Lecture 2 : Formal definitions

Def 1 (Block code): An (n, M) block code is a subset $\mathcal{C} \subseteq \mathcal{X}^n$ with $|\mathcal{C}| = M$.

Blocklength

We hence have rate $R = \frac{\log M}{n}$.

Egs: (i) Rate of repetition code $\mathcal{C} = \{000, 111\}$ is $1/3$ and $\mathcal{C}$ is a $(3,2)$ block code.

(ii) Rate of single parity-check code $\mathcal{C} \subseteq \{0,1\}^n$ is ? and it is a $(\_\_, \_\_)$ block code.

(Recall): From basic communication theory, for a given noisy channel $W = \left(P(\underline{y}|\underline{c}) : \underline{c} \in \mathcal{C}, \underline{y} \in \mathcal{X}^n\right)$, the decoder that minimizes the error probability is the "maximum a-posteriori probability" (MAP) decoder:

$$\hat{\underline{c}} = \arg\max_{\underline{c} \in \mathcal{C}} P(\underline{c} | \underline{y}).$$

HW①: Provide / read up a proof of this fact.

Now, suppose that $\underline{c} \sim \mathrm{Unif}(\mathcal{C})$. Then,

$$\hat{\underline{c}} = \underset{\underline{c} \in \mathcal{C}}{\arg\max} \; \frac{P(\underline{y}|\underline{c})P(\underline{c})}{P(\underline{y})}$$

$$= \underset{\underline{c} \in \mathcal{C}}{\arg\max} \; \underbrace{\frac{1}{M P(\underline{y})}}_{\text{constant!}} \cdot P(\underline{y}|\underline{c}) = \underset{\underline{c} \in \mathcal{C}}{\arg\max} \; P(\underline{y}|\underline{c})$$

$\triangleq$ ML decoder ("maximum likelihood").

What is ML decoding for the BSC ?

For a fixed $\underline{c} \in \mathcal{C}$ and any $\underline{y} \in \{0,1\}^n$,

$$P(\underline{y}|\underline{c}) = p^{d(\underline{c},\underline{y})} (1-p)^{n-d(\underline{c},\underline{y})},$$

where $d(\underline{c},\underline{y}) = d(\underline{y},\underline{c}) \triangleq d$ is the "Hamming distance" b/w $\underline{c}$ and $\underline{y}$,

i.e., the # positions where $\underline{c}$ & $\underline{y}$ differ.

Hence, $\mathrm{ML}(\underline{y}) = \underset{\underline{c} \in \mathcal{C}}{\arg\max} \; P(\underline{y}|\underline{c}) = \underset{\underline{c} \in \mathcal{C}}{\arg\min} \; d(\underline{c},\underline{y})$, when $p < 1/2$.

Lemma 1 : $d(\cdot,\cdot)$ is a metric over $\mathcal{X}^n$.

Pf : HW.

Def 2 (Minimum distance) : The minimum distance of a block code $\mathcal{C}$

$$d(\mathcal{C}) = d_{min}(\mathcal{C}) = \min_{\substack{\underline{c}_1, \underline{c}_2 \in \mathcal{C}, \\ \underline{c}_1 \neq \underline{c}_2}} d(\underline{c}_1, \underline{c}_2).$$

An (n, M) block code with min. dist. d is written as an (n, M, d) block code.

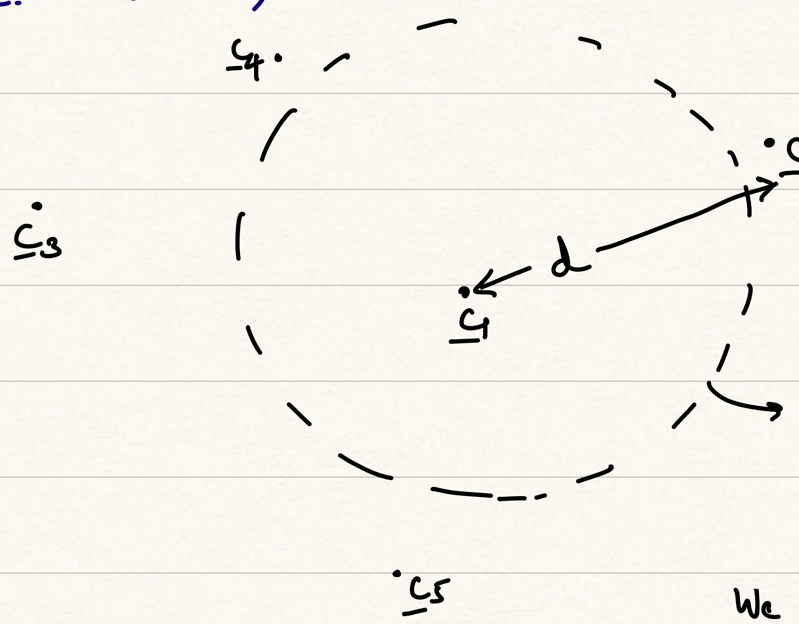
Error detection/correction tied to distance

Thm 1: For an (n, M, d) block code $\mathcal{C}$, $\exists$ a decoder that detects up to $d-1$ bit-flip errors.

This decoder obeys

$$\mathcal{D}(y) = \begin{cases} y, & \text{if } y \in \mathcal{C}, \\ \text{ERROR}, & \text{o.w.} \end{cases}$$

Pf: (Picture)



Ball $B(\underline{c}_1, t)$ of radius t around $\underline{c}_1$.
$\cap$
$\mathcal{X}^n$

We must have $B(\underline{c}_1, t) \cap \mathcal{C} = \{\underline{c}_1\}$, if $t \le d-1$; else, contradiction

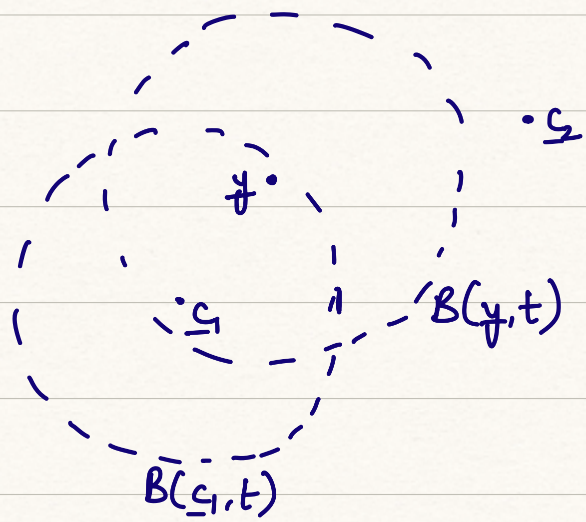
$\blacksquare$

Thm 2: For an (n, M, d) block code $\mathcal{C}$, $\exists$ a decoder that corrects $t \leq \left\lfloor \frac{d-1}{2} \right\rfloor$ bit-flip errors.

This decoder is the minimum distance decoder

$$\mathcal{D}(\underline{y}) = \underline{c}, \text{ if } B(\underline{y}, t) \cap \mathcal{C} = \{\underline{c}\}.$$

Pf: (Picture)



We cannot have $\underline{c}_2 \neq \underline{c}_1$ s.t. $\underline{c}_2 \in B(\underline{y}, t)$, as then

$$\begin{aligned} d(\underline{c}_1, \underline{c}_2) &\leq d(\underline{c}_1, \underline{y}) + d(\underline{c}_2, \underline{y}) \\ &\leq 2t \leq 2 \cdot \left\lfloor \frac{d-1}{2} \right\rfloor < d, \end{aligned}$$

a contradiction. $\blacksquare$

An aside: Erasures are channel noise that behave as follows:

$$\underline{c} = (c_1, c_2, \cdots, c_{n-1}, c_n) \xrightarrow[\text{channel}]{\text{Erasure}} \underline{y} = (c_1, ?, c_3, \cdots, c_{n-1}, ?)$$

- Selected codeword symbols are replaced with a '?'.

Thm: For an (n, M, d) block code $\mathcal{C}$, there is a decoder that <u>corrects</u> up to $d-1$ erasures.

<u>Pf:</u> HW. (state a decoder and prove its erasure-correction property).