whether the high capacity factor makes it attractive enough to make up for the low availability.

3. Combined power production of two wind turbines

a) Expected power output

Calculate the expected amount of combined power generated by both turbines, i.e. $\mathbb{E}(P(V1) + P(V2))$. *This actually reduces to a one dimensional problem, properly explain why this is true and use this fact in the estimation.*

The reason that this reduces to a one dimensional problem is the fact that expectation is a linear operator. For two arbitrary random variables X and Y, we in general have that

$$\begin{aligned}
\mathbb{E}(X+Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)f_{X,Y}(x,y)dxdy && (64)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xf_{X,Y}(x,y)dydx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}yf_{X,Y}(x,y)dxdy && (65)\\
&= \int_{-\infty}^{\infty}x\int_{-\infty}^{\infty}f_{X,Y}(x,y)dydx + \int_{-\infty}^{\infty}y\int_{-\infty}^{\infty}f_{X,Y}(x,y)dxdy && (66)\\
&= \int_{-\infty}^{\infty}xf_X(x)dx + \int_{-\infty}^{\infty}yf_Y(y)dy && (67)\\
&= \mathbb{E}(X) + \mathbb{E}(Y) && (68)
\end{aligned}$$

Now, since $P(V_1)$ and $P(V_2)$ are functions of random variables, they too are random variables and can therefore replace X and Y in the proof above. Since they are also equally distributed and thus have the same expected value, we have

$$\begin{aligned}
\mathbb{E}(P(V1) + P(V2)) &= \mathbb{E}(P(V_1)) + \mathbb{E}(P(V_2)) && (69)\\
&= 2\mathbb{E}(P(V_1)) && (70)
\end{aligned}$$

Thus, to get an estimate of this expectation we simulate 1000 samples using importance sampling as in 2c), with parameters set to $k = 1.96$ and $\lambda = 9.13$, and multiply the estimate by 2. Our estimate then becomes

$$\hat{\mathbb{E}}(P(V1) + P(V2)) = 7.6086 \cdot 10^6 \tag{71}$$

b) Covariance between two plants

Calculate the covariance $\mathbb{C}(P(V1), P(V2))$ *between the produced power in two identical wind power plants receiving dependent winds as above.*

Here, the problem is two-dimensional since the covariance is not a linear operator in the same manner as

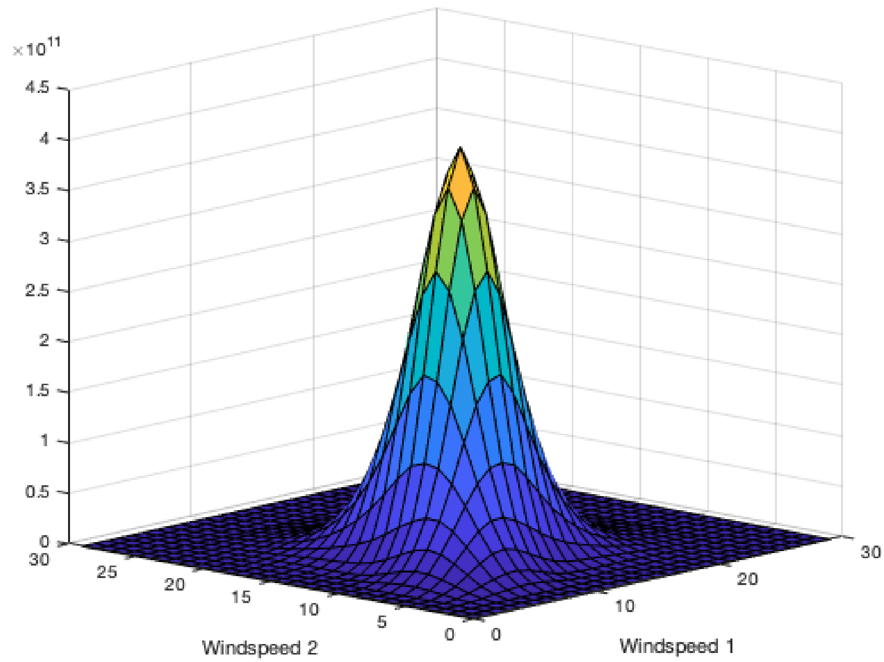

Figure 6: Surface plot of the expected product of power outputs at each simultaneous windspeed level.

the expectation, but is rather defined by

$$\mathbb{C}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) \tag{72}$$

which cannot be simplified into a one-dimensional problem. However, the factors in the right-hand term can easily be estimated one-dimensionally as we have done previously. What is left is to simulate the left-hand term, which can be written as

$$\mathbb{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)dxdy \tag{73}$$

To estimate this expectation, we use a two-dimensional version of our approach in 2c), trying to find a good instrumental distribution to flatten the surface of the power output product multiplied by the importance weight function $P(V_1)P(V_2)\frac{f_{V_1,V_2}}{g_{V_1,V_2}} = P(V_1)P(V_2)\omega_{V_1,V_2}$. A surface plot of $P(V_1)P(V_2)f_{V_1,V_2}$ may be seen in figure 6. We judge that this looks much like a two-dimensional Gaussian curve, and since the marginal modes seem to be 11 for both wind speeds, we set $\boldsymbol{\mu} = [11, 11]$ as our mean vector for the Gaussian instrumental distribution $g_X(x)$. The covariance matrix $\boldsymbol{\Sigma}$ of g_X is hard to choose properly from figure 6 alone, and we instead opt to make a similar surface plot for $P(V_1)P(V_2)\frac{f_{V_1,V_2}}{g_{V_1,V_2}}$, using different $\boldsymbol{\Sigma}$, and finding where the surface is reasonably flat.

We found that this surface plot was very sensitive to our choice of $\boldsymbol{\Sigma}$. Setting the covariance σ_{12} too high, or setting the variances σ_{11} or σ_{22} either too low or too high, led to the surface plot having extreme peaks seeming almost like singularities. The opposite problem was when the central peak was too high compared

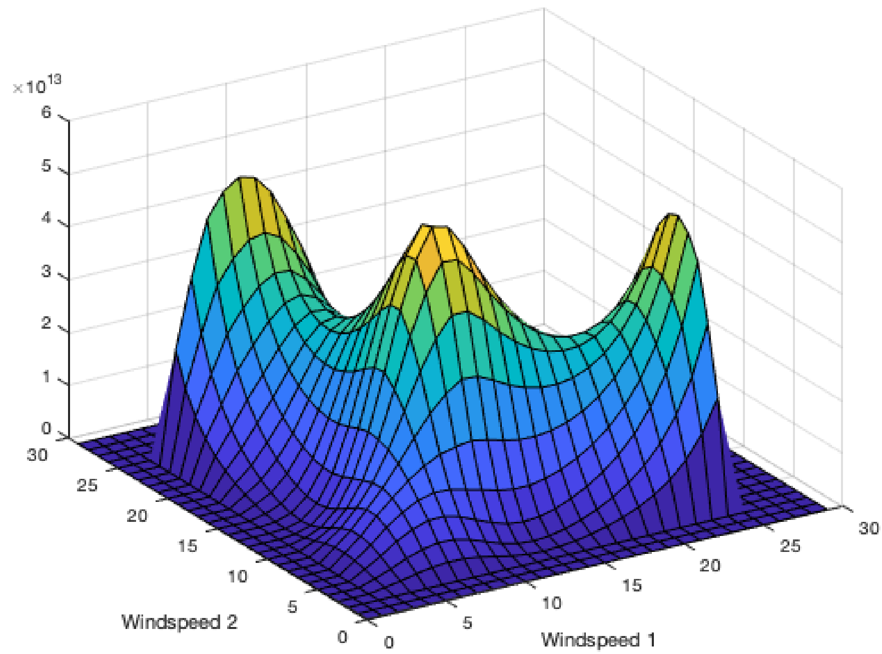

Figure 7: Surface plot of the expected product of power outputs at each simultaneous windspeed level, divided by $g_{V_1,V_2}(v_1,v_2) \sim \mathcal{N}_2\left([11,11], \begin{bmatrix} 21 & 0 \\ 0 & 21 \end{bmatrix}\right)$.

to the other points on the surface. We found a reasonable shape of the surface plot, with both the central peak and the edge peaks being of acceptable height, for

$$\boldsymbol{\Sigma} = \begin{bmatrix} 21 & 6 \\ 6 & 21 \end{bmatrix}$$

which may be seen in figure 7.

Having found a satisfactory instrumental distribution, we simulate a 1000×2 output matrix from this distribution, take the product of each row and multiply it by the importance weight function, $P(V_1)P(V_2)\omega_{V_1,V_2}(v_1,v_2) = P(V_1)P(V_2)\frac{f_{V_1,V_2}(v_1,v_2)}{g_{V_1,V_2}(v_1,v_2)}$, to get an output with the same expected value as the target distribution $P(V_1)P(V_2)f_{V_1,V_2}(v_1,v_2)$. We find the estimate of this expectation to be

$$\hat{\mathbb{E}}(P(V_1)P(V_2)) = 2.1568 \cdot 10^{13} \tag{74}$$

which we may then use, along with estimates of $\mathbb{E}(P(V_1))$ and $\mathbb{E}(P(V_2))$, to calculate the covariance. The expectations of the individual power outputs was estimated identically to the estimation of the power output in 2c), and the estimated covariance becomes

$$\hat{\mathbb{C}}(P(V_1), P(V_2)) = \hat{\mathbb{E}}(P(V_1)P(V_2)) - \mathbb{E}(P(V_1))\mathbb{E}(P(V_2)) \tag{75}$$

$$= 2.1568 \cdot 10^{13} - 4.3198 \cdot 10^{6} \cdot 4.2650 \cdot 10^{6} \tag{76}$$

$$= 3.1437 \cdot 10^{12} \tag{77}$$

c) Variance of the total power output

Estimate the variability $\mathbb{V}(P(V1) + P(V2))$ *in the amount of combined power generated by both turbines as well as the standard deviation* $\mathbb{D}(P(V1) + P(V2))$.

In order to find an estimate of this variance, we lack two expectations, namely $\mathbb{E}(P^2(V_1))$ and $\mathbb{E}(P^2(V_2))$. These, however, can easily be estimated by adjusting the simulation code in 2c) to produce samples of $P^2(V_1)$ instead of $P(V_1)$. Unlike the product expectation, this is a one-dimensional problem and simply required minor adjustments to the parameters of our instrumental truncated Weibull distribution.

This gave variance estimates

$$\begin{aligned}
\hat{\mathbb{V}}(P(V_1)) &= \hat{\mathbb{E}}(P^2(V_1)) - \hat{\mathbb{E}}^2(P(V_1)) && (78)\\
&= 2.5802 \cdot 10^{13} - (4.1398 \cdot 10^6)^2 && (79)\\
&= 7.1417 \cdot 10^{12} && (80)\\
\hat{\mathbb{V}}(P(V_2)) &= \hat{\mathbb{E}}(P^2(V_2)) - \hat{\mathbb{E}}^2(P(V_2)) && (81)\\
&= 2.5886 \cdot 10^{13} - (4.2650 \cdot 10^6)^2 && (82)\\
&= 7.6960 \cdot 10^{12} && (83)
\end{aligned}$$

which finally allow us to calculate an estimated variance of the total power output:

$$\begin{aligned}
\hat{\mathbb{V}}(P(V_1) + P(V_2)) &= \hat{\mathbb{V}}(P(V_1)) + \hat{\mathbb{V}}(P(V_2)) + 2\hat{\mathbb{C}}(P(V_1), P(V_2)) && (84)\\
&= 7.1417 \cdot 10^{12} + 7.6960 \cdot 10^{12} + 2 \cdot 3.1437 \cdot 10^{12} && (85)\\
&= 2.1125 \cdot 10^{13} && (86)
\end{aligned}$$

Finally, we take the square root of this estimate to get an estimate of the standard deviation:

$$\begin{aligned}
\hat{\mathbb{D}}(P(V_1) + P(V_2)) &= \sqrt{\hat{\mathbb{V}}(P(V_1) + P(V_2))} && (87)\\
&= \sqrt{2.1125 \cdot 10^{13}} && (88)\\
&= 4.5962 \cdot 10^6 && (89)
\end{aligned}$$

d) Probability estimation

Find an approximate 95% confidence interval for the probability $\mathbb{P}(P(V1) + P(V2) > 9.5MW)$ *that the combined power generated by the two turbines is greater than half of their installed capacity and for* $\mathbb{P}(P(V1)+ P(V2) < 9.5MW)$. *(use importance sampling or other variance reduction techniques for both probabilities) Do the probabilities sum to one or not? Why?*

The difference between this problem and our previous two-dimensional problem is that we are not trying to estimate the mean $\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P(V_1)P(V_2)f_{V_1,V_2}(v_1, v_2)dxdy$, but rather the probability that the sum of the powers is above or below a certain value. We choose to express this as an indicator function which

Probability	Estimate	Variance	CI lower	CI upper
$\mathbb{P}(P(V1) + P(V2) > 9.5MW)$	0.3747	0.0911	0.3560	0.3934
$\mathbb{P}(P(V1) + P(V2) < 9.5MW)$	0.5987	2.0652	0.5097	0.6878

Table 9: Estimates of the probabilities that the total power output is above and below 9.5 MW.

becomes 1 if the inequality is fulfilled or zero otherwise, and thus do estimates of

$$\tau_{greater} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{1}_{[P(V_1)+P(V_2)>9.5]} f_{V_1,V_2}(v_1, v_2)dxdy \tag{90}$$

$$\tau_{less} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{1}_{[P(V_1)+P(V_2)<9.5]} f_{V_1,V_2}(v_1, v_2)dxdy \tag{91}$$

$$\tag{92}$$

Since these τ also depend on the bivariate Weibull density function which we are not able to invert, we used the same two-dimensional importance sampling method as in problem 3b). Our instrumental distribution was the same two-dimensional normal distribution with mean vector $\boldsymbol{\mu} = \begin{bmatrix} 11 & 11 \end{bmatrix}$ and covariance matrix $\boldsymbol{\Sigma} = \begin{bmatrix} 21 & 5 \\ 5 & 21 \end{bmatrix}$ as in problem 3b). After sampling from the instrumental distribution, we run the resulting wind speeds through the power function and then run the sum through the indicator function. Finally, this indicator function is multiplied with the importance weight function and we take the mean of this product vector as our estimate.

This gives us the estimates of the probabilities featured in table 9.

Something we wish to address is that the probability of the sum being less than 9.5 MW has much higher variance than the probability that it is greater. This is reasonably due to the fact that the instrumental distribution more often produces simulated values which give total power greater than 9.5 MW. This means that the importance weight function gives large weight each to few observations when estimating the probability of being less than 9.5 MW, and less weight each to many observations when estimating the probability of being greater than 9.5 MW. For one estimator, we thus have many zeroes and some very large values, leading to a much larger sample variance.

Another thing to observe is that the point estimates do not sum to 1, but rather 0.9734. Normally this should never be the case for continuous distributions or functions thereof, since the probability of any outcome is zero. However, in this case there is a non-zero probability that the power output of one turbine is 9.5 MW while the power output of the other is zero. Since this exact power output can arise for intervals of wind speeds rather than for single values, it is non-zero. It can in fact be calculated explicitly as

$$
\begin{aligned}
\mathbb{P}(P(V_1)+P(V_2)=9.5\cdot 10^6) &= \mathbb{P}(P(V_1)=0)\cdot\mathbb{P}(P(V_2)=9500|P(V_1)=0) && (93)\\
&+\mathbb{P}(P(V_2)=0)\cdot\mathbb{P}(P(V_1)=9500|P(V_2)=0) && (94)\\
&=2\cdot\mathbb{P}(P(V_1)=0,P(V_2)=9500) && (95)\\
&=2\cdot\mathbb{P}((V_1>25\cup V_1<3.5)\cap(14<V_2<25)) && (96)\\
&=2(\mathbb{P}((V_1<3.5)\cap(14<V_2<25))+\mathbb{P}((V_1>25)\cap(14<V_2<25))) && (97)\\
&=2(\mathbb{P}((V_1<3.5)\cap(14<V_2<25))+\mathbb{P}(14<V_2<25) && (98)\\
&-\mathbb{P}((V_1<25)\cap(14<V_2<25))) && (99)\\
&=2(F_{V_1,V_2}(3.5,25)-F_{V_1,V_2}(3.5,14)+F_{V_2}(25)-F_{V_2}(14) && (100)\\
&-(F_{V_1,V_2}(25,25)-F_{V_1,V_2}(25,14))) && (101)\\
&=2\bigg(F_{V_1}(3.5)F_{V_2}(25)[1+0.638(1-F_{V_1}(3.5)^3)^{1.5}(1-F_{V_2}(25)^3)^{1.5}] && (102)\\
&-F_{V_1}(14)F_{V_2}(25)[1+0.638(1-F_{V_1}(14)^3)^{1.5}(1-F_{V_2}(25)^3)^{1.5}] && (103)\\
&+F_{V_2}(25)-F_{V_2}(14)-F_{V_2}^2(25)[1+0.638(1-F_{V_2}(3.5)^3)^3] && (104)\\
&+F_{V_1}(25)F_{V_2}(14)[1+0.638(1-F_{V_1}(25)^3)^{1.5}(1-F_{V_2}(14)^3)^{1.5}]\bigg) && (105)\\
&=2\bigg((1-e^{-(\frac{3.5}{9.13})^{1.96}})(1-e^{-(\frac{25}{9.13})^{1.96}})[1+0.638(1-(1-e^{-(\frac{3.5}{9.13})^{1.96}})^3)^{1.5} && (106)\\
&(1-(1-e^{-(\frac{25}{9.13})^{1.96}})^3)^{1.5}]-(1-e^{-(\frac{14}{9.13})^{1.96}})(1-e^{-(\frac{25}{9.13})^{1.96}})[1+0.638 && (107)\\
&(1-(1-e^{-(\frac{14}{9.13})^{1.96}})^3)^{1.5}(1-(1-e^{-(\frac{14}{9.13})^{1.96}})^3)^{1.5}]+(1-e^{-(\frac{25}{9.13})^{1.96}}) && (108)\\
&-(1-e^{-(\frac{14}{9.13})^{1.96}})-(1-e^{-(\frac{25}{9.13})^{1.96}})^2[1+0.638(1-(1-e^{-(\frac{25}{9.13})^{1.96}})^3)^3] && (109)\\
&+(1-e^{-(\frac{25}{9.13})^{1.96}})(1-e^{-(\frac{14}{9.13})^{1.96}})[1+0.638(1-(1-e^{-(\frac{25}{9.13})^{1.96}})^3)^{1.5} && (110)\\
&(1-(1-e^{-(\frac{14}{9.13})^{1.96}})^3)^{1.5}]\bigg) && (111)\\
&=2\bigg(0.141616\cdot 0.999255[1+0.638(1-0.141616^3)^{1.5}(1-0.999255^3)^{1.5}] && (112)\\
&-0.141616\cdot 0.900885[1+0.638(1-0.141616^3)^{1.5}(1-0.900885^3)^{1.5}] && (113)\\
&+0.999255-0.900885-0.999255^2[1+0.638(1-0.999255^3)^3] && (114)\\
&+0.999255\cdot 0.900885[1+0.638(1-0.999255^3)^{1.5}(1-0.900885^3)^{1.5}]\bigg) && (115)\\
&=2(0.141520-0.127588+0.999255-0.900885-0.998511+0.900222) && (116)\\
&=0.028 && (117)
\end{aligned}
$$

which agrees quite well with the sampled difference $1-0.9734=0.0266$.