

POINTS OF CONVERGENCE - MUSIC MEETS MATHEMATICS

LASSE REMPE

ABSTRACT. *Phase-locking* is a fundamental phenomenon in which coupled or periodically forced oscillators synchronise. The *Arnold family* of circle maps, which describes a forced oscillator, is the simplest mathematical model of phase-locking and has been studied intensively since its introduction in the 1960s. The family exhibits regions of parameter space where phase-locking phenomena can be observed. A long-standing question asked whether *hyperbolic* parameters – those whose behaviour is dominated by periodic attractors, and which are therefore stable under perturbation – are dense within the family. A positive answer was given in 2015 by van Strien and the author, which implies that, no matter how chaotic a map within the family may behave, there are always systems with stable behaviour nearby. This research was a focal point of a pioneering collaboration with composer Emily Howard, commencing with Howard’s residency in Liverpool’s mathematics department in 2015. The collaboration generated impacts on creativity, culture and society, including several musical works by Howard, and lasting influence on artistic practice through a first-of-its-kind centre for science and music. We describe the research and the collaboration, and reflect on the factors that contributed to the latter’s success.

INTRODUCTION

In the 17th century, the Dutch scientist Christiaan Huygens discovered that two pendulum clocks, coupled by being mounted on the same wooden beam, synchronised their movements. This phenomenon, described by Huygens as “a miraculous sympathy,” is called *phase locking* (also mode locking, or entrainment): interacting periodic oscillators tend to synchronise their movements (with the same period or with periods that are related by an integer multiple). Phase locking is near-ubiquitous in physical oscillators: Examples include the fact that the revolution period and orbital period of the moon are identical and the synchronisation of fireflies. A similar effect occurs when one oscillator is *forced* by another, for example in bowed musical instruments: When a violin string is plucked by hand, it exhibits non-harmonic overtones; when bowed, all of these overtones are forced onto the harmonic series.

A goal of “pure” (i.e., theoretical) mathematics is to investigate interesting phenomena in their most fundamental settings, to understand the underlying principles. In a 1961 paper [1, § 12], Vladimir Arnold introduced what may be the simplest model for phase-locking behaviour: a family of self-maps of the circle, motivated by the movement (in discrete time-steps) of a forced periodic oscillator. Known as the *Arnold family* of circle maps, it can be described by the formula

$$f_{\alpha,b}(\theta) := \theta + \alpha + b \sin(\theta) \pmod{2\pi}. \quad (1)$$

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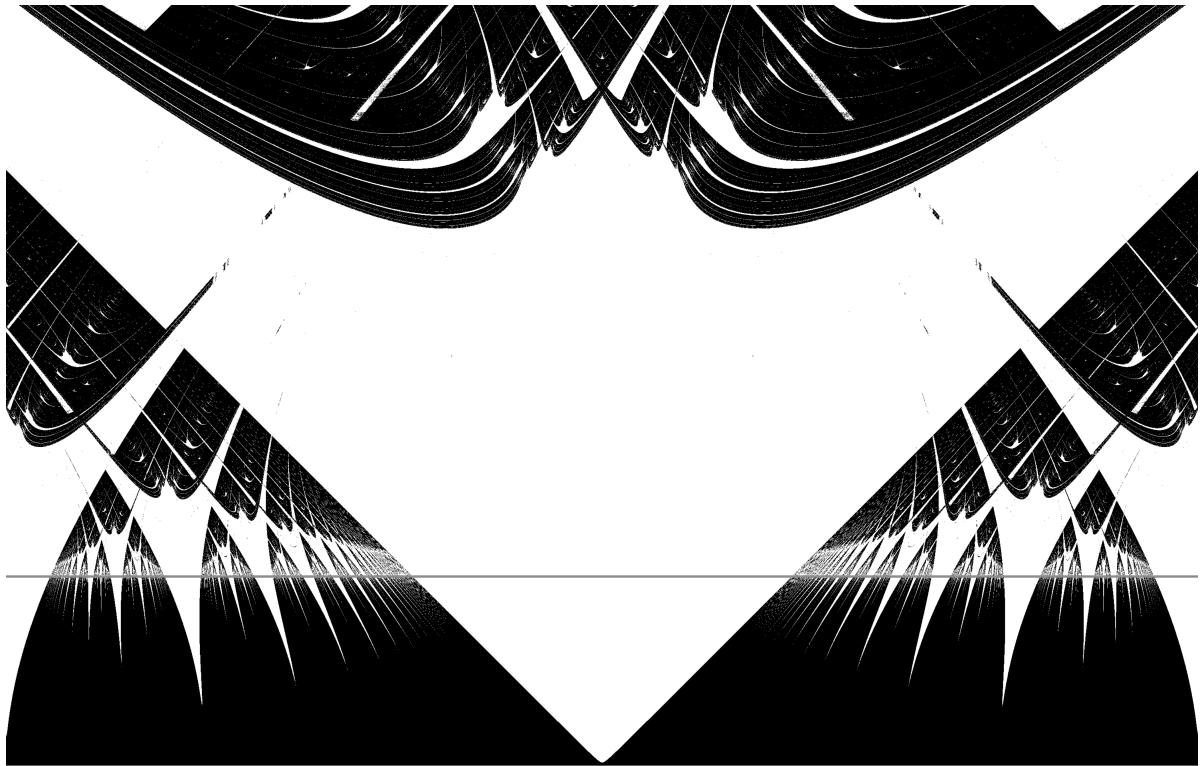


FIGURE 1. Parameter space of the Arnold family. The horizontal and vertical directions correspond, respectively, to the parameters α , ranging from $-\pi$ to π and b , ranging from 0 to 4. The critical line $b = 1$ is shown in grey; white regions correspond to stable (hyperbolic) maps. Theorem 1 states that these regions are dense in parameter space; the apparent presence of solid black regions arises from the fact that stable regions may be very thin.

The angle θ represents a point on the circle $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. The number α is a rotation parameter (also an element of S^1); if $b = 0$, then $f_{\alpha,b}$ is simply the rotation by an angle of α . The final term in (1) is a forcing term, which must also be a function of θ , and thus periodic. We use the simplest possible periodic forcing term, namely a multiple of \sin .¹ The parameter $b \in [0, \infty)$ determines the strength of the forcing.

Let us fix parameters α and b and write $f = f_{\alpha,b}$. Given a starting state $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$, we think of $\theta_1 = f(\theta_0)$ as determining the state of our dynamical system after one time step. So $\theta_2 = f(\theta_1)$ is the state after two steps, and after n steps:

$$\theta_n = f(\theta_{n-1}) = f(f(\dots f(\theta_0) \dots)) \quad (n \text{ times}).$$

The sequence $(\theta_n)_{n=0}^\infty$ is called the *orbit* of θ_0 under f . If $\theta_n = \theta_0$ for some $n > 0$, then the (finite) orbit is called a *periodic cycle*; this cycle is *stable* (under perturbation of θ_0) if the orbits of all nearby starting values converge to it.

¹Arnold used \cos instead of \sin , which is equivalent.

For $b < 1$ (the bottom quarter of Figure 1), the map $f_{\alpha,b}$ is a diffeomorphism of the circle. It indeed exhibits phase-locking phenomena, which are well-understood: For parameters in certain regions, called “Arnold tongues” (the white regions), all but finitely many orbits tend to a stable periodic cycle. Moreover, these maps are stable under perturbations of f within the Arnold family: a small change of the parameter leads to a small change in the overall behaviour of orbits, so $f_{\alpha,b}$ is “phase-locked” to the periodic orbit.

For $b > 1$, the map $f_{\alpha,b}$ is no longer invertible and has two distinct critical points on S^1 . The Arnold tongues begin to intersect, and the periodic orbit for a given tongue may bifurcate and become unstable. This can lead to *chaotic* behaviour, where an arbitrarily small change of the starting state θ_0 could lead to completely different long-term behaviour. It is thus natural to ask whether for $b > 1$, there is still a dense set of parameters whose behaviour is stable under perturbations of the parameter.

More precisely, $f = f_{\alpha,b}$ with $b > 1$ is called *hyperbolic* if the orbits of both critical points tend to stable periodic cycles. Such f is stable under perturbations of the parameter – all nearby maps are also hyperbolic – and almost every orbit converges to a stable cycle. Density of hyperbolic maps in parameter spaces is a central question of one-dimensional dynamics. For real polynomials, it was established by Kozlovski, Shen and van Strien [2] in 2007, answering part (b) of Smale’s 11th problem. However, this does not resolve the question in the Arnold family: a key issue is that f , when extended to the complex plane, is transcendental rather than algebraic (see below). This problem was overcome by van Strien and the author [4]:²

Theorem 1 (Density of hyperbolicity). *Hyperbolic maps are dense in the Arnold family. That is, given any $\alpha \in S^1$ and $b > 1$, and every $\varepsilon > 0$, there exist perturbed parameters $\alpha' \in S^1$ and $b' > 1$ with $|\alpha - \alpha'|, |b - b'| < \varepsilon$ such that $f_{\alpha',b'}$ is hyperbolic.*

1. IDEAS IN THE PROOF OF THE THEOREM

To prove Theorem 1, we consider the *complex extension* of $f_{\alpha,b}$, allowing the state θ in (1) to be a complex number. Applying the change of variable $z = \exp(i\theta)$, we obtain a self-map of the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (see Figure 2):

$$g_{\alpha,b}(z) = \exp(i(\theta + \alpha + b \sin(\theta))) = e^{i\alpha} \cdot z \cdot \exp\left(\frac{1}{2}\left(z - \frac{1}{z}\right)\right).$$

This function $g_{\alpha,b}$ has isolated singularities at 0 and ∞ , which are *essential* (neither removable nor poles). Consequently, the behaviour near these points is very complicated; for example, the preimages of every $z \notin \{0, \infty\}$ accumulate both at 0 and at ∞ . There is a set of starting values of positive area which are *escaping*; i.e., their orbits accumulate only on the essential singularities 0 and ∞ .

A crucial step in the proof is to establish *rigidity*: a map in the Arnold family cannot be deformed, by a small change of parameters, to one with the same qualitative dynamical behaviour, except through certain well-understood mechanisms. In our setting, a new difficulty arises: We must exclude the existence of deformations arising from the set of

²In fact, the results of [4] are more general, covering many families of transcendental entire functions and circle maps.



FIGURE 2. The dynamics of $g_{\alpha,b}$ in the complex plane, for one choice of parameters. As usual, the horizontal and vertical directions correspond to the real and imaginary parts of the starting value. The map is chaotic on the unit circle (visible in the central part of the figure), and almost every orbit is escaping. Different shades of grey distinguish different patterns with which orbits tend to the essential singularities at 0 and ∞ , highlighting the intricate dynamics of the complex extension.

escaping points. This issue does not arise for polynomials, as treated in [2], which have no essential singularities. It is overcome by using techniques developed for studying the behaviour of transcendental entire functions near ∞ [3].

To prove the theorem, we then begin with a map $f_{\alpha,b}$ that is not hyperbolic; recall that this map has two critical points and two critical values. It follows from the above-mentioned rigidity statement that we may perturb (α, b) slightly to a parameter (α_1, b_1) for which the orbit of one critical value passes through a critical point. The set of parameters satisfying this critical relation forms an analytic variety of dimension 1. We can make another perturbation, *within this variety*, to create a second critical relation. (Otherwise, maps within this variety would have the same qualitative dynamical behaviour as f_{α_1, b_1} , which contradicts rigidity.)

Thus we obtain parameters (α', b') , arbitrarily close to (α, b) , for which both critical values are eventually mapped to a critical point. This means that each critical point is mapped to a periodic cycle containing a critical point; such a cycle is necessarily stable. Thus f is hyperbolic, and the proof of the theorem is complete.

2. A MUSICAL COLLABORATION

In 2015, Liverpool's mathematics department hosted award-winning composer Emily Howard as a Leverhulme Artist in Residence. Howard previously used mathematical and scientific ideas in her compositions, but the mathematical discussions during the residency led to the use of frontier mathematical research, rather than classical mathematical principles, in her creative process for the first time.

A particular focus of discussions during the residency were the article [4] and Theorem 1. Howard challenged the dynamics group to create sets of numeric data, encapsulating key ideas of the work. Two datasets were created by the author and Alexandre

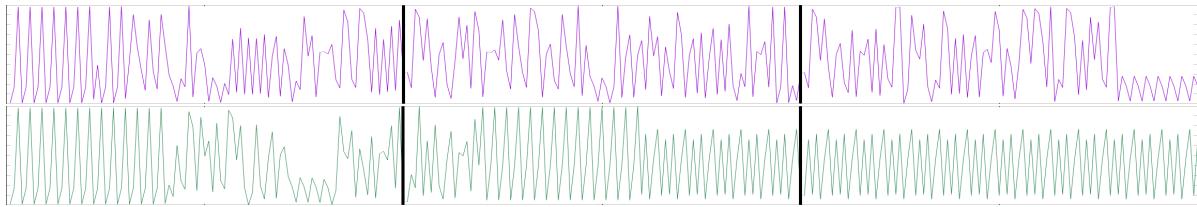


FIGURE 3. The data set used in Howard’s composition *Orbit 2a* first follows two orbits of an unstable function in the Arnold family, starting near an unstable periodic cycle, but quickly separating due to the chaotic dynamics. As in the proof of Theorem 1, the function is slightly perturbed once, changing the parameters by less than 10^{-5} , to create an attracting cycle. A second, even smaller, perturbation yields another attracting cycle, and thus a hyperbolic function. (A member of the Arnold family with two attracting cycles is necessarily hyperbolic.)

Dezotti (see Figure 3); the resulting composition, *Orbits* [5], is a direct creative response to the data. Other pieces (*Leviathan*, *Threnos* and *Chaos or Chess*) written during this period also used the discussions around [4] as pivotal creative input, establishing enduring principles for Howard’s approach to composition.

For example, *Torus* was a BBC Proms co-commission with the Royal Liverpool Philharmonic Orchestra, first performed in 2016 at the Royal Albert Hall to a sell-out audience. It was inspired by Howard’s discussions with mathematicians in Liverpool, including those with the dynamical systems group, as well as conversations with Liverpool geometer Anna Pratoussevitch. The ideas of chaotic motion, and of a small perturbation that changes the fundamental nature of a system, formed an important part of Howard’s compositional approach for this piece. *Torus* was hailed by the Guardian as ‘one of this year’s finest new works’, and recognised with a 2017 British Composer Award. Two further geometry-inspired orchestral pieces followed: *sphere* (2017) was commissioned by the Bamberg Symphony Orchestra and broadcast on BBC Radio 3, and *Antisphere* was commissioned to open the London Symphony Orchestra’s 2019/20 season at the Barbican under Sir Simon Rattle.

In 2017, the Royal Northern College of Music established a new dedicated centre for Practice and Research in Science and Music (PRiSM), with Howard as Director. PRiSM is building on the approach developed during the Liverpool residency, bringing together scientists, composers and performers for mutual benefit.

Due to the specialised nature of mathematical research, it can be challenging to communicate non-superficial ideas and concepts to those outside the specific area of research, let alone outside of mathematics. There are two factors that contributed strongly to successfully creating a dialogue between music and research mathematics. We shall discuss them briefly here, as they may be instructive for collaborative projects of a similar nature.

The first is the presence of a shared language: Howard is an Oxford graduate in mathematics and computing, while the author is an amateur orchestral musician. Howard’s familiarity with mathematical terminology allowed for detailed discussions of mathematical ideas, which were reflected in the resulting compositions. For instance, in the

data underlying *Orbits*, there appear repeating patterns arising from both stable and unstable cycles. Though they are indistinguishable from the data alone, Howard chose to represent their differing nature musically as a result of the mathematical discussions. Likewise, the perturbations of parameters are imperceptible in the data; knowing about their significance, Howard marked them as recognisable musical events. It appears likely that, for a collaboration like this to succeed, a common language, if not already present, needs to be carefully developed.

A second key factor in the success of the collaboration was the careful choice of a specific piece of research as the basis of discussions. There are several reasons why Theorem 1 provided fertile ground for developing new creative thinking:

- (1) It lends itself to expression in simple and general terms: no matter how chaotic the system, there is always stability nearby.
- (2) Phase-locking is known and relevant to musicians, e.g. through the synchronisation of linked metronomes or the elimination of non-harmonic overtones.
- (3) Aspects of the proof are philosophically intriguing: the complex plane – invisible in the formulation of the problem – plays a crucial part in the proof.
- (4) The systems in question can be used to design data series as input into and inspiration for the creative process.

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