

A non-contextual hidden variable model for quantum mechanics

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We propose a non-contextual hidden variable model, consistent with all predictions of quantum mechanics (QM). A careful scrutiny of consistency requirements of any hidden variable model with quantum mechanics and the corresponding no-go theorems, leads us to the conclusion that the notion of contextuality is not a necessary feature of QM. The alternative view that emerges, hinges on identifying a new classical property which we call “multiplicativity”. It turns out that by relaxing this condition of “multiplicativity” on hidden variable models, they can be made consistent with QM. The connection of non-multiplicativity to Bohm’s model and to apparent contextuality is also discussed.

Quantum mechanics has been one of the most successful theories in physics so far, however, there has not yet been a final word on its completeness and interpretation. Einstein’s [1] work on the incompleteness of QM and the subsequent seminal work of Bell [2], assessing the compatibility of a more complete model involving hidden variables (HV) and locality with QM, has provided deep insights into how the quantum world differs from its classical counterpart. In recent times, these insights have been of pragmatic utility in the areas of quantum information processing, where EPR pairs are fundamental motifs of entanglement [3–5]. The work of Kochen Specker [6] and Gleason et. al. [7–10]) broadened the schism between HV models and QM. They showed that it was contextuality and not non-locality which was at the heart of this schism and the incompatibility between HV models and QM can arise even for a single quantum system. Contextuality has thus been identified as a fundamental non-classical feature of the quantum world and experiments have also been proposed and conducted to this effect [11, 12]. Contextuality, on the one hand has led to investigations on the foundational aspects of QM including attempts to prove completeness of QM [13, 14], and on the other hand has been harnessed for computation and cryptography [15, 16]. However, not all HV models are incompatible with QM, and a very important case in point is Bohm’s model based on precise trajectories for quantum systems [17, 18].

A closer look at the aforesaid no-go theorems reveals that contextuality is in fact not a necessary feature of QM (see Figure 1). While this has been shown earlier by taking disturbances due to measurements into consideration [19], we take a different approach and construct a HV model whose predictions match with those of QM for systems described by arbitrary finite-dimensional Hilbert spaces. The algebraic constraints obeyed by

quantum observables may not necessarily be obeyed by the predictions of an HV model at the level of individual outcomes. Those HV models that obey such algebraic constraints possess a special property which we call ‘multiplicativity’. Enlarging the scope of HV models to include non-multiplicative models, permits us to construct an explicit non-contextual model consistent with QM. The re-examination of the very proof of ‘contextuality’ is used to illustrate non-multiplicativity, an intrinsic quantum feature, which we propose as a possible alternative to contextuality.

Notation. (a) $\psi \in \mathcal{H}$ represents a pure quantum mechanical state of the system in the Hilbert space \mathcal{H} , (b) $\hat{\mathcal{H}}$ is defined to mean the set of Hermitian observables for the system, (c) $[\mathcal{H}]$ is defined to mean $(\mathcal{H}, \mathbb{R}^\otimes)$, which represents the state of the system including HVs, (d) $[\psi] \in [\mathcal{H}]$ will represent the state of the system, including HVs, (e) a prediction map is $M : \{\hat{\mathcal{H}}, [\mathcal{H}]\} \rightarrow \mathbb{R}$, (f) a sequence map is $S : \{\hat{\mathcal{H}}, [\mathcal{H}], \mathbb{R}\} \rightarrow [\mathcal{H}]$, (g) f is an arbitrary map from $\{\hat{\mathcal{H}}, \hat{\mathcal{H}}, \dots \hat{\mathcal{H}}\} \rightarrow \hat{\mathcal{H}}$ constructed using multiplication and addition of the observables, and multiplication with complex numbers¹, (h) \hat{f} is a map constructed by replacing observables in f with real numbers.

Definition 1. A theory is non-contextual, if it provides a map $M : \{\hat{\mathcal{H}}, [\mathcal{H}]\} \rightarrow \mathbb{R}$ to explain measurement outcomes. A theory which is not non-contextual is contextual.

Remark 1.1. The notion of non-contextuality may be extended to maps. A map is non-contextual if it is of the form $M : \{\hat{\mathcal{H}}, [\mathcal{H}]\} \rightarrow \mathbb{R}$; prediction maps as defined above, for individual observables, are non-contextual by construction.

Remark 1.2. Broader definitions in the literature have been suggested which declare a larger set of theories as non-contextual. For our purposes however, this restricted definition will suffice. The term contextual is used

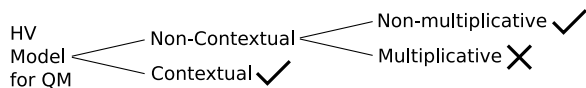


Figure 1. Non-contextuality is not inconsistent with QM

¹ Strictly, the map f would depend on the operators to ensure preservation of Hermiticity; thus a given f will be defined only for a subset of $\{\hat{\mathcal{H}}, \hat{\mathcal{H}}, \dots \hat{\mathcal{H}}\}$.

to suggest that the value an operator takes might depend on which other compatible observable it is being measured with.

Before we re-examine the result of Kochen and Specker [6] that non-contextual models cannot be compatible with quantum mechanics and show that multiplicativity can provide an alternative to contextuality, we define multiplicativity more precisely.

Definition 2. A prediction map M is *multiplicative* iff

$$M(f(\hat{B}_1, \hat{B}_2, \dots, \hat{B}_N), [\psi]) = \tilde{f}(M(\hat{B}_1, [\psi]), M(\hat{B}_2, [\psi]), \dots, M(\hat{B}_N, [\psi])),$$

where $\hat{B}_i \in \hat{\mathcal{H}}$ are arbitrary mutually commuting observables and $[\psi] \in [\mathcal{H}]$. A *non-multiplicative* map is one that is not multiplicative.

Note that if M is taken to represent the measurement outcome (in QM), then for states of the system which are simultaneous eigenkets of \hat{B}_i s, M must clearly be multiplicative. It is, however, not obvious that this property must always hold. For example, consider two spin-half particles in the state $|1\rangle \otimes |1\rangle$ written in the computational basis and take operators $\hat{B}_1 = \hat{\sigma}_x \otimes \hat{\sigma}_x$, $\hat{B}_2 = \hat{\sigma}_y \otimes \hat{\sigma}_y$ and $\hat{C} = \hat{B}_1 \hat{B}_2 = -\hat{\sigma}_z \otimes \hat{\sigma}_z$ written in terms of Pauli operators. We must have $M(\hat{C}) = -1$ while $M(\hat{B}_1) = \pm 1$ and $M(\hat{B}_2) = \pm 1$ independently, according to QM, with probability half. Here multiplicativity does not seem to hold. Antithetically, it is clear that if one first measures \hat{B}_1 and subsequently measures \hat{B}_2 , then the product of the results must be -1 . This is consistent with measuring \hat{C} . To capture this property we define *sequential multiplicativity* as follows.

Definition 3. A prediction map m is *sequentially multiplicative* for a given sequence map s , iff

$$M(f(\hat{B}_1, \hat{B}_2, \dots, \hat{B}_N), [\psi_1]) = \tilde{f}(M(\hat{B}_1, [\psi_{k_1}]), M(\hat{B}_2, [\psi_{k_2}]), \dots, M(\hat{B}_N, [\psi_{k_N}])),$$

where $\mathbf{k} = (k_1, k_2, \dots, k_N) \in \{N!\text{ permutations of } k'\text{'s}\}$, $\hat{B}_i \in \hat{\mathcal{H}}$ are arbitrary mutually commuting observables, $[\psi_i] \in [\mathcal{H}]$ and $[\psi_{k+1}] := S(\hat{B}_k, [\psi_k], M(\hat{B}_k, [\psi_k]))$, $\forall [\psi_i]$.

With these definitions we are now ready to discuss the ‘proof of contextuality’. We first state the contextuality theorem in our notation:

Theorem. Let a map $M : \hat{\mathcal{H}} \rightarrow \mathbb{R}$, be s.t. (a) $M(\hat{\mathbb{I}}) = 1$, (b) $M(f(\hat{B}_1, \hat{B}_2, \dots)) = \tilde{f}(M(\hat{B}_1), M(\hat{B}_2), \dots)$, for any arbitrary function f , where \hat{B}_i are mutually commuting Hermitian operators. If m is assumed to describe the outcomes of measurements, then no M exists which is consistent with all predictions of QM.

Proof. Peres Mermin [PM] ($|\mathcal{H}| \geq 4$) [9, 10]: For a system composed of two spin-half particles consider the following set of operators

$$\hat{A}_{ij} \doteq \begin{bmatrix} \hat{\mathbb{I}} \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\mathbb{I}} & \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y & \hat{\sigma}_y \otimes \hat{\sigma}_y \\ \hat{\sigma}_y \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\sigma}_y & \hat{\sigma}_z \otimes \hat{\sigma}_z \end{bmatrix}$$

which have the property that all operators along a row (or column) commute. Further, the product of rows (or columns) yields $\hat{R}_i = \hat{\mathbb{I}}$ and $\hat{C}_j = \hat{\mathbb{I}}$ ($j \neq 3$), $\hat{C}_3 = -\hat{\mathbb{I}}$, ($\forall i, j$) where $\hat{R}_i := \prod_j \hat{A}_{ij}$, $\hat{C}_j := \prod_i \hat{A}_{ij}$. Let us assume that M exists. From property (b) of the map, to get $M(\hat{C}_3) = -1$ (as required by property (a)), we must have an odd number of -1 assignments in the third column. In the remaining columns, the number of -1 assignments must be even (for each column). Thus, in the entire square, the number of -1 assignments must be odd. Let us use the same reasoning, but along the rows. Since each $M(\hat{R}_i) = 1$, we must have even number of -1 assignments along each row. Thus, in the entire square, the number of -1 assignments must be even. We have arrived at a contradiction and therefore we conclude that our assumption that a consistent M exists, must be wrong. \square

Remark. One could in principle assume M , to be s.t. (a) $M(\hat{\mathbb{I}}) = 1$, (b) $M(\alpha \hat{B}_i) = \alpha M(\hat{B}_i)$, for $\alpha \in \mathbb{R}$, (c) $M(\hat{B}_i^2) = M(\hat{B}_i)^2$, (d) $M(\hat{B}_i + \hat{B}_j) = M(\hat{B}_i) + M(\hat{B}_j)$, to deduce (d') $M(\hat{B}_i \hat{B}_j) = M(\hat{B}_i)M(\hat{B}_j)$ and that $M(\hat{B}_i) \in \text{spectrum of } \hat{B}_i$. Effectively then, condition (b) listed in the theorem is satisfied as a consequence. Therefore, assuming (a)-(d) as listed here, rules out a larger class of M [6].

Here M maybe viewed as a specific class of prediction maps, that implicitly depends on the state $[\psi]$. It is clear that according to the theorem, non-contextual maps which are *multiplicative* must be incompatible with QM. However, non-contextual maps which are *non-multiplicative* might be consistent with QM. Indeed it is and we will show this by constructing an explicit model.

Before proceeding we note, however, that QM can enforce *sequential multiplicativity*. This result will be used to assess accuracy of any proposal of a HV model.

Proposition. Let a quantum mechanical system be in a state, s.t. measurement of \hat{C} yields repeatable results (same result each time). Then according to QM, sequential multiplicativity holds, where $\hat{C} := f(\hat{B}_1, \hat{B}_2, \dots, \hat{B}_n)$, and \hat{B}_i are as defined (in Definition 3)

Proof. Without loss of generality we can take $\hat{B}_1, \hat{B}_2, \dots, \hat{B}_n$ to be mutually compatible and a complete set of observables (operators can be added to make the set complete if it not so). It follows that $\exists |\mathbf{b} = (b_1, b_2, \dots, b_n)\rangle$ s.t. $\hat{B}_i |\mathbf{b}\rangle = b_i |\mathbf{b}\rangle$, and that

$\sum_{\mathbf{b}} |\mathbf{b}\rangle \langle \mathbf{b}| = \hat{\mathbb{I}}$. Let the state of the system $|\psi\rangle$ be s.t. $\hat{C}|\psi\rangle = c|\psi\rangle$. For the statement to follow, one need only show that $|\psi\rangle$ must be made of only those $|\mathbf{b}\rangle$ s, which satisfy $c = \tilde{f}(b_1, b_2, \dots, b_n)$. This is the crucial step and proving this is straightforward. We start with $\hat{C}|\psi\rangle = c|\psi\rangle$ and take its inner product with $\langle \mathbf{b}|$ to get

$$\begin{aligned} \langle \mathbf{b} | \hat{C} | \psi \rangle &= c \langle \mathbf{b} | \psi \rangle, \\ \langle \mathbf{b} | f(\hat{B}_1, \hat{B}_2, \dots, \hat{B}_n) | \psi \rangle &= c \langle \mathbf{b} | \psi \rangle, \\ \tilde{f}(b_1, b_2, \dots, b_n) \langle \mathbf{b} | \psi \rangle &= c \langle \mathbf{b} | \psi \rangle. \end{aligned}$$

Also, we have $|\psi\rangle = \sum_{\mathbf{b}} \langle \mathbf{b} | \psi \rangle |\mathbf{b}\rangle$, from completeness. If we consider $|\mathbf{b}\rangle$ s for which $\langle \mathbf{b} | \psi \rangle \neq 0$, then we can conclude that indeed $c = \tilde{f}(b_1, b_2, \dots, b_n)$. However, when $\langle \mathbf{b} | \psi \rangle = 0$, viz. $|\mathbf{b}\rangle$ s that are orthogonal to $|\psi\rangle$, then nothing can be said but that is irrelevant. We can thus conclude that $|\psi\rangle$ is made only of those $|\mathbf{b}\rangle$ s that satisfy the required relation. \square

It is worth noting that in the Peres Mermin case, where \hat{R}_i and \hat{C}_j are just $\pm \hat{\mathbb{I}}$, it follows that all states are their eigenstates. Consequently, for these operators *sequential multiplicativity* must always hold. We are now ready to describe our explicit model. Let the state of a finite dimensional quantum system be $|\chi\rangle$. We wish to assign a value to an arbitrary observable $\hat{A} = \sum_a a |a\rangle \langle a|$, which has eigenvectors $\{|a_j\rangle\}$. The corresponding ordered eigenvalues are $\{a_j\}$ such that $a_{\min} = a_1$ and $a_{\max} = a_n$.

Our HV model for QM assigns values in the following three steps:

1. **Initial HV:** Pick a number $c \in [0, 1]$, from a uniform random distribution.
2. **Assignment or Prediction:** The value assigned to \hat{A} is given by finding the smallest a s.t. $c \leq \sum_{a'=a_{\min}}^a |\langle a' | \chi \rangle|^2$, viz. we have specified a prediction map, $M(\hat{A}, [\chi]) = a$.

3. **Update:** After measuring an operator, the state is updated (collapsed) in accordance with the rules of QM. This completely specifies the sequence map S .

To see how this works consider a spin-half particle in the state $|\chi\rangle = \cos\theta |0\rangle + \sin\theta |1\rangle$ and the observable $\hat{A} = \hat{\sigma}_z = |0\rangle \langle 0| - |1\rangle \langle 1|$. Now, according to the postulates of this theory, $M(\hat{A}, [\chi]) = +1$. If the randomly generated value for $c \leq \cos^2\theta$ else \hat{A} is assigned -1 , it then follows, from c being uniformly random in $[0, 1]$ that the statistics agree with the predictions of QM.

Note that the assignment described by the prediction map M is non-contextual since, given an operator and a state (+ the HV c), the value is uniquely assigned. The map M is however, non-multiplicative.

To bring out the non-multiplicativity of our model and to convince the reader of its generic validity, we apply it to the Peres Mermin situation of two spin-half particles. Consider the initial state of the system $|\psi_1\rangle =$

$|00\rangle$. Assume we obtained $c = 0.4$ as a random choice. To arrive at the assignments, note that $|00\rangle$ is an eigenket of only \hat{R}_i, \hat{C}_j and $\hat{A}_{33} = \hat{\sigma}_z \otimes \hat{\sigma}_z$. Thus, in the first iteration, all these should be assigned their respective eigenvalues. The remaining operators must be assigned -1 as one can readily verify by explicitly finding the smallest a as described in postulate 2 of the model (see Table I). This model is manifestly *non-multiplicative*, for $M_1(\hat{C}_3) = 1 \neq M_1(\hat{A}_{13})M_1(\hat{A}_{23})M_1(\hat{A}_{33}) = -1$, where the subscript refers to the iteration number. More precisely, $M_1(\hat{O}) := M(\hat{O}, [|\psi_1\rangle = |00\rangle])$ where the complete state $[|\psi_1\rangle]$ implicitly refers to both the quantum state $|00\rangle$ and the HV $c = 0.4$. The model however obeys the *sequential multiplicativity* as required for consistency, namely, $M_1(\hat{C}_3) = M_1(\hat{A}_{33})M_2(\hat{A}_{23})M_3(\hat{A}_{13})$, where $M_2 := M(\hat{O}, [|\psi_2\rangle])$, $M_3 := M(\hat{O}, [|\psi_3\rangle])$ and $|\psi_2\rangle, |\psi_3\rangle$ are obtained from postulate 3. Note that for each iteration, a new HV is generated. To illustrate *sequential multiplicativity*, we must choose to measure \hat{A}_{33} . According to step 3, since $|00\rangle$ is an eigenstate of \hat{A}_{33} , the final state remains $|00\rangle$.

For the next iteration, $i = 2$, after the first measurement is over say the random number generator yielded the value $c = 0.1$. Since $|\psi\rangle$ is also unchanged the assignment remains invariant (in fact any of $c < 0.5$ would yield the same result as evident from the previous exercise). For the final step we choose to measure $\hat{A}_{23} (= \hat{\sigma}_y \otimes \hat{\sigma}_y)$, to proceed with sequentially measuring \hat{C}_3 . To simplify calculations, we note $|00\rangle = \frac{1}{\sqrt{2}} ((|\tilde{+}\tilde{-}\rangle + |\tilde{-}\tilde{+}\rangle)/\sqrt{2} + (|\tilde{+}\tilde{+}\rangle + |\tilde{-}\tilde{-}\rangle)/\sqrt{2})$, with $|\tilde{\pm}\rangle = |0\rangle \pm i|1\rangle$ (eigenkets of $\hat{\sigma}_y$). Since $|00\rangle$ is manifestly not an eigenket of \hat{A}_{23} , we must find an appropriate eigenket $|a_{23}^-\rangle$ s.t. $\hat{A}_{23}|a_{23}^-\rangle = -|a_{23}^-\rangle$, since $c = 0.1$ and $\langle a_{23}^- | 00 \rangle$ is already > 0.1 . It is evident that $|a_{23}^-\rangle = (|\tilde{+}\tilde{-}\rangle + |\tilde{-}\tilde{+}\rangle)/\sqrt{2} = (|00\rangle + |11\rangle)/\sqrt{2}$, which becomes the final state.

For the final iteration, $i = 3$, say we obtain $c = 0.7$. So far, we have $M_1(\hat{A}_{33}) = 1$ and $M_2(\hat{A}_{23}) = -1$. We must obtain $M_3(\hat{A}_{13}) = 1$, independent of the value of c , to be consistent. Let us check that. Indeed, according to postulate 2, since $\hat{\sigma}_x \otimes \hat{\sigma}_x (|00\rangle + |11\rangle)/\sqrt{2} = 1(|00\rangle + |11\rangle)/\sqrt{2}$, $M_3(\hat{A}_{13}) = 1$ for all allowed values of c . It is to be noted that $M_2(\hat{A}_{33}) = M_3(\hat{A}_{33})$ and $M_2(\hat{A}_{23}) = M_3(\hat{A}_{23})$, which essentially expresses the compatibility of these observables; i.e. once measured, the values of observables compatible with \hat{A}_{13} are not affected by the measurement of \hat{A}_{13} .

This has implications to Bohm's HV model (BHVM), where we have the notion of precise trajectories for particles guided by the wavefunction. In BHVM if the initial conditions are precisely known the entire trajectory of a particle can be predicted including the individual outcome of measurements. For the case of a single spin-half particle in a Stern-Gerlach experiment, BHVM can predict opposite results for two measurements of the same operator. This observation is often

$i = 1, c = 0.4, \psi_{\text{init}}\rangle = 00\rangle$	$i = 2, c = 0.1, \psi_{\text{init}}\rangle = 00\rangle$	$i = 3, c = 0.7, \psi_{\text{init}}\rangle = \frac{ 00\rangle + 11\rangle}{\sqrt{2}}$
$M_1(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$	$M_2(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$	$M_3(\hat{A}_{ij}) \doteq \begin{bmatrix} +1 & +1 & +1 \\ +1 & +1 & -1 \\ +1 & +1 & +1 \end{bmatrix}$
$M_1(\hat{R}_i), M_1(\hat{C}_j) = +1 (j \neq 3)$ $M_1(\hat{C}_3) = -1$	$M_2(\hat{R}_i), M_2(\hat{C}_j) = +1 (j \neq 3)$ $M_2(\hat{C}_3) = -1$	$M_3(\hat{R}_i), M_3(\hat{C}_j) = +1 (j \neq 3)$ $M_3(\hat{C}_3) = -1$
$\hat{A}_{33} = \hat{\sigma}_z \otimes \hat{\sigma}_z; M_1(\hat{A}_{33}) = +1$ $ \psi_{\text{final}}\rangle = 00\rangle$	$\hat{A}_{23} = \hat{\sigma}_y \otimes \hat{\sigma}_y; M_2(\hat{A}_{23}) = -1$ $ \psi_{\text{final}}\rangle = \frac{ 00\rangle + 11\rangle}{\sqrt{2}}$	$\hat{A}_{13} = \hat{\sigma}_x \otimes \hat{\sigma}_x; M_3(\hat{A}_{13}) = +1$ $ \psi_{\text{final}}\rangle = \frac{ 00\rangle + 11\rangle}{\sqrt{2}}$

Table I. HV model applied to the Peres Mermin situation

used as a demonstration of ‘contextuality’ in BM, although we did not even speak of compatible measurements [20]. It turns out, however, that if one restricts oneself to a one-one map between an experiment and an observable (by following a certain convention), the so-called ‘contextuality’ can be removed from BHVM. One finds that the map so obtained from observables to their predictions turns out to be non-multiplicative, suggesting that at least in this case, non-multiplicativity is a more suitable explanation of non-classicality.

We have constructed an explicit non-contextual HV model consistent with QM which is non-multiplicative. We illustrate how non-multiplicativity gives rise to situations which could get confused with the presence of contextuality.

Imagine $\hat{B}_1 = \hat{\sigma}_z \otimes \hat{\mathbb{I}} = |00\rangle\langle 00| + |01\rangle\langle 01| - |10\rangle\langle 10| + |11\rangle\langle 11|$, $\hat{B}_2 = \hat{\mathbb{I}} \otimes \hat{\sigma}_z = |10\rangle\langle 10| + |11\rangle\langle 11| - |00\rangle\langle 00| + |01\rangle\langle 01|$, while we define $\hat{C} = f(\{\hat{B}_i\}) = 0 \cdot |00\rangle\langle 00| + 1 \cdot |01\rangle\langle 01| + 2 \cdot |10\rangle\langle 10| + 3 \cdot |11\rangle\langle 11|$. \hat{C} maybe viewed as a function of \hat{B}_1 , \hat{B}_2 and other operators \hat{B}_i which are constructed to obtain a maximally commuting set. A measurement of \hat{C} , will collapse the state into one of the states which are simultaneous eigenkets of B_1 and B_2 . Consequently, from the observed value of \hat{C} , one can deduce the values of \hat{B}_1 and \hat{B}_2 . Now consider $\sqrt{2}|\chi\rangle = |10\rangle + |01\rangle$, for which $M_1(\hat{B}_1) = 1$, and $M_1(\hat{B}_2) = 1$, using our model, with $c < 0.5$. However, $M_1(\hat{C}) = 1$, from which one can deduce that B_1 was $+1$, while B_2 was -1 . This property itself, one may be tempted call contextuality, viz. the value of B_1 depends on whether it is measured alone or with the remaining $\{B_i\}$. However, it must be noted that B_1 has a well defined value, and so does \hat{C} . Thus by our accepted definition, there is no contextuality. It is just that $M_1(\hat{C}) \neq \tilde{f}(M_1(\hat{B}_1), M_1(\hat{B}_2), \dots)$, viz. the theory is non-multiplicative. Note that after measuring \hat{C} however, $M_2(\hat{B}_1) = +1$ and $M_2(\hat{B}_2) = -1$ (for any value of c) consistent with those deduced by measuring \hat{C} .

Evidently, multiplicativity must hold for the common eigenkets of \hat{B}_i . Consequently, any violation of multiplicativity must arise from states that are superpositions or linear combinations of these common eigenkets.

Note, however, that entanglement is not necessary to show a violation; for instance the Peres Mermin test is a state independent test where a separable state can be used to arrive at a contradiction. Entanglement can be used to show non-locality of non-multiplicative prediction maps using Bell’s theorem however, the notion of non-multiplicativity is more basic.

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