Bridging Informal and Formal Mathematical Reasoning with AI

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April 14, 2025

Carnegie Mellon University

AI agents in expert domains

AI agents in expert domains

- Finance
- Medicine
- Mathematics

AI agents in expert domains

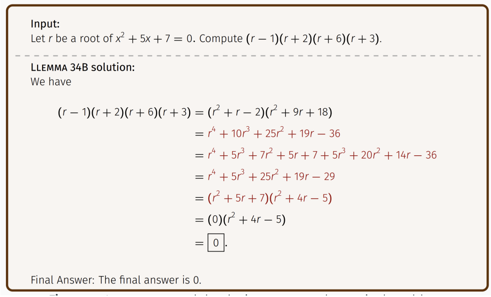
AI agents in expert domains

- Finance
- Medicine
- Mathematics
    - Open-ended dialogue
    - Come up with counterexamples
    - Help write proofs
    - ...

“Informal” mathematics

Math as raw data (text, images, ...)

- Flexible
- Widely used
- Difficult to check



Input:
Let r be a root of $x^2 + 5x + 7 = 0$. Compute $(r-1)(r+2)(r+6)(r+3)$.

Llemma 34B solution:
We have

$$\begin{aligned}(r-1)(r+2)(r+6)(r+3) &= (r^2+r-2)(r^2+9r+18)\\ &= r^4+10r^3+25r^2+19r-36\\ &= r^4+5r^3+7r^2+5r+7+5r^3+20r^2+14r-36\\ &= r^4+5r^3+25r^2+19r-29\\ &= (r^2+5r+7)(r^2+4r-5)\\ &= (0)(r^2+4r-5)\\ &= \boxed{0}.\end{aligned}$$

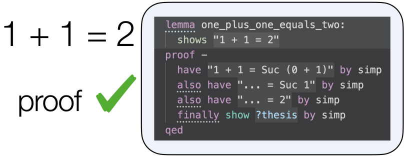
Final Answer: The final answer is 0.

Language model solution.

Formal mathematics

Math as **source code**

- Write a specification (e.g., $1+1=2$)
- Write a proof
- **Automatically checked**
    - Code compiles $\equiv$ correct proof



```
lemma one_plus_one_equals_two:
  shows "1 + 1 = 2"
proof -
  have "1 + 1 = Suc (0 + 1)" by simp
  also have "... = Suc 1" by simp
  also have "... = 2" by simp
  finally show ?thesis by simp
qed
```

Math as source code.

Formal mathematics

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Lean Isabelle Coq

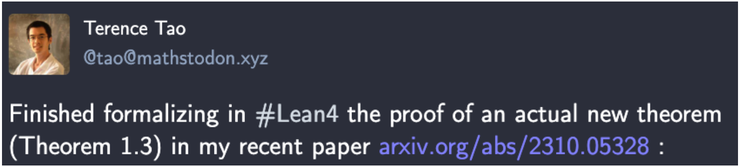
Theorem proving languages

Formal mathematics (Demo)

If $R \subseteq S$ and $S \subseteq T$ then $R \subseteq T$

How is formal math used in practice?

Growing use in mathematics:



Terence Tao

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Finished formalizing in #Lean4 the proof of an actual new theorem (Theorem 1.3) in my recent paper arxiv.org/abs/2310.05328 :

Terence Tao's Lean formalization project (October 2023)

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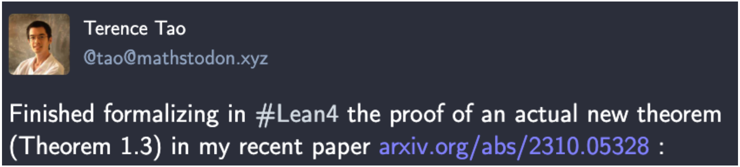

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Terence Tao's Lean formalization project (October 2023)

- **Lean Mathlib** project: 1+ million lines of code, 300+ contributors

How is formal math used in practice?

Why?[1]

- **Collaboration**
    - Break down a big problem into multiple pieces
    - Anyone can submit code to solve a piece
    - We know we can trust the code since it is automatically checked!

---

[1] See e.g., *Mathematics and the formal turn*, AFM Aims and Scope

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- **Guaranteed correctness**

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- ...

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Why is AI $\cap$ formal math important?

Formal math for AI

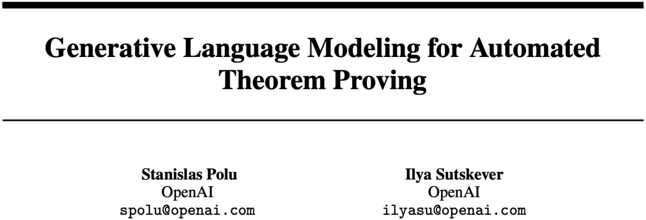
- **Verifiable**
  - Prevent incorrect math and code generation
  - Feedback signal for learning

Why is AI ∩ formal math important?

Formal math for AI

- **Verifiable**
    - Prevent incorrect math and code generation
    - Feedback signal for learning
- Tests **reasoning**
    - From easy: $1+1 = 2$
    - To hard: Fermat's Last Theorem

LLMs ∩ formal math



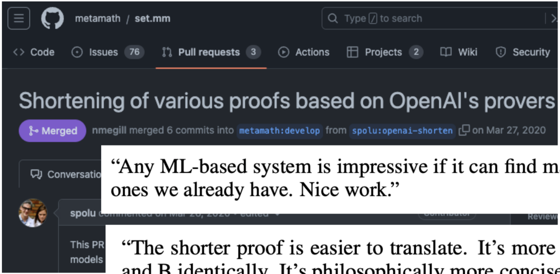
Generative Language Modeling for Automated Theorem Proving

Stanislas Polu
OpenAI
spolu@openai.com

Ilya Sutskever
OpenAI
ilyasu@openai.com

gpt-f (2020)

LLMs ∩ formal math



Shortening of various proofs based on OpenAI's provers

Merged nmegill merged 6 commits into `metamath:develop` from `spolu:openai-shorten` on Mar 27, 2020

This PR

models

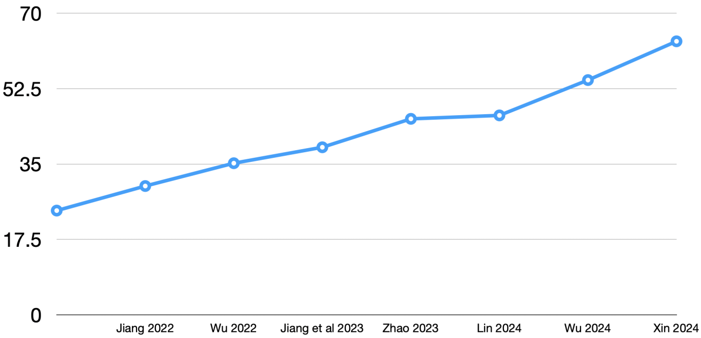
“Any ML-based system is impressive if it can find many shorter proofs than the ones we already have. Nice work.”

“The shorter proof is easier to translate. It’s more symmetric in that it treats A and B identically. It’s philosophically more concise in that it doesn’t rely on the existence of a universal class of all sets.”

gpt-f (2020)

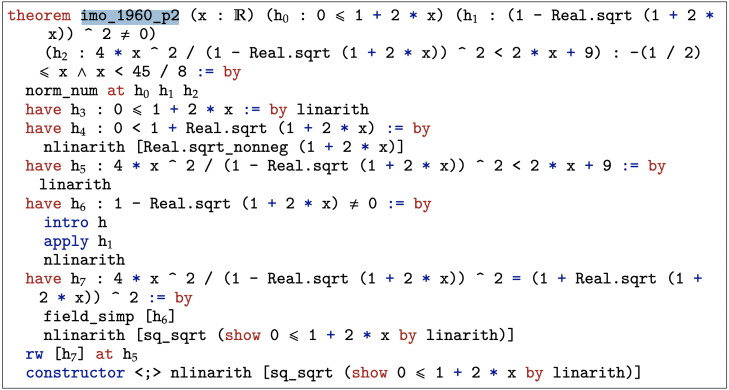
LLMs ∩ formal math

Rapid progress in methods based on language models:



miniF2F benchmark performance, 2022-2024

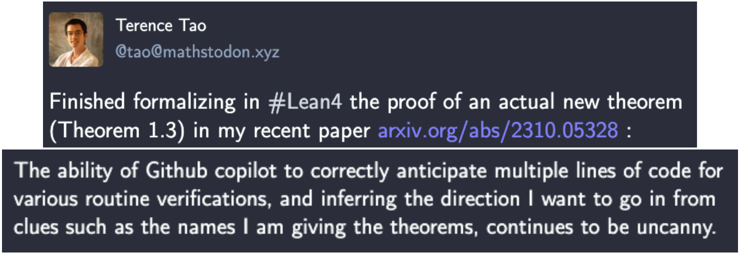
LLMs ∩ formal math



```
theorem imo_1960_p2 (x : ℝ) (h₀ : 0 ⩽ 1 + 2 * x) (h₁ : (1 - Real.sqrt (1 + 2 *
    x)) ^ 2 ≠ 0)
    (h₂ : 4 * x ^ 2 / (1 - Real.sqrt (1 + 2 * x)) ^ 2 < 2 * x + 9) : -(1 / 2)
   ⩽ x ∧ x < 45 / 8 := by
  norm_num at h₀ h₁ h₂
  have h₃ : 0 ⩽ 1 + 2 * x := by linarith
  have h₄ : 0 < 1 + Real.sqrt (1 + 2 * x) := by
    nlinarith [Real.sqrt_nonneg (1 + 2 * x)]
  have h₅ : 4 * x ^ 2 / (1 - Real.sqrt (1 + 2 * x)) ^ 2 < 2 * x + 9 := by
   linarith
  have h₆ : 1 - Real.sqrt (1 + 2 * x) ≠ 0 := by
    intro h
    apply h₁
    nlinarith
  have h₇ : 4 * x ^ 2 / (1 - Real.sqrt (1 + 2 * x)) ^ 2 = (1 + Real.sqrt (1 +
   2 * x)) ^ 2 := by
    field_simp [h₆]
    nlinarith [sq_sqrt (show 0 ⩽ 1 + 2 * x by linarith)]
  rw [h₇] at h₅
  constructor <;> nlinarith [sq_sqrt (show 0 ⩽ 1 + 2 * x by linarith)]
```

Generated International Math Olympiad solution in Lean
(DeepSeek Prover-1.5B, Xin et al 2024)

LLMs ∩ formal math



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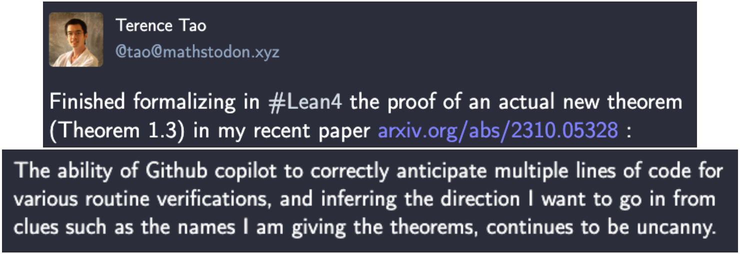
The ability of Github copilot to correctly anticipate multiple lines of code for various routine verifications, and inferring the direction I want to go in from clues such as the names I am giving the theorems, continues to be uncanny.

Terence Tao's Lean formalization project (October 2023)

LLMs ∩ formal math



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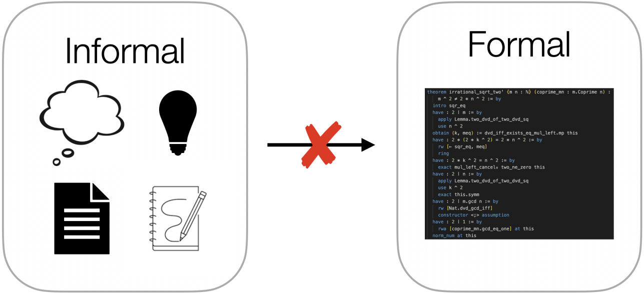
So...why don't people and AI always use formal math?

Key challenge: the informal-formal gap

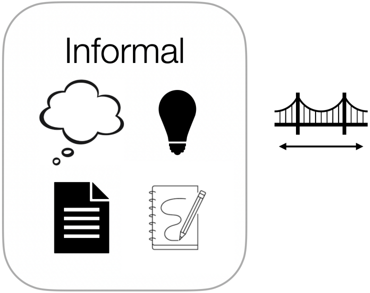
Informal ideas, intuitions, and even proofs are difficult to express formally:



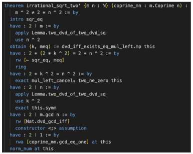
```
theorem irrational_sqrt_two' {m n : ℕ} (coprime_mn : m.Coprime n) :
    m ^ 2 ≠ 2 * n ^ 2 := by
  intro sqr_eq
  have : 2 ∣ m := by
    apply Lemma.two_dvd_of_two_dvd_sq
    use n ^ 2
  obtain ⟨k, meq⟩ := dvd_iff_exists_eq_mul_left.mp this
  have : 2 * (2 * k ^ 2) = 2 * n ^ 2 := by
    rw [← sqr_eq, meq]
    ring
  have : 2 * k ^ 2 = n ^ 2 := by
    exact mul_left_cancel₀ two_ne_zero this
  have : 2 ∣ n := by
    apply Lemma.two_dvd_of_two_dvd_sq
    use k ^ 2
    exact this.symm
  have : 2 ∣ m.gcd n := by
    rw [Nat.dvd_gcd_iff]
    constructor <;> assumption
  have : 2 ∣ 1 := by
    rwa [coprime_mn.gcd_eq_one] at this
  norm_num at this
```

- Each step of reasoning needs to be specified in detail
- Requires a deep knowledge of the formal system

Bridging Informal and Formal Mathematical Reasoning



Formal



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    constructor <;> assumption
  have : 2 ∣ 1 := by
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  norm_num at this
```

Flexible

Easy to check

This talk: Bridging Informal and Formal Mathematical Reasoning with AI

This talk: Bridging Informal and Formal

1. Informal thoughts
2. Informal provers
3. Research-level mathematics

This talk: Bridging Informal and Formal

1. Informal thoughts
    - Training models to think informally
        - Lean-STaR
2. Informal provers
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1. Informal thoughts
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2. Informal provers
    - Sketching proofs and filling in the gaps
        - Draft, Sketch, Prove
        - LeanHammer

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This talk: Bridging Informal and Formal

1. Informal thoughts
    - Training models to think informally
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2. Informal provers
    - Sketching proofs and filling in the gaps
        - Draft, Sketch, Prove
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3. Research-level mathematics
    - Assisting in research-level projects
        - Practical tools
        - MiniCTX

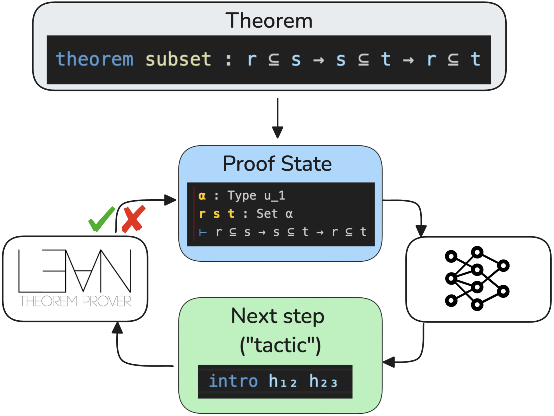
I: Informal thoughts

1. Training models to “think” — Lean-STaR

Lean-STaR: Learning to Interleave Thinking and Proving
Haohan Lin, Zhiqing Sun, Yiming Yang, Sean Welleck
ICLR 2025 (**Spotlight**)

1. Training models to “think” — Neural theorem proving

Neural theorem proving



Theorem

```
theorem subset : r ⊆ s → s ⊆ t → r ⊆ t
```

Proof State

```
α : Type u_1
r s t : Set α
⊢ r ⊆ s → s ⊆ t → r ⊆ t
```

Next step ("tactic")

```
intro h₁₂ h₂₃
```

- Math as checkable code
- Proof: sequence of (state, step)

1. Training models to “think” — Neural theorem proving

Language model-based proving:

- **Train** a model $p_\theta(y|x)$ on a dataset $\mathcal{D} = \{(x, y)\}$, e.g.,
  - x: proof state
  - y: next tactic (next “step”)
  - $\mathcal{D}$: extracted from theorems and proofs

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Language model-based proving:

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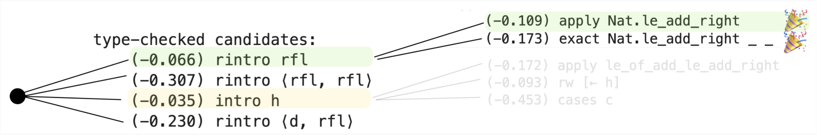
    - x: proof state
    - y: next tactic (next "step")
    - $\mathcal{D}$: extracted from theorems and proofs
- **Generate** proofs:



```
type-checked candidates:
(-0.066) rintro rfl
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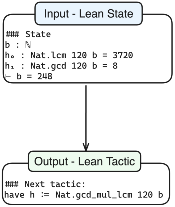
(-0.093) rw [← h]
(-0.453) cases c
```

Best-first search

1. Training models to “think” — Lean-STaR

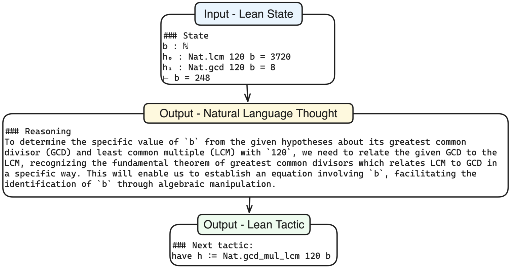


Input - Lean State

```
### State
b : ℕ
h₀ : Nat.lcm 120 b = 3720
h₁ : Nat.gcd 120 b = 8
⊢ b = 248
```

Output - Lean Tactic

```
### Next tactic:
have h := Nat.gcd_mul_lcm 120 b
```



Input - Lean State

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Output - Natural Language Thought

```
### Reasoning
To determine the specific value of `b` from the given hypotheses about its greatest common
divisor (GCD) and least common multiple (LCM) with `120`, we need to relate the given GCD to the
LCM, recognizing the fundamental theorem of greatest common divisors which relates LCM to GCD in
a specific way. This will enable us to establish an equation involving `b`, facilitating the
identification of `b` through algebraic manipulation.
```

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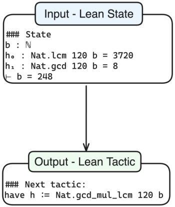
Can we train a model to “think” before each step of formal reasoning?

1. Training models to “think” — Lean-STaR



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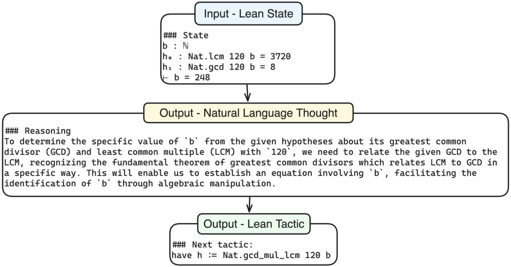

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Why?

- Plan proof steps
- Diversify search space
- More tokens can give more computational capacity

1. Training models to “think” — Lean-STaR

Lean-STaR (Self-taught reasoner[2])

Learn to generate thoughts via reinforcement learning

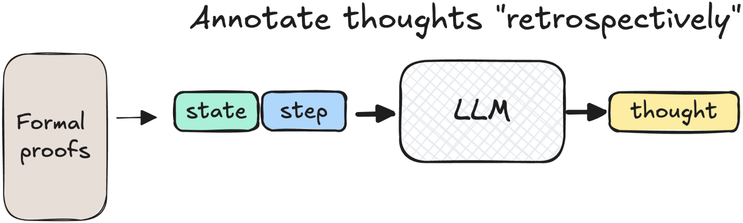
1. Initialization
2. Reinforcement learning

---

[2] Inspired by *STaR: Bootstrapping Reasoning with Reasoning*, Zelikman et al 2022

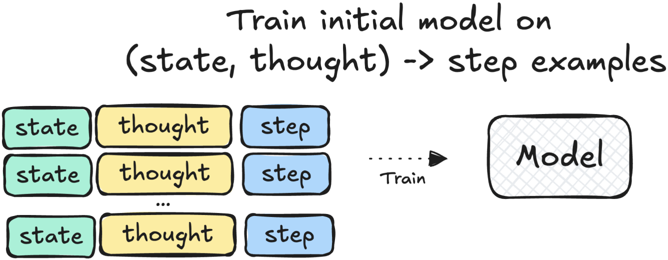
1. Training models to “think” — Lean-STaR

1. **Initialization**



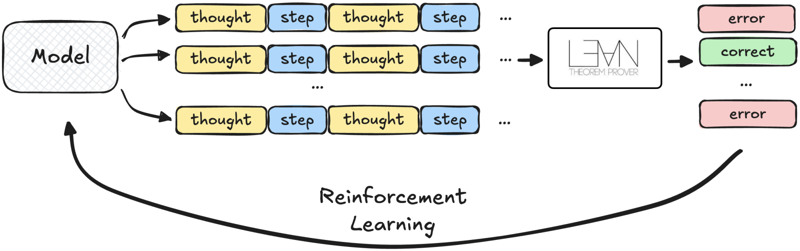
1. Training models to "think" — Lean-STaR

1. **Initialization**



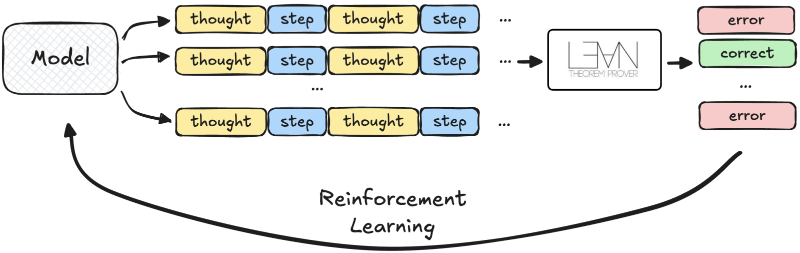
1. Training models to "think" — Lean-STaR

2: **Reinforcement learning**



1. Training models to “think” — Lean-STaR

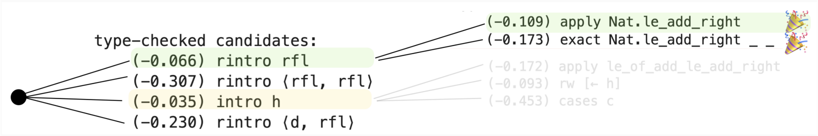
2: **Reinforcement learning**



Need:

- Method to generate proofs
- Learning algorithm

1. Training models to “think” — Lean-STaR



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(-0.453) cases c
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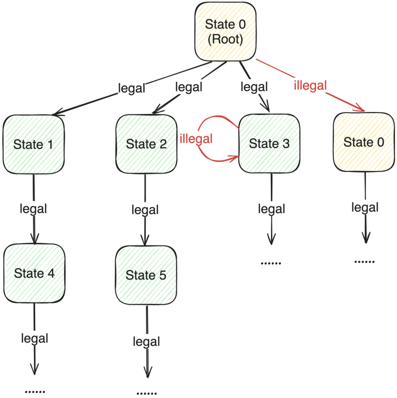
Best-first search: difficult to score (thought, tactic) candidates

1. Training models to "think" — Lean-STaR



New sampling method

Training models to “think” — Lean-STaR

Algorithm: train on the successful proofs, and repeat:[3]

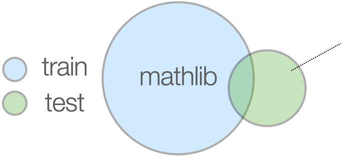
- Collect (state, thought, tactic) from successful proofs
- Train a new model $p_\theta^1(\mathit{thought}, \mathit{tactic} | \mathit{state})$
- Generate proofs
- ...

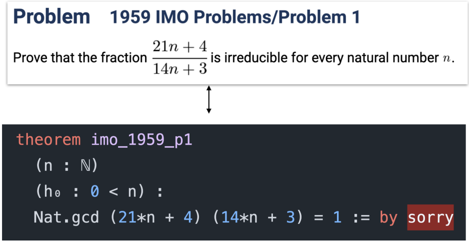
---

[3] I.e. Expert Iteration [Polu et al 2022], Rest-EM [Singh et al 2024]

Training models to “think” — Lean-STaR

- **miniF2F**: competition problems (AMC, AIME, IMO)



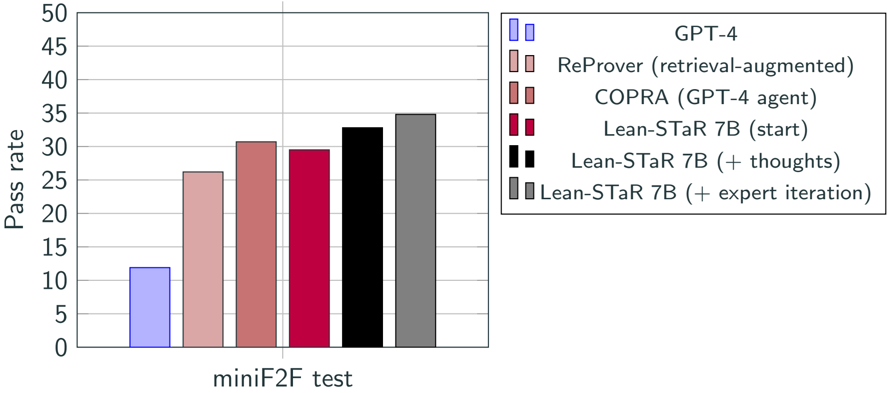


Problem **1959 IMO Problems/Problem 1**

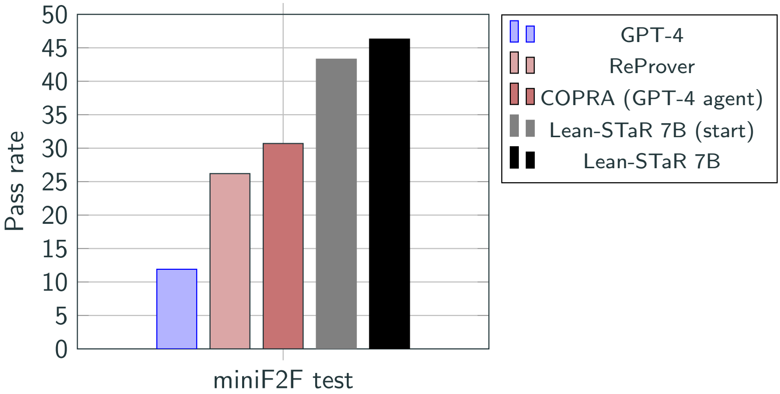
Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

```
theorem imo_1959_p1
  (n : ℕ)
  (h₀ : 0 < n) :
  Nat.gcd (21*n + 4) (14*n + 3) = 1 := by sorry
```

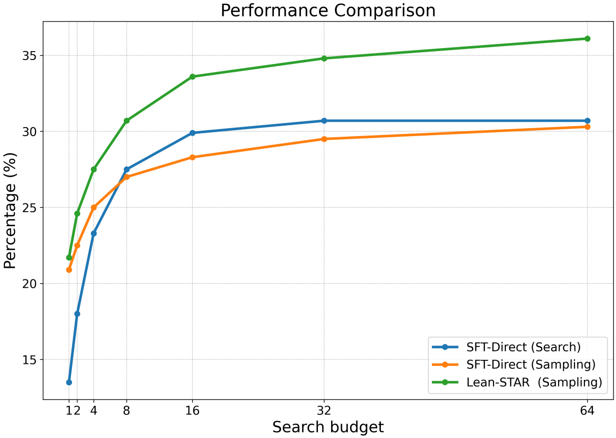
Training models to “think” — Lean-STaR



Training models to “think” — Lean-STaR



Training models to “think” — Lean-STaR



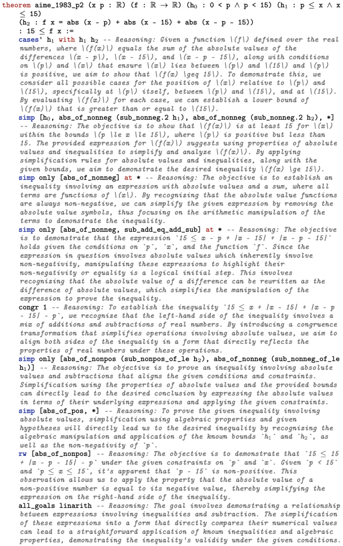
Increasing the search budget is more effective with thoughts

Training models to “think” — Lean-STaR

```
theorem amc12a_2002_p6 (n : ℕ) (h₀ : 0 < n)
    : ∃ m, m > n ∧ ∃ p, m * p ≤ m + p :=
    use n + 1, by linarith only [h₀] -- Reasoning: To demonstrate the existence of
    such an `m` and `p` that satisfy the given inequalities, we leverage the
    property of natural numbers that ensures an increase in `n` from `0` leads to
    the possibility of finding such `m` and `p`. This approach is supported by the
    fundamental property of natural numbers and their arithmetic operations,
    suggesting that for a given `n` greater than `0`, there exists a successor `m`
    and a number `p` that satisfy the conditions outlined.
    exact ⟨0, by simp ⟩ -- Reasoning: To demonstrate that there exists an \(n \in ℕ
    \) such that \((n + 1) * p \leq n + 1 + p\) holds for some \(p \in ℕ\), we
    recognize that setting \(p = 0\) satisfies the inequality by the properties of
    zero in arithmetic. This is because \(n + 1 > 0\) by assumption, ensuring the
    left-hand side is non-negative and the right-hand side simplifies to \(n +
    1\), making the inequality trivially true.
```

Example generated thoughts and proof from Lean-STaR

Training models to “think” — Lean-STaR

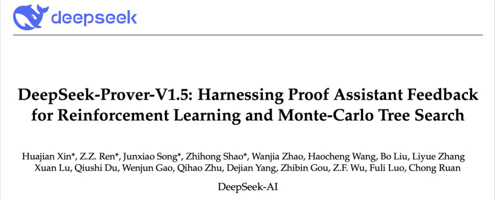


```
theorem aime_1983_p2 (x p : ℝ) (f : ℝ → ℝ) (h₀ : 0 < p ∧ p < 15) (h₁ : p ≤ x ∧ x
    ≤ 15)
    (h₂ : f x = abs (x - p) + abs (x - 15) + abs (x - p - 15))
    : 15 ≤ f x :=
    cases' h₁ with h₁ h₂ -- Reasoning: Given a function \(f\) defined over the real
    numbers, where \(f(x)\) equals the sum of the absolute values of the
    differences \(x - p\), \(x - 15\), and \(x - p - 15\), along with conditions
    on \(p\) and \(x\) that ensure \(x\) lies between \(p\) and \(15\) and \(p\)
    is positive, we aim to show that \(f(x) \geq 15\). To demonstrate this, we
    consider all possible cases for the position of \(x\) relative to \(p\) and
    \(15\), specifically at \(p\) itself, between \(p\) and \(15\), and at \(15\).
    By evaluating \(f(x)\) for each case, we can establish a lower bound of
    \(f(x)\) that is greater than or equal to \(15\).
    simp [h₀, abs_of_nonneg (sub_nonneg.2 h₁), abs_of_nonneg (sub_nonneg.2 h₂), *]
    -- Reasoning: The objective is to show that \(f(x)\) is at least 15 for \(x\)
    within the bounds \(p \le x \le 15\), where \(p\) is positive but less than
    15. The provided expression for \(f(x)\) suggests using properties of absolute
    values and inequalities to simplify and analyze \(f(x)\). By applying
    simplification rules for absolute values and inequalities, along with the
    given bounds, we aim to demonstrate the desired inequality \(f(x) \ge 15\).
    simp only [abs_of_nonneg] at * -- Reasoning: The objective is to establish an
    inequality involving an expression with absolute values and a sum, where all
    terms are functions of \(x\). By recognizing that the absolute value functions
    are always non-negative, we can simplify the given expression by removing the
    absolute value symbols, thus focusing on the arithmetic manipulation of the
    terms to demonstrate the inequality.
    simp only [abs_of_nonneg, sub_add_eq_add_sub] at * -- Reasoning: The objective
    is to demonstrate that the expression `15 ≤ x - p + |x - 15| + |x - p - 15|`
    holds given the conditions on `p`, `x`, and the function `f`. Since the
    expression in question involves absolute values which inherently involve
    non-negativity, manipulating these expressions to highlight their
    non-negativity or equality is a logical initial step. This involves
    recognizing that the absolute value of a difference can be rewritten as the
    difference of absolute values, which simplifies the manipulation of the
    expression to prove the inequality.
    congr 1 -- Reasoning: To establish the inequality `15 ≤ x + |x - 15| + |x - p
    - 15| - p`, we recognize that the left-hand side of the inequality involves a
    mix of additions and subtractions of real numbers. By introducing a congruence
    transformation that simplifies operations involving absolute values, we aim to
    align both sides of the inequality in a form that directly reflects the
    properties of real numbers under these operations.
    simp only [abs_of_nonpos (sub_nonpos_of_le h₂), abs_of_nonneg (sub_nonneg_of_le
    h₁)] -- Reasoning: The objective is to prove an inequality involving absolute
    values and subtractions that aligns the given conditions and constraints.
    Simplification using the properties of absolute values and the provided bounds
    can directly lead to the desired conclusion by expressing the absolute values
    in terms of their underlying expressions and applying the given constraints.
    simp [abs_of_pos, *] -- Reasoning: To prove the given inequality involving
    absolute values, simplification using algebraic properties and given
    hypotheses will directly lead us to the desired inequality by recognizing the
    algebraic manipulation and application of the known bounds `h₁` and `h₂`, as
    well as the non-negativity of `p`.
    rw [abs_of_nonpos] -- Reasoning: The objective is to demonstrate that `15 ≤ 15
    + |x - p - 15| - p` under the given constraints on `p` and `x`. Given `p < 15`
    and `p ≤ x ≤ 15`, it's apparent that `p - 15` is non-positive. This
    observation allows us to apply the property that the absolute value of a
    non-positive number is equal to its negative value, thereby simplifying the
    expression on the right-hand side of the inequality.
    all_goals linarith -- Reasoning: The goal involves demonstrating a relationship
    between expressions involving inequalities and subtraction. The simplification
    of these expressions into a form that directly compares their numerical values
    can lead to a straightforward application of known inequalities and algebraic
    properties, demonstrating the inequality's validity under the given conditions.
```

Example generated thoughts and proof from Lean-STaR

Training models to “think” — After Lean-STaR

After Lean-STaR, incorporating thoughts became a widely-used component of LLM-based theorem proving:

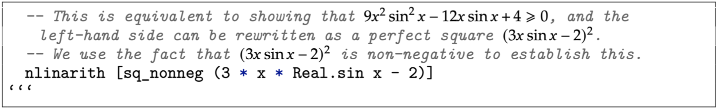

deepseek

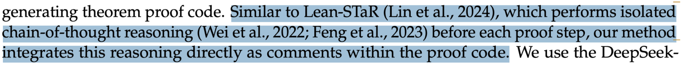
DeepSeek-Prover-V1.5: Harnessing Proof Assistant Feedback for Reinforcement Learning and Monte-Carlo Tree Search

Huajian Xin*, Z.Z. Ren*, Junxiao Song*, Zhihong Shao*, Wanjia Zhao, Haocheng Wang, Bo Liu, Liyue Zhang
Xuan Lu, Qiushi Du, Wenjun Gao, Qihao Zhu, Dejian Yang, Zhibin Gou, Z.F. Wu, Fuli Luo, Chong Ruan

DeepSeek-AI

Output:

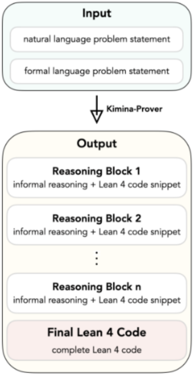

```
  -- This is equivalent to showing that $9x^2\sin^2 x - 12x\sin x + 4 \geqslant 0$, and the
    left-hand side can be rewritten as a perfect square $(3x\sin x - 2)^2$.
  -- We use the fact that $(3x\sin x - 2)^2$ is non-negative to establish this.
  nlinarith [sq_nonneg (3 * x * Real.sin x - 2)]
‘‘‘
```

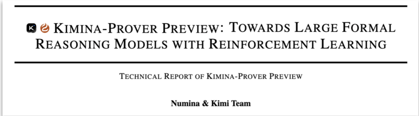

generating theorem proof code. Similar to Lean-STaR (Lin et al., 2024), which performs isolated chain-of-thought reasoning (Wei et al., 2022; Feng et al., 2023) before each proof step, our method integrates this reasoning directly as comments within the proof code. We use the DeepSeek-

Informal thoughts in DeepSeek Prover 1.5

Training models to "think" — After Lean-STaR

After Lean-STaR, incorporating thoughts became a widely-used component of LLM-based theorem proving:

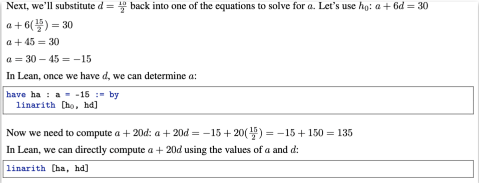

Input
natural language problem statement
formal language problem statement
Kimina-Prover
Output
Reasoning Block 1
informal reasoning + Lean 4 code snippet
Reasoning Block 2
informal reasoning + Lean 4 code snippet
⋮
Reasoning Block n
informal reasoning + Lean 4 code snippet
Final Lean 4 Code
complete Lean 4 code



KIMINA-PROVER PREVIEW: TOWARDS LARGE FORMAL REASONING MODELS WITH REINFORCEMENT LEARNING

TECHNICAL REPORT OF KIMINA-PROVER PREVIEW

Numina & Kimi Team



Next, we'll substitute $d = \frac{15}{2}$ back into one of the equations to solve for a. Let's use h_0: $a + 6d = 30$

$a + 6(\frac{15}{2}) = 30$

$a + 45 = 30$

$a = 30 - 45 = -15$

In Lean, once we have d, we can determine a:

```
have ha : a = -15 := by
  linarith [h₀, hd]
```

Now we need to compute $a + 20d$: $a + 20d = -15 + 20(\frac{15}{2}) = -15 + 150 = 135$

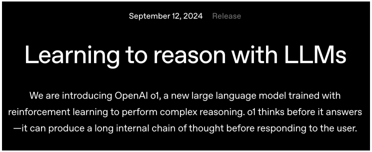
In Lean, we can directly compute $a + 20d$ using the values of a and d:

```
linarith [ha, hd]
```

Informal thoughts in Kimina Prover (April 14 2025)

Training models to "think" — After Lean-STaR

More broadly, *reasoning models* that generate long chains-of-thought have subsequently begun to revolutionize LLM reasoning:



September 12, 2024 Release

Learning to reason with LLMs

We are introducing OpenAI o1, a new large language model trained with reinforcement learning to perform complex reasoning. o1 thinks before it answers—it can produce a long internal chain of thought before responding to the user.

OpenAI o1 reasoning model

deepseek

DeepSeek-R1: Incentivizing Reasoning Capability in LLMs via Reinforcement Learning

DeepSeek-AI

research@deepseek.com

DeepSeek R1 reasoning model

Training models to “think” — Lean-STaR

Recap: **Lean-STaR**

- Training on formal code may be insufficient to learn the underlying thought process needed to produce the code
- Learn to generate thoughts via reinforcement learning

This talk: Bridging Informal and Formal

1. Informal thoughts

2. **Informal provers**
    - Sketching proofs and filling in the gaps
        - Draft, Sketch, Prove
        - LeanHammer

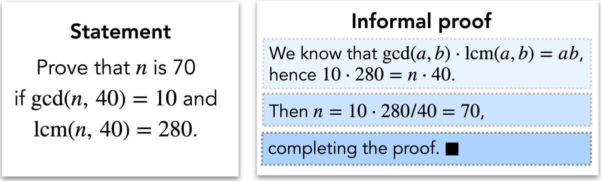
3. Research-level mathematics

II: Informal and formal provers

Combining informal and formal provers

Overall goal: combine high-level reasoning and low-level reasoning

Motivation: informal proofs and formal proofs



Statement

Prove that n is 70 if $\gcd(n, 40) = 10$ and $\operatorname{lcm}(n, 40) = 280$.

Informal proof

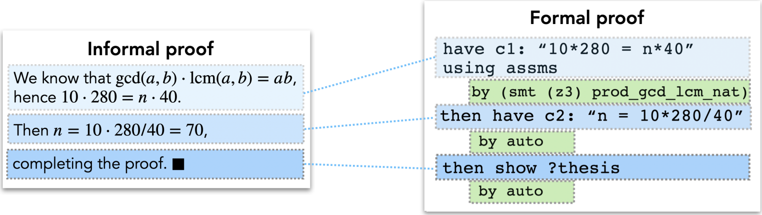
We know that $\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$, hence $10 \cdot 280 = n \cdot 40$.

Then $n = 10 \cdot 280/40 = 70$,

completing the proof. ■

How would we write this as a formal proof?

Motivation: informal proofs and formal proofs



Informal proof

We know that $\gcd(a, b) \cdot \mathrm{lcm}(a, b) = ab$, hence $10 \cdot 280 = n \cdot 40$.

Then $n = 10 \cdot 280/40 = 70$,

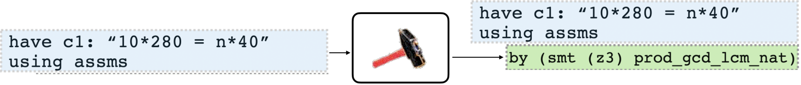
completing the proof. ■

Formal proof

```
have c1: "10*280 = n*40"
using assms
  by (smt (z3) prod_gcd_lcm_nat)
then have c2: "n = 10*280/40"
  by auto
then show ?thesis
  by auto
```

A proof with a high-level sketch and low-level proof steps.

Low-level provers: Sledgehammer



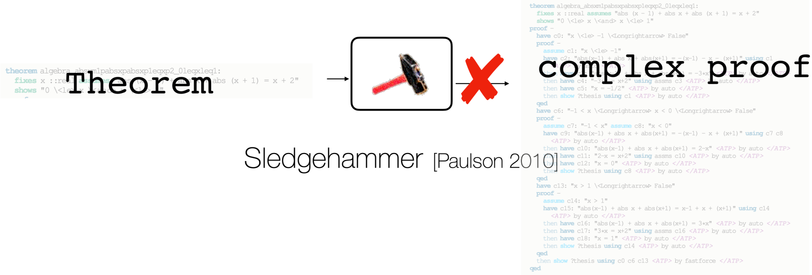
```
have c1: “10*280 = n*40”
using assms
```

```
have c1: “10*280 = n*40”
using assms
by (smt (z3) prod_gcd_lcm_nat)
```

Sledgehammer [Paulson 2010] calls out to external automated provers.

- First-order logic, higher-order logic, SMT

Low-level provers: Sledgehammer



Theorem

```
theorem algebra_absxm1pabsxpabsxp1eqxp2_0leqxleq1:
  fixes x ::real assumes "abs (x - 1) + abs x + abs (x + 1) = x + 2"
  shows "0 \<le> x \<and> x \<le> 1"
```

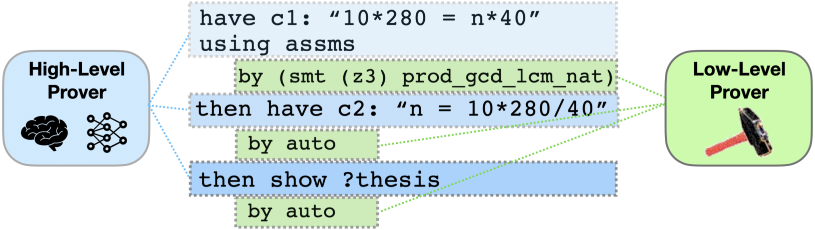
complex proof

```
theorem algebra_absxm1pabsxpabsxp1eqxp2_0leqxleq1:
  fixes x ::real assumes "abs (x - 1) + abs x + abs (x + 1) = x + 2"
  shows "0 \<le> x \<and> x \<le> 1"
proof -
  have c0: "x \<le> -1 \<Longrightarrow> False"
  proof -
    assume c1: "x \<le> -1"
    have c2: "abs(x-1) + abs x + abs(x+1) = -(x-1) - x - (x+1)" using c1
      <ATP> by auto </ATP>
    then have c3: "abs(x-1) + abs x + abs(x+1) = -3*x" <ATP> by auto </ATP>
    then have c4: "-3*x = x+2" using assms c3 <ATP> by auto </ATP>
    then have c5: "x = -1/2" <ATP> by auto </ATP>
    then show ?thesis using c1 <ATP> by auto </ATP>
  qed
  have c6: "-1 < x \<Longrightarrow> x < 0 \<Longrightarrow> False"
  proof -
    assume c7: "-1 < x" assume c8: "x < 0"
    have c9: "abs(x-1) + abs x + abs(x+1) = -(x-1) - x + (x+1)" using c7 c8
      <ATP> by auto </ATP>
    then have c10: "abs(x-1) + abs x + abs(x+1) = 2-x" <ATP> by auto </ATP>
    then have c11: "2-x = x+2" using assms c10 <ATP> by auto </ATP>
    then have c12: "x = 0" <ATP> by auto </ATP>
    then show ?thesis using c8 <ATP> by auto </ATP>
  qed
  have c13: "x > 1 \<Longrightarrow> False"
  proof -
    assume c14: "x > 1"
    have c15: "abs(x-1) + abs x + abs(x+1) = x-1 + x + (x+1)" using c14
      <ATP> by auto </ATP>
    then have c16: "abs(x-1) + abs x + abs(x+1) = 3*x" <ATP> by auto </ATP>
    then have c17: "3*x = x+2" using assms c16 <ATP> by auto </ATP>
    then have c18: "x = 1" <ATP> by auto </ATP>
    then show ?thesis using c14 <ATP> by auto </ATP>
  qed
  then show ?thesis using c0 c6 c13 <ATP> by fastforce </ATP>
qed
```

Sledgehammer [Paulson 2010]

Struggles due to the large search space of possible proofs

Idea: combine high-level and low-level proving



```
have c1: “10*280 = n*40”
using assms
    by (smt (z3) prod_gcd_lcm_nat)
then have c2: “n = 10*280/40”
    by auto
then show ?thesis
    by auto
```

Idea: combine high-level (human, LLM) and low-level proving

Draft-Sketch-Prove

Draft, Sketch, Prove: Guiding Formal Theorem Provers with Informal Proofs

Albert Q. Jiang, Sean Welleck, Jin Peng Zhou
Jiacheng Liu, Wenda Li, Mateja Jamnik
Guillaume Lample, Yuhuai Wu
*ICLR 2023 (**Oral**)*

Draft-Sketch-Prove

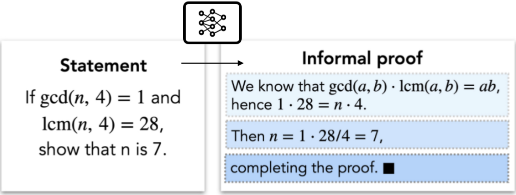
Idea:

Draft an informal proof, translate it into a formal **sketch**, then use a low-level **prover** to fill in the gaps.

Draft, Sketch, Prove

Given informal theorem x_I,
formal theorem x_F

1. **Draft** $y_I \sim p(\cdot|x_I)$



Statement

If $\gcd(n, 4) = 1$ and $\mathrm{lcm}(n, 4) = 28$, show that n is 7.

Informal proof

We know that $\gcd(a, b) \cdot \mathrm{lcm}(a, b) = ab$, hence $1 \cdot 28 = n \cdot 4$.

Then $n = 1 \cdot 28/4 = 7$,

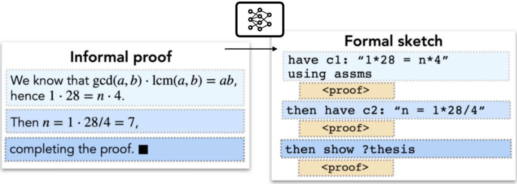
completing the proof. ■

Human-written or LLM-generated draft

Draft, Sketch, Prove

Given informal theorem x_I,
formal theorem x_F

1. Draft $y_I \sim p(\cdot|x_I)$
2. **Sketch** $z_F \sim p(\cdot|x_F, x_I, y_I)$



Informal proof

We know that $\gcd(a,b) \cdot \mathrm{lcm}(a,b) = ab$, hence $1 \cdot 28 = n \cdot 4$.

Then $n = 1 \cdot 28/4 = 7$,

completing the proof. ■

Formal sketch

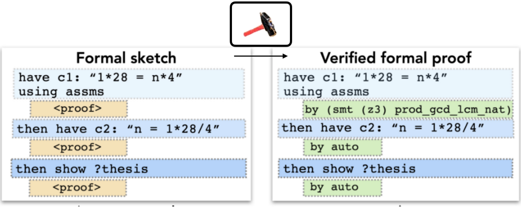
```
have c1: "1*28 = n*4"
using assms
    <proof>
then have c2: "n = 1*28/4"
    <proof>
then show ?thesis
    <proof>
```

LLM-generated sketch

Draft, Sketch, Prove

Given informal theorem x_I,
formal theorem x_F

1. Draft $y_I \sim p(\cdot|x_I)$
2. Sketch $z_F \sim p(\cdot|x_F, x_I, y_I)$
3. **Prove** $y_F = f(x_F, z_F)$



Formal sketch

```
have c1: "1*28 = n*4"
using assms
    <proof>
then have c2: "n = 1*28/4"
    <proof>
then show ?thesis
    <proof>
```

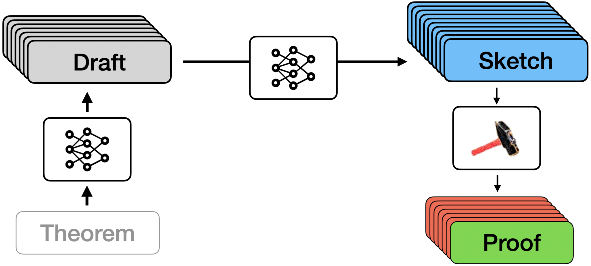
Verified formal proof

```
have c1: "1*28 = n*4"
using assms
    by (smt (z3) prod_gcd_lcm_nat)
then have c2: "n = 1*28/4"
    by auto
then show ?thesis
    by auto
```

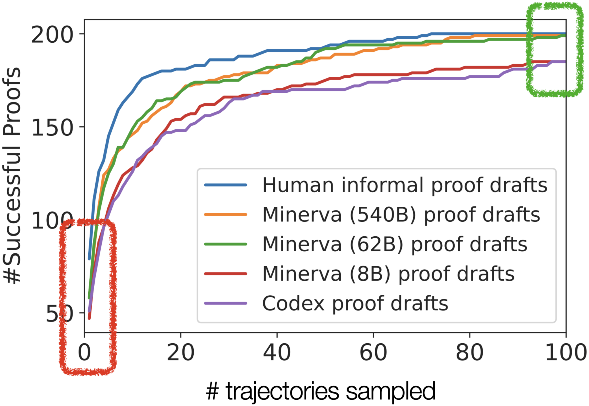
Low-level prover (*Sledgehammer*) fills in the gaps

Draft-sketch-prove

Proof search

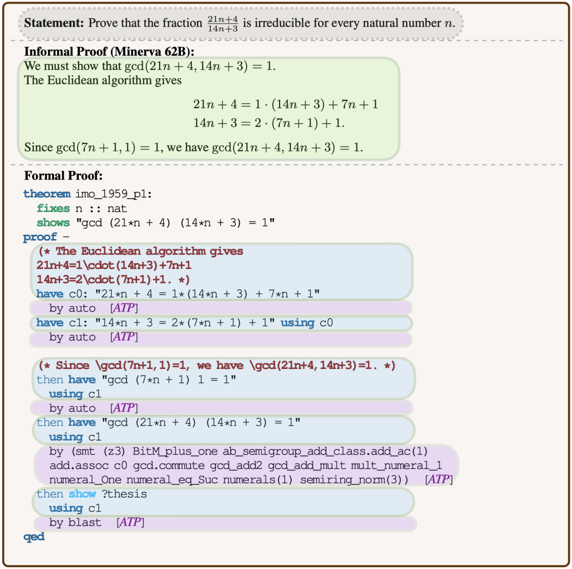


Draft-sketch-prove



Inference-time proof search scaling

Draft-sketch-prove



Statement: Prove that the fraction $\frac{21n+4}{14n+3}$ is irreducible for every natural number n.

Informal Proof (Minerva 62B):

We must show that $\gcd(21n+4, 14n+3) = 1$.
The Euclidean algorithm gives

$$\begin{aligned} 21n+4 &= 1 \cdot (14n+3) + 7n+1 \\ 14n+3 &= 2 \cdot (7n+1) + 1. \end{aligned}$$

Since $\gcd(7n+1, 1) = 1$, we have $\gcd(21n+4, 14n+3) = 1$.

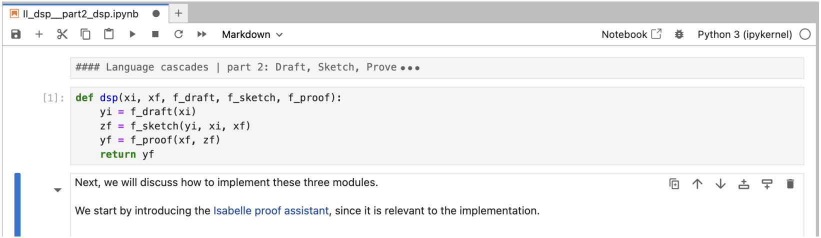
Formal Proof:

```
theorem imo_1959_p1:
  fixes n :: nat
  shows "gcd (21*n + 4) (14*n + 3) = 1"
proof -
  (* The Euclidean algorithm gives
  21n+4=1\cdot(14n+3)+7n+1
  14n+3=2\cdot(7n+1)+1. *)
  have c0: "21*n + 4 = 1*(14*n + 3) + 7*n + 1"
    by auto   [ATP]
  have c1: "14*n + 3 = 2*(7*n + 1) + 1" using c0
    by auto   [ATP]

  (* Since \gcd(7n+1,1)=1, we have \gcd(21n+4,14n+3)=1. *)
  then have "gcd (7*n + 1) 1 = 1"
    using c1
    by auto   [ATP]
  then have "gcd (21*n + 4) (14*n + 3) = 1"
    using c1
    by (smt (z3) BitM_plus_one ab_semigroup_add_class.add_ac(1)
    add.assoc c0 gcd.commute gcd_add2 gcd_add_mult mult_numeral_1
    numeral_One numeral_eq_Suc numerals(1) semiring_norm(3))   [ATP]
  then show ?thesis
    using c1
    by blast   [ATP]
qed
```

International Math Olympiad problem

Draft, Sketch, Prove



```
#### Language cascades | part 2: Draft, Sketch, Prove •••
```

```python
def dsp(xi, xf, f_draft, f_sketch, f_proof):
    yi = f_draft(xi)
    zf = f_sketch(yi, xi, xf)
    yf = f_proof(xf, zf)
    return yf
```

Next, we will discuss how to implement these three modules.

We start by introducing the Isabelle proof assistant, since it is relevant to the implementation.

Demo notebook: github.com/cmu-l3/ntptutorial-II

Next: building a low-level prover for Lean

Recap:

- Draft-Sketch-Prove: generate high-level sketches and fill in gaps
- Isabelle's *Sledgehammer* calls out to external provers to fill in gaps

Next: building a low-level prover for Lean

Recap:

- Draft-Sketch-Prove: generate high-level sketches and fill in gaps
- Isabelle's *Sledgehammer* calls out to external provers to fill in gaps

Next: can we build a Sledgehammer for Lean?

LeanHammer

Premise Selection for a Lean Hammer

Thomas Zhu, Joshua Clune
Jeremy Avigad, Albert Q. Jiang, Sean Welleck
Under Review 2025

What is a hammer?

A **hammer** integrates an automated theorem prover into an interactive theorem prover

What is a hammer?

A **hammer** integrates an automated theorem prover into an interactive theorem prover

- Interactive theorem prover: Lean, Isabelle, Coq
    - A programming language that checks proofs

What is a hammer?

A **hammer** integrates an automated theorem prover into an interactive theorem prover

- Interactive theorem prover: Lean, Isabelle, Coq
    - A programming language that checks proofs

- Automated theorem prover: higher-order logic provers, SMT solvers
    - Tries to automatically find proofs

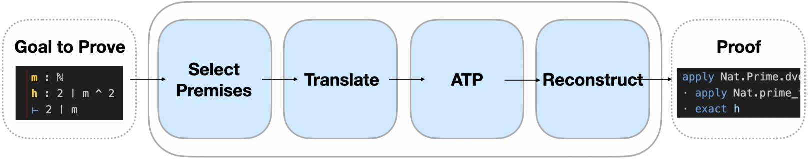
Key challenge: Premise selection

Automated theorem provers (ATPs) struggle with the large search space of possible proofs.

- **Premise selection**: select a small subset of theorems and definitions that are likely to be useful for proving a given theorem.
  - Around $250,000$ premises in Lean's Mathlib
  - Cuts down the prover's search space

A hammer pipeline

A standard hammer pipeline:



Goal to Prove

```
m : ℕ
h : 2 ∣ m ^ 2
⊢ 2 ∣ m
```

Select Premises

Translate

ATP

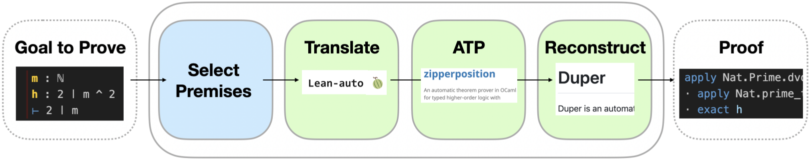
Reconstruct

Proof

```
apply Nat.Prime.dv
· apply Nat.prime_
· exact h
```

A hammer pipeline

A standard hammer pipeline:



```
m : ℕ
h : 2 ∣ m ^ 2
⊢ 2 ∣ m
```

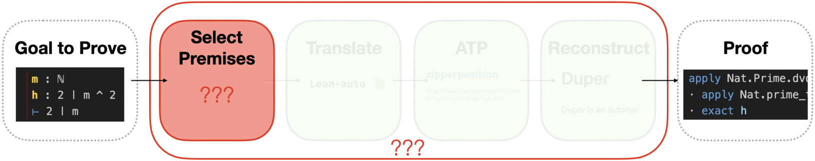
```
apply Nat.Prime.dv
· apply Nat.prime_
· exact h
```

Pre-existing components:

- **Translation**: LeanAuto [Qian et al 2025]
- **ATP**: Zipperposition [Cruanes et al 2015]
- **Reconstruction**: Duper [Clune et al 2024]

A hammer pipeline

A standard hammer pipeline:



```
m : ℕ
h : 2 ∣ m ^ 2
⊢ 2 ∣ m
```

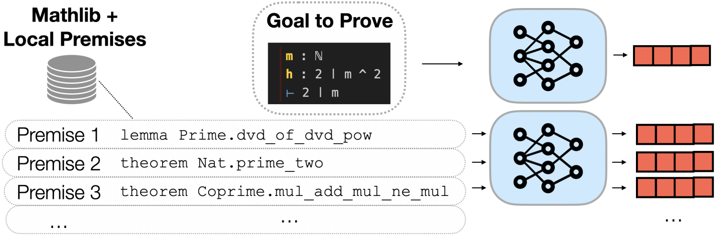
```
apply Nat.Prime.dvd
· apply Nat.prime_
· exact h
```

Our challenge:

- **Premise selection**
- **Put it all together** to create LeanHammer

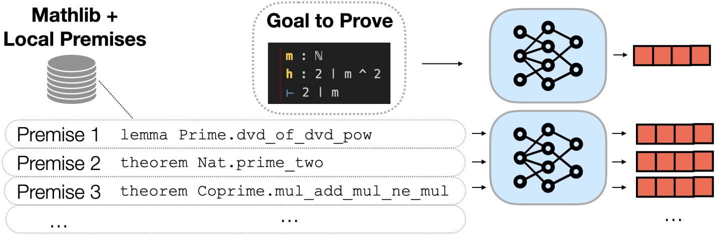
LeanHammer — Neural premise selection

Idea: frame premise selection as retrieval with a neural language model


Mathlib +
Local Premises
Goal to Prove
m : ℕ
h : 2 ∣ m ^ 2
⊢ 2 ∣ m
Premise 1 lemma Prime.dvd_of_dvd_pow
Premise 2 theorem Nat.prime_two
Premise 3 theorem Coprime.mul_add_mul_ne_mul
…
…
…

LeanHammer — Neural premise selection

Idea: frame premise selection as retrieval with a neural language model



Mathlib + Local Premises

Goal to Prove

```
m : ℕ
h : 2 ∣ m ^ 2
⊢ 2 ∣ m
```

Premise 1 `lemma Prime.dvd_of_dvd_pow`

Premise 2 `theorem Nat.prime_two`

Premise 3 `theorem Coprime.mul_add_mul_ne_mul`

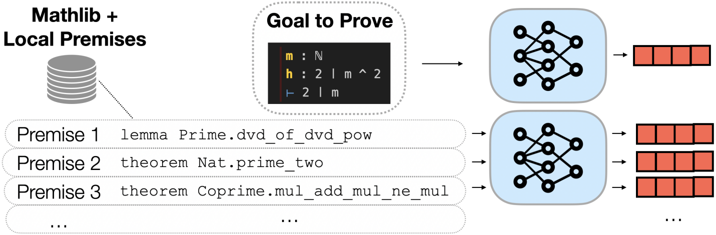
… …

…

- Transformer encoder embeds the state and candidate premises

LeanHammer — Neural premise selection

Idea: frame premise selection as retrieval with a neural language model



Mathlib +
Local Premises

Goal to Prove

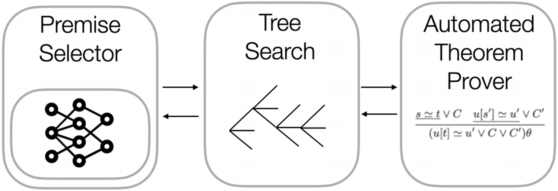
```
m : ℕ
h : 2 ∣ m ^ 2
⊢ 2 ∣ m
```

Premise 1	`lemma Prime.dvd_of_dvd_pow`
Premise 2	`theorem Nat.prime_two`
Premise 3	`theorem Coprime.mul_add_mul_ne_mul`
…	…

- Transformer encoder embeds the state and candidate premises
- Contrastive loss on (state, $\{\text{premise}^+\}$, $\{\text{premise}^-\}$) examples
    - Nuance in how to collect and format examples

LeanHammer — Putting it all together

Idea: combine the premise selector and ATP with a tree search



Premise Selector

Tree Search

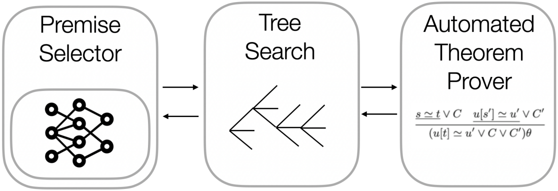
Automated Theorem Prover

$$\frac{\underline{s \simeq t} \vee C \quad \underline{u[s']} \simeq u' \vee C'}{(u[t] \simeq u' \vee C \vee C')\theta}$$

Tree search: *Aesop* [Limperg & From 2023]

LeanHammer — Putting it all together

Idea: combine the premise selector and ATP with a tree search


Premise Selector
Tree Search
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$\frac{s \simeq t \vee C \quad u[s'] \simeq u' \vee C'}{(u[t] \simeq u' \vee C \vee C')\theta}$

Tree search: *Aesop* [Limperg & From 2023]

1. Queries the automated theorem prover using the premises
2. Applies tactics (e.g. `apply`, `simp_all`) using the premises

LeanHammer — Putting it all together

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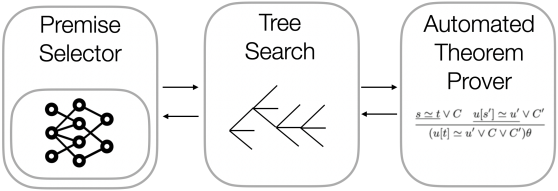
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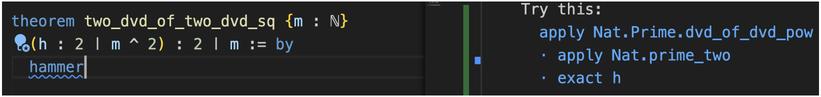
Tree search: *Aesop* [Limperg & From 2023]

1. Queries the automated theorem prover using the premises
2. Applies tactics (e.g. `apply`, `simp_all`) using the premises

Goes beyond the standard hammer pipeline!

LeanHammer

As a user, simply issue `hammer` at any step of a proof:



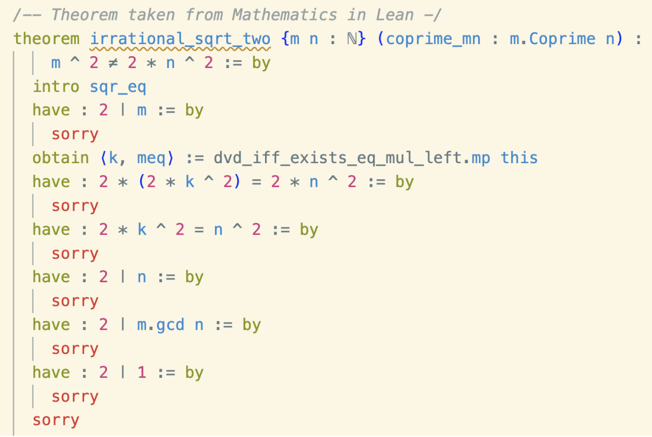
```
theorem two_dvd_of_two_dvd_sq {m : ℕ}
(h : 2 ∣ m ^ 2) : 2 ∣ m := by
  hammer
```

```
Try this:
  apply Nat.Prime.dvd_of_dvd_pow
  · apply Nat.prime_two
  · exact h
```

LeanHammer in action

LeanHammer — Demo

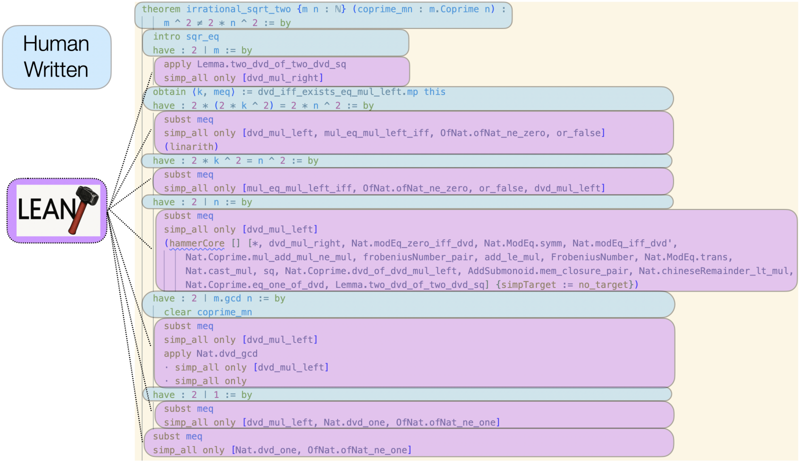
Demo: start with human-written proof sketch (from *Mathematics in Lean*)



```
/-- Theorem taken from Mathematics in Lean -/
theorem irrational_sqrt_two {m n : ℕ} (coprime_mn : m.Coprime n) :
    m ^ 2 ≠ 2 * n ^ 2 := by
  intro sqr_eq
  have : 2 ∣ m := by
    sorry
  obtain ⟨k, meq⟩ := dvd_iff_exists_eq_mul_left.mp this
  have : 2 * (2 * k ^ 2) = 2 * n ^ 2 := by
    sorry
  have : 2 * k ^ 2 = n ^ 2 := by
    sorry
  have : 2 ∣ n := by
    sorry
  have : 2 ∣ m.gcd n := by
    sorry
  have : 2 ∣ 1 := by
    sorry
  sorry
```

LeanHammer — Demo

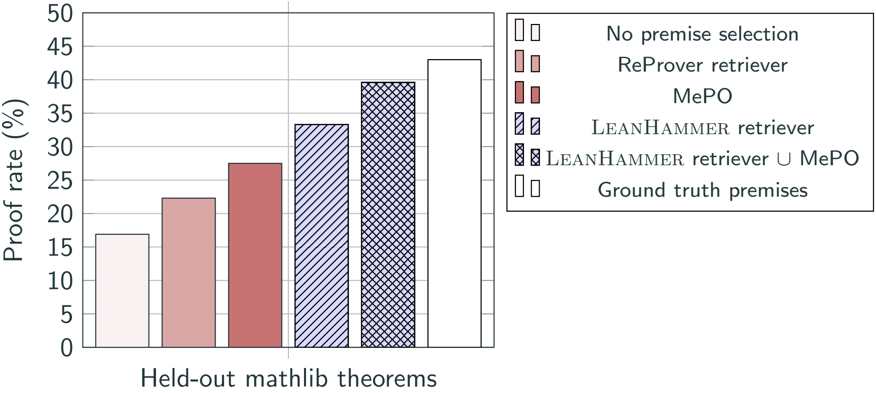
Demo: fill in the gaps (`sorry`s) with LeanHammer



```lean
theorem irrational_sqrt_two {m n : ℕ} (coprime_mn : m.Coprime n) :
    m ^ 2 ≠ 2 * n ^ 2 := by
  intro sqr_eq
  have : 2 ∣ m := by
    apply Lemma.two_dvd_of_two_dvd_sq
    simp_all only [dvd_mul_right]
  obtain ⟨k, meq⟩ := dvd_iff_exists_eq_mul_left.mp this
  have : 2 * (2 * k ^ 2) = 2 * n ^ 2 := by
    subst meq
    simp_all only [dvd_mul_left, mul_eq_mul_left_iff, OfNat.ofNat_ne_zero, or_false]
    (linarith)
  have : 2 * k ^ 2 = n ^ 2 := by
    subst meq
    simp_all only [mul_eq_mul_left_iff, OfNat.ofNat_ne_zero, or_false, dvd_mul_left]
  have : 2 ∣ n := by
    subst meq
    simp_all only [dvd_mul_left]
    (hammerCore [] [*, dvd_mul_right, Nat.modEq_zero_iff_dvd, Nat.ModEq.symm, Nat.modEq_iff_dvd',
        Nat.Coprime.mul_add_mul_ne_mul, frobeniusNumber_pair, add_le_mul, FrobeniusNumber, Nat.ModEq.trans,
        Nat.cast_mul, sq, Nat.Coprime.dvd_of_dvd_mul_left, AddSubmonoid.mem_closure_pair, Nat.chineseRemainder_lt_mul,
        Nat.Coprime.eq_one_of_dvd, Lemma.two_dvd_of_two_dvd_sq] {simpTarget := no_target})
  have : 2 ∣ m.gcd n := by
    clear coprime_mn
    subst meq
    simp_all only [dvd_mul_left]
    apply Nat.dvd_gcd
    · simp_all only [dvd_mul_left]
    · simp_all only
  have : 2 ∣ 1 := by
    subst meq
    simp_all only [dvd_mul_left, Nat.dvd_one, OfNat.ofNat_ne_one]
  subst meq
  simp_all only [Nat.dvd_one, OfNat.ofNat_ne_one]
```

LeanHammer — Quantitative results

Varying the premise selector within LeanHammer:



Recap

Two approaches for combining informal and formal provers:

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- **Draft-Sketch-Prove (DSP)**
  - LLM drafts informal proof, generates formal sketch
  - Low-level prover (Sledgehammer) fills in low-level details

Recap

Two approaches for combining informal and formal provers:

- **Draft-Sketch-Prove (DSP)**
  - LLM drafts informal proof, generates formal sketch
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- **LeanHammer**
  - Brings hammer functionality to Lean
  - Neural premise selection + tree search + automated theorem proving
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Recap

Two approaches for combining informal and formal provers:

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Even small neural networks are powerful! (retriever: $<$ 100M params)

This talk: Bridging Informal and Formal

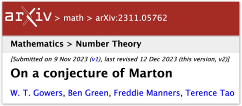
1. Informal thoughts

2. Informal provers

3. **Research-level mathematics**
    - Assisting in research-level projects
    - Practical tools
    - MiniCTX

III: Research-level mathematics

What does it look like to formalize research-level math?[4]



arXiv > math > arXiv:2311.05762

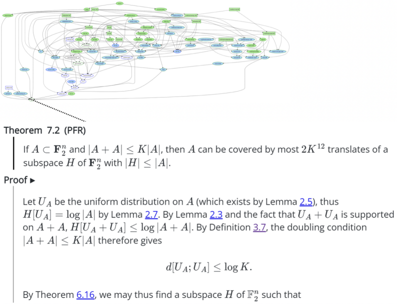
Mathematics > Number Theory

[Submitted on 9 Nov 2023 (v1), last revised 12 Dec 2023 (this version, v2)]

On a conjecture of Marton

W. T. Gowers, Ben Green, Freddie Manners, Terence Tao

arXiv



Theorem 7.2 (PFR)

If $A \subset \mathbf{F}_2^n$ and $|A+A| \leq K|A|$, then A can be covered by most $2K^{12}$ translates of a subspace H of $\mathbf{F}_2^n$ with $|H| \leq |A|$.

Proof ▸

Let U_A be the uniform distribution on A (which exists by Lemma 2.5), thus $H[U_A] = \log |A|$ by Lemma 2.7. By Lemma 2.3 and the fact that $U_A + U_A$ is supported on $A+A$, $H[U_A + U_A] \leq \log |A+A|$. By Definition 3.7, the doubling condition $|A+A| \leq K|A|$ therefore gives

$$d[U_A; U_A] \leq \log K.$$

By Theorem 6.16, we may thus find a subspace H of $\mathbb{F}_2^n$ such that

Informal "Blueprint"

---

[4] *Formalizing the proof of PFR in Lean4 using Blueprint: a short tour* by Terence Tao

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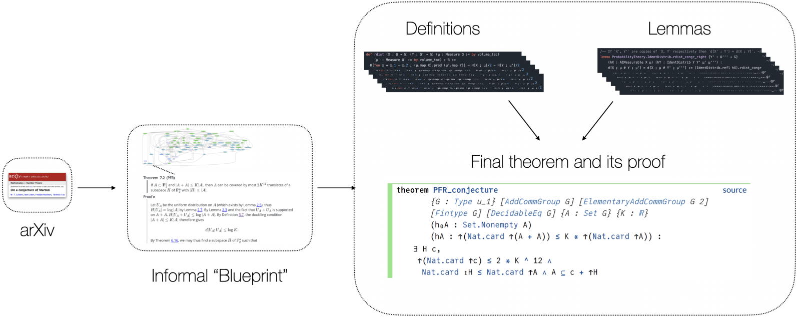

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By Theorem 6.16, we may thus find a subspace H of $\mathbb{F}_2^n$ such that

Informal “Blueprint”

Definitions

Lemmas

Final theorem and its proof

```
theorem PFR_conjecture                                              source
        {G : Type u_1} [AddCommGroup G] [ElementaryAddCommGroup G 2]
        [Fintype G] [DecidableEq G] {A : Set G} {K : ℝ}
        (h₀A : Set.Nonempty A)
        (hA : ↑(Nat.card ↑(A + A)) ≤ K * ↑(Nat.card ↑A)) :
    ∃ H c,
     ↑(Nat.card ↑c) ≤ 2 * K ^ 12 ∧
      Nat.card ↥H ≤ Nat.card ↑A ∧ A ⊆ c + ↑H
```

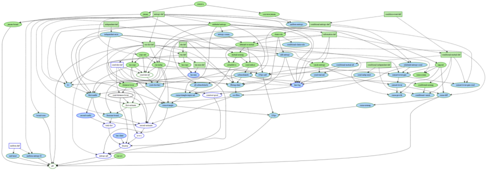
Formal
Definitions, Theorems, Proofs

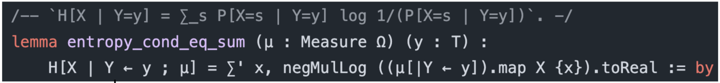
---

[4] *Formalizing the proof of PFR in Lean4 using Blueprint: a short tour* by Terence Tao

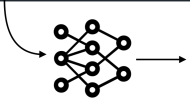
Where can AI help?

As a start, can AI help with filling in small parts of the blueprint?





```
/-- `H[X | Y=y] = ∑_s P[X=s | Y=y] log 1/(P[X=s | Y=y])`. -/
lemma entropy_cond_eq_sum (μ : Measure Ω) (y : T) :
    H[X | Y ← y ; μ] = ∑' x, negMulLog ((μ[|Y ← y]).map X {x}).toReal := by
```



???

Where can AI help? — Challenges

Accessibility gap:

- Some methods are hard to integrate into tools
    - Not open-source (AlphaProof, ...)
    - Expensive to run (MCTS, ...)

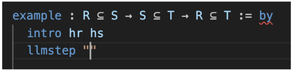
Where can AI help? — Challenges

Accessibility gap:

- Some methods are hard to integrate into tools
    - Not open-source (AlphaProof, ...)
    - Expensive to run (MCTS, ...)

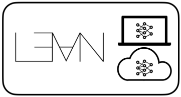
However, there **are** already tools available!

Where can AI help? — Existing tools

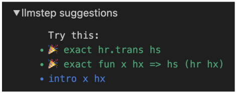


```
example : R ⊆ S → S ⊆ T → R ⊆ T := by
  intro hr hs
  llmstep ""
```

Send proof state +
doc context



LLMLean



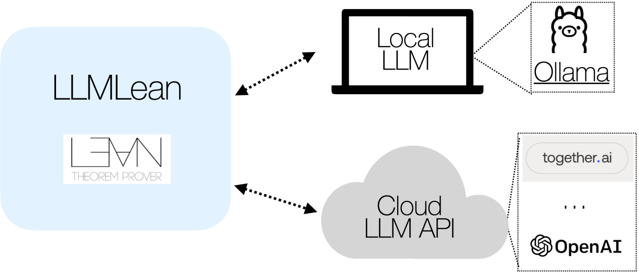
```
▼llmstep suggestions

    Try this:
  • 🎉 exact hr.trans hs
  • 🎉 exact fun x hx => hs (hr hx)
  • intro x hx
```

Verified suggestions

LLMLean: https://github.com/cmu-l3/llmlean

Where can AI help? — Existing tools



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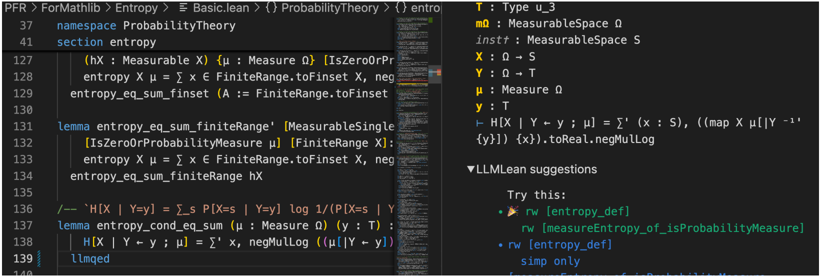
Where can AI help? — Existing tools


PFR > ForMathlib > Entropy > Basic.lean > {} ProbabilityTheory > {} entro

```
namespace ProbabilityTheory
section entropy
    (hX : Measurable X) {μ : Measure Ω} [IsZeroOrPr
    entropy X μ = ∑ x ∈ FiniteRange.toFinset X, neg
  entropy_eq_sum_finset (A := FiniteRange.toFinset

lemma entropy_eq_sum_finiteRange' [MeasurableSingle
    [IsZeroOrProbabilityMeasure μ] [FiniteRange X]:
    entropy X μ = ∑ x ∈ FiniteRange.toFinset X, neg
  entropy_eq_sum_finiteRange hX

/-- `H[X | Y=y] = ∑_s P[X=s | Y=y] log 1/(P[X=s | Y
lemma entropy_cond_eq_sum (μ : Measure Ω) (y : T) :
    H[X | Y ← y ; μ] = ∑' x, negMulLog ((μ[|Y ← y])
  llmqed
```

```
T : Type u_3
mΩ : MeasurableSpace Ω
inst† : MeasurableSpace S
X : Ω → S
Y : Ω → T
μ : Measure Ω
y : T
⊢ H[X | Y ← y ; μ] = ∑' (x : S), ((map X μ[|Y ⁻¹'
{y}]) {x}).toReal.negMulLog
```

▼LLMLean suggestions

```
    Try this:
  • 🎉 rw [entropy_def]
      rw [measureEntropy_of_isProbabilityMeasure]
  • rw [entropy_def]
      simp only
```

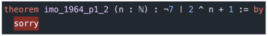
LLMLean example on Polynomial Freiman Rusza Conjecture project

Where can AI help? — Benchmarking gap

Math competition problems

INTERNATIONAL MATHEMATICAL OLYMPIAD
IMO 2024
IMO 2025
Timeline • Countries • Results • Search • Problems • Hall of fame • About IMO

Theorem to prove



```
theorem imo_1964_p1_2 (n : ℕ) : ¬7 ∣ 2 ^ n + 1 := by
  sorry
```

- Self-contained
- Uses standard results

Where can AI help? — Benchmarking gap

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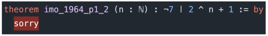

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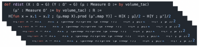
```

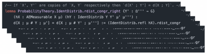
Real projects

Definitions

Lemmas

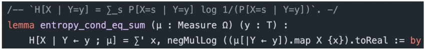


```
def rdist (X : Ω → G) (Y : Ω' → G) (μ : Measure Ω := by volume_tac)
    (μ' : Measure Ω' := by volume_tac) : ℝ :=
  H[fun x ↦ x.1 - x.2 ; (μ.map X).prod (μ'.map Y)] - H[X ; μ]/2 - H[Y ; μ']/2
```



```
/-- If `X', Y'` are copies of `X, Y` respectively then `d[X' ; Y'] = d[X ; Y]`. -/
lemma ProbabilityTheory.IdentDistrib.rdist_congr_right {Y' : Ω''' → G}
    (hX : AEMeasurable X μ) (hY : IdentDistrib Y Y' μ' μ''') :
    d[X ; μ # Y ; μ'] = d[X ; μ # Y' ; μ'''] := (IdentDistrib.refl hX).rdist_congr
```

Theorem to prove



```
/-- `H[X | Y=y] = ∑_s P[X=s | Y=y] log 1/(P[X=s | Y=y])`. -/
lemma entropy_cond_eq_sum (μ : Measure Ω) (y : T) :
    H[X | Y ← y ; μ] = ∑' x, negMulLog ((μ[|Y ← y]).map X {x}).toReal := by
```

- Self-contained
- Uses standard results

- Part of a project
- Uses new definitions and lemmas

miniCTX

miniCTX: Neural Theorem Proving with (Long-)Contexts
Jiewen Hu, Thomas Zhu, Sean Welleck
*ICLR 2025 (**Oral**)*

miniCTX

Research-level theorems depend on newly-formalized **context**

- (context, theorem) $\rightarrow$ proof
    - Context: repository of code, new definitions, auxiliary lemmas

miniCTX

miniCTX:

Test models on real Lean projects:[5]

- "Future mathlib": theorems added after a time cutoff
- Recent projects: PFR, PrimeNumberTheorem, ...

---

[5] + tools for easily adding new projects: https://github.com/cmu-l3/ntp-toolkit

miniCTX

miniCTX:

Test models on real Lean projects:[5]

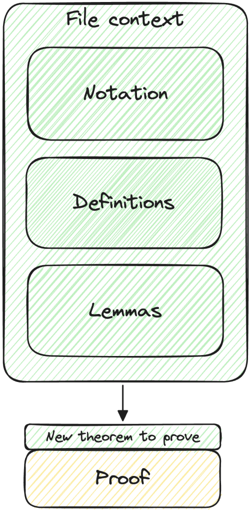
- "Future mathlib": theorems added after a time cutoff
- Recent projects: PFR, PrimeNumberTheorem, ...

Periodically updated with new projects to stay ahead of LLM training cutoffs.

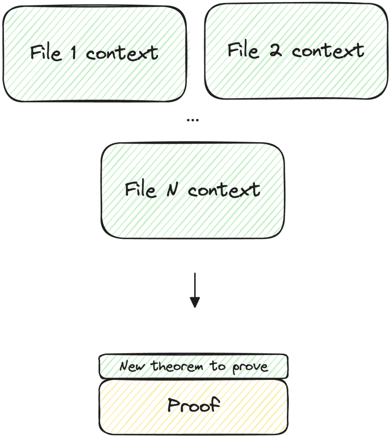
---

[5] + tools for easily adding new projects: https://github.com/cmu-l3/ntp-toolkit

miniCTX



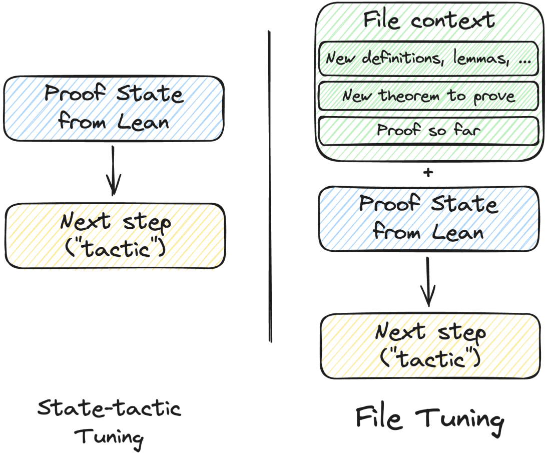
In-file: dependencies appear in the file



Cross-file: dependencies span files

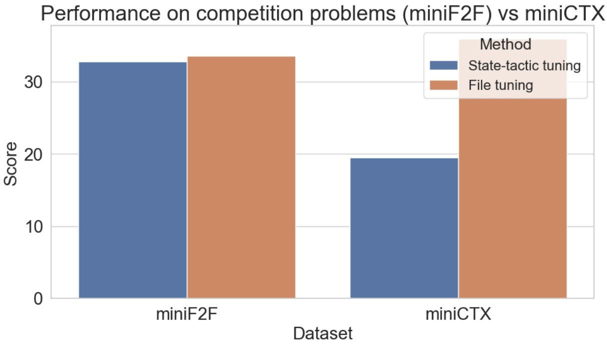
miniCTX — preceding code context

Does context actually matter? A simple experiment.



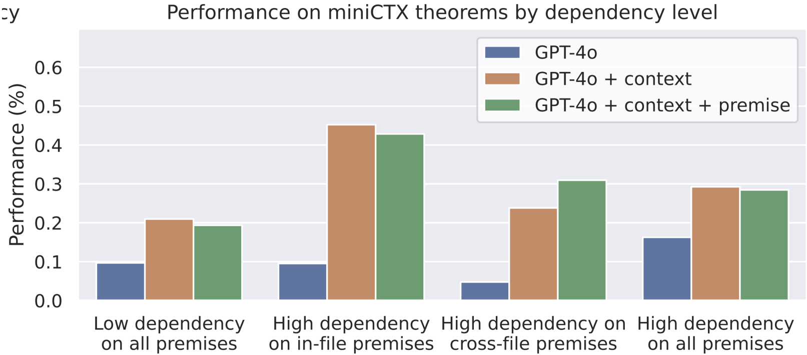
"File tuning": train on (preceding code, state, next-tactic) examples

miniCTX



Two methods can have similar performance on competition problems, but vastly difference performance on actual projects

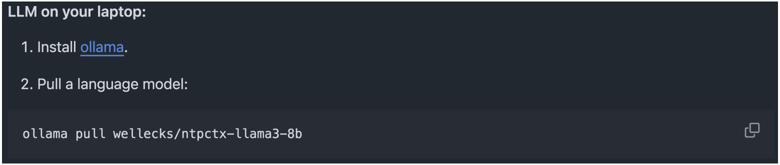
miniCTX



Premise selection helps with cross-file dependencies.

miniCTX — LLMLean

File-tuned model is deployed in LLMLean:



LLM on your laptop:

1. Install ollama.
2. Pull a language model:

```
ollama pull wellecks/ntpctx-llama3-8b
```

https://github.com/cmu-l3/llmlean

miniCTX

Benchmark, models, code are open-source:

- Data/models: `https://huggingface.co/l3lab`
- Data extraction: `https://github.com/cmu-l3/ntp-toolkit`
- Evaluation: `https://github.com/cmu-l3/minictx-eval`

Recap: Towards AI for Research-Level Formalization

Formalizing research-level math has unique challenges

- Accessibility gap exists between AI advances and real-world formalization
    - But we have some tools available!
- Benchmarking gap exists between competition problems and real-world formalization
    - miniCTX tests the ability to generalize to new, real-world projects

This talk: Bridging Informal and Formal

1. Informal thoughts
    - Training models to think informally
        - Lean-STaR

2. Informal provers
    - Sketching proofs and filling in the gaps
        - Draft, Sketch, Prove
        - LeanHammer

3. Research-level mathematics
    - Assisting in research-level projects
    - Practical tools
    - MiniCTX

Thank you!

Collaborators on works in this talk (alphabetical by last name):

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- Joshua Clune (CMU)
- Jiewen Hu (CMU)
- Mateja Jamnik (Cambridge)
- Albert Q. Jiang (Cambridge, Mistral)
- Timothee Lacroix (Meta, Mistral)
- Guillaume Lample (Meta, Mistral)
- Haohan Lin (Tsinghua)
- Wenda Li (Edinburgh)
- Jiacheng Liu (Washington)
- Zhiqing Sun (CMU, OpenAI)
- Yuhuai (Tony) Wu (Google, X.ai)
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