

QUANTUM DYNAMICAL BOUNDS FOR LONG-RANGE OPERATORS WITH SKEW-SHIFT POTENTIALS

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ABSTRACT. We employ Weyl’s method and Vinogradov’s method to analyze skew-shift dynamics on semi-algebraic sets. Consequently, we improve the quantum dynamical upper bounds of Jitomirskaya-Powell, Liu, and Shamis-Sodin for long-range operators with skew-shift potentials.

1. INTRODUCTION

In this paper, we are interested in studying the quantum dynamics of long-range operators on the integer lattice \mathbb{Z} . That is, we study bounded self-adjoint operators H acting on $\ell^2(\mathbb{Z})$ in the following way:

$$(Hu)_n = \sum_{n' \in \mathbb{Z}} A(n, n') u_{n'} + V(n) u_n$$

where $V(n)$ is a sequence of real numbers and $A(n, n')$ satisfies, for any $n, n' \in \mathbb{Z}$,

$$\begin{aligned} |A(n, n')| &\leq C e^{-c|n-n'|}, \\ A(n, n') &= \overline{A(n', n)}, \\ A(n+k, n'+k) &= A(n, n'), \quad \text{for all } k \in \mathbb{Z}. \end{aligned}$$

Operators such as H are commonly associated with the Hamiltonians of quantum particles that evolve according to Schrödinger dynamics. For $p > 0$ and $\phi \in \ell^2(\mathbb{Z})$, one of the primary objects of interest for such operators is the p th moment of the position operator, given by

$$\langle |X_H|_\phi^p \rangle(T) = \sum_{n \in \mathbb{Z}} |n|^p |(e^{-iT H} \phi, \delta_n)|^2$$

and its time average,

$$\langle |\tilde{X}_H|_\phi^p \rangle(T) = \frac{2}{T} \int_0^\infty e^{-2t/T} \langle |X_H|_\phi^p \rangle(t) dt.$$

These moments relate to the spread of the wavepacket $e^{-itH}\phi$, which is in turn closely related to the spectral measure μ_ϕ . For example, the celebrated RAGE Theorem of Ruelle, Amrein, Georgescu, and Enss says that

$$\lim_{T \rightarrow \infty} \langle |\tilde{X}_H|_\phi^p \rangle(T) = \infty$$

if the spectral measure μ_ϕ is not pure point. Refinements of this relation have been obtained in various works; we highlight two: one by Last [Las96] and another more recently by Landrian-Powell [LP22] both indicate that continuity of μ_ϕ with respect to a (generalized) Hausdorff measure implies a lower bound on $\langle |\tilde{X}_H|_\phi^p \rangle(T)$, typically called “quasi-ballistic transport” in the literature, while upper bounds on $\langle |\tilde{X}_H|_\phi^p \rangle(T)$ imply that μ_ϕ must be singular with respect to a particular (generalized) Hausdorff measure. The converse is not true in general: the singularity of μ_ϕ alone does not imply anything about the behavior of $\langle |\tilde{X}_H|_\phi^p \rangle(T)$ (see e.g. [Las96] or [dRJLS96]).

It is well-known that the motion of the quantum particle cannot be faster than ballistic:

$$\langle |X_H|_\phi^p \rangle(T) \leq C^p T^p.$$

More refined (upper or lower) bounds typically have to be obtained on a case-by-case basis (see, e.g. [DT07, DT08, GSB02, Hae24, HJ19, JP22, JZ22, Las96, Liu23, SS23], see also the survey [DMY24] and references therein). Two fruitful methods, broadly speaking, for obtaining lower bounds include: (1) a careful study of the spectral measures and/or solutions to an eigenvalue equation (see, e.g. [GSB02, JZ22, Las96] see also [DT07] and the references therein) and (2) approximation via operators with ballistic transport (c.f. [Hae24, Las96]). On the other hand, upper bounds have been obtained using various methods often inspired by localization proofs [HJ19, Liu23, JP22, SS23] or involving complex analytic methods [DT07, DT08]. The focus of this paper will be to obtain upper bounds for a large class of operators.

Let us consider a particular family of operators $\{H_x\}_{x \in \Omega}$ with a dynamically-defined potential. Let (Ω, f, μ) be a measure-preserving dynamical system, and suppose $V(n) = \lambda v(f^n x)$ for some μ -measurable function $v : \Omega \rightarrow \mathbb{R}$ and some $\lambda \neq 0$, which we call the coupling constant. Such families are particularly well-studied:

- (1) When Ω is a sequence space and f is a Bernoulli shift, we obtain a long-range Anderson model;
- (2) When $\Omega = \mathbb{T}^b$, $(1, \omega)$ rationally dependent, and f is the shift by ω , we obtain the periodic operators;
- (3) When $\Omega = \mathbb{T}^b$, $(1, \omega)$ rationally independent, and f is the shift by ω , we obtain the quasi-periodic operators;
- (4) When $\Omega = \mathbb{T}^b$, $\omega \in \mathbb{T} \setminus \mathbb{Q}$, and f is the skew-shift:

$$fx = (x_1 + \omega, x_2 + x_1, x_3 + x_2, \dots, x_b + x_{b-1}),$$

we obtain a model closely related to the kicked-rotor problem.

When investigating quasi-periodic and skew-shift models, it is frequently necessary to impose an arithmetic constraint on the admissible frequencies ω , since

the spectral properties of these models depend sensitively on the arithmetic properties of ω . One such constraint we will employ here is the Diophantine condition $DC(\gamma, \tau)$ with $\gamma > 0, \tau > 1$. Specifically, we say $\omega \in DC(\gamma, \tau)$ if

$$\|k\omega\|_{\mathbb{T}} \geq \frac{\gamma}{|k|^{\tau}}, \text{ for any } k \in \mathbb{Z} \setminus \{0\}.$$

For Anderson models with a large coupling constant, or at a spectral edge, dynamical localization holds ($\langle |X_H|_{\phi}^p \rangle(T)$ is bounded uniformly in T). For periodic models, the spectral measures are absolutely continuous, leading to $\langle |X_H|_{\phi}^p \rangle(T)$ being arbitrarily close to ballistic behavior. As stated above, however, quasi-periodic models with v being real analytic, the dynamics exhibit a delicate dependence on the arithmetic properties of both ω and x . Specifically, when $\omega \in DC(\gamma, \tau)$ and λ is sufficiently large (depending only on v), dynamical localization is known to hold for a.e. (but not every) x . Despite the intricate dynamics, bounds on $\langle |X_{H_{x,\omega}}|_{\phi}^p \rangle(T)$ exhibit greater stability. It is known that under the conditions $\omega \in DC(\gamma, \tau)$ and λ sufficiently large (depending only on v), the inequality

$$\langle |X_{H_{x,\omega}}|_{\phi}^p \rangle(T) \leq C(\log T)^{p\sigma+\varepsilon}$$

holds uniformly in x , where $\sigma = \sigma(b, \tau)$ [JP22, Liu23, SS23]. For skew-shift models, the story is even more delicate (see [HJ19, JP22] for the Schrödinger case and [SS23] for the long-range case). For v real analytic and $\omega \in DC(\gamma, \tau)$, there is $\lambda_0 = \lambda_0(v, \omega)$ such that

$$\langle |X_{H_{x,\omega}}|_{\phi}^p \rangle(T) \leq C(\log T)^{p\sigma+\varepsilon}$$

holds uniformly in x for $\lambda > \lambda_0$, with

$$\sigma = 4^{b-1} b^3 \tau^2.$$

Quasi-periodic and skew-shift models have been studied extensively; localization (see [BG00, Jit99] and references therein for quasi-periodic results and [Bou02, BGS01, HLS20b, HLS20a, Kle14] and references therein for skew-shift results) and quantum dynamics (see [BJ00, Fil17, GK23, JZ22, Liu23, ZZ17, Zha17] and references therein for quasi-periodic results and [HJ19, JP22, SS23] and references therein for skew-shift results) are of particular interest.

The main idea of [HJ19] and [JP22], which are specifically applicable to Schrödinger operators with short-range potentials, involves the combination of an LDT (large deviation theorem) with a sublinear bound and a relation derived by Damanik and Tcheremchantsev [DT07, JP22]. It was remarked in [DT07, JP22] that the quantum dynamical bounds could be improved if either the LDT or the sublinear bound were improved. The idea of [SS23], which applies to long-range operators of the form we consider here, was to combine an LDT with integration along a suitable contour.

Here, we will take an approach which is closer in spirit to Liu [Liu23]. There, one of the authors developed a new method inspired by Anderson localization proofs for quasi-periodic Schrödinger operators to show that, for long-range quasi-periodic operators, quantum dynamical upper bounds follow from a suitable sublinear bound of the bad Green's functions. We extend this argument to the skew-shift setting and prove novel sublinear bounds for the skew-shift to obtain better quantum dynamical upper bounds.

Denote

$$\psi(b) = \begin{cases} 2^{b-1}, & \text{if } 2 \leq b \leq 5, \\ b(b-1), & \text{if } b \geq 6. \end{cases}$$

We prove the following:

Theorem 1.1. *Suppose $\omega \in DC(\gamma, \tau)$. Let*

$$(H_{x,\omega} u)_n = \sum_{n' \in \mathbb{Z}} A(n, n') u_{n'} + v(f^n x) u_n$$

where $x \in \mathbb{T}^b$, v is real analytic on \mathbb{T}^b , and f is the skew-shift on \mathbb{T}^b . Suppose $H_{x,\omega}$ satisfies the LDT (see Section 5 for the precise definition). Then for any ϕ with compact support and $p > 0$ there exists $C = C(\varepsilon, v, A, b, \gamma, \tau, \phi, p)$ such that

$$\langle |\tilde{X}_{H_{x,\omega}}|_\phi^p \rangle(T) \leq C(\log T)^{\frac{p}{\delta} + \varepsilon}, \quad (1)$$

$$\langle |X_{H_{x,\omega}}|_\phi^p \rangle(T) \leq C(\log T)^{\frac{p}{\delta} + \varepsilon}, \quad (2)$$

where $\delta = \frac{1}{\tau b \psi(b)}$.

Corollary 1.2. *Suppose $\omega \in DC(\gamma, \tau)$. Let*

$$(H_{x,\omega} u)_n = \sum_{n' \in \mathbb{Z}} A(n, n') u_{n'} + \lambda v(f^n x) u_n$$

where $x \in \mathbb{T}^b$, v is real analytic on \mathbb{T}^b , and f is the skew-shift on \mathbb{T}^b . Then there is $\lambda_0 = \lambda_0(v, \omega)$ such that if $\lambda > \lambda_0$, (1) and (2) hold.

Proof. By Theorem 3.14 from [Liu22], there is $\lambda_0 = \lambda_0(v, \omega)$ such that the LDT holds for $\lambda > \lambda_0$. The corollary now follows immediately from Theorem 1.1. \square

For long-range operators, as far as we know, the previous best upper bound on $\langle |X_{H_{x,\omega}}|_\phi^p \rangle(T)$ in our framework, due to Shamis and Sodin [SS23], was $(\log T)^{p/\delta + \varepsilon}$ with $\delta = (4^{b-1} b^3 \tau^2)^{-1}$. Here we have tightened this estimate to

$$(\log T)^{p\tau b \psi(b) + \varepsilon} < (\log T)^{p4^{b-1} b^3 \tau^2 + \varepsilon}.$$

For Schrödinger operators, to the best of our knowledge, our upper bounds are also best. Han and Jitomirskaya [HJ19] obtained a bound of the discrepancy $N^{-1/(\tau(2^b-1)) + \varepsilon}$. From the discrepancy, it is possible to derive a sublinear bound $N^{1-\delta}$ (combining with the dimension b loss) with $\delta = (\tau b(2^b - 1))^{-1}$. Plugging

this into the machinery developed by Liu [Liu23] (see also Section 5), a weaker bound $(\log T)^{p/\delta+\varepsilon}$ than that in Theorem 1.1 arises. Later, Jitomirskaya and Powell [JP22] improved the upper bound with $\delta = (\tau b^2(2^{b-1}))^{-1}$ (in [JP22], a similar Diophantine condition was used with $\tau = 1 + \varepsilon$). According to the following relation:

$$\tau b\psi(b) \leq \min \left\{ \tau b^2 2^{b-1} \text{ [JP22]}, \tau b(2^b - 1) \text{ [HJ19]} \right\},$$

Theorem 1.1 and Corollary 1.2 are, to the best of our knowledge, better than any previous bounds.

Let us now say a few words about our argument. Our central object is the so-called sublinear bound: suppose $\mathcal{S} \subseteq \mathbb{T}^b$ is a semi-algebraic set of degree B , then we say \mathcal{S} satisfies a sublinear bound if for any $N \in \mathbb{N}$ with $\log B \lesssim \log N < -\log(\text{Leb}(\mathcal{S}))$, the following inequality

$$\#\{1 \leq n \leq N : f^n x \in \mathcal{S}\} \leq N^{1-\delta}$$

holds for all $x \in \mathbb{T}^b$.

First, we establish an abstract result (i.e. Theorem 5.2) relating sublinear bounds to upper bounds on $\langle |X_{H_{x,\omega}}|_\phi^p \rangle(T)$ by essentially following the argument from [Liu23] (see Section 5 for details). This method reduces the problem to proving a sublinear bound and is also a major source of improvement over the works of Han-Jitomirskaya, Jitomirskaya-Powell, and Shamis-Sodin [HJ19, JP22, SS23]. In fact, combining this result with the sublinear bound proved in [JP22] or in [Liu22] yields an improvement.

Our next step is to improve on the sublinear bounds established in [JP22] and [Liu22]. One fruitful way to obtain sublinear bounds on $\mathcal{S} \subseteq \mathbb{T}^b$ is to cover \mathcal{S} by ϵ -balls B_ϵ and estimate the following

$$\#\{1 \leq n \leq N : f^n x \in B_\epsilon\}.$$

This can be reduced, via Fourier analysis (see Lemma 2.12) to estimating an exponential sum of the form

$$\sum_{|k_1| < R} \cdots \sum_{|k_b| < R} \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, f^n x \rangle}.$$

We employ two different number-theoretic arguments to estimate these exponential sums which are optimal in different situations. Specifically, we use the classic method of Weyl's method (see, e.g. [Mon94]) when considering \mathbb{T}^b , $2 \leq b \leq 5$ (see Section 3 for details), and we use Vinogradov's method (see, e.g. [Mon94]) and recent proof of Vinogradov's mean value theorem by Bourgain-Demeter-Guth [BDG16] when $b > 5$ (see Section 4 for details). We refer readers to [GZ19] for the history and recent developments related to solutions of the Vinogradov system. Han and Jitomirskaya also employed Weyl's method in [HJ19] to derive

their upper bounds, but we introduce some techniques (square lattice decomposition) which improve on those estimates (c.f. [HJ19, Section 5] and Section 3 below).

The rest of our paper is organized as follows. In Section 2, we provide useful definitions and prove estimates which are used throughout this paper. In Section 3, we use Weyl's method to obtain a sublinear bound for semi-algebraic sets. In Section 4, we use the Vinogradov's method to obtain a sublinear bound for semi-algebraic sets. Finally, in Section 5, we detail how to relate a particular sublinear bound to an upper bound on $\langle |X_{H_{x,\omega}}|_\phi^p \rangle(T)$ and prove Theorem 1.1.

2. PRELIMINARY

We use $\|\cdot\|_{\mathbb{T}^b}$ to represent the distance to the nearest integer lattice in \mathbb{Z}^b . For $\mathbf{k} \in \mathbb{Z}^b$, define $\|\mathbf{k}\| := \max\{|k_1|, |k_2|, \dots, |k_b|\}$. Throughout the paper, $A \lesssim B$ means $A \leq CB$ for some constant $C > 0$. Denote by $\deg(P)$ the degree of the polynomial P .

2.1. Exponential sums: Weyl's method. Denote

$$S = S(\alpha) = \sum_{n=1}^N e^{2\pi i P(n;\alpha)},$$

where $P(x; \alpha) = \sum_{j=0}^b \alpha_j x^j$ is a polynomial with real coefficients.

Lemma 2.1. [Mon94, page 42] *For any $b \geq 2$,*

$$|S|^{2^{b-1}} \lesssim N^{2^{b-1}-1} + N^{2^{b-1}-b} \sum_{h_1=1}^N \cdots \sum_{h_{b-1}=1}^N \min\left(N, \frac{1}{\|b!h_1 \cdots h_{b-1}\alpha_b\|_{\mathbb{T}}}\right).$$

Remark 2.2. Note that $\min(N, \frac{1}{\|b!h_1 \cdots h_{b-1}\alpha_b\|_{\mathbb{T}}}) \geq 1$, thus Lemma 2.1 implies

$$|S|^{2^{b-1}} \lesssim N^{2^{b-1}-b} \sum_{h_1=1}^N \cdots \sum_{h_{b-1}=1}^N \min\left(N, \frac{1}{\|b!h_1 \cdots h_{b-1}\alpha_b\|_{\mathbb{T}}}\right).$$

Multi-sums often appear in this paper. The following estimate is used frequently to reduce the multi-sums to a single sum.

Lemma 2.3. [Kor92, Lemma 13, page 71] *Let M and m_1, \dots, m_n be positive integers. Denote by $\tau_n(M)$ the number of solutions of the equation $m_1 \cdots m_d = M$. Then for any $0 < \varepsilon \leq 1$,*

$$\tau_n(M) \leq C(\varepsilon, n)M^\varepsilon.$$

2.2. Exponential sums: Vinogradov's method. Next, we recall the Vinogradov's Mean Value Theorem. It is obvious that $|S|$ is independent of α_0 . We put $\alpha = (\alpha_1, \dots, \alpha_b) \in \mathbb{T}^b$ and reset $P(x; \alpha) = \sum_{j=1}^b \alpha_j x^j$. Denote

$$J_b(N; \rho) = \int_{\mathbb{T}^b} |S(\alpha)|^{2\rho} d\alpha.$$

We see that $J_b(N; \rho)$ is the number of solutions of the systems

$$\begin{aligned} m_1 + \dots + m_\rho &= n_1 + \dots + n_\rho, \\ m_1^2 + \dots + m_\rho^2 &= n_1^2 + \dots + n_\rho^2, \\ &\dots \\ m_1^b + \dots + m_\rho^b &= n_1^b + \dots + n_\rho^b, \end{aligned}$$

where $1 \leq m_j, n_j \leq N$.

Lemma 2.4 (Vinogradov's Mean Value Theorem). [BDG16, Theorem 1.1] *For each $\rho \geq 1$ and $b, N \geq 2$, the following upper bound holds:*

$$J_b(N; \rho) \leq C(\varepsilon, b, \rho)(N^{\rho+\varepsilon} + N^{2\rho - \frac{b(b+1)}{2} + \varepsilon}).$$

Lemma 2.5. [Mon94, pages 79–81] *For any $\rho \geq 1$ and $b \geq 3$,*

$$|S|^{2\rho} \lesssim N^{\frac{(b-1)(b-2)}{2}-1} J_{b-1}(3N; \rho) \sum_{|h| \leq 2\rho b N^{b-1}} \min\left(N, \frac{1}{\|h\alpha\|_{\mathbb{T}}}\right).$$

Remark 2.6. For readers' convenience, we provide the proof of Lemma 2.5 in the Appendix A.

2.3. Diophantine condition.

Lemma 2.7. *Let $\alpha \in DC(\gamma, \tau)$ and $p/q \in \mathbb{Q}$. There exists $\tilde{\gamma} = \tilde{\gamma}(p, q, \gamma)$ such that $\alpha p/q \in DC(\tilde{\gamma}, \tau)$.*

Lemma 2.8. *Suppose that $\alpha \in DC(\gamma, \tau)$. Let $\{p_n/q_n\}$ be the best approximation of α . Then for any $N \in \mathbb{N}$, there exists q_n such that*

$$(\gamma N)^{\frac{1}{\tau}} < q_n \leq N.$$

Proof. On the one hand, for any $N \in \mathbb{N}$, there exist q_n and q_{n+1} such that

$$q_n \leq N < q_{n+1}.$$

On the other hand, since $\alpha \in DC(\gamma, \tau)$, then

$$\frac{\gamma}{q_n^\tau} \leq \|q_n \alpha\|_{\mathbb{T}} < \frac{1}{q_{n+1}},$$

which gives

$$q_n > (\gamma q_{n+1})^{\frac{1}{\tau}} > (\gamma N)^{\frac{1}{\tau}}.$$

□

The following estimate of the sum will be used frequently.

Lemma 2.9. *Suppose that $\alpha \in DC(\gamma, \tau)$. Then*

$$\sum_{k=1}^H \min\left(N, \frac{1}{\|k\alpha\|_{\mathbb{T}}}\right) \leq \gamma^{-\frac{1}{\tau}} H N^{1-\frac{1}{\tau}} + H \log N + N + N \log N.$$

Proof. Let us first recall the following classic estimate (c.f. [Mon94, page 41]). If $|\alpha - p/q| \leq 1/q^2$, then

$$\sum_{k=1}^H \min\left(N, \frac{1}{\|k\alpha\|_{\mathbb{T}}}\right) \leq \frac{HN}{q} + H \log q + N + q \log q. \quad (3)$$

Now let $\{p_n/q_n\}$ be the best approximation of α . It is well-known that

$$\|q_n\alpha\|_{\mathbb{T}} < \frac{1}{q_{n+1}} < \frac{1}{q_n}, \text{ for any } n \geq 1,$$

and thus $|\alpha - p_n/q_n| \leq 1/q_n^2$. Then the proof of Lemma 2.9 is finished by combining (3) with Lemma 2.8. \square

2.4. Semi-algebraic set.

Definition 2.10 (Semi-algebraic set). We say $\mathcal{S} \subseteq \mathbb{R}^b$ is a semi-algebraic set if it is a finite union of sets defined by a finite number of polynomial inequalities. More precisely, let $\{P_1, P_2, \dots, P_s\}$ be a family of real polynomials to the variables $x = (x_1, x_2, \dots, x_b)$ with $\deg(P_i) \leq d$ for $i = 1, 2, \dots, s$. A (closed) semi-algebraic set \mathcal{S} is given by the expression

$$\mathcal{S} = \cup_j \cap_{\ell \in \mathcal{L}_j} \{x \in \mathbb{R}^b : P_\ell(x) \leq_{j\ell} 0\}, \quad (4)$$

where $\mathcal{L}_j \subseteq \{1, 2, \dots, s\}$ and $\leq_{j\ell} \in \{\geq, \leq, =\}$. Then we say that the degree of \mathcal{S} , denoted by $\deg(\mathcal{S})$, is at most sd . In fact, $\deg(\mathcal{S})$ means the smallest sd overall representation as in (4).

Lemma 2.11. [Bou05, Corollary 9.6] *Let $\mathcal{S} \subseteq [0, 1]^b$ be a semi-algebraic set of degree B . Let $\epsilon > 0$ be a small number and $\text{Leb}(\mathcal{S}) \leq \epsilon^b$. Then \mathcal{S} can be covered by a family of ϵ -balls with total number less than $B^{C(b)} \epsilon^{1-b}$.*

2.5. Fourier analysis. Let $\chi_\epsilon(\cdot)$ be the characteristic function of the ball in \mathbb{T}^b of radius ϵ centered at 0. That is,

$$\chi_\epsilon(x) = \begin{cases} 1, & \|x\|_{\mathbb{T}^b} < \epsilon, \\ 0, & \text{others.} \end{cases}$$

The following result is a part of the calculation in [JP22]. For convenience, we state it as a lemma and provide the proof.

Lemma 2.12. [JP22, pages 189–190] Suppose that $\{x_n\}_{n=1}^N \subseteq \mathbb{T}^b$. Let $R = [\epsilon^{-1}/10]$. Then

$$\sum_{n=1}^N \chi_\epsilon(x_n) \leq C(b) R^{-b} \sum_{\|\mathbf{k}\| < R} \left| \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, x_n \rangle} \right|.$$

Remark 2.13. In this lemma and the rest of this paper, $[x]$ denotes the integer part of $x \in \mathbb{R}$.

Proof. Let $F(\cdot)$ be the usual Fejér kernel on \mathbb{R} :

$$F(x) = \frac{1}{R} \left(\frac{\sin \pi Rx}{\sin \pi x} \right)^2 = \sum_{|k| < R} \left(1 - \frac{|k|}{R} \right) e^{2\pi i k x} =: \sum_{|k| < R} \widehat{F}(k) e^{2\pi i k x}.$$

For any $x = (x_1, \dots, x_b) \in \mathbb{T}^b$, if $\chi_\epsilon(x) = 0$, then $\chi_\epsilon(x) \leq R^{-b} \prod_{j=1}^b F(x_j)$ holds trivially. On the other hand, if $\chi_\epsilon(x) = 1$, then $R/2 \leq F(x_j) \leq 2R$ for every $1 \leq j \leq b$, and we also have $\chi_\epsilon(x) \leq 2^b R^{-b} \prod_{j=1}^b F(x_j)$. Thus

$$\chi_\epsilon(x) \lesssim R^{-b} \prod_{j=1}^b F(x_j), \text{ for any } x \in \mathbb{T}^b.$$

Set $\mathbf{k} = (k_1, \dots, k_b) \in \mathbb{Z}^b$ and let $\widehat{F}(\mathbf{k}) = \prod_{j=1}^b \widehat{F}(k_j)$. Then

$$\begin{aligned} \prod_{j=1}^b F(x_j) &= \prod_{j=1}^b \sum_{|k_j| < R} \widehat{F}(k_j) e^{2\pi i k_j x_j} \\ &= \sum_{\|\mathbf{k}\| < R} \widehat{F}(\mathbf{k}) e^{2\pi i \langle \mathbf{k}, x \rangle}. \end{aligned}$$

Note that $|\widehat{F}(\mathbf{k})| \leq \prod_{j=1}^b |\widehat{F}(k_j)| \leq 1$, then

$$\sum_{n=1}^N \chi_\epsilon(x_n) \lesssim R^{-b} \sum_{n=1}^N \sum_{\|\mathbf{k}\| < R} \widehat{F}(\mathbf{k}) e^{2\pi i \langle \mathbf{k}, x_n \rangle} \lesssim R^{-b} \sum_{\|\mathbf{k}\| < R} \left| \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, x_n \rangle} \right|.$$

This finishes the proof. \square

3. SUBLINEAR BOUND: WEYL'S METHOD

Theorem 3.1. Let $\mathbf{P} = (P_1, \dots, P_b)$ be a vector ($b \geq 2$) of real polynomials and α_i be the leading coefficient of P_i for $1 \leq i \leq b$. Suppose $1 \leq \deg(P_1) < \dots < \deg(P_b) \leq m$ and $\alpha_i \in DC(\gamma, \tau)$ with $\gamma > 0, \tau > 1$ for $1 \leq i \leq b$. Let $\mathcal{S} \subseteq [0, 1]^b$ be a semi-algebraic set of degree B and $\text{Leb}(\mathcal{S}) < \eta$. Let $N \in \mathbb{N}$ such that

$$\log B \lesssim \log N < 2^{m-1} \tau \log \frac{1}{\eta}.$$

Then for any $0 < \varepsilon \leq 1$,

$$\#\{1 \leq n \leq N : \mathbf{P}(n) \bmod \mathbb{Z}^b \in \mathcal{S}\} \leq C(\varepsilon, b, m, \gamma, \tau) B^{C(b)} N^{1 - \frac{1}{\tau b 2^{m-1}} + \varepsilon}.$$

Proof. Let $\epsilon = N^{-\frac{1}{\tau b 2^{m-1}}}$ and $R = [\epsilon^{-1}/10]$. Then

$$\text{Leb}(\mathcal{S}) < \eta < N^{-\frac{1}{\tau 2^{m-1}}} = \epsilon^b.$$

Thus by Lemma 2.11, \mathcal{S} can be covered by at most $B^{C(b)} \epsilon^{1-b}$ many ϵ -balls.

Consider one such ball; without loss of generality, we may assume the ball is centered at 0. Let $\chi_\epsilon(\cdot)$ be the characteristic function of the ball in \mathbb{T}^b with radius ϵ centered at 0. Apply Lemma 2.12 we have

$$\sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) \lesssim R^{-b} \sum_{\|\mathbf{k}\| < R} \left| \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, \mathbf{P}(n) \rangle} \right|. \quad (5)$$

To estimate (5), we decompose the square lattice $\{\mathbf{k} : \|\mathbf{k}\| < R\}$ such that

$$\{\mathbf{k} : \|\mathbf{k}\| < R\} = \cup_i \mathcal{K}^i, \text{ and } \mathcal{K}^i \cap \mathcal{K}^{i'} = \emptyset, \text{ for any } i \neq i',$$

where

$$\begin{aligned} \mathcal{K}^b &= \{\|\mathbf{k}\| < R : k_b \neq 0\}, \\ \mathcal{K}^i &= \{\|\mathbf{k}\| < R : k_i \neq 0, k_{i'} = 0 \text{ for any } i' > i\}, \quad 1 \leq i < b, \\ \mathcal{K}^0 &= \{k_b = k_{b-1} = \dots = k_1 = 0\}. \end{aligned} \quad (6)$$

Thus it follows from (5) and (6) that

$$\sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) \lesssim NR^{-b} + R^{-b} \sum_{i=1}^b \sum_{\mathbf{k} \in \mathcal{K}^i} \left| \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, \mathbf{P}(n) \rangle} \right|. \quad (7)$$

Denote

$$S = \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, \mathbf{P}(n) \rangle}.$$

Let us consider one such \mathcal{K}^i . Denote $\deg(P_i) = D_i$. We need to deal with two cases:

Case 1: $D_i = 1$. Obviously, by the monotonicity of D_i to i and $1 \leq D_i \leq m$, it must holds that $i = 1$. Then for $\mathbf{k} \in \mathcal{K}^1$,

$$\langle \mathbf{k}, \mathbf{P}(n) \rangle = k_1 \alpha_1 n + O(1).$$

Combining the estimate of the geometric series with Lemma 2.9 shows

$$\sum_{\mathbf{k} \in \mathcal{K}^1} |S| \leq 2 \sum_{k_1=1}^R \min \left(N, \frac{1}{\|k_1 \alpha_1\|} \right) \lesssim RN^{1-\frac{1}{\tau}} \lesssim R^b N^{1-\frac{1}{\tau 2^{m-1}}}. \quad (8)$$

Case 2: $D_i \geq 2$. By the assumption of \mathbf{P} and $\mathbf{k} \in \mathcal{K}^i$, we know

$$\langle \mathbf{k}, \mathbf{P}(n) \rangle = k_i \alpha_i n^{D_i} + O(n^{D_i-1}).$$

Since $\#\mathcal{K}^i \lesssim R^i$, by the Hölder inequality,

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S| \leq R^{i(1-\frac{1}{2^{D_i-1}})} \left(\sum_{\mathbf{k} \in \mathcal{K}^i} |S|^{2^{D_i-1}} \right)^{1/2^{D_i-1}}. \quad (9)$$

By Lemma 2.1 and Remark 2.2, we have

$$|S|^{2^{D_i-1}} \lesssim N^{2^{D_i-1}-D_i} \sum_{h_1=1}^N \cdots \sum_{h_{D_i-1}=1}^N \min \left(N, \frac{1}{\|(D_i)!h_1 \cdots h_{D_i-1} k_i \alpha_i\|_{\mathbb{T}}} \right).$$

We apply Lemma 2.3 to reduce the multi-sums to a single sum: for any $0 < \varepsilon \leq 1$,

$$|S|^{2^{D_i-1}} \lesssim N^{2^{D_i-1}-D_i+\varepsilon} \sum_{h=1}^{(D_i)!N^{D_i-1}} \min \left(N, \frac{1}{\|hk_i \alpha_i\|_{\mathbb{T}}} \right).$$

Similarly, combining the above inequality with Lemma 2.3 again, so that

$$\begin{aligned} \sum_{\mathbf{k} \in \mathcal{K}^i} |S|^{2^{D_i-1}} &\lesssim N^{2^{D_i-1}-D_i+\varepsilon} R^{i-1} \sum_{k_i=1}^R \sum_{h=1}^{(D_i)!N^{D_i-1}} \min \left(N, \frac{1}{\|hk_i \alpha_i\|_{\mathbb{T}}} \right) \\ &\lesssim N^{2^{D_i-1}-D_i+\varepsilon} R^{i-1} \sum_{k=1}^{(D_i)!RN^{D_i-1}} \min \left(N, \frac{1}{\|k \alpha_i\|_{\mathbb{T}}} \right). \end{aligned} \quad (10)$$

Since $\alpha_i \in DC(\gamma, \tau)$ with $\tau > 1$, we apply Lemma 2.9 with $H = (D_i)!RN^{D_i-1}$,

$$\sum_{k=1}^{(D_i)!RN^{D_i-1}} \min \left(N, \frac{1}{\|k \alpha_i\|_{\mathbb{T}}} \right) \lesssim RN^{D_i-\frac{1}{\tau}}. \quad (11)$$

Substituting (11) into (10) shows that

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S|^{2^{D_i-1}} \lesssim R^i N^{2^{D_i-1}-\frac{1}{\tau}+\varepsilon}. \quad (12)$$

Now we combine (12) with (9), one can see that

$$\begin{aligned} \sum_{\mathbf{k} \in \mathcal{K}^i} |S| &\lesssim R^{i(1-\frac{1}{2^{D_i-1}})} (R^i N^{2^{D_i-1}-\frac{1}{\tau}+\varepsilon})^{\frac{1}{2^{D_i-1}}} \\ &\lesssim R^i N^{1-\frac{1}{\tau 2^{D_i-1}}+\varepsilon} \lesssim R^b N^{1-\frac{1}{\tau 2^m-1}+\varepsilon}, \end{aligned} \quad (13)$$

where the last step uses the assumption that $D_i \leq m$.

Thus, **Case 1** and **Case 2** imply that

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S| \lesssim R^b N^{1-\frac{1}{\tau 2^m-1}+\varepsilon}, \text{ for any } 1 \leq i \leq m.$$

Combine the above inequality with (7), we have

$$\begin{aligned} \sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) &\lesssim NR^{-b} + R^{-b} \sum_{i=1}^b R^b N^{1-\frac{1}{\tau 2^{m-1}}+\varepsilon} \\ &\lesssim NR^{-b} + N^{1-\frac{1}{\tau 2^{m-1}}+\varepsilon}. \end{aligned}$$

According to the definition of R , we know $R^{-b} \lesssim N^{-\frac{1}{\tau 2^{m-1}}}$ and thus

$$\sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) \lesssim N^{1-\frac{1}{\tau 2^{m-1}}+\varepsilon}.$$

Since we only need $B^{C(b)} \epsilon^{1-b}$ many ϵ -balls to cover \mathcal{S} , we get

$$\begin{aligned} \#\{1 \leq n \leq N : \mathbf{P}(n) \bmod \mathbb{Z}^b \in \mathcal{S}\} &\leq B^{C(b)} \epsilon^{1-b} \sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) \\ &\leq C(\varepsilon, b, m, \gamma, \tau) B^{C(b)} N^{1-\frac{1}{\tau b 2^{m-1}}+\varepsilon}. \end{aligned}$$

This finishes the proof. \square

4. SUBLINEAR BOUND: VINOGRADOV'S METHOD

Theorem 4.1. *Let $\mathbf{P} = (P_1, \dots, P_b)$ be a vector ($b \geq 2$) of real polynomials and α_i be the leading coefficient of P_i for $1 \leq i \leq b$. Suppose $1 \leq \deg(P_1) < \dots < \deg(P_b) \leq m$ and $\alpha_i \in DC(\gamma, \tau)$ with $\gamma > 0, \tau > 1$ for $1 \leq i \leq b$. Let $\mathcal{S} \subseteq [0, 1]^b$ be a semi-algebraic set of degree B and $\text{Leb}(\mathcal{S}) < \eta$. Let $N \in \mathbb{N}$ such that*

$$\log B \lesssim \log N < m(m-1)\tau \log \frac{1}{\eta}.$$

Then for any $0 < \varepsilon \leq 1$,

$$\#\{1 \leq n \leq N : \mathbf{P}(n) \bmod \mathbb{Z}^b \in \mathcal{S}\} \leq C(\varepsilon, b, m, \gamma, \tau) B^{C(b)} N^{1-\frac{1}{\tau b m(m-1)}+\varepsilon}.$$

Proof. Let $\epsilon = N^{-\frac{1}{\tau b m(m-1)}}$ and $R = [\epsilon^{-1}/10]$. Then

$$\text{Leb}(\mathcal{S}) < \eta < N^{-\frac{1}{\tau m(m-1)}} = \epsilon^b.$$

Thus by Lemma 2.11, \mathcal{S} can be covered by at most $B^{C(b)} \epsilon^{1-b}$ many ϵ -balls.

Consider one such ball; without loss of generality, we may assume the ball is centered at 0. Let $\chi_\epsilon(\cdot)$ be the characteristic function of the ball in \mathbb{T}^b with radius ϵ centered at 0. Apply Lemma 2.12 we have

$$\sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) \lesssim R^{-b} \sum_{\|\mathbf{k}\| < R} \left| \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, \mathbf{P}(n) \rangle} \right|. \quad (14)$$

Just as we did in Section 3, the square lattice $\{\mathbf{k} : \|\mathbf{k}\| < R\}$ are decomposed as the disjoint sets $\mathcal{K}^i, 0 \leq i \leq b$. That is, (14) can be rewritten as

$$\sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) \lesssim NR^{-b} + R^{-b} \sum_{i=1}^b \sum_{\mathbf{k} \in \mathcal{K}^i} \left| \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, \mathbf{P}(n) \rangle} \right|, \quad (15)$$

where \mathcal{K}^i is defined in (6). Denote

$$S := \sum_{n=1}^N e^{2\pi i \langle \mathbf{k}, \mathbf{P}(n) \rangle}.$$

Fix i and consider one such \mathcal{K}^i . Denote $\deg(P_i) = D_i$. We need to deal with two cases:

Case 1: $1 \leq D_i \leq 2$. As similar as (8) and (13), the following holds:

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S| \lesssim R^i N^{1 - \frac{1}{\tau_2 D_i - 1} + \varepsilon} \lesssim R^b N^{1 - \frac{1}{\tau_m(m-1)} + \varepsilon}.$$

Case 2: $D_i \geq 3$. By the assumption of \mathbf{P} and $\mathbf{k} \in \mathcal{K}^i$, we know

$$\langle \mathbf{k}, \mathbf{P}(n) \rangle = k_i \alpha_i n^{D_i} + O(n^{D_i-1}).$$

Since $\#\mathcal{K}^i \lesssim R^i$, by the Hölder inequality, for any $\rho \in \mathbb{Z}^+$ (ρ to be determined),

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S| \leq R^{i(1 - \frac{1}{2\rho})} \left(\sum_{\mathbf{k} \in \mathcal{K}^i} |S|^{2\rho} \right)^{1/(2\rho)}. \quad (16)$$

By Lemma 2.5, for $\mathbf{k} \in \mathcal{K}^i$,

$$|S|^{2\rho} \lesssim N^{\frac{(D_i-1)(D_i-2)}{2}-1} J_{D_i-1}(3N; \rho) \sum_{|h| \leq 2\rho N^{D_i-1}} \min \left(N, \frac{1}{\|hk_i \alpha_i\|_{\mathbb{T}}} \right). \quad (17)$$

Notice that $\min(N, \frac{1}{\|hk_i \alpha_i\|_{\mathbb{T}}}) \geq 1$, then the inner sum in (17) satisfies that

$$\begin{aligned} \sum_{|h| \leq 2\rho N^{D_i-1}} \min \left(N, \frac{1}{\|hk_i \alpha_i\|_{\mathbb{T}}} \right) &= N + 2 \sum_{h=1}^{2\rho N^{D_i-1}} \min \left(N, \frac{1}{\|hk_i \alpha_i\|_{\mathbb{T}}} \right) \\ &\lesssim \sum_{h=1}^{2\rho N^{D_i-1}} \min \left(N, \frac{1}{\|hk_i \alpha_i\|_{\mathbb{T}}} \right), \end{aligned} \quad (18)$$

where the last step uses that $D_i \geq 3$. Combine (17) and (18) with Lemma 2.3, for any $0 < \varepsilon \leq 1$, we have

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S|^{2\rho} \lesssim R^{i-1} N^{\frac{(D_i-1)(D_i-2)}{2}-1+\varepsilon} J_{D_i-1}(3N; \rho) \sum_{k=1}^{2\rho RN^{D_i-1}} \min \left(N, \frac{1}{\|k \alpha_i\|_{\mathbb{T}}} \right).$$

Thus it follows from Lemma 2.9 and $\alpha_i \in DC(\gamma, \tau)$ with $\tau > 1$ that

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S|^{2\rho} \lesssim R^i N^{\frac{D_i(D_i-1)}{2} - \frac{1}{\tau} + \varepsilon} J_{D_i-1}(3N; \rho).$$

Recall the Vinogradov's Mean Value Theorem (see Lemma 2.4):

$$J_{D_i-1}(3N; \rho) \lesssim N^{\rho+\varepsilon} + N^{2\rho - \frac{D_i(D_i-1)}{2} + \varepsilon}.$$

Choose $2\rho = D_i(D_i - 1)$. Then we have

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S|^{2\rho} \lesssim R^i N^{D_i(D_i-1) - \frac{1}{\tau} + \varepsilon}.$$

Substituting the above inequality into (16) shows

$$\begin{aligned} \sum_{\mathbf{k} \in \mathcal{K}^i} |S| &\lesssim R^{i(1 - \frac{1}{D_i(D_i-1)})} (R^i N^{D_i(D_i-1) - \frac{1}{\tau} + \varepsilon})^{\frac{1}{D_i(D_i-1)}} \\ &\lesssim R^i N^{1 - \frac{1}{\tau D_i(D_i-1)} + \varepsilon} \lesssim R^b N^{1 - \frac{1}{\tau m(m-1)} + \varepsilon}, \end{aligned}$$

where the last step uses the assumption that $D_i \leq m$.

Therefore, regardless of **Case 1** or **Case 2**, it always holds that

$$\sum_{\mathbf{k} \in \mathcal{K}^i} |S| \lesssim R^b N^{1 - \frac{1}{\tau m(m-1)} + \varepsilon}, \text{ for any } 1 \leq i \leq m.$$

Combine the above estimate with (15), one can see that

$$\begin{aligned} \sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) &\lesssim NR^{-b} + R^{-b} \sum_{i=1}^b R^b N^{1 - \frac{1}{\tau m(m-1)} + \varepsilon} \\ &\lesssim NR^{-b} + N^{1 - \frac{1}{\tau m(m-1)} + \varepsilon}. \end{aligned}$$

According to the definition of R , we know $R^{-b} \lesssim N^{-\frac{1}{\tau m(m-1)}}$, thus

$$\sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) \lesssim N^{1 - \frac{1}{\tau m(m-1)} + \varepsilon}.$$

Finally, we cover \mathcal{S} by $B^{C(b)} \epsilon^{1-b}$ many ϵ -balls so that

$$\begin{aligned} \#\{1 \leq n \leq N : \mathbf{P}(n) \bmod \mathbb{Z}^b \in \mathcal{S}\} &\leq B^{C(b)} \epsilon^{1-b} \sum_{n=1}^N \chi_\epsilon(\mathbf{P}(n) \bmod \mathbb{Z}^b) \\ &\leq C(\varepsilon, b, m, \gamma, \tau) B^{C(b)} N^{1 - \frac{1}{\tau b m(m-1)} + \varepsilon}. \end{aligned}$$

This finishes the proof. \square

5. PROOF OF THEOREM 1.1

5.1. Large deviation, sublinear bound, and quantum dynamics. We begin this section with an abstract result (i.e. Theorem 5.2) that applies to bounded long-range operators

$$(Hu)_n = \sum_{n' \in \mathbb{Z}^d} A(n, n') u_{n'} + V(n) u_n$$

on $\ell^2(\mathbb{Z}^d)$ which satisfy

(1) for any $n, n' \in \mathbb{Z}^d$,

$$|A(n, n')| \leq C_1 e^{-c_1 \|n - n'\|}, \quad C_1 > 0, c_1 > 0;$$

(2) for any $n, n' \in \mathbb{Z}^d$,

$$A(n, n') = \overline{A(n', n)};$$

(3) for any $n, n', k \in \mathbb{Z}^d$,

$$A(n + k, n' + k) = A(n, n').$$

Theorem 5.2 may be of independent interest and could find applications in the future.

Let us explain the setting. We first recall the concept of elementary region. For $d = 1$, the elementary region of size N centered at 0 is given by

$$Q_N = [-N, N].$$

For $d \geq 2$, denote by Q_N an elementary region of size N centered at 0, which is one of the following regions,

$$Q_N = [-N, N]^d$$

or

$$Q_N = [-N, N]^d \setminus \{n \in \mathbb{Z}^d : n_i \leq 0, 1 \leq i \leq d\},$$

where for $i = 1, 2, \dots, d$, $\varsigma_i \in \{\langle, \rangle, \emptyset\}$ and at least two ς_i are not \emptyset . Denote by \mathcal{E}_N^0 the set of all elementary regions of size N centered at 0. Let

$$\mathcal{E}_N := \{n + Q_N : n \in \mathbb{Z}^d, Q_N \in \mathcal{E}_N^0\}.$$

We call the elements in \mathcal{E}_N elementary regions.

Next, we define the Green's function. Let R_Λ be the operator of restriction (projection) to $\Lambda \subseteq \mathbb{Z}^d$. Define the Green's function at z by

$$G_\Lambda(z) = (R_\Lambda(H - zI)R_\Lambda)^{-1}.$$

Set $G(z) = (H - zI)^{-1}$. Clearly, both $G_\Lambda(z)$ and $G(z)$ are always well-defined for $z \in \mathbb{C}_+ \equiv \{z \in \mathbb{C} : \Im z > 0\}$. Fixed $0 < \sigma_1 < 1$, we say an elementary region

$\Lambda \in \mathcal{E}_N$ is in class SG_N (strongly good with size N) if

$$\begin{aligned} \|G_\Lambda(z)\| &\leq e^{N\sigma_1}, \\ |G_\Lambda(z)(n, n')| &\leq e^{-c_2\|n-n'\|}, \text{ for } \|n - n'\| \geq N/10, \end{aligned}$$

Finally, we note that the self-adjoint operator H is bounded, so there exists a large $K > 0$ such that $\sigma(H) \subseteq [-K + 1, K - 1]$.

Theorem 5.1. [Liu23, Corollary 2.3] Define \mathcal{B}_{N,N_1} as

$$\mathcal{B}_{N,N_1} = \{n \in [-N, N]^d : \text{there exists } Q_{N_1} \in \mathcal{E}_{N_1}^0 \text{ such that } n + Q_{N_1} \notin SG_{N_1}\}.$$

Assume that there exists $\epsilon_0 > 0$ such that for any $z = E + i\epsilon$ with $|E| \leq K$ and $0 < \epsilon \leq \epsilon_0$, and arbitrarily small $\varepsilon > 0$,

$$\#\mathcal{B}_{N,[N^\varepsilon]} \leq N^{1-\delta} \text{ when } N \geq N_0 \quad (19)$$

(N_0 may depend on ε). Then for any ϕ with compact support and any $\varepsilon > 0$ there exists $T_0 > 0$ (depending on $d, p, \phi, K, \sigma_1, \delta, \epsilon_0, c_1, c_2, C_1, N_0$ and ε) such that for any $T \geq T_0$,

$$\begin{aligned} \langle |\tilde{X}_H|^p_\phi \rangle(T) &\leq (\log T)^{\frac{p}{\delta}+\varepsilon}, \\ \langle |X_H|^p_\phi \rangle(T) &\leq (\log T)^{\frac{p}{\delta}+\varepsilon}. \end{aligned}$$

Theorem 5.1 can be applied to dynamically-defined operators. Let f be a function from $\mathbb{Z}^d \times \mathbb{T}^b$ to \mathbb{T}^b . Assume for any $m, n \in \mathbb{Z}^d$,

$$f(m+n, x) = f(m, f(n, x)).$$

Sometimes, we write down $f^n x$ for $f(n, x)$ for convenience, where $n \in \mathbb{Z}^d$ and $x \in \mathbb{T}^b$. Define a family of operators $\{H_x\}_{x \in \mathbb{T}^b}$ on $\ell^2(\mathbb{Z}^d)$:

$$(H_x u)_n = \sum_{n' \in \mathbb{Z}^d} A(n, n') u_{n'} + v(f(n, x)) u_n,$$

where v is a real analytic function on \mathbb{T}^b .

In fact, (19) in Theorem 5.1 can be deduced from the assumption of the LDT (large deviation theorem) and the sublinear bound for the semi-algebraic set. More precisely, we say the Green's function of an operator H_x satisfies the LDT in complexified energies (sometimes just say LDT for short) if there exist $\epsilon_0 > 0$ and $N_0 > 0$ such that for any $N \geq N_0$, there exists a subset $X_N \subseteq \mathbb{T}^b$ such that

$$\text{Leb}(X_N) \leq e^{-N\sigma_2}, \quad (20)$$

and for any $x \notin X_N \text{ mod } \mathbb{Z}^b$ and $Q_N \in \mathcal{E}_N^0$,

$$\begin{aligned} \|G_{Q_N}(z)\| &\leq e^{N\sigma_1}, \\ |G_{Q_N}(z)(n, n')| &\leq e^{-c_2\|n-n'\|}, \text{ for } \|n - n'\| \geq N/10, \end{aligned}$$

where $z = E + i\epsilon$ with $E \in [-K, K]$ (recall that $\sigma(H_x) \subseteq [-K + 1, K - 1]$) and $0 < \epsilon \leq \epsilon_0$.

Theorem 5.2. Let $\mathcal{S} \subseteq [0, 1]^b$ be a semi-algebraic set of degree B and $\text{Leb}(\mathcal{S}) < \eta$. Assume that for any N with

$$\log B \lesssim \log N < \log \frac{1}{\eta},$$

we have the following sublinear bound:

$$\#\{n \in [-N, N]^d : f^n x \in \mathcal{S}\} \leq C_2 N^{1-\delta}. \quad (21)$$

Assume LDT holds. Then for any ϕ with compact support and any $\varepsilon > 0$ there exists $T_0 > 0$ (depending on $d, b, p, \phi, K, \sigma_1, \sigma_2, \delta, \epsilon_0, c_1, c_2, C_1, C_2, N_0$ and ε) such that for any $T \geq T_0$,

$$\begin{aligned} \langle |\tilde{X}_{H_x}|_\phi^p \rangle(T) &\leq (\log T)^{\frac{p}{\delta} + \varepsilon}, \\ \langle |X_{H_x}|_\phi^p \rangle(T) &\leq (\log T)^{\frac{p}{\delta} + \varepsilon}. \end{aligned}$$

Proof. By approximating the analytic function with trigonometric polynomials, employing Taylor expansions, and applying standard perturbation arguments, we can assume that X_N (as defined in (20)) is a semi-algebraic set with degree less than N^C . Let $N_1 = [N^\varepsilon]$. Applying $\mathcal{S} = X_{N_1}$, $\eta = e^{-N_1^{\sigma_2}}$ to (21), one has (19) holds. Now Theorem 5.2 follows from Theorem 5.1. \square

5.2. Proof of Theorem 1.1. Recall the skew-shift $f : \mathbb{T}^b \rightarrow \mathbb{T}^b$ is defined as

$$fx = f(x_1, x_2, \dots, x_b) = (x_1 + \omega, x_2 + x_1, \dots, x_b + x_{b-1}).$$

By the direct calculation, the n step iteration of the skew-shift is

$$\begin{aligned} f^n x &= f^n(x_1, x_2, \dots, x_b) \\ &= (x_1 + C_n^1 \omega, x_2 + C_n^1 x_1 + C_n^2 \omega, \dots, x_b + C_n^1 x_{b-1} + \dots + C_n^b \omega). \end{aligned}$$

Obviously, for any $1 \leq i \leq b$, the projection onto the i th coordinate for the n step skew-shift is a polynomial in n of degree i whose highest degree term is $(\omega/i!)n^i$. In particular, the projection onto the b th coordinate is

$$(f^n x)_b = x_b + nx_{b-1} + \frac{n(n-1)}{2}x_{b-2} + \dots + \frac{n(n-1)\dots(n-b+1)}{b!}\omega.$$

Now Theorem 1.1 follows as a corollary of our semi-algebraic set estimates.

Proof of Theorem 1.1. By Lemma 2.7 and $\omega \in DC(\gamma, \tau)$, we know that for any $1 \leq i \leq b$, $\omega/i! \in DC(\tilde{\gamma}, \tau)$ for some $\tilde{\gamma} = \tilde{\gamma}(b, \gamma)$.

For $2 \leq b \leq 5$, we apply Theorem 3.1 (for $b \geq 6$, we apply Theorem 4.1) with

$$m = b, \mathbf{P}(n) \bmod \mathbb{Z}^b = f^n x, \deg(P_i) = i, \alpha_i = \omega/i!,$$

so that

$$\#\{n \in [-N, N] : f^n x \in \mathcal{S}\} \leq C(\varepsilon, b, \gamma, \tau) N^{1 - \frac{1}{\tau b^{2(b-1)}} + \varepsilon}. \quad (22)$$

and

$$\#\{n \in [-N, N] : f^n x \in \mathcal{S}\} \leq C(\varepsilon, b, \gamma, \tau) N^{1 - \frac{1}{\tau b^2(b-1)} + \varepsilon}. \quad (23)$$

Recall

$$\psi(b) = \begin{cases} 2^{b-1}, & \text{if } 2 \leq b \leq 5, \\ b(b-1), & \text{if } b \geq 6. \end{cases} \quad (24)$$

Hence the theorem follows from Theorem 5.2, (22), (23), and (24). \square

APPENDIX A. PROOF OF LEMMA 2.5

Denote $c_n = e^{2\pi i P(n;\alpha)}$. For any $\rho \in \mathbb{Z}^+$, we can rewrite

$$S^\rho = \sum_{\mathbf{n}} c_{n_1} c_{n_2} \cdots c_{n_\rho}, \quad (25)$$

where \mathbf{n} runs over $\{1, 2, \dots, N\}^\rho$. For $\mathbf{n} = (n_1, n_2, \dots, n_\rho)$, we let $s_j(\mathbf{n}) = n_1^j + n_2^j + \dots + n_\rho^j$. Now we classify the set of \mathbf{n} according to the value of $\mathbf{s} = (s_1(\mathbf{n}), s_2(\mathbf{n}), \dots, s_{b-2}(\mathbf{n}))$. Let

$$\mathfrak{S} = \{1, 2, \dots, \rho N\} \times \{1, 2, \dots, \rho N^2\} \times \dots \times \{1, 2, \dots, \rho N^{b-2}\}.$$

Since $1 \leq n_i^j \leq N^j$, we may restrict our attention to points $\mathbf{s} \in \mathfrak{S}$. For $\mathbf{s} \in \mathfrak{S}$, we stratify the set of \mathbf{n} by letting

$$\mathcal{N}(\mathbf{s}) = \{\mathbf{n} \in \{1, \dots, N\}^\rho : s_j(\mathbf{n}) = s_j, 1 \leq j \leq b-2\}.$$

Then the sum in (25) can be partitioned into subsums:

$$S^\rho = \sum_{\mathbf{s} \in \mathfrak{S}} \sum_{\mathbf{n} \in \mathcal{N}(\mathbf{s})} c_{n_1} c_{n_2} \cdots c_{n_\rho}.$$

Since

$$\#\mathfrak{S} = \rho N \times \rho N^2 \times \dots \times \rho N^{b-2} = \rho^{b-2} N^{(b-1)(b-2)/2},$$

by the Hölder inequality, we have

$$\begin{aligned} |S|^{2\rho} &\leq \rho^{b-2} N^{(b-1)(b-2)/2} \sum_{\mathbf{s} \in \mathfrak{S}} \left| \sum_{\mathbf{n} \in \mathcal{N}(\mathbf{s})} c_{n_1} c_{n_2} \cdots c_{n_\rho} \right|^2 \\ &= \rho^{b-2} N^{(b-1)(b-2)/2} \sum_{\substack{\mathbf{m}, \mathbf{n} \\ s_j(\mathbf{m}) = s_j(\mathbf{n}) \\ 1 \leq j \leq b-2}} c_{m_1} \cdots c_{m_\rho} \overline{c_{n_1}} \cdots \overline{c_{n_\rho}}. \end{aligned} \quad (26)$$

To estimate (26), we only need to estimate

$$\mathcal{Z} := \sum_{\substack{\mathbf{m}, \mathbf{n} \\ s_j(\mathbf{m}) = s_j(\mathbf{n}) \\ 1 \leq j \leq b-2}} c_{m_1} \cdots c_{m_\rho} \overline{c_{n_1}} \cdots \overline{c_{n_\rho}}. \quad (27)$$

In the following, we estimate \mathcal{Z} . By the elimination from the symmetry of \mathbf{m} and \mathbf{n} , we can see that

$$\mathcal{Z} = e^{2\pi i (\alpha_b(s_b(\mathbf{m}) - s_b(\mathbf{n})) + \alpha_{b-1}(s_{b-1}(\mathbf{m}) - s_{b-1}(\mathbf{n})))}. \quad (28)$$

Now we shift the every component of \mathbf{m} and \mathbf{n} by m_1 . More precisely, we let $m_i = m_1 + u_i$ and $n_i = m_1 + v_i$ for $1 \leq i \leq \rho$. Then by Binomial Theorem,

$$s_j(\mathbf{u}) = \sum_{i=1}^{\rho} (m_i - m_1)^j = \sum_{r=0}^j C_j^r s_r(\mathbf{m})(-m_1)^{j-r},$$

and similarly,

$$s_j(\mathbf{v}) = \sum_{i=1}^{\rho} (n_i - m_1)^j = \sum_{r=0}^j C_j^r s_r(\mathbf{n})(-m_1)^{j-r}.$$

One can see that for $1 \leq j \leq b$,

$$s_j(\mathbf{u}) - s_j(\mathbf{v}) = \sum_{r=0}^j C_j^r \left(s_r(\mathbf{m}) - s_r(\mathbf{n}) \right) (-m_1)^{j-r}. \quad (29)$$

Hence the relations $s_j(\mathbf{m}) = s_j(\mathbf{n})$ for $1 \leq j \leq b-2$ imply that $s_j(\mathbf{u}) = s_j(\mathbf{v})$ for $1 \leq j \leq b-2$. In addition, we observe from (29) that

$$s_{b-1}(\mathbf{u}) - s_{b-1}(\mathbf{v}) = s_{b-1}(\mathbf{m}) - s_{b-1}(\mathbf{n}),$$

and that

$$s_b(\mathbf{u}) - s_b(\mathbf{v}) = s_b(\mathbf{m}) - s_b(\mathbf{n}) - b m_1 (s_{d-1}(\mathbf{m}) - s_{b-1}(\mathbf{n})).$$

For brevity we let $t_j = t_j(\mathbf{u}, \mathbf{v}) = s_j(\mathbf{u}) - s_j(\mathbf{v})$. Then (28) may be written as

$$\mathcal{Z} = \sum_{\substack{\mathbf{u}, \mathbf{v} \\ t_j=0 \\ 1 \leq j \leq b-2}} e^{2\pi i(t_b \alpha_b + t_{b-1} \alpha_{b-1})} \sum_{m_1=1}^N e^{2\pi i(b t_{b-1} m_1 \alpha_b)}. \quad (30)$$

It is obvious that the geometric series above satisfies

$$\left| \sum_{m_1=1}^N e^{2\pi i(b t_{b-1} m_1 \alpha_b)} \right| \lesssim \min \left(N, \frac{1}{\|b t_{b-1} \alpha_b\|_{\mathbb{T}}} \right). \quad (31)$$

Substituting (31) into (30) showing that

$$\mathcal{Z} \lesssim \sum_{\substack{\mathbf{u}, \mathbf{v} \\ t_j=0 \\ 1 \leq j \leq b-2}} \min \left(N, \frac{1}{\|b t_{b-1} \alpha_b\|_{\mathbb{T}}} \right). \quad (32)$$

Let $h \in \mathbb{Z}$ be a parameter with $|h| \leq 2\rho N^{b-1}$. Let $R_1(h)$ be the number of solutions of the system of equations

$$\begin{aligned} u_2^j + \cdots + u_{\rho}^j &= v_1^j + \cdots + v_{\rho}^j, \quad 1 \leq j \leq b-2, \\ u_2^{b-1} + \cdots + u_{\rho}^{b-1} &= h + v_1^{b-1} + \cdots + v_{\rho}^{b-1}, \end{aligned}$$

in integer variables for which $|u_i| \leq N$ and $|v_i| \leq N$. Then by (32) and the definition of $R_1(h)$,

$$\mathcal{Z} \lesssim \sum_{|h| \leq 2\rho N^{b-1}} R_1(h) \min\left(N, \frac{1}{\|b h \alpha_b\|_{\mathbb{T}}}\right). \quad (33)$$

Now we recover the variables \mathbf{m}, \mathbf{n} from \mathbf{u}, \mathbf{v} . Treat m_1 as a parameter. We let $m_i = m_1 + u_i$ and $n_i = m_1 + v_i$ for $1 \leq i \leq \rho$. Then $R_1(h)$ is the number of solutions of the system

$$\begin{aligned} s_j(\mathbf{m}) &= s_j(\mathbf{n}), \quad 1 \leq j \leq b-2, \\ s_{b-1}(\mathbf{m}) &= h + s_{b-1}(\mathbf{n}), \end{aligned} \quad (34)$$

in integer variables for which

$$\begin{aligned} m_1 - N &\leq m_i \leq m_1 + N, \quad 1 \leq i \leq \rho, \\ m_1 - N &\leq n_i \leq m_1 + N, \quad 1 \leq i \leq \rho. \end{aligned}$$

Since $1 \leq m_i, n_i \leq N$, we only need to consider $N+1 \leq m_1 \leq 2N$. Recall that Hardy–Ramanujan–Littlewood circle method relates the number of solutions to the integral on the circle. To apply the circle method, we let $R_2(h)$ be the number of solutions of (34) subject to the weaker constraints:

$$\begin{aligned} 1 &\leq m_i \leq 3N, \quad 1 \leq i \leq \rho, \\ 1 &\leq n_i \leq 3N, \quad 1 \leq i \leq \rho. \end{aligned}$$

Clearly, by the standard circle method,

$$R_2(h) = \int_{\mathbb{T}^{d-1}} \left| \sum_{n=1}^{3N} e^{2\pi i \tilde{P}(n; \beta)} \right|^{2\rho} e^{-2\pi i h \beta_{b-1}} d\beta_1 \cdots d\beta_{b-1}, \quad (35)$$

where $\tilde{P}(n; \beta) := \sum_{j=1}^{b-1} \beta_j n^j$ is a polynomial of degree of $b-1$. Since for each $N+1 \leq m_1 \leq 2N$ we have $R_2(h) \geq R_1(h)$, thus $R_2(h) \geq NR_1(h)$. It is evident that by taking the absolute value in (35),

$$R_2(h) \leq R_2(0), \quad \text{for all } h. \quad (36)$$

But by (2.2) and (2.2) we know $R_2(0) = J_{b-1}(3N; \rho)$. So by (33), (35), and (36), we have

$$\mathcal{Z} \lesssim \frac{1}{N} J_{b-1}(3N; \rho) \sum_{|h| \leq 2\rho N^{b-1}} \min\left(N, \frac{1}{\|b h \alpha_b\|_{\mathbb{T}}}\right). \quad (37)$$

Finally, we substitute (37) and (27) into (26), one can get

$$|S|^{2\rho} \lesssim N^{(b-1)(b-2)/2} \frac{1}{N} J_{b-1}(3N; \rho) \sum_{|h| \leq 2\rho b N^{b-1}} \min\left(N, \frac{1}{\|h \alpha_b\|_{\mathbb{T}}}\right).$$

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STATEMENTS AND DECLARATIONS

Conflict of Interest The authors declare no conflicts of interest.

Data Availability Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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