Chapter

Bisociation: Creativity of an Aha! Moment

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Abstract

This chapter presents a new theory of creativity in mathematics education. The theory has been anchored in two foundations: that of teaching practice of Vrunda Prabhu which occasioned surprisingly many Aha! Moments experienced by students in her remedial arithmetic classes in college and that of the Arthur Koestler's volume The Act of Creation. The Act of Creation introduces term bisociation describing an Aha! Moment and Eureka Experience as "the spontaneous leap of insight which connects previously unconnected frames of reference" by unearthing "hidden analogies". Whereas Koestler formulated the concept of bisociation within humor, scientific discover and art, we focus primarily on the bisociative creativity in mathematics. We abstract the concepts of the bisociative frame as the two unconnected frames of reference, useful method for (1) identification of heightened probability of creative insight in a given mathematical situation and (2) identification of possible creativity within different theories of learning. The chapter explores Koestler/Prabhu theory of learning through Aha! Moment and applies the bisociative frame to investigate its processes of interaction with several different theories of learning. Uncovered processes of interaction suggest that creativity should be the basis of contemporary learning and teaching.

Keywords: creativity, Aha! Moment, bisociative frame, interiorization, internalization, appropriation, Koestler/Prabhu theory of learning, constructivist, socio-cultural

1. Introduction

This chapter presents a new theory of creativity in mathematics education. The theory has been built from two sources: Prabhu's [1] teaching experiment, which brought about a surprising number of Aha! Moments in students enrolled in her developmental math classes and Arthur Koestler's volume The Act of Creation [2]. The research, or rather the teaching–research pathway that led us to formulating the new theory of creativity in mathematics education, started during the 2010/2011 teaching-experiment conducted by Prabhu and supported by CUNY grant C^3IRG 7, *Problem Solving in Remedial Mathematics—a Jumpstart for the Reform*. The grant was awarded to members of the Teaching-Research (TR) Team of the Bronx anchored in Hostos Community College, New York City. The work on creativity of Aha! Moment took two creative cycles of work each lasting several years, during which we realized we have here a new distinct approach to creativity through Aha! Moment. Since Aha! or epiphanies are common occurrences among the general public in many domains of activity, the presented theory applies equally to the "underserved and underrepresented" as well as to the "talented and gifted".

These separate research ideas coalesce into a whole presented in the work by Czarnocha, Baker, Dias and Prabhu in [3], where the first sketches of the new theory appeared in different chapters. The volume introduced the theory of creativity in its discussion of teaching–research methodology, showing that TR methodology is intrinsically creative. Because teaching and research constitute two matrices of thought, generally and unfortunately not connected with each other, working at the teaching–research interface enables bisociative processes to create/build Aha! bridges between them.

During the second teaching–research creative cycle, we investigated processes of facilitation of an Aha! Moment assessment of the depth of knowledge gained during the insight, and constraints imposed on development of creativity in mathematics classrooms. The full exploration of the phenomenon Aha! Moment took place at the completion of the second creative teaching–research cycle with the volume Creativity of an Aha! Moment and Mathematics Education, Czarnocha [4] which collected our own research as well as that of international experts on Aha! Moments. The Koestler/Prabhu theory of creativity through Aha! Moment was formulated in that volume for the first time. The reflection on this work provided the basis on which to build the Philosophy of Creativity in Mathematics Education [5]. In this chapter, we present the Koestler/Prabhu theory in Section 2 together with some of its application into creativity research. We also provide a method of assessment for the increase of knowledge occasioned by Aha! Moment insight in Section 3. Section 4 presents our investigations into the relationship between the bisociative frame and sociocultural framework. In Section 5, we summarize and connect different threads of the paper.

2. Koestler theory of bisociation

2.1 Elements of the Koestler theory of bisociation

The work presented here on the creative moments of insight popularly called Eureka experience or Aha! Moment is based on [2] where the author introduced a new concept/term of bisociation, which is the spontaneous act of thought which combines or dialectally synthesizes the information from two different, generally unconnected domains. Bisociation is seen here as distinct from association, which produces knowledge within a single domain. Koestler makes a clear distinction between more routine or habitual thinking (association) operating within a single plane or a matrix of thought, referred to here as the exercise of understanding and the more creative bisociative mode of thinking that connects independent autonomous matrices called progress in understanding [6].

What is a matrix of thought also called a frame of reference?

> *The matrix is the pattern before you, representing the ensemble of permissible moves. The code which governs the matrix can be put into simple mathematical equations… or it can be expressed in words. The code is the fixed invariable factor in a skill or*

> *habit, the matrix is the variable part. The two words do not refer to different entities, they refer to different aspects of the same activity.* ([2], p. 40).

A clear example of a matrix and a code is given in chess where the matrix is the full collection of available moves, while the code are the rules for the movement and interaction among the chess pieces. In high school algebra, the matrix is the full set of polynomials and power series in one or two variables; the code are the established rules of operations on these mathematics objects.

In our effort to apply Koestler's theoretical framework in the creativity process within learning mathematics, we focus on two related questions: (1) How can you describe the genesis of a new code-matrix? and (2) How can you characterize moments of creative insight that lead to new structure-matrices and ultimately new codes? Koestler considered the formation of new structure-matrices, as well as the hierarchy of matrices serving the organism as "[s]ymbolic models of the external world" (p. 506) that govern how we interpret and react to a situation in a predictable manner, as the central role of cognition. For Koestler, an important component in characterizing moments of insight is expressed in his notion of the degree of originality, which is inherent in our discussion of the depth of knowledge acquired during a moment of insight. In learning theory, an individual's development of structure is a focal point of constructivist research and cognitive science.

Koestler translates the term matrix very broadly into such diverse domains as literature, art, drama, motor skills and humor. He notes that matrix in cognitive psychology would be termed a schema.

> *The concept of matrices with fixed codes and adaptable strategies, proposed as a unifying formula, appears to be equally applicable to, perceptual, cognitive, and motor skills and to the psychological structures variously called, 'frames of reference', 'associate contexts', 'universes of discourse', 'mental sets', or 'schemata' etc.* (p. 96)

Cognitive psychologists use the term schema to describe mental structures that guide our response to life situations; thus, there is a schema for work, family relationships, commuting, religious services, etc. The particular schema that interests us is an individual's schema for mathematics problem-solving.

Constructivists use the term scheme to describe a mental process for resolving problem situations. We loosely translate schema as a hierarchical structure of schemes. Thus, the building of schema begins in the second stage of the Piaget Garcia Triad [7] through the connection of schemes into hierarchical collections (mental toolboxes for a given domains of math). The third stage begins connections between schemata. The term code translates into the constructivist notion of an invariant relationships, which can be understood as the automated principle of a conceptual relationship that underlies an activity–effect relationship in a problem situation (scheme).

Creativity or a Eureka moment occurs when an idea is suddenly understood to exist simultaneously in two previously unrelated frame (**Figure 1**).

> *The perceiving of a situation, or idea, L, in two self-consistent but previously incompatible frames of reference (fig. 1) The event L in which the two intersect is made to vibrate simultaneously on two different wavelengths, as it were. While this unusual situation lasts, L, is not merely linked to one associative context, but bisociated with two*. ([2], p. 35).

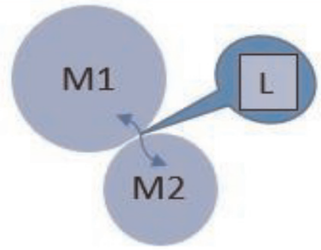


Figure 1.
Bisociated concept L between matrices M1 and M2.

2.2 Koestler: progress in understanding new codes

In the first stage of the Piaget Garcia Triad, the birth of a conceptual relationship begins with interiorization. For Koestler the genesis of a new code is referred to as progress in understanding; the equivalent constructivist term is accommodation. In constructivist theory, all acts of accommodation are the result of what Piaget refers to as reflective abstraction (abstraction of processes). Interiorization is understood as the foundational type or process of such abstraction that translates externally directed activity into an internal process. Koester describes the genesis of codes, or progress in understanding, as due to moments of insight, within the learning process, or moments of creativity insight that lead to the synthesis of two matrices or, "bisociation"

> *[p]rogress in understanding – the acquisition of new insights … is achieved by the formulation of new codes by … empirical induction, abstraction, and discrimination, bisociation*. (p. 619).

The Transition from Empirical to Abstract reasoning.

Concepts are born in the first stage of the Piaget & Garcia Triad. Constructivist research based on Piaget's work and social constructivism based on Vygotsky; both view the development of a child's cognition as transitioning from empirical to more abstract reasoning. Constructivists view the transition from empirical reasoning, during which one's intuitive solution activity is directed by the situation, as interiorization. This process leads to an abstract activity–effect relationship based on understanding the relevant concept and thus no longer dependent on the situation. They view this first-stage transition as the way human cognition evolved historically as well as the pathway for child development.

Vygotsky refers to the transition from empirical to abstract thought as one from spontaneous to scientific concepts and considers it essential for the dawning of the child's ability to engage in self-reflection. Vygotsky considers such internalization of exterior activity as social-based, primarily guided by adult communication, and in large part based on imitation of adult behavior.

In this debate, Koestler is strictly a constructivist. Indeed in his chapter on science and emotion, he titled one of the sections "The Boredom of Science" lamenting how direct instruction of theorems and repetition have made math and science "antihuman". In contrast to the social-cultural approach, in his view individual moments of insight lead to a transcendence experience, and only from this can we begin to appreciate math and science. Like Vygotsky, Koestler understands objective abstract thought as essential for human self-reflection, but like constructivists he situates such reflection, as based upon transcend moments of insight that are individualistic. Thus, Koestler like constructivist researchers is primarily focused on bisociation within the learning process of an individual and not on moments of insight within social discourse.

The phenomenon under investigation is moments of insight within the learning process, which lead to progress in understanding, or concept development. As teacher researchers our contention is that such moments occur both during social discourse-internalization as well as during reflection on our own solution activity. Furthermore, the novel concepts and structure developed through the connections established at this moment of insight (progress in understanding) can be assessed through the Piaget-Garcia Triad. Before continuing with these themes, we review Eureka or Aha! Moments and creativity theory.

2.3 Eureka experience/Aha! Moment

The phenomenon under investigation of the proposed theory is the act of creation in mathematics and science called Eureka experience or Aha! Moment. It is that moment when suddenly, after a long period of trying to solve a problem or understand a new concept without success, the solution comes in a flash, generally with a good doze of satisfaction. It is a very particular form of creativity that appears as an insight, as a discrete insight in that it appears instantaneously at separate moments of time.

Gestalt creativity approaches the Eureka experience as the stage of illumination within the sequence of preparation, incubation, illumination and verification stages suggested by [8, 9]. The sequence represents the stages through which the formation of the creative idea takes place. While the first three stages came from psychological research, the fourth stage was added as the necessary part of the creative process at the Poincare insistence [8]. Why would Poincare insist on verification as the components of creativity? Because Poincare in [10] says: *It never happens that unconscious work supplies a ready-made result of a lengthy calculation in which we have only to apply the fixed rules… All that we can hope from these inspirations, which are the fruits of unconscious work, is to obtain points of departure for such calculations. As for calculations themselves, they must be made in the second period of conscious work, which follows the inspiration and in which the results of inspiration are verified and the consequences deduced* (pp. 62–63).

The more so, of course, because sometimes the Aha! Moment is false. Consequently, the verification stage plays a dual role: On one hand as a possible completion of the creative act and as the check on the correctness of the logical-causal structure of its content. Recent examination of Wallas's work suggests in [11] the fifth stage of intimation to be placed between incubation and illumination. Intimation is, for Wallas, the

> *"fringe of consciousness" which surrounds our "focal" consciousness as the Sun's 'corona' surrounds the disk of full luminosity", Wallas continues: "This fringe consciousness may last up to the flash instance, may accompany it, and in some cases may continue beyond that."*

In such a case intimation can be an excellent "point of departure" for calculations in the verification stage with certain though unclear anticipations for its results. The three-process theory formulated by [12] helps in identifying different types of insights. Selective combination takes place when someone suddenly puts together elements of the problem situation in a way that previously was not obvious to the individual. Selective encoding occurs when a person suddenly sees ... one or more features that previously have not been obvious [12]. Selective comparison occurs when a person suddenly discovers a nonobvious relationship between new and old information. Koestler's Act of Creation occupies the central illumination stage of those 4 or 5 stages pathway. His definition of bisociation, as a generalization of Aha! Moment and Eureka experience, allows to deepen our knowledge of the illumination stage into cognitive (and affective) aspects of the insight.

The second approach to creativity is via [13, 14] who were interested in the development of creative characteristic human attributes centered on divergent thinking. It is the creative product theory in distinction to the process-oriented Gestalt approach. In mathematics education its central qualities have been established to be fluency and flexibility of thinking assessed by speed and precision of thinking and the number of different solutions to the same problem. The third quality is originality measured by comparing individual solutions with those of the whole set of participants.

Originality of thought or action displayed through an Aha! Moment is the quality that joins Koestler's bisociation theory with Guilford's approach. However, this spontaneous originality is presented within the tension of automatization of a habit. In fact, Koestler sees in [2] the creative act of the insight as "an act of liberation—the defeat of habit by originality". He of course realizes that "habits are indispensable core of stability and ordered behavior, [yet] they also have a tendency to become mechanized and reduce man to state of conditions automata" (p. 96). This tension has a direct bearing on the mathematics classroom. Till recently learning of mathematics involved learning the procedures or codes and, actually, trying to make them automatic exactly as Koestler describes, because it increases the fluency. The habits are condensation of codes (procedures) learning. Recent curricular changes with the emphasis on problem solving have as a goal precisely lessening the grip of habits to create the conditions for creative solutions. If as Czarnocha in [15] suggests, creativity should be the foundation of learning, in particular of learning in mathematics, then we have a serious philosophical didactical problem to solve: What should be route of integration of creativity with the necessity of knowing and mastering procedures?

An important issue that arises in the debate between conceptual or relational instruction versus a procedural orientation is the nature of the concept that arises during a moment of insight. In social-constructivist theory, this issue can be understood within the context of Vygotsky's statement that unlike procedural knowledge, conceptual knowledge cannot be taught. Thus, it is the individual, guided by a mentor, who gives meaning to the cultural artifact being presented. In constructivist theory the concept-process is first interiorized and then reflected upon. This reflection upon the interiorized activity leads, through the emerging action scheme, to further structural development, which can be assessed with the help of Piaget and Garcia Triad. This structural development is called here the depth of knowledge (DoK) gained in the moment of insight.

3. The development of theoretical framework underlying the assessment of the depth of knowledge reached during the insight

3.1 The Triad of Piaget and Garcia

The Koestler-based definition of an Aha! Moment or Eureka experience insight of bisociation as a spontaneous leap of insight that connects unconnected frames of reference, matrices of thought is very informative. This suggests that the creativity of an Aha! Moment is in building new connections between different matrices of thought or more precisely between different schemas of thinking (that is, between networks of concepts and connections/relationships between them). Consequently, to understand a so defined creative process, we need to use a theory of schema development. The Triad of Piaget and Garcia developed in [7] turns out to be an excellent constructivist model of schema development whose three stages, -intra, -inter and -trans, allow us to characterize the cognitive levels of different Aha! Insights.

The -intra stage of the development of a concept in the mind of a learner consists of isolated instances, concrete manifestations of the concept as actions, rules or operations, whose performance requires an external help. The process of making connections between isolated instances of the concept means entering the -inter stage of the development. The transition between -intra and -inter case that is from isolated instances to connected instances of the developing concept is one of the sources of Aha! Moments because the transition between the two creates the necessary bisociative context. The -inter stage is rich in possibilities of different connections, from two concepts, single connection schema to multi-connections schemas, and each new connection made can be the effect of the creative insight.

The second important transition in the development of a concept is from the -inter stage connectivity to the -trans stage of generality or abstraction, finding the unifying principle or unifying structure of the concept. This transition is also good source for Aha! Moments. The thinking mechanism which propels the learner along these developmental pathways is called a reflective abstraction, that is, a learner's reflection on its own solution activity. Baker [16] has shown that a bisociative frame can be identified within the reflective abstraction process composed of two parts: projection and constructive generalization indicating a close relationship between bisociation and reflective abstraction.

Our aim is to create assessment of the depth of knowledge reached during the particular insight by the analysis of the nature of created schema of thinking. We take for that purpose numbers of connections made, the conceptual distance between the involved concepts and the quality of connections made. As the measure of that quality, we will take verbal or written description of discovered relationships.

3.2 Koestler/Prabhu theory of creativity

The presentation of the Koestler/Prabhu theory follows Redford, who in [17] formulates requirements for a theory in mathematics education more precisely by suggesting that "a theory can be seen as a way of producing understanding and ways of action based on:

- A system of basic principles, which includes implicit views and explicit statements which delineate the frontier of what will be the universe of discourse and the adopted research perspective.

- A methodology, M, that includes collection of data interpretation as supported by P.

- A set Q of paradigmatic research questions."

The phenomenon under investigation is moments of insight in the learning process: those leading to progress in understanding, or concept development. As teacher researchers our contention is that such moments occur both during social discourse, internalization, as well as during reflection on solution activity.

The basic principle underlying the Koestler/Prabhu theory of learning through creativity of Aha! Moment is the conviction that creativity is the cornerstone of that learning, which leads to progress in understanding. Aha! Moment as a *spontaneous leap of insight, which connects unconnected matrices of thought,* termed by Koestler *bisociation,* is the insight, which leads to progress in understanding through creation of the new conceptual connection, the new entities that is new (for the learner) knowledge.

The second principle of the theory is the ***bisociative frame:*** that is, those originally disconnected matrices of experience within which the Aha! Moment insight takes, or may take, place. Such moments of insight help transcend the learner or solver to a more universal plane within their community of peers and the mathematics classroom, producing positive affect. Since, as we show below, bisociative frame can be identified within different theories of learning, creativity can be seen as an irreducible component of learning. The bisociative frame is the possible creativity detector; it is the bisociative frame that is responsible for the coordination of creativity with different theories of learning, both on the practical level of the classroom as well as on the theoretical research level. The ease and the method with which a bisociative frame can be identified within the "host" theory is described by Czarnocha and Baker in ([4], Introduction). This process of identification has become the tool with the help of which we can indicate the areas of heightened creativity within the cognitive structure of the "host" theory; see examples in [18, 19]. The strength of this tool can be seen in the process of identification of creativity within Vygotsky's sociocultural theory within the process of internalization [20].

Koestler/Prabhu's theory of learning investigates the nature of creativity of Aha! Moment insight in several dimensions paradigmatic research horizons:

- In its relation to student affect and its relationship with conation, that is, with the possibility of satisfying deep human needs of the individual student;

- In its cognitive dimension with the hope of understanding the intrinsic structure of creativity as progress in understanding; it studies the nature and the development of the schema of thinking that is created during the Aha! Moment insight.

- In its theory of learning dimension, it studies the changes in understanding of mathematical concepts occasioned by Aha! Moments, the relationship between interiorization and internalization and the role of creativity in abstraction.

- In its networking theories' dimension, it investigates the processes of integration with different theories of learning during which bisociation can express itself in terms of the host theory.

The third principle of the theory is measurability of creativity, whose aim is to establish the depth of knowledge (DoK) reached during Aha! Moment insight; however, that needs to be done in a way that does not disturb creativity itself, or as [21] suggests, it is measured by internal variables. We use the theory of the Piaget and Garcia Triad [7], which allows us to assess the dynamic development of the new schema from the learner's or a scientist's description of the Aha! Moment insight coordinated with the researcher's view of the mathematical situation. Important disclaimer: Although we use a constructivist Piagetian concept here, we are not necessarily ascribing to the full scope of constructivist approach, although Koestler's point of view in his Act of Creation is not positioned very far. What is important, however, is that we can undertake an analysis of creativity within the sociocultural approach with the help of the same concept of the theory, the bisociative frame, which has been developed in the individual context. We see that Koestler/Prabhu theory has a consistent, independent relationships with both the Piagetian and the Vygotskian approaches. Investigation of that strange fact and its consequences is the research theme for a next investigation.

While the central paradigmatic question of the theory is two-fold

- identification of acts of creation in students thinking.
- identification of enhanced possibility of acts of creation within different theories of learning,

each of the horizons of the theory has its own paradigmatic questions, and the results of recent inquiries into the horizon of learning are presented in Section 4.

The methodology of the theory can be viewed through several angles, including the angle of teaching, of research and of assessment. Within teaching, we attempt to create mathematical situations that contain gaps in student understanding. For instance, we frame the questions, hints or assessment process using bisociative frames, which we suspect will be unconnected in student minds. Or sometimes we add a component that we suspect is needed for the student to gain an insight. In its research aspect, the aim of our methodology is to identify bisociative frames and hence to identify creative possibilities within the "host" theories of learning. Having done that, we have the possibility of using the methodology of a host theory to express the new forms of creativity within it. In its assessment aspect, we assess the change in the development of the relevant schema with the help of PG Triad.

An excellent example of such a coordination between bisociation and modes of reflective abstraction is presented by [18], who demonstrates the presence of a bisociative frame in two critical stages of his framework called "reflection on activity–effect relationship," the Ref* AER. The framework Ref*AER is the elaboration of a Piaget mechanism of reflective abstraction, within which Tzur [18] identifies two different stages of development, participatory and anticipatory. In the participatory stage, he finds six different categories of reflection, each equipped with a bisociative frame, suggesting the presence of sufficient cognitive conditions for facilitation of six, cognitively different Aha! Moments. That work immediately suggested to him a host of new paradigmatic research question. It is instructive how identification of bisociative frame within Ref*AER theory has broadened and deepened the theory itself.

Similar investigations have also been undertaken in [19] where identifying bisociation within different modes of attention allowed the introduction of dynamics within its structure. This bisociative dynamic results in a shift of attention from dyadic to triadic mode of attention, that is, from two components of the bisociative frame to both components together with the new relationship provided by the Aha! Moment.

This unusual capacity of a bisociative frame to be identified within different theories of learning and therefore in corresponding teaching practices provides an important opportunity to conceptually unify the constructivists and the sociocultural theories of learning. That pathway of inquiry has been initiated by Baker in [16] where he identified bisociative frames within the constructive generalization of Piaget, and in [20], he explored creativity within central sociocultural concepts of internalization and appropriation. One major effort of the theory is clarification of the relationship between the concept of interiorization arising in Piagetian theories and internalization of the sociocultural theory to which we devote Section 4.

4. The assessment of the depth of knowledge reached during the Aha! Moment insight

The definition of bisociation, whose cognitive content is to construct a connection between two unconnected frames of reference or matrices of thought, suggests the theory of the schema understood as the network of concepts to be the tool with the help of which we can trace the developmental aspect of the insight. DoK in this case will be the progress in understanding the relevant concept or the difference between student understanding of the mathematical situation before and after the insight. The Aha! Moments analyzed below are taken from the Collection of Aha! Moments in [4]. We established there three levels of DoK, namely mild, normal and strong.

4.1 Calculus AHA! Moment: mild

During my Calculus 1, the teacher gave us an example to solve: $\lim_{X \longrightarrow 0} \frac{\sqrt{1+X}}{X} - \frac{\sqrt{1-X}}{X} =$

1. I verify if the limit is defined when X approaching to 0. It is not.

2. I asked myself "how can I do and find a way for this limit can be defined?"

   I remember in my previews class math 150 when the teacher gave us a rational fraction to solve, he said that we must eliminate the radical in the denominator by multiplicated by the conjugate. But for this equation we do not have radical in the denominator but in the numerator.

   I'm a little bit struggling. What can I do?

3. I was looking at the limit and said to myself why not apply the same rule for the fraction when we have the radical in the denominator.

$$\lim_{X \longrightarrow 0} \frac{\sqrt{1+X}}{X} - \frac{\sqrt{1-X}}{X} = \lim_{x \to 0} \left(\frac{\left(\sqrt{1+x} - \sqrt{1-x}\right)\left(\sqrt{1+x} + \sqrt{1-x}\right)}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)} \right) \tag{1}$$

$$= \lim_{x \to 0} \left(\frac{\left(\sqrt{1+x}\right)^2 - \left(\sqrt{1-x}\right)^2}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)} \right) \tag{2}$$

$$= \lim_{x \to 0} \left(\frac{1 + x - 1 + x}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)} \right) \tag{3}$$

$$= \lim_{x \to 0} \left(\frac{1 + x - 1 + x}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)} \right) \tag{4}$$

$$= \lim_{x \to 0} \left(\frac{2x}{x\left(\sqrt{1+x} + \sqrt{1-x}\right)} \right) \tag{5}$$

$$= \lim_{x \to 0} \left(\frac{2}{\left(\sqrt{1+x} + \sqrt{1-x}\right)} \right) \tag{6}$$

$$= 1$$

4. Now the limit is defined. I can solve it and finish.

Analysis

The student reports the experience of an Aha! Moment in the Calculus class while solving a limit problem. The content of the student insight is the discovery of the analogy (hidden analogy) between the algebraic expression involved in the limit problem with the algebraic expression she learned in the previous algebra class topic on rationalization of algebraic fractions. The similarity suggests to her the method of conjugates used in the rationalization case as the method of solution for the limit problem.

Thus, this single-connection schema element is constructed from two very close yet separate (in the student's mind) representations of the pre-conjugated algebraic expression: one positioned in the denominator of the fraction, and the other in the numerator of the fraction. By applying conjugates method the student learned in the past for the rationalization purposes to the calculation of the limit she has understood (and verbalized) that the application of the method does not depend on the position of the pre-conjugated algebraic expression within a fraction.

We assess the DoK of this insight as Mild for two reasons: (1) the increase of understanding was of just one new connection and (2) the two unconnected initially frames of reference were conceptually very close to each other. This classification is based on the local nature of the search process leading to the moment of insight. This moment of insight takes place in second level of P&G Triad, as the moment of bisociation connects two existing matrices one introduced in calculus the other earlier in algebra. Finally, the concept formed was a new understanding of the limit process.

This moment of insight occurred as the student reflected on one scheme she was learning in calculus class M1 with another she had learned previously in college algebra course M2. Thus, it was mostly constructivist in nature, or a moment of untutored bisociation as Koestler would phrase it. However, it contains distinct relationships to insights within the internalization process, as the student begins with replicating a matrix M1 modeled in class: internalization.

4.2 Fir tree AHA! Moment: normal

Consider the following function that generates the geometric pattern of a reverse growing fir tree (**Table 1**):

Instruction: Draw and describe the Stage 5 of the patterns in terms of its shape and the number of unit squares.

ANSWERS of the student

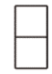
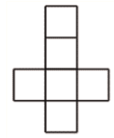
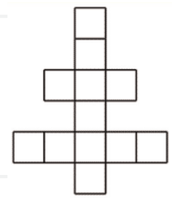
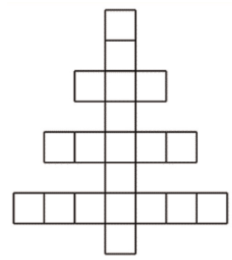
Stage 1	 2 unit squares
Stage 2	 6 unit squares
Stage 3	 12 unit squares
Stage 4	

Table 1.
Stages of the fir tree assignment.

1. Stage 5's shape is bigger than the previous ones; it grows horizontally and follows the patterns of stage 4. It has 30 unit squares, meaning it increased by 10.

2. Describe how the patterns are growing.

   The pattern is growing by increasing 1 unit square to the left, 1 unit square to the left, one additional unit in the center and 1 unit down compared to stage 4.

3. How many unit squares are needed to build Stage 10 of the fir tree?

   Answer #3. It needs 110 unit squares. I developed this answer from [the discussion] next page

4. Given any stage **n** determine the close form of equation to determine the number of unit squares needed to build the tree.

   The formula is: $n(n + 1)$.

   Any stage number multiply by its following number equals the number of unit square, for instance: stage number: 3 multiplied by its following number meaning 4 equals 12 (unit squares.)

5. Your mate tells you that exactly 274 unit squares make up a fir tree. He is wrong. Explain to him why he is wrong.

Answer #5.
It is really wrong because:

Stage 16 = 16(16 + 1) = 272.
Stage 17 = 17(17 + 1) = 306.
The fir tree would never have 274 unit squares.

> *I had a tremendous Aha! Moment. I just realized that the formula I got from the patterns was a factorized expression and if I multiply it "n(n+1)= units square" I would have something like an algebraic expression exactly a trinomial expression that can be factorized as well, and it equals real numbers for example: n(n+1)= 12 n^2+n= 12 n^2+n-12=0 n^2+n-20=0 n^2+n-56=0 n^2+n-90 (n+4)(n-3)=0 (n+5)(n-4)=0 (n+8)(n-7)=0 (n+10)(n-9) And when it comes to n^2+n-274 it cannot be factorized.*

Analysis

The Aha! Moment took place after the student solved the assigned problem. From that solution, we know about the student's cognitive capabilities: She obtained the general formula using the variable n by generalization from the table of results for each term. She solved the question 5 by trial and error and demonstrated from the logic of these calculations that proposed result is incorrect. These are the components of the knowledge schema before the insight. The insight provided new conceptual solution to the same problem (**Table 2**).

It connected the factorized quadratic algebraic expression, a binomial with the corresponding quadratic trinomial by incorporating data of the problem. The second constructed connection is between quadratic trinomial and solutions of quadratic equation via its factorization. It is interesting that both connections are made through the process of factorization, and the concepts they connect are close to each other conceptually. However, the concept of quadratic equation is significantly further apart conceptually from factorization of trinomials, so much that the student does not recognize it within the mathematical situation. Yet the fact that student recognizes that constant terms in the binomials obtained through factorization are the solutions of unrecognized quadratic equation and have bearing upon the solution of the whole given problem indicates a larger conceptual distance within the second connection of the constructed schema. What also adds to the depth of the new schema is that from

Stage 1	1(1 + 1) = 2
Stage 2	2(2 + 1) = 6
Stage 3	3(3 + 1) = 12
Stage 4	4(4 + 1) = 20
Stage 5	5(5 + 1) = 30
Stage 6	6(6 + 1) = 42
Stage 7	7(7 + 1) = 56
Stage 8	8(8 + 1) = 72
Stage 9	9(9 + 1) = 90
Stage 10	10(10 + 1) = 110

Table 2.
Numerical relationships discovered at each stage of the fir tree Aha! Moment.

the language used it is clear the student (despite some holes in the overall schema of the situation) has a control over the "gestalt" of the schema—the student owns it—that is that it has been interiorized as well as internalized, with evidence for both contained in the student's description. The evidence of internalization is in the ability to explain the logical connections verbally, and interiorization is evidenced by the ability to provide the second solution, hence independent action upon the problem. We assess this progress of understanding at the second level of schema construction and name it as normal.

4.3 The domain AHA! Moment: strong

The problem starts with the function f(x)= $\sqrt{X+3}$. The teacher asked the students during the review: "Can all real values of x be used for the domain of the function f(x) = $\sqrt{X+3}$?"

1. Student (S): No, negative x's cannot be used.
2. Teacher (T): How about x = -5?
3. S: No good.
4. T: How about x = -4?
5. S: No good either.
6. T: How about x = -3?

   Student, after a minute of thought:
7. S: It works here.
8. T: How about x = -2?
9. S: It works here too.

   A moment later the student adds:
10. S: Those x's which are smaller than -3 cannot be used here.
11. T: How about g(x) = f(x)= $\sqrt{X+1}$?

    Student, after a minute of thought:
12. S: "Smaller than 1 cannot be used."
13. T: In that case, how about h(x) = f(x)= $\sqrt{X-a}$?
14. S: Smaller than a cannot be used.

Analysis

This Aha! Moment clarifies student misconception concerning the domain of the $\sqrt{X+3}$. The original and habitual student answer in line 1 represents student misconception. The intent of the teacher's question was to direct the student's attention to the contradiction of her answer with the situation at hand, what led the student into cognitive conflict. The Aha! Moment takes place as a resolution of that conflict. The student's short reflection on the previous verbal interaction results in the correct reorganization of her approach.

Note that student's insight engages in her the domain of the function f(x)= $\sqrt{X+3}$ and order on the real number line. The bisociative frame of this bisociation were the axes of the cognitive conflict: the habit of well-established misconception and the data brought to the student's attention by focusing her attention on the relevant details. The conceptual distance between these components is not very large – just the change of the parameter but to traverse it the student had to engage her schema of addition for integers, which makes student thinking a bit more complex and places this insight on the second level.

Her language is straightforward, does not convey any doubts and places understanding on Normal level of DoK. However, follow-up questions of the instructor reveal deeper levels of understanding. For instance, the student can easily transfer her understanding to a different related example (line 11). More important, she can abstract and generalize the answer for arbitrary parameter a (line 14). This is the second-order reflection on the "family type" expressions, which leads to abstraction with generalization of the -trans stage. That indicates significantly larger conceptual distance between the components of the bisociative frame and places the Domain Aha! Moment on the third Strong level.

5. Koestler theory of creativity, Aha! Moment and learning

For Koestler creativity and indeed learning what is subjectively new for an individual, which he refers to as progress in understanding, takes place through the synthesis of two distinct and previously incompatible frames of reference. He refers to these as matrices, each of which has its own rules of the game or codes that govern appropriate activity.

What is a matrix of thought, or a frame of reference?

> *The matrix is the pattern before you, representing the ensemble of permissible moves. The code which governs the matrix can be put into simple mathematical equations… or it can be expressed in words. The code is the fixed invariable factor in a skill or habit, the matrix is the variable part. The two words do not refer to different entities, they refer to different aspects of the same activity*. (Koestler ([2], p. 40)).

Referring to individual moments of insight within the learning process, Koestler notes that "*Minor, subjective bisociative processes do occur on all levels and are the main vehicle for untutored learning*"(p. 658). This raises another important question addressed in [16] and in [20], how does one describe interiorization in Koestler bisociative frame? The more general question of how Piaget's notion of reflective abstraction fits into Koestler's bisociative frame is discussed in [22]. Since constructivist research methodology can be described as minimally guided, it certainly classifies as untutored and therefore should be bisociative in nature. Can the same be said of internalization?

Our analysis of the genesis or birth of a code during a moment of insight, whether during interiorization, internalization or bisociation, is conducted through the lens of three defining characteristics of such moments. (We will use here Koestler's term "blocked situation" as one in which routine matrices fail to accomplish the desired goal: to solve the problem). The first is the search process in a blocked situation. The second is the connection realized during the moment of insight between the blocked situation and a matrix that provides conceptual reasoning that allows for resolution of the blocked features. The third is the novel concept and process based on the concept that acts on the formerly blocked features to obtain the goal.

5.1 Discovery of a hidden analogy, interiorization and internalization

Koestler describes a blocked situation as one in which routine matrices fail to accomplish the desired goal. Constructivists use the term non-assimilatory situation. Koestler describes the process of trying to resolve a blocked situation by searching for a connection to an analogous matrix as the discovery of a hidden analogy, as a search for something that is unknown:

> *[T]he subject looks for a clue, the nature of which he does not know, except that it should be a 'clue' … a link to a type of problem familiar to him … [H]e must try out one frame after another … until he finds the frame into which it fits, … an analogy with past experience and allows him to come to grips with it*. (pp. 653-654).

This search process in such a blocked situation to find a connection to an appropriate matrix-scheme in our repertoire can itself be considered a matrix-schema Mo. The search process follows in its general outlines the steps of [8, 9]. We may consider the code to contain some or all of the following guiding principles. First, identify relevant features of a problem situation that are blocked. Second, review our toolbox or collection of matrices that are even remotely associated with these features. Finally, select and proceed-verify the one most appropriate. If this fails, Koestler notes that mathematicians recommend to sit tight and wait for inspiration.

We note that contemporary instructional methodology of teachers encourages students to seek external assistance during the process of these transitions. As a result, the search process from constructivist pedagogy turns to internalization. This search process is highly subjective. Individuals vary in their ability to discriminate or identify what objects are relevant and abstract the conceptual relationship between these objects and appropriate activity. There is also a wide variety of motivation. Patience on the part of individuals during this search and finally even the motivation to seek assistance varies. All these factors impact the success or failure of this process.

5.2 Interiorization

Interiorization occurs during a child's transition from empirical-spontaneous to abstract reasoning, and thus, M_1 is an intuitive matrix and hence limited to situations where input present low cognitive demand. When a situation involves objects with more structure or cognitive load, the subject becomes perturbed, their attention has two foci—it is so-called dyadic attention. One is their Mo search to understand the structural objects that cannot be assimilated, and the other is that part of the M_1 matrix that acted upon the analogous spontaneous concepts. In a moment of bisociative insight between the search matrix and their intuitive matrix when the shift of attention shifts from dyadic to triadic attention of seeing the whole [19], the subject abstracts the spontaneous concepts.

This allows the formerly intuitive reasoning to become conceptual-based reasoning, and a pseudocode is born, through the bisociation of Mo and M_1. This particular description of interiorization in creativity theory is known as selective encoding. In this, the subject's attention is on a relevant matrix-scheme or tool that they previously could use only in a limited context and their insight allows them to use it more fully. Thus, interiorization as described consists of bisociation through selecting an object(s) that cannot be assimilated, search to connect them with the spontaneous concepts of a person's intuitive scheme, and the result is scientific concepts simultaneous with a process or conscious scheme.

5.3 Internalization and interiorization: the search matrix Mo

The distinction between interiorization and internalization as it relates to moments of insight in the learning process begins with the nature of the search matrix Mo. In Koestler's description of searching for a hidden analogy, the search matrix Mo is focused on the subject's collection of even remotely related schemes in the hope of finding an unknown connection. The second tenet of constructivism is that all prerequisite knowledge should be present in a learning situation.

Thus, the search process is typically on solution activity determined by an existing scheme (M_1) that is deemed appropriate and yet remains insufficient to resolve the non-assimilatory situation. As such, interiorization involves selective comparisons between the non-assimilatory objects in Mo and an intuitive M_1 leading to an abstraction of the underlying conceptual-invariant relationship. Reflective abstraction is the name given by Piaget to the mechanism of accommodation that includes interiorization. It has two steps. In the first step, an appropriate M_1 is projected into the search matrix Mo for a non-assimilatory situation. The second constructive generalization step involves perturbation between and reflection upon these two matrices. The creativity of Aha! Moment arises during the search within unconscious or semi-conscious Incubation Gestalt stage through the shift of attention from the separate matrices to the whole revealing the new structure.

Social constructivists use the term reflective thinking, instead of reflective abstraction, to include any form of conscious reflection on collective discourse, for example, shared activities, including observation of, or communication with, an exterior source of knowledge. Thus, the search process Mo in internalization includes co-creation of knowledge in all its forms, making it ideal for analysis of classroom discourse. Social constructivists use the term socially mediated activity to describe solution activity that involves assistance through any form of cultural artifacts, or guided communication, for example, written text, pre-recorded videos, other internet search sources, peer-mentor, or teacher etc. As students attempt to internalize material presented in the classroom, their primary motive is to understand or assimilate externally directed activity. Thus, they may have a motive but not realize the problem goal.

5.4 Moments of insight: connection

The moment of insight has a transcendent nature as a student is led to this connection by intuition not reason. As a result, in Koestler's view, pedagogy should be (re)structured to support such moments of insight into the guided discovery as math and science cannot be appreciated outside such experiences.

In the constructivist frame, during interiorization the moment of insight occurs as the individual struggles to employ an intuitive situation-dependent scheme M_1 in a non-assimilatory situation, that is due to the presence of Mo objects beyond the capacity of spontaneous thought. The perturbation between Mo and M_1 is resolved in a moment of insight or, shift of attention which abstracts the formally intuitive reasoning. This experience transcends the reasoning from the situation as it becomes an abstracted activity–effect relationships. This moment of insight or abstraction allows the individual to have conscious control over the process and thus to act independently. Thus, for constructivists, Mo contains an existing M_1 scheme, Situation/Activity/Effect (S-A-E) triad, which in a moment of insight is transformed into an abstracted Activity/Effect (A-E) dyad. That is one not dependent upon the situation. The pedagogy is guided discovery to promote reflection and abstraction upon such an M_1. In Koestler's search for a hidden analogy, the focus is on the formation of a S-A-E triad. During internalization, as in Koestler's frame, there is typically no existing S-A-E link within the search matrix Mo.

However, the pedagogy of internalization is not centered on guided discovery but instead on the mentoring of students as they accommodate new situations at the upper limit of their zone of proximal development. That is the upper limit of what they can realize with assistance. So, where is their own individual creativity? It must be beyond the assistance: beyond the upper limit of their individual ZPD. Where is the bisociative frame?

5.5 Creativity as deviation during the process of appropriation

Here, we can use an important suggestion provided by [23] who informs that internalization encounters difficulties in the proper description of the concept formation, also from the sociocultural points view. To address these problems, we need to introduce the concept of appropriation. The term appropriation is often used to analyze learning within social discourse; it has its origins in the work of the Russian psychologist M.M. Bahktin to understand how children learn language.

> *The word in language is half someone else's. It becomes "one's own" only when the speaker populates it with his own intention, his own accent, when he appropriates the work, adapting it to his own semantic and expressive intention.* [24]

When analyzed in sociocultural terms as the interaction between learners in the learning community, appropriation facilitates the development of concepts of individual learners. Let me continue here with the words of [23]:

> *"Where is creativity related to the appropriation process? This is its own deviation. Table 2 lists some of the possibilities of a wide variety of deviations. This includes the gap between the concepts of the learners interacting with each other and between the concepts of the learning community as seen by each learner. The cause of these deviations lies in the learners' historical and cultural constraints. By interacting according to*

> *these deviations, each learner may misunderstand what he/she is talking about. Simultaneously, the learner may create a new concept not included in the speaker's concept. In other words, the deviation of appropriation can be the source of creativity that leads to new ideas that the speaker did not intend."*

In other words, creativity in sociocultural theory originates in the differences between the learner's and the community's matrices of thought owing to different historical and cultural constraints. Hence, we have recovered the bisociative frame between the two matrices of thought at exactly the unique site of creativity within sociocultural theory. If one of two matrices is that of the community and another of the learner, it might be natural to call creativity a deviation from the community's point of view. It is interesting to trace out how that particular process of appropriation of the concept interacts with the relevant inner spontaneous concepts of the learner. We leave that subject as an open research question: how do appropriation and interiorization interact?

6. Conclusions

This work presented a new Koestler/Prabhu theory of creativity within learning and provided some investigations suggested by the theory. One of them was to investigate whether the bisociation frame as the central component of the theory of the creativity of Aha! Moment can be identified within the constructivist and the sociocultural approaches to learning. Our discussion in the first section where Koestler's theory is discussed indicates that the term and the concept of bisociation derive from the observation and analysis of individual insights. On the other hand, sociocultural approach places the emphasis on learning within collaborative teams rather than on the individual activity. Consequently, to find the space within that theory where social creativity can be understood with the help of bisociation as the theory of the individual insight is very important especially from the point of view of classroom teachers of mathematics, where both individual as well as collaborative learning are taking place.

Section 1 examined bisociation theory, interweaving it with the constructivist framework and followed by the preliminary discussion of the nature of creativity of Aha! Moment/Eureka experience. In Section 2, we presented Koestler/Prabhu's theory of creativity in mathematics following [17] who formulated the requirements for a theory in mathematics education. Koestler/Prabhu's theory stands on three principles: (1) the definition of an Aha! Moment as the object of our investigations, (2) the bisociative frame and (3) the measurability of creativity as the assessment of DoK reached during the insight.

At present, the theory has established four horizons of its investigations:

- A relationship of the insight with the affect and conation.
- The cognitive dimension, which studies the nature and the development of the schema of thinking created during the insight. We see this dimension of research in Section 3 where the cognitive mechanism of thinking called PG Triad is used for the analysis of DoK.
- In its learning horizon, we study changes in understanding of concepts occasioned by Aha! Moments. In particular, Section 4 explains the central

investigation of the relationship between interiorization and internalization in the context of bisociative creativity. As interiorization is a constructivist concept while internalization is a Vygotskian, sociocultural concept, this section has a bearing upon the goal of unifying individual and sociocultural approaches to creativity in mathematics education. In the section, we show the method of our approach and note its limitation due to it being too rigidly bound with the concept of ZPD as a framework for mentor–mentee social interaction. We discovered that the much more useful concept to investigate bisociative creativity within that approach is the concept of appropriation, and in particular that of deviation during the process of appropriation. As we indicated there within deviation of the concept of the individual learner from that of the community, the bisociative frame can be identified as two matrices of thought, which are different due to their historical and cultural constraints.

- In the networking theories dimension, we investigate processes of integration of Koestler/Prabhu theory with theories of learning what enables bisociativity express itself in terms of host theories. We have touched upon interaction of bisociativity with the theory of reflective abstraction of Piaget, theory of attention of John Mason and participation/anticipation approach of Tzur. Intimate connections of bisociative creativity with learning theories enables us to introduce creativity into different pedagogies of learning.

- Section 3 presented the method of DoK assessment on three different types of Aha! Insights, taken from the Collection in [4]. Section 4 explored our investigations leading to the identification of the bisociative frame exactly where it should be according to sociocultural approach, in the deviation within the process of appropriation.

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