A non-contextual hidden variable model for quantum mechanics

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We propose a non-contextual hidden variable model, consistent with all predictions of quantum mechanics (QM). A careful scrutiny of consistency requirements between any hidden variable model and quantum mechanics and the corresponding no-go theorems, leads us to the conclusion that the notion of contextuality is not a necessary feature of QM. The alternative view that emerges, hinges on identifying a new classical property, which we call "multiplicativity". It turns out that, by relaxing this condition of "multiplicativity" on hidden variable models, they can be made consistent with QM. Advantages of this view are illustrated by considering the implications of non-multiplicativity to locality, entanglement, superposition, and apparent contextuality.

Quantum mechanics has been one of the most successful theories in physics so far, however, there has not yet been a final word on its completeness and interpretation [?]. Einstein's [1] work on the incompleteness of QM and the subsequent seminal work of Bell [2], assessing the compatibility of a more complete model involving hidden variables (HV) and locality with QM, has provided deep insights into how the quantum world differs from its classical counterpart. In recent times, these insights have been of pragmatic utility in the areas of quantum information processing, where EPR pairs are fundamental motifs of entanglement [3, 4]. The work of Kochen Specker [5] and Gleason et. al. [6–9]) broadened the schism between HV models and QM. They showed that it was contextuality and not non-locality which was at the heart of this schism and the incompatibility between HV models and QM can arise even for a single quantum system. Contextuality has thus been identified as a fundamental non-classical feature of the quantum world and experiments have also been proposed and conducted to this effect [10–13]. Contextuality, on the one hand has led to investigations on the foundational aspects of QM [14?], and on the other hand has been harnessed for computation and cryptography [15? , 16]. Quantitative measures of contextuality based on memory [17] have also been used to demonstrate the completeness of QM [18]. However, not all HV models are incompatible with QM, and a very important case in point is Bohm's model based on precise trajectories for quantum systems [19, 20].

A closer look at the aforesaid no-go theorems reveals that contextuality is in fact not a necessary feature of QM (see figure 1). While this has been shown earlier by taking disturbances due to measurements into consideration [21, 22], we take a different approach and construct a HV model consistent with QM. The predictions of the aforesaid model match with those of QM for systems described by arbitrary finite-dimensional Hilbert spaces. The algebraic constraints obeyed by quantum observables may not necessarily be obeyed by the predictions of an HV model at the level of individual outcomes. Those HV models that obey such algebraic constraints possess a special property which we call 'multiplicativity'. Enlarging the scope of HV models to include non-multiplicative models, permits us to construct an explicit non-contextual model consistent with QM. The re-examination of the very proof of 'contextuality' is used to illustrate non-multiplicativity, an intrinsic quantum feature, which we propose as a possible alternative to contextuality.

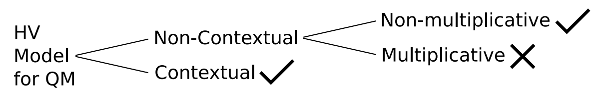

Figure 1. Non-contextuality is not inconsistent with QM

Notation: (a) $\psi \in \mathcal{H}$ represents a pure quantum mechanical state of the system in the Hilbert space $\mathcal{H}$, (b) $\hat{\mathcal{H}}$ is defined to mean the set of Hermitian observables for the system, (c) $[\mathcal{H}]$ is defined to mean $(\mathcal{H}, \mathbb{R}^{\otimes})$, which represents the state of the system including HVs, (d) $[\psi] \in [\mathcal{H}]$ will represent the state of the system, including HVs, (e) a prediction map is $m : \hat{\mathcal{H}}, [\mathcal{H}] \to \mathbb{R}$, (f) a sequence map is $s : \hat{\mathcal{H}}, [\mathcal{H}], \mathbb{R} \to [\mathcal{H}]$, (g) f is an arbitrary map from $\{\hat{\mathcal{H}}, \hat{\mathcal{H}}, \dots \hat{\mathcal{H}}\} \to \hat{\mathcal{H}}$ constructed using multiplication and addition of the observables, and multiplication with complex numbers[1], (h) $\tilde{f}$ is a map constructed by replacing observables in f with real numbers.

Definition 1. A theory is non-contextual, if it provides a map $m : \{\hat{\mathcal{H}}, [\mathcal{H}]\} \to \mathbb{R}$ to explain measurement outcomes. A theory which is not non-contextual is contextual.

Remark 1.1. The notion of non-contextuality may be extended to maps. A map is non-contextual if it is of the form $m : \{\hat{\mathcal{H}}, [\mathcal{H}]\} \to \mathbb{R}$; prediction maps Comment: Is this really needed "for individual observables" are non-contextual by construction.

Remark 1.2. Broader definitions in the literature have been suggested which declare a larger set of theories

[1] Strictly, the map f would depend on the operators to ensure preservation of Hermiticity; thus a given f will be defined only for a subset of $\{\hat{\mathcal{H}}, \hat{\mathcal{H}}, \dots \hat{\mathcal{H}}\}$.

as non-contextual. For our purposes however, this restricted definition will suffice. The term contextual is used to suggest that the value an operator takes might depend on which other compatible observable it is being measured with.

Before we re-examine the result of Kochen and Specker [5] that non-contextual models cannot be compatible with quantum mechanics and show that multiplicativity can provide an alternative to contextuality, we define multiplicativity more precisely.

Definition 2. A prediction map m is *multiplicative* iff

$$m(f(\hat{B}_1, \hat{B}_2, \dots \hat{B}_N), [\psi]) = \tilde{f}(m(\hat{B}_1, [\psi]), m(\hat{B}_2, [\psi]), \dots m(\hat{B}_N, [\psi])),$$

where $\hat{B}_i \in \hat{\mathcal{H}}$ are arbitrary mutually commuting observables and $[\psi] \in [\mathcal{H}]$. A *non-multiplicative* map is one that is not multiplicative.

Note that if m is taken to represent the measurement outcome (in QM), then for states of the system which are simultaneous eigenkets of $\hat{B}_i$s, m must clearly be multiplicative. It is, however, not obvious that this property must always hold. For example, consider two spin-half particles in the state $|1\rangle \otimes |1\rangle$ written in the computational basis and take operators $\hat{B}_1 = \hat{\sigma}_x \otimes \hat{\sigma}_x$, $\hat{B}_2 = \hat{\sigma}_y \otimes \hat{\sigma}_y$ and $\hat{C} = \hat{B}_1\hat{B}_2 = -\hat{\sigma}_z \otimes \hat{\sigma}_z$ written in terms of Pauli operators. We must have $m(\hat{C}) = -1$ while $m(\hat{B}_1) = \pm 1$ and $m(\hat{B}_2) = \pm 1$ independently, according to QM, with probability half. Here multiplicativity does not seem to hold. Antithetically, it is clear that if one first measures $\hat{B}_1$ and subsequently measures $\hat{B}_2$, then the product of the results must be -1. This is consistent with measuring $\hat{C}$. To capture this property we define *sequential multiplicativity* as follows.

Definition 3. A prediction map m is *sequentially multiplicative* for a given sequence map s, iff

$$m(f(\hat{B}_1, \hat{B}_2, \dots \hat{B}_N), [\psi_1]) = \tilde{f}(m(\hat{B}_1, [\psi_{k_1}]), m(\hat{B}_2, [\psi_{k_2}]), \dots, m(\hat{B}_N, [\psi_{k_N}])),$$

where $\mathbf{k} = (k_1, k_2, \dots, k_N) \in \{N! \text{ permutations of } k'\text{s}\}$, $\hat{B}_i \in \hat{\mathcal{H}}$ are arbitrary mutually commuting observables, $[\psi_i] \in [\mathcal{H}]$ and $[\psi_{k+1}] := s(\hat{B}_k, [\psi_k], m(\hat{B}_k, [\psi_k]))$, $\forall\, [\psi_i]$.

With these definitions we are now ready to discuss the 'proof of contextuality'. We first state the contextuality theorem in our notation:

Theorem 1. *Let a map $m : \hat{\mathcal{H}} \to \mathbb{R}$, be s.t. (a) $m(\hat{\mathbb{I}}) = 1$, (b) $m(f(\hat{B}_1, \hat{B}_2, \dots)) = \tilde{f}(m(\hat{B}_1), m(\hat{B}_2), \dots)$, for any arbitrary function f, where $\hat{B}_i$ are mutually commuting Hermitian operators. If m is assumed to describe the outcomes of measurements, then no m exists which is consistent with all predictions of QM.*

Proof. Peres Mermin ($|\mathcal{H}| \geq 4$) [8, 9]: For a system composed of two spin-half particles consider the following set of operators

$$\hat{A}_{ij} \doteq \begin{bmatrix} \hat{\mathbb{I}} \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\mathbb{I}} & \hat{\sigma}_x \otimes \hat{\sigma}_x \\ \hat{\sigma}_y \otimes \hat{\mathbb{I}} & \hat{\mathbb{I}} \otimes \hat{\sigma}_y & \hat{\sigma}_y \otimes \hat{\sigma}_y \\ \hat{\sigma}_y \otimes \hat{\sigma}_x & \hat{\sigma}_x \otimes \hat{\sigma}_y & \hat{\sigma}_z \otimes \hat{\sigma}_z \end{bmatrix}$$

which have the property that all operators along a row (or column) commute. Further, the product of rows (or columns) yields $\hat{R}_i = \mathbb{I}$ and $\hat{C}_j = \mathbb{I}\,(j \neq 3)$, $\hat{C}_3 = -\mathbb{I}$, $(\forall\, i, j)$ where $\hat{R}_i := \prod_j \hat{A}_{ij}$, $\hat{C}_j := \prod_i \hat{A}_{ij}$. Let us assume that m exists. From property (b) of the map, to get $m(\hat{C}_3) = -1$ (as required by property (a)), we must have an odd number of -1 assignments in the third column. In the remaining columns, the number of -1 assignments must be even (for each column). Thus, in the entire square, the number of -1 assignments must be odd. Let us use the same reasoning, but along the rows. Since each $m(\hat{R}_i) = 1$, we must have even number of -1 assignments along each row. Thus, in the entire square, the number of -1 assignments must be even. We have arrived at a contradiction and therefore we conclude that our assumption that a consistent m exists, must be wrong. □

Remark 1.1. One could in principle assume m, to be s.t. (a) $m(\hat{\mathbb{I}}) = 1$, (b) $m(\alpha\hat{B}_i) = \alpha m(\hat{B}_i)$, for $\alpha \in \mathbb{R}$, (c) $m(\hat{B}_i^2) = m(\hat{B}_i)^2$, (d) $m(\hat{B}_i + \hat{B}_j) = m(\hat{B}_i) + m(\hat{B}_j)$, to deduce (d') $m(\hat{B}_i\hat{B}_j) = m(\hat{B}_i)m(\hat{B}_j)$ and that $m(\hat{B}_i) \in$ spectrum of $\hat{B}_i$. Effectively then, condition (b) listed in the theorem is satisfied as a consequence. Therefore, assuming (a)-(d) as listed here, rules out a larger class of m. [5]

Here m maybe viewed as a specific class of prediction maps, that implicitly depends on the state $[\psi]$. It is clear that according to Theorem 1, non-contextual maps which are *multiplicative* must be incompatible with QM. However, non-contextual maps which are non-*multiplicative* might be consistent with QM. Indeed it is and we will show this by constructing an explicit model.

Before proceeding we note, however, that QM can enforce *sequential multiplicativity.* This result will be used to assess accuracy of any proposal of a HV model.

Proposition. *Let a quantum mechanical system be in a state, s.t. measurement of $\hat{C}$ yields repeatable results (same result each time). Then according to QM,* sequential multiplicativity *holds, where $\hat{C} := f(\hat{B}_1, \hat{B}_2, \dots \hat{B}_n)$, and $\hat{B}_i$ are as defined (in Definition 3)*

Proof. Without loss of generality we can take $\hat{B}_1, \hat{B}_2, \dots \hat{B}_n$ to be mutually compatible and a complete set of operators (Operators can be added to make the set complete if it not so). It follows that $\exists\ |\mathbf{b} = (b_1, b_2, \dots b_n)\rangle$ s.t. $\hat{B}_i\,|\mathbf{b}\rangle = b_i\,|\mathbf{b}\rangle$, and that

$\sum_{\mathbf{b}} |\mathbf{b}\rangle\langle\mathbf{b}| = \hat{\mathbb{I}}$. Let the state of the system $|\psi\rangle$ be s.t. $\hat{C}|\psi\rangle = c|\psi\rangle$. For the statement to follow, one need only show that $|\psi\rangle$ must be made of only those $|\mathbf{b}\rangle$s, which satisfy $c = \tilde{f}(b_1, b_2, \dots b_n)$. This is the crucial step and proving this is straightforward. We start with $\hat{C}|\psi\rangle = c|\psi\rangle$ and take its inner product with $\langle\mathbf{b}|$ to get

$$\begin{aligned}\langle\mathbf{b}|\hat{C}|\psi\rangle &= c\langle\mathbf{b}|\psi\rangle,\\ \langle\mathbf{b}| f(\hat{B}_1, \hat{B}_2, \dots \hat{B}_n)|\psi\rangle &= c\langle\mathbf{b}|\psi\rangle,\\ \tilde{f}(b_1, b_2, \dots b_n)\langle\mathbf{b}|\psi\rangle &= c\langle\mathbf{b}|\psi\rangle.\end{aligned}$$

Also, we have $|\psi\rangle = \sum_{\mathbf{b}}\langle\mathbf{b}|\psi\rangle|\mathbf{b}\rangle$, from completeness. If we consider $|\mathbf{b}\rangle$s for which $\langle\mathbf{b}|\psi\rangle \neq 0$, then we can conclude that indeed $c = \tilde{f}(b_1, b_2, \dots b_n)$. However, when $\langle\mathbf{b}|\psi\rangle = 0$, viz. $|\mathbf{b}\rangle$s that are orthogonal to $|\psi\rangle$, then nothing can be said but that is irrelevant. We can thus conclude that $|\psi\rangle$ is made only of those $|\mathbf{b}\rangle$s that satisfy the required relation. □

It is worth noting that in the Peres Mermin case, where $\hat{R}_i$ and $\hat{C}_j$ are just $\pm\hat{\mathbb{I}}$, it follows that all states are their eigenstates. Consequently, for these operators *sequential multiplicativity* must always hold. We are now ready to describe our explict model. Let the state of a finite dimensional quantum system be $|\chi\rangle$. We wish to assign a value to an arbitrary observable $\hat{A} = \sum_a a|a\rangle\langle a|$, which has eigenvectors $\{|a_j\rangle\}$. The corresponding ordered eigenvalues are $\{a_j\}$ such that $a_{\min} = a_1$ and $a_{\max} = a_n$.

Our HV model for QM assigns values in the following three steps:

1. **Initial HV:** Pick a number $c \in [0, 1]$, from a uniform random distribution.

2. **Assignment or Prediction:** The value assigned to $\hat{A}$ is given by finding the smallest a s.t. $c \leq \sum_{a'=a_{\min}}^{a} |\langle a'|\chi\rangle|^2$, viz. we have specified a prediction map, $m(\hat{A}) = a$.

3. **Update:** After measuring an operator, the state is updated (collapsed) in accordance with the rules of QM. This completely specifies the sequence map s.

To see how this works consider a spin-half particle in the state $|\chi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$ and the observable $\hat{A} = \hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$. Now, according to the postulates of this theory, $m(\hat{A}) = +1$, if the randomly generated value for $c \leq \cos^2\theta$ else $\hat{A}$ is assigned -1. It follows then, from c being uniformly random in $[0, 1]$ that the statistics agree with predictions of QM.

Note that the assignment, described by the prediction map m, is non-contextual since given an operator and a state (+ the HV c), the value is uniquely assigned. The map m is, however, non-multiplicative.

To bring out the non-multiplicativity of our model and to convince the reader of its generic validity, we apply it to the Peres Mermin situation of two spin-half particles. Consider the initial state of the system $|\psi_1\rangle = |00\rangle$. Assume we obtained $c = 0.4$ as a random choice. To arrive at the assignments, note that $|00\rangle$ is an eigenket of only $\hat{R}_i, \hat{C}_j$ and $\hat{A}_{33} = \hat{\sigma}_z \otimes \hat{\sigma}_z$. Thus, in the first iteration, all these should be assigned their respective eigenvalues. The remaining operators must be assigned -1 as one can readily verify by explicitly finding the smallest a as described in postulate 2 of the model (see Table I). Two remarks are in order. First, this model is manifestly *non-multiplicative*, for $m_1(\hat{C}_3) = 1 \neq m_1(\hat{A}_{13})m_1(\hat{A}_{23})m_1(\hat{A}_{33}) = -1$, where the subscript has been introduced to index the iteration. More precisely, $m_1(\hat{O}) := m(\hat{O}, [|\psi_1\rangle = |00\rangle])$ where the complete state $[|\psi_1\rangle]$ implicitly refers to both the quantum state $|00\rangle$ and the HV $c = 0.4$. Second, we must impose *sequential multiplicativity* as a consistency check of the model, which in particular entails that $m_1(\hat{C}_3) = m_1(\hat{A}_{33})m_2(\hat{A}_{23})m_3(\hat{A}_{13})$, where $m_2 := m(\hat{O}, [|\psi_2\rangle])$, $m_3 := m(\hat{O}, [|\psi_3\rangle])$ and $|\psi_2\rangle, |\psi_3\rangle$ are obtained from postulate 3. Note that for each iteration, a new HV is generated. To illustrate *sequential multiplicativity*, we must choose to measure $\hat{A}_{33}$. According to step 3, since $|00\rangle$ is an eigenstate of $\hat{A}_{33}$, the final state remains $|00\rangle$.

For the next iteration, $i = 2$, viz. after the first measurement has been performed, say the random number generator yielded $c = 0.1$. Since $|\psi\rangle$ is also unchanged the assignment remains invariant (in fact any of $c < 0.5$ would yield the same result), which should be evident from the previous exercise. For the final step we choose to measure $\hat{p} := \hat{A}_{23} (= \hat{\sigma}_y \otimes \hat{\sigma}_y)$, to proceed with sequentially measuring $\hat{C}_3$. To simplify calculations, we note

$$|00\rangle = \frac{(|\tilde{+}\tilde{-}\rangle + |\tilde{-}\tilde{+}\rangle)/\sqrt{2} + (|\tilde{+}\tilde{+}\rangle + |\tilde{-}\tilde{-}\rangle)/\sqrt{2}}{\sqrt{2}},$$

where $|\tilde{\pm}\rangle = |0\rangle \pm i|1\rangle$ (eigenkets of $\hat{\sigma}_y$). Since $|00\rangle$ is manifestly not an eigenket of $\hat{p}$, we must find an appropriate eigenket $|p_-\rangle$ s.t. $\hat{p}|p_-\rangle = -|p_-\rangle$, since $c = 0.1$ and $\langle p_-|00\rangle$ is already > 0.1. It is immediate that $|p_-\rangle = \left(|\tilde{+}\tilde{-}\rangle + |\tilde{-}\tilde{+}\rangle\right)/\sqrt{2} = \left(|00\rangle + |11\rangle\right)/\sqrt{2}$, which becomes the final state.

For the final iteration, $i = 3$, say we obtain $c = 0.7$. So far, we have $m_1(\hat{A}_{33}) = 1$ and $m_2(\hat{A}_{23}) = -1$. We must obtain $m_3(\hat{A}_{13}) = 1$, independent of the value of c, to be consistent. Let's check that. Indeed, according to postulate 2, since $\hat{\sigma}_x \otimes \hat{\sigma}_x\left(|00\rangle + |11\rangle\right)/\sqrt{2} = 1\left(|00\rangle + |11\rangle\right)/\sqrt{2}$, $m_3(\hat{A}_{13}) = 1$ for all allowed values of c. As a remark, it maybe emphasised that the $m_2(\hat{A}_{33}) = m_3(\hat{A}_{33})$ and $m_2(\hat{A}_{23}) = m_3(\hat{A}_{23})$, which essentially expresses compatibility of these observables, viz. measurement of $\hat{A}_{13}$ doesn't affect the result one would obtain by measuring operators compatible to it (granted they have been measured once before).

Will work on CC after everthin else is done

We have already constructed an explicit non-contextual model, which is consistent with QM. This

$i=1, c=0.4, \|\psi_{\text{init}}\rangle = \|00\rangle$	$i=2, c=0.1, \|\psi_{\text{init}}\rangle = \|00\rangle$	$i=3, c=0.7, \|\psi_{\text{init}}\rangle = \frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$
$m_1(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$	$m_2(\hat{A}_{ij}) \doteq \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & +1 \end{bmatrix}$	$m_3(\hat{A}_{ij}) \doteq \begin{bmatrix} +1 & +1 & +1 \\ +1 & +1 & -1 \\ +1 & +1 & +1 \end{bmatrix}$
$m_1(\hat{R}_i), m_1(\hat{C}_j) = +1\,(j \neq 3)$	$m_2(\hat{R}_i), m_2(\hat{C}_j) = +1\,(j \neq 3)$	$m_3(\hat{R}_i), m_3(\hat{C}_j) = +1\,(j \neq 3)$
$m_1(\hat{C}_3) = -1$	$m_2(\hat{C}_3) = -1$	$m_3(\hat{C}_3) = -1$
$\hat{A}_{13} = \hat{\sigma}_z \otimes \hat{\sigma}_z; m_1(\hat{A}_{13}) = +1$	$\hat{A}_{23} = \hat{\sigma}_y \otimes \hat{\sigma}_y; m_2(\hat{A}_{23}) = -1$	$\hat{A}_{33} = \hat{\sigma}_x \otimes \hat{\sigma}_x; m_3(\hat{A}_{33}) = +1$
$\|\psi_{\text{final}}\rangle = \|00\rangle$	$\|\psi_{\text{final}}\rangle = \frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$	$\|\psi_{\text{final}}\rangle = \frac{\|00\rangle+\|11\rangle}{\sqrt{2}}$

Table I. HV model applied to the Peres Mermin situation

model we knew had to be non-multiplicative. We will see how non-multiplicativity gives rise to what one might confuse to mean contextuality. Imagine $\hat{B}_1 = \hat{\sigma}_z \otimes \hat{\mathbb{I}} = |00\rangle\langle 00| + |01\rangle\langle 01| - [|10\rangle\langle 10| + |11\rangle\langle 11|]$, $\hat{B}_2 = \hat{\mathbb{I}} \otimes \hat{\sigma}_z = |10\rangle\langle 10| + |11\rangle\langle 11| - [|00\rangle\langle 00| + |01\rangle\langle 01|]$, while we define $\hat{C} = f(\{\hat{B}_i\}) = 0.\,|00\rangle\langle 00| + 1.\,|01\rangle\langle 01| + 2.\,|10\rangle\langle 10| + 3.\,|11\rangle\langle 11|$.

$\hat{C}$ maybe viewed as a function of $\hat{B}_1$, $\hat{B}_2$ and other operators $\hat{B}_i$ which are constructed to obtain a maximally commuting set. A measurement of $\hat{C}$, will collapse the state into one of the states which are simultaneous eignkets of B_1 and B_2. Consequently, from the observed value of $\hat{C}$, one can deduce the values of $\hat{B}_1$ and $\hat{B}_2$. Now consider $\sqrt{2}\,|\chi\rangle = |10\rangle + |01\rangle$, for which $m_1(\hat{B}_1) = 1$, and $m_1(\hat{B}_2) = 1$, using the C-ingle model, with $c < 0.5$. However, $m_1(\hat{C}) = 1$, from which one can deduce that B_1 was $+1$, while B_2 was -1. This property itself, one may be tempted call contextuality, viz. the value of B_1 depends on whether it is measured alone or with the remaining $\{B_i\}$. However, it must be noted that B_1 has a well defined value, and so does $\hat{C}$. Thus by our accepted definition, there's no contextuality. It is just that $m_1(\hat{C}) \neq \tilde{f}(m_1(\hat{B}_1), m_1(\hat{B}_2), \dots)$, viz. the theory is non-multiplicative. Note that after measuring $\hat{C}$ however, $m_2(\hat{B}_1) = +1$ and $m_2(\hat{B}_2) = -1$ (for any value of c) consistent with those deduced by measuring $\hat{C}$.

It is known that for simultaneous eigenstates of $\hat{B}_i$, multiplicativity must hold. Consequently, any violation of multiplicativity must arise from states that are superpositions (of the simultaneous eigenkets). Note however, that entanglement is not necessary to show a violation, since, for example, the PM test is a state independent test, where a separable state can be used to arrive at a contradiction. For demonstrating non-locality using Bell's proof, however, it can be shown that entanglement is necessary.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] J. S. Bell, Physics **1**, 195 (1964).
[3] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[4] S. Pironio, A. Acín, S. Massar, A. B. de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Nature **464**, 1021 (2010).
[5] S. Kochen and E. P. Specker, Journal of Mathematics and Mechanics **17**, 59 (1967).
[6] A. Gleason, Indiana Univ. Math. J. **6**, 885 (1957).
[7] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[8] A. Peres, Journal of Physics A: Mathematical and General **24**, L175 (1991).
[9] N. D. Mermin, Phys. Rev. Lett. **65**, 3373 (1990).
[10] C. Simon, M. Żukowski, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **85**, 1783 (2000).
[11] Y.-F. Huang, C.-F. Li, Y.-S. Zhang, J.-W. Pan, and G.-C. Guo, Phys. Rev. Lett. **90**, 250401 (2003).
[12] T. Yang, Q. Zhang, J. Zhang, J. Yin, Z. Zhao, M. Żukowski, Z.-B. Chen, and J.-W. Pan, Phys. Rev. Lett. **95**, 240406 (2005).
[13] Y. Hasegawa, R. Loidl, G. Badurek, M. Baron, and H. Rauch, Phys. Rev. Lett. **97**, 230401 (2006).
[14] P. Blasiak, Annals of Physics **353**, 326 (2015).
[15] M. Howard, J. Wallman, V. Veitch, and J. Emerson, Nature **510**, 351 (2014), article.
[16] A. Cabello, V. D'Ambrosio, E. Nagali, and F. Sciarrino, Phys. Rev. A **84**, 030302 (2011).
[17] M. Kleinmann, O. Gühne, J. R. Portillo, J. Åke Larsson, and A. Cabello, New Journal of Physics **13**, 113011 (2011).
[18] A. Cabello, Foundations of Physics **42**, 68 (2012).
[19] D. Bohm, Phys. Rev. **85**, 166 (1952).
[20] D. Bohm, Phys. Rev. **85**, 180 (1952).
[21] B. R. La Cour, Phys. Rev. A **79**, 012102 (2009).
[22] B. R. La Cour, Foundations of Physics **44**, 1059 (2014).