

# Information-Gain Computation

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**Abstract.** Despite large incentives, correctness in software remains an elusive goal. Declarative programming techniques, where algorithms are derived from a specification of the desired behavior, offer hope to address this problem, since there is a combinatorial reduction in complexity in programming in terms of specifications instead of algorithms, and arbitrary desired properties can be expressed and enforced in specifications directly.

However, limitations on performance have prevented programming with declarative specifications from becoming a mainstream technique for general-purpose programming. To address the performance bottleneck in deriving an algorithm from a specification, I propose information-gain computation, a framework where an adaptive evaluation strategy is used to efficiently perform a search which derives algorithms that provide information about a query via the most efficient routes. Within this framework, opportunities to compress the search space present themselves, which suggest that information-theoretic bounds on the performance of such a system might be articulated and a system designed to achieve them.

In a preliminary empirical study of adaptive evaluation for a simple test program, the evaluation strategy adapts successfully to evaluate a query efficiently.

## 1 Introduction and Motivation

Despite large incentives, correctness in software remains an elusive goal. Declarative programming techniques, where algorithms are derived from a specification of the desired behavior, offer hope to address this problem, since there is a combinatorial reduction in complexity in programming in terms of specifications instead of algorithms, and arbitrary desired properties can be expressed and enforced in specifications directly. Additionally, giving an explicit specification preserves information about program semantics and programmer intent that is lost by forcing the programmer to manually translate an explicit or implicit specification into an algorithm that implicitly and usually only partially satisfies that specification, information that may be used by automated systems to implement correct behavior in a performant way.

However, limitations on performance have prevented programming with declarative specifications from becoming a mainstream technique for general-purpose programming. Without domain-specific knowledge, default evaluation strategies must strike a sophisticated balance among efficiency, the semantic properties of soundness and completeness, and simplicity, which is relevant both to implementation effort and to comprehensibility by the programmer, if the programmer must be relied upon to implicitly influence the behavior of the search to achieve efficiency, which is in practice how efficiency is achieved in general-purpose declarative languages such as Prolog. Furthermore, because of the combinatorial nature of the searches involved in evaluation, exponential or worse reductions in efficiency can result from deviations from the best evaluation strategy, meaning these efficiency concerns are often decisive in whether a program is practical to use at all; they are concerns of the highest order. This, of course, undermines the status of such languages as declarative, since the task of programming still involves understanding and influencing the evaluation of the program at an algorithmic level, and it is insufficient to program only in terms of the declarative semantics of the problem.

Kowalski's framing of algorithm = logic + control in the paper of the same name[5] provides guidance here. Accordingly, to obtain an algorithm that implements the logic of a specification, we must add a control component (that is, a choice of order of evaluation of the declarative logic) that uses that logic to produce the desired result efficiently.

To address the performance bottleneck in deriving an algorithm from a specification in this way, I propose *information-gain computation*, a framework where an adaptive evaluation strategy is used to efficiently perform a search which derives algorithms that provide information about a query most directly. The key aspect is to measure information gain about a goal when a certain control choice is made in a certain context, and adapt these choices to increase the rate of information gain. Information gain provides a meaningful measure of progress for a computation, which in turn provides an objective for optimization for the adaptive strategy. Measuring the progress of a computation in general has proven difficult because unbounded effort can be expended without any indication as to whether the computation will halt if allowed to continue. Here, halting is replaced with yielding information, and execution proceeds nondeterministically to find paths that yield information at the highest rate, sidestepping these issues. Also, the factoring of the program logic into recursive predicates makes it possible to share information about effective control choices to achieve statistical efficiency.

Within this framework, opportunities to compress the search space present themselves, first, by identifying the traces that contribute the most information to answering a given query, and then, by compressing these traces. This framing suggests that information-theoretic bounds on the performance of such a system might be articulated and a system designed to achieve them, holding promise for a definitive solution to the problem of deriving an algorithm from a specification, and thus to that of declarative software.

To test the core idea, adaptation of the evaluation strategy with respect to information gain, adaptive evaluation of a fixed Prolog program was implemented and the efficiency measured and the induced control structure described. The program was adapted from an elementary Prolog programming example for beginners where ancestorhip is computed recursively in a directed graph. The program was modified from its usual form to introduce an extremely large amount of sparsity in the search space, but in a form where sharing of statistical strength due to the explicitly recursive structure of the description of the search space should permit quickly learning a bias away from the costly diversions. The program confounds Prolog’s default evaluation strategy even on problems of trivial size but the adaptive evaluation strategy succeeds in evaluating the program efficiently.

The rest of this paper is organized as follows. Section 2 presents a focused review of related work. In Section 3, I propose an architecture for a system implementing information-gain-driven evaluation. Section 4 describes the design of an experiment which uses an adaptive evaluation strategy on a test program. Section 5 discusses the results of the experiments. Section 6 presents conclusions and discusses future work.

## 2 Related Work

Prolog is the most well-established and widely-used logic programming language. Its recursive, top-down decomposition of queries (goals) into subgoals will be retained and used to advantage here.[4]

Blog[6] and Problog[9] are examples of languages that combine logical and probabilistic semantics, however they do not attempt to take advantage of this to improve their evaluation strategies or realize true declarative semantics, instead relying on standard logic program evaluation and extending it with a variety of ad-hoc techniques for probabilistic portions of the program. To the best of my knowledge, using probabilistic semantics to achieve efficient evaluation in a general-purpose setting is a novel approach, and it yields benefits for purely logical programs as well as advantageously framing the problem of integrating probabilistic information into a logical framework.

Schmidhuber’s adaptive history compression[10] will form the basis for the proposed trace compression, along with Info-clustering[2], both of which will be discussed in that context.

Bandit algorithms will play the central role in adapting the evaluation strategy. Bandit algorithms address the problem of balancing exploration and exploitation in choosing actions with unknown rewards; this is the problem we face in trying to choose a branch to explore in evaluation of a program in hopes of gaining information. The literature on bandit algorithms is vast; a highlight relevant to the present and future work here is Thompson sampling, a Bayesian framework[7]. The present work uses UCB1, a simple rule that orchestrates exploration and exploitation according to a bound on an expected reward estimate modified with an additive uncertainty term which decreases as a choice is sam-

pled according to a simple error model. In Powley and Cowling’s work[8] there is some precedent for using UCB1 in a similar setting, to explore an unbounded tree that yields reward only at its leaves.

### 3 Design for Information-Driven Evaluation

This section describes an architecture for a system to perform adaptive evaluation and suggests a strategy for a concrete implementation.

The proposed design has three main aspects. First, an adaptive technique is used to identify control choices that tend to yield the most information about the query given the data. Second, the results of executing these control choices are examined to determine the most frequently executed traces (equivalently, information propagation paths from data to the query). Third, these traces are recursively compressed in a way that creates new, shorter information propagation paths from the data to the query.

Before considering the design we begin with a review of the aspects of the usual evaluation strategy in a relational language with emphasis on the features that lead to difficulties the current design addresses.

#### 3.1 Review of Evaluation in Relational Languages

In a relational language such as Prolog, a program consists of rules that refer to other rules recursively, and which ultimately may match facts (data). Evaluation proceeds from a *query*, which is a top-level goal that is recursively expanded to subgoals in a left-to-right, depth-first search. When a fact is matched, it is added to an answer set for the goal it matches, and these sets are combined according to the logic of the predicates of the goals they appear in until they produce a final answer set for the query.

Depth-first search requires the least state for a search; only a stack of previous goals and position in them must be maintained. However, one drawback of this approach is that it is incomplete; for example, a left-recursive rule would prevent the search from terminating since the rule would expand into itself ad infinitum before any other branch is taken.

Each predicate thus implicitly represents a discrete joint space of variation, and, given data, we can construct the joint spaces they represent explicitly by evaluating the program.

The Prolog strategy described above is a *backward-chaining* strategy; it is also possible to use *forward-chaining*, which applies rules in a bottom-up fashion, starting from facts that match rules and then recursively invoking rules those rules appear in as subgoals until the query is reached. The identification and use of traces described below uses adaptive backward-chaining to identify when forward-chaining would be advantageous and apply it in those cases.

### 3.2 Adaptive Evaluation With Bandit Algorithms

As mentioned in the introduction, we take the information about the query as the objective of evaluation, or rather more specifically the information gained per unit of computational effort expended, and treat the choice of which subgoal to evaluate in each goal as a *multi-armed bandit* problem.[13] Briefly, in each goal we face a choice of which subgoal’s tree to explore, in hopes it will match and yield information about facts at some point. Whenever we encounter subgoals that correspond to the same predicate, we can share information about subgoal choice, since each predicate is identically an implicit representation of its own joint membership space.

To measure progress in gaining information about a goal, we measure the decrease in the *total correlation*[12], which is the sum of the entropies of the individual variables minus the entropy of their joint distribution. For random variables  $X_1, X_2, \dots, X_n$ , the expression is this:

$$\sum_{i=1}^n H(X_i) - H(X_1, X_2, \dots, X_n)$$

This is equivalent to the Kullback-Leibler divergence from the joint distribution to the independent distribution of the variables. In the set of variables in this calculation, we include the variables appearing in the goal, as well as a variable representing a prior over the joint space from which the variables in the goal may be drawn. Thus under this measure, information is gained whenever variables are distinguished from one another or when the joint distribution is distinguished from the prior. For logic programming, we may use the uniform prior over the discrete set of facts.

Algorithms for multi-armed bandit problems admit several desiderata. Those relevant here are succeeding with high probability vs. only in expectation, and contextuality, or the ability to condition decisions on side information. The ability to succeed under an adversarial choice of rewards rather than only with i.i.d. rewards may be important in problems that closely model an adversarial setting.

The ability to take context into account can be used to take the current state of information about the goal into account. The choice of branch to explore next can be conditioned on the amount of effort spent exploring each branch so far, or a summary statistic thereof, together with any contextual restrictions on the search space resulting from top-down unification, because together they are a sufficient statistic for the current state of knowledge. It may also be possible to condition on the current state of knowledge directly, for example by representing the joint distribution representing the current beliefs in an efficient basis and conditioning on its parameters.

Bearing in mind these desiderata, we can proceed with Exp4.P,[1] an algorithm which works in both stochastic and adversarial conditions, accepts contextual information, succeeds with high probability, and achieves a regret bound with a square root of a log factor of the optimal bound.

We associate the state of a bandit choice algorithm with each predicate in the program, and measure the information gained per unit effort in exploring a

branch, using a budget for further expansions that may occur during that exploration to ensure that the exploration terminates and the effort can be accounted for properly. This budget may be incrementally increased as in an iterative-deepening search strategy.

### 3.3 Hot Traces and Optimistic Forward-Chaining

The most frequently taken control paths in evaluating the program are now determined by a near-optimal procedure for selecting those that yield the most information with the least computational effort, which we may view as a first, most basic optimization in evaluating the program. We can identify these control paths either by sampling control stacks to gather statistics during execution or by examining the weights of the bandit choice algorithms for a query and its constituent subgoals.

Identifying these traces brings several benefits. First, they can be optimistically executed in forward-chaining mode whenever a fact that matches one is known, since they have already been determined to belong to a near-optimal algorithm for taking that fact into account. Second, they may be heavily optimized by techniques from tracing JITs, being inherently in the appropriate form. Third, since they are lacking in control flow and specify the type of data they consume explicitly, they are especially amenable to execution on efficient dataflow-oriented hardware such as GPUs, FPGAs, and vector processors.

Another potential benefit, perhaps the most significant one because it holds the promise of a tight bound on the effort required to answer the query, is compression of the search/inference space, described below.

### 3.4 Compressing Traces

We can attempt to compress traces in the following way, a generalization of Schmidhuber’s history compression[10] for sequences which does not require a total order, i.e. to space-like rather than time-like relations.

In history compression, a hierarchy of representations of a sequence is learned by learning a predictor of the sequence, and constructing a new sequence that consists of the indices of mispredicted symbols along with the correct symbol, and so on recursively with that sequence as desired or until no more compression is obtained.

To generalize beyond sequences, I propose using *info-clustering*[2] to learn dependency structures within each predicate’s empirical joint distribution, and to learn a probabilistic model of each of those joint distributions. The parameters of these models, along with their residual errors, are analogous to the models and their mispredictions used by history compression. Now, to build up a recursive hierarchy of representations as in history compression, we take the model parameters and residual errors and consider the joint spaces of those which appear adjacent to one another on a trace, and recursively apply info-clustering and joint-space modeling on those until no more compression can be obtained.

The result is a tree of reduced descriptions of the joint spaces, which may provide exponentially shorter information propagation paths from facts to query. Facts entering can be transformed through a number of models logarithmic in the length of the trace to yield information about the query efficiently.

## 4 Experiments

### 4.1 Simulating Adaptive Evaluation

Due to time restrictions, modifying existing Prolog implementations and implementing a simple Prolog-like language from scratch proved prohibitively complex and involved many aspects outside the scope of the present study, so to investigate the fundamental feasibility of the strategy arising from the framing here we simulate the relevant aspects of the search for solutions that would be used by a relational language with the proposed adaptive evaluation strategy. To simulate evaluation, we fix a program and data and implement by hand the search that corresponds to applying the evaluation strategy to that program and data.

### 4.2 Example Preliminaries

As an elementary example, consider the following code to compute the transitive closure of a graph (phrased in terms of the elementary Prolog programming example about recursively finding ancestors of a person given a set of parent-child relationships):

```
ancestor(A, B) :- parent(A, B).
ancestor(A, B) :- parent(A, X),
    ancestor(X, B).
```

That is, A is an ancestor of B either if A is a parent of B, or if A is a parent of another X and X is an ancestor of B.

We can use this as the basis of a simple example to test the effects of an adaptive evaluation strategy. To represent the effects of sparsity of the search space, which is the main obstacle that adaptive evaluation is meant to address, we add additional rules that confound the search, like so:

```
ancestor(A, B) :- parent(A, B).
ancestor(A, B) :- deadend(A, B).
deadend(A, B) :- deadend(A, B,
    1000000000).
deadend(A, B, N) :- N1 is N - 1, N > 0
    -> deadend(A, B, N1);
    fail.
ancestor(A, B) :- parent(A, X),
    ancestor(X, B).
```

Further we assume these data:

```

parent(tom, fred).
parent(fred, jill).

```

Under the default Prolog evaluation strategy, the deadend rule is preferred to the informative rules for pursuing the ancestor search, because it appears first in the program text, resulting in the program counting down from 100 million before resuming a productive branch of the search, and causing a delay of about 10 seconds to answer the query `?- ancestor(tom, jill).` on SWI-Prolog 7.4.2 on a 2.8 GHz Intel Xeon CPU E3-1505M v5. An adaptive search should be able to detect that a great deal of work is being performed without yielding information on this branch and should thereafter strongly prefer another.

### 4.3 Implementation

This Prolog program under an evaluation strategy that uses UCB1[8] (for the sake of simplicity of implementation) to choose each subgoal was implemented in Python as a program that selects a branch to explore with UCB1 in a loop and updates the answer set and UCB1 parameters as facts are encountered. Two sets of benchmark data are used: one corresponding to the Tom, Fred, Jill example above, and one that creates a 1000x1000 upper-triangular matrix where elements above the diagonal are 1 with  $p = 1/8$  for a 1 entry,  $p = 7/8$  for a 0 to represent parenthood. The query in this case asks for the indices of nodes with node 0 as an ancestor. The information accounting ignores structure in the joint space and assumes the uniform over pairs in the joint space as the prior, gaining a bit of information whenever a descendant is identified, and neglecting to check when it becomes known that an element cannot possibly be a descendant. The time, UCB1 expected information gain estimates, and number of times each predicate was branched to were reported.

## 5 Evaluation

For the Tom, Fred, and Jill example, a negligible amount of time is taken, and the deadend branch is taken once (which, however, requires it to be played 100,000,000 times because it recurs only into itself that many times,) while the informative branch is taken 3 times, to yield the correct answer. The UCB1 weights learned were about 0.0006 for the deadend branch and 3.037 for the informative branch, showing that approximately all 2 bits of information were estimated to have come from the informative branch, which was the case. Since reward was about one bit per branch down the informative branch, the much higher weight showed that in only 3 searches of that branch, the uncertainty bound could not be improved to even within the same order of magnitude as the true expected utility value, and the presence of the number 3 in both the leading digit of the branch weight and the number of times it was taken is a coincidence.

For the 1000x1000 random matrix example, execution took about 40 seconds, and again took the deadend branch once by choice and 100,000,000 times total, thereafter taking the informative branch 145,205 times and again learning a

weight of about 0.0006 for the deadend branch but a weight of about 1.0092 for the informative branch, which is very close to the true expected value of one bit per branch taken, the uncertainty bound penalty in the weight term having been reduced to a negligible level by the large number of tries of the branch.

## 6 Conclusions, Comments, and Future Work

The empirical test of the bandit-algorithm-driven evaluation branch choice proved successful, even with an example program that can be made to take arbitrarily long in Prolog with a choice of the parameter setting the depth of the pathological branch.

This suggests that the additional work to build information-gain-driven adaptive evaluation into a relational language is justified. For a pure-logic language, the technique of defaulting to uniform prior over a discrete domain of facts used in the simulated evaluation here may suffice, but to use non-uniform priors, to represent correlations and information-sharing in the joint spaces in predicates properly, and to move forward with ideas for search space compression, a full probabilistic-relational language is called for.

So far, only declarative computation - lacking in side effects - has been considered. To embrace interaction with an environment with mutable state into the same framework in a way that preserves its strengths, we can associate program fragments with predicates, and thread program fragments together along traces to produce plans as in Pyke, a Prolog-like relational language implemented in Python.[3] This will introduce additional concerns in working with compressed traces.

The potential for information-theoretic optimality of the evaluation strategy induced by the proposed methods was mentioned in passing but not discussed in toto. The claim is that the bandit algorithm generates efficient evaluation plans, and that relatively few traces out of all possible traces will contribute the majority of the obtainable information. Then, compression of these few traces in such a way that new information propagation paths of length proportional to the information content of the joint relation space along the entire trace will reduce evaluation effort along these traces to costs of the order of an information-theoretic bound. Practical difficulties may arise in the expense of transforming through the hierarchy of models. Likewise, practical difficulties in optimizing and parallelizing traces may arise in residual control components inherent in the semantics of the individual base predicates in the system, and in accounting for interactions with side-effecting plans as mentioned in the previous paragraph.

In a probabilistic-relational language that had implemented the recursive joint-space modeling in trace compression, data could be supplied with an uninformative predicate structure, similar to that used in a convolutional neural network, where the same relationship is assumed in each local neighborhood in a larger joint space. The trace compression would then automatically build a hierarchy of autoassociators to model the data as they relate to the query. This suggests computational interpretations of neuroanatomical structures that may

be explored in future work - cortex as a hierarchy of autoassociating compressors attempting to learn short information propagation paths from senses to query-like structures directing evaluation in the basal ganglia, emitting motor plans along these traces to effect attention and interaction with the environment, and optimizing motor plans and maximizing their dependence directly on data in the cerebellum.

For choices with large numbers of arms (such as for selection of a discretized continuous parameter,) the CEMAB method[11] may be applicable. It is derived from the cross-entropy method, an importance-sampling technique for rare-event simulation later adapted to hard optimization problems.

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