

Optimal risk sharing, equilibria, and welfare with empirically realistic risk attitudes*

Jean-Gabriel Lauzier[†] Liyuan Lin[‡] Peter Wakker[§] Ruodu Wang[¶]

October 6, 2025

Abstract

This paper examines optimal risk sharing for empirically realistic risk attitudes, providing results on Pareto optimality, competitive equilibria, utility frontiers, and the first and second theorems of welfare. Contrary to common theoretical assumptions, empirical studies find prevailing risk seeking in particular subdomains, in particular for losses. We first allow for some risk-seeking agents, still assuming expected utility. Yet more empirical realism is obtained by allowing agents to be neither risk averse nor risk seeking and by generalizing expected utility. Here we provide first results, pleading for future research. Our main new tool is a counter-monotonic improvement theorem.

Keywords: Gambling behaviour, counter-monotonicity, competitive equilibrium, convex utility functions, rank-dependent utility

*This paper was previously circulated under the title “Negatively dependent optimal risk sharing”.

[†]Dept. of Economics, Memorial University of Newfoundland, Canada. \boxtimes jlauzier@mun.ca

[‡]Dept. of Econometrics & Business Statistics, Monash University, Australia. \boxtimes liyuan.lin@monash.edu

[§]Dept. of Economics, Erasmus University Rotterdam, the Netherlands. \boxtimes wakker@ese.eur.nl

[¶]Dept. of Statistics and Actuarial Science, University of Waterloo, Canada. \boxtimes wang@uwaterloo.ca

1 Introduction

This paper studies optimal risk sharing for empirically realistic risk attitudes. Classical studies invariably assumed universal risk aversion (Mas-Colell et al., 1995, Chapter 10). However, modern empirical studies found much risk seeking, so much that it deserves systematic study. Thus, in the loss domain, as in cost sharing problems and in times of economic crises, risk seeking is even prevailing rather than risk aversion.¹ We, therefore, extend classical results on Pareto optimality, competitive equilibria, and utility frontiers, and the first and second welfare theorems, to the case where (partial) risk seeking is allowed. Our main tool is a new counter-monotonic improvement theorem (Theorem 1).

Marshall (1890), the first to show that risk aversion is equivalent to concave utility under expected utility, pleaded for universal risk aversion, dismissing risk seekers as follows:

*since experience shows that they are likely to engender a restless, feverish character,
unsuited for steady work as well as for the higher and more solid pleasures of life.*

The common classical belief holds that risk seeking is yet more implausible in stable markets. After all, as soon as there are two or more risk seekers, they can engage in mutual risky zero-sum games, using any randomization device. It will end only if some boundary restriction is reached, such as bankruptcy for all but one. Accordingly, fundamental laws of economics have all been established under the assumption of universal risk aversion (Arrow and Debreu, 1954). Whereas this common classical belief has guided economics for centuries, the supposed fate of risk seekers was never formally stated.

Since the 1980s, economics has become empirically oriented. It then became widely understood that there is much risk seeking. The first formal result on extreme fates of risk seekers was by Ebert and Strack (2015): repeated individual decisions can lead to unstoppable gambling unless bankruptcy, assuming risk seeking as found in prospect theory. However, implementations of nonexpected utility theories in such dynamic settings are controversial

¹See Kühberger's (1998) meta-analysis of 136 studies, Edwards's (1966) review in finance, l'Haridon and Vieider's (2019) analysis of a world-wide representative student sample, Myagkov and Plott (1997) for competitive equilibria, Abdellaoui et al. (2013) for financial professional traders, Laughhunn et al. (1980) for managers, Olsen (1997) for professional investors, and Shen and Zhong (2025) for 109,658 Chinese subjects. Risk seeking for losses underlies the disposition effect in finance (Barberis and Huang, 2008; Heimer et al., 2025). Netzer (2009) gave evolutionary arguments for risk seeking for losses. Crinich et al. (2013) wrote, on classical studies: "Risk seekers seem to be forgotten."

([Machina, 1989](#)). We therefore focus on static decisions. Further, we will study risk sharing rather than individual decisions.

[Araujo et al. \(2017\)](#) provided the first formal statement about risk seekers in optimal risk sharing. They considered the special case of infinite sequences with an extra assumption of “reasonable strict optimism”. Building on that, [Araujo et al. \(2018\)](#) established a competitive equilibrium for markets that have enough risk averters with total prior endowment large enough to clear the market. [Herings and Zhan \(2025\)](#) provided generalizations to incomplete markets. We provide an exact statement of the classically believed fate of risk seekers in full generality, without any assumption about the remaining market (Theorems 3 and 5). This and other results in our paper essentially invoke external randomizations, providing a new rationalization for using [Cass and Shell’s \(1983\)](#) sunspots. [Dillenberger and Segal \(2025\)](#) assumed multidimensional discrete (indivisible) outcomes and general utility functions over them, which do not restrict risk attitudes. They focused on deviations from expected utility (increasing empirical realism) and assumed universal risk aversion in those deviations through quasiconvexity with respect to probability mixing. They found that, in many situations, only lotteries over two dimensions of outcomes are part of a Pareto-efficient allocation mechanism.

[Landsberger and Meilijson’s \(1994\)](#) comonotonic improvement theorem is an important tool for classical risk sharing: optimality is only possible for risk-averse agents if their allocations are mutually comonotonic. Comonotonicity means that the individual risks of the agents are maximally aligned, so that all mutual hedging possibilities have been used up. Our new counter-monotonic improvement theorem provides the analogous tool for risk seeking: optimality and stability are only possible for risk-seeking agents if their allocations are mutually counter-monotonic. Counter-monotonicity means that the individual risks of the agents are minimally aligned, so that all mutual leveraging possibilities have been used up. We use this theorem to derive our formal results.

Counter-comonotonicity, also called anti-comonotonicity and first studied by [Dall’Aglio \(1972\)](#), appears as an optimal structure in risk sharing under quantile models ([Embrechts et al., 2018, 2020](#)). Quantile models are important in finance (“Value-at-Risk”), but do not provide empirically realistic decision models. This paper focuses on the latter. [Lauzier et al. \(2023\)](#) obtained a stochastic representation of counter-monotonicity, used in Section 3 to introduce jackpot allocations (“winner-take-all”) and their duals, scapegoat allocations. These allocations formalize the above boundary restrictions for risk-seeking agents, to some

extent confirming Marshall’s pessimistic view and the classical economic views on risk seeking, but showing what remains possible. For expected utility agents, some of whom are risk seeking, Theorems 2–3 in Section 4 analyze Pareto optimality and Theorems 4–7 in Section 5 establish welfare and competitive equilibria under different conditions.

To achieve full empirical realism, another refinement is desirable. All decision models discussed so far assumed for each agent either entire (for all lotteries) risk aversion or entire risk seeking. However, the prevailing empirical finding is that agents do not have such “global” domain-independent risk attitudes. Risk aversion is prevailing for gains of moderate and high probability, but risk seeking is prevailing for small-probability gains (Fehr-Duda and Epper, 2012; l’Haridon and Vieider, 2019), with these phenomena reflected for losses. Such probability dependence cannot be accommodated by expected utility, and generalized models are called for.² This probability dependence explains, for instance, the coexistence of gambling and insurance, a paradox for classical EU (Friedman and Savage, 1948).

We will consider the most popular generalization of EU, Quiggin’s (1982) rank-dependent utility (RDU), which for gains agrees with Tversky and Kahneman’s (1992) prospect theory.³ Unlike with the classical models considered as yet, for behavioral models there is a special role for one outcome, formalized as the reference outcome and scaled as outcome 0. Better, positive outcomes are gains and worse, negative, outcomes are losses. In our analysis of RDU we will only study gain outcomes, leaving extensions to mixed and loss random variables to future studies. We thus study domain-dependent risk attitudes only for gains. Still, questions arise that are too difficult to handle in full generality with current techniques. We, nevertheless, provide some first results in Section 6, for homogeneous agents with empirically prevailing risk attitudes. It turns out that some natural jackpot allocations are Pareto-optimal in this case (Theorem 8), even though the agents are mostly risk averse. Moreover, under some specific assumptions, we show that for small-scale total payoffs (aggregate endowments) it is Pareto optimal to gamble, whereas for large-scale total payoffs it is better, or even optimal, to share proportionally, in agreement with empirical findings (e.g., Jullien and Salanié, 2000). We also obtain a competitive equilibrium under the assumption of

²Friedman and Savage’s (1948) famous attempt to incorporate partial risk seeking into EU did not work empirically; see Moscati (2018, p. 227).

³The quantile models mentioned above are special cases of RDU, and they are neither universally risk averse nor universally risk seeking. But they are not empirically realistic for decision making. Beissner and Werner (2023) also did not need universal risk aversion or seeking for their first-order optimality conditions for risk sharing. However, their results give conditions in terms of preference functionals, rather than preferences, and only apply to interior solutions. The new jackpot and scapegoat allocations in this paper are not interior.

no aggregate uncertainty on the small-scale total endowment (Proposition 7). Finally, proofs are in Appendices A and B, and the supplementary material contains Appendices S.1–S.5 on additional background, details, and results.

2 Model setting

We consider a one-period economy, with uncertainty realized at the end of the period. By $(\Omega, \mathcal{F}, \mathbb{P})$ we denote a probability space, with \mathcal{F} the σ -algebra of events, and by \mathbb{E} we denote expectation under \mathbb{P} . Let \mathcal{X} be a set of random variables on Ω , referred to as *payoffs*, which represent random monetary payoffs at the end of the period. We assume that \mathcal{X} is a convex cone, i.e., it is closed under addition and positive scalar multiplication. For instance, \mathcal{X} may be the space L^1 of integrable random variables. Two random variables X and Y are almost surely equal if $\mathbb{P}(X = Y) = 1$, and we identify them, omitting “almost surely” in equalities unless we want it emphasized. Let $\mathbb{R}_+ = [0, \infty)$. We assume n agents for some $n > 0$ and write $[n] = \{1, \dots, n\}$.⁴ Let

$$\Delta_n = \left\{ (\theta_1, \dots, \theta_n) \in \mathbb{R}_+^n : \sum_{i=1}^n \theta_i = 1 \right\}$$

be the standard simplex in \mathbb{R}^n . We write $\Delta_n(v) = v\Delta_n = \{v\boldsymbol{\theta} : \boldsymbol{\theta} \in \Delta_n\}$ for $v > 0$. Throughout, for a scalar z and a vector $\mathbf{y} = (y_1, \dots, y_n)$, we write $z\mathbf{y}$ for the vector (zy_1, \dots, zy_n) . Denote by $\mathbf{0}$ and $\mathbf{1}$ the vectors $(0, \dots, 0)$ and $(1, \dots, 1)$ in \mathbb{R}^n . Thus, $y\mathbf{1} = (y, \dots, y)$ and $\mathbf{1}/y = (1/y, \dots, 1/y)$ for $y > 0$. We use boldface capital letters for (possibly random) n -dimensional vectors. Throughout, $0/0 = 0$.

Our setting of risk sharing concerns n agents who share a random variable $X \in \mathcal{X}$, the *total payoff*, interpreted as the total random wealth to be allocated among the agents. The set of all *allocations* of $X \in \mathcal{X}$ is

$$\mathbb{A}_n(X) = \left\{ (X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}.$$

That is, an allocation of X is a random vector whose components sum to X . This means the wealth is completely redistributed among the agents without any transfers outside the

⁴Our assumption of finitely many agents avoids cases where risk-seeking agents are negligible in the market, analyzed by [Aumann \(1966\)](#).

group. Note that the choice of \mathcal{X} is important in the definition of \mathbb{A}_n as it restricts the possible allocations. An allocation is *nontrivial* if it has at least two non-zero components.

For each agent i , her preference relation \succsim_i is represented by a preference functional \mathcal{U}_i , that is,

$$X \succsim_i Y \iff \mathcal{U}_i(X) \geq \mathcal{U}_i(Y).$$

The value $\mathcal{U}_i(X)$ is the *utility* of X for agent i . We assume that if X and Y are equally distributed, denoted by $X \stackrel{\text{d}}{=} Y$, then $\mathcal{U}_i(X) = \mathcal{U}_i(Y)$. This means that \mathcal{U}_i represents a decision model under risk, and all agents agree on the probability measure \mathbb{P} . For instance, \mathcal{U}_i may be an EU preference functional $\mathcal{U}_i : X \mapsto \mathbb{E}[u_i(X)]$ for some increasing function $u_i : \mathbb{R} \rightarrow \mathbb{R}$ (called a *utility function*); such agents are called *EU agents* or *EU maximizers*. Throughout, the terms “increasing” and “decreasing” are in the non-strict sense. We will study two types of optimality in risk sharing: Pareto optimality and Arrow-Debreu competitive equilibria, explained next.

Pareto optimality. For two allocations $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$ in $\mathbb{A}_n(X)$, we say that \mathbf{X} *dominates* \mathbf{Y} if $\mathcal{U}_i(X_i) \geq \mathcal{U}_i(Y_i)$ for all i , and the domination is *strict* if at least one of the inequalities is strict. The allocation \mathbf{X} is *Pareto optimal* if it is not strictly dominated by any allocation in $\mathbb{A}_n(X)$. Pareto optimality is closely connected to the optimization of a linear combination of the utilities. For $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, an allocation \mathbf{X} is $\boldsymbol{\lambda}$ -optimal in $\mathbb{A}_n(X)$ if $\sum_{i=1}^n \lambda_i \mathcal{U}_i(X_i)$ is maximized over $\mathbb{A}_n(X)$. Here, the vector $\boldsymbol{\lambda}$ is called a *Negishi weight vector* (Negishi, 1960). We use the term sum optimality for the case $\boldsymbol{\lambda} = \mathbf{1}$. It is well known and straightforward to check that $\boldsymbol{\lambda}$ -optimality for $\boldsymbol{\lambda}$ with positive components implies Pareto optimality. The converse holds under some additional conditions (Mas-Colell et al., 1995, Chapter 16).

Competitive equilibria. Suppose that each agent has an initial endowment, summarized by the vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n) \in \mathbb{A}_n(X)$. Consider the individual optimization problem for agent i :

$$\text{maximize } \mathcal{U}_i(X_i) \quad \text{over } X_i \in \mathcal{X}_i \quad \text{subject to } \mathbb{E}^Q[X_i] \leq \mathbb{E}^Q[\xi_i], \tag{1}$$

where Q is a probability measure representing a linear pricing mechanism,⁵ \mathbb{E}^Q is the expected value under Q , and $\mathbb{E}^Q[X_i] \leq \mathbb{E}^Q[\xi_i]$ is the budget condition. For each i , \mathcal{X}_i is the set of

⁵It is without loss of generality to assume that Q is absolutely continuous with respect to \mathbb{P} , because Q can be arbitrary on events with 0 probability under \mathbb{P} , which does not affect equilibria.

possible choices X_i for agent i . In Section 5, we will choose \mathcal{X}_i to be the set of all random variables Y satisfying $0 \leq Y \leq X$, where X is assumed to be nonnegative.

The tuple (X_1, \dots, X_n, Q) is a *competitive equilibrium* if (a) individual optimality holds: X_i solves (1) for each i ; and (b) market clearance holds: $\sum_{i=1}^n X_i = X$. Then (X_1, \dots, X_n) is an *equilibrium allocation*, and Q is an *equilibrium price*. We then often do not mention the initial endowments, meaning that (1) is solved for some ξ .

Individual rationality. For an initial endowment vector $\xi \in \mathbb{A}_n(X)$, an allocation $\mathbf{X} \in \mathbb{A}_n(X)$ is *individually rational* if it dominates ξ . Then risk sharing is beneficial for each agent.

An equilibrium allocation is always individually rational because of individual optimality. Pareto-optimal allocations and equilibrium allocations are intimately connected through the two fundamental theorems of welfare economics. Under certain conditions, the first welfare theorem states that every equilibrium allocation is Pareto optimal, and the second welfare theorem states that every Pareto-optimal allocation is an equilibrium allocation for some initial endowments and equilibrium price.

3 Counter-monotonic improvement

This section introduces our new tools for analyzing risk sharing. We assume $\mathcal{X} = L^1$.

3.1 Convex order, risk aversion, and comonotonicity

A random variable X is *smaller than* a random variable Y in *convex order*, denoted $X \leq_{\text{cx}} Y$, if $\mathbb{E}[\phi(X)] \leq \mathbb{E}[\phi(Y)]$ for every convex function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ provided that both expectations exist (Rüschendorf, 2013; Shaked and Shanthikumar, 2007). That is, X is less risky than Y in the sense of Rothschild and Stiglitz (1970). If $X \leq_{\text{cx}} Y$, then $\mathbb{E}[X] = \mathbb{E}[Y]$, meaning that the relation of convex order compares random variables with the same mean; hence, Y is also called a mean-preserving spread of X . Preference functional \mathcal{U} is (*strongly*) *risk averse* (Rothschild and Stiglitz, 1970) if

$$X \leq_{\text{cx}} Y \implies \mathcal{U}(X) \geq \mathcal{U}(Y).$$

We usually omit “strongly” because we do not consider weak versions. *Strict risk aversion* holds if $X <_{\text{cx}} Y$ (meaning $X \leq_{\text{cx}} Y$ and $Y \not\leq_{\text{cx}} X$) implies $\mathcal{U}(X) > \mathcal{U}(Y)$. Similarly,

$$X \leq_{\text{cx}} Y \implies \mathcal{U}(X) \leq \mathcal{U}(Y)$$

defines *risk seeking*, and the strict version is analogous. Under EU, (strict) risk aversion is equivalent to (strictly) concave utility, and (strict) risk seeking is equivalent to (strictly) convex utility.

Two random variables X, Y are *comonotonic* if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \text{ for } (\mathbb{P} \times \mathbb{P})\text{-almost every } (\omega, \omega') \in \Omega^2.$$

An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is *comonotonic* if every pair of its components is. The random variables X_1, \dots, X_n are comonotonic if and only if there exists a random variable Z such that each X_i is an increasing transformation of Z . We can take $Z = \sum_{i=1}^n X_i$ ([Denneberg, 1994](#), Proposition 4.5).

The comonotonic improvement theorem ([Rüschedorf, 2013](#), Theorem 10.50) states that, for every $X \in \mathcal{X} = L^1$ and every $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, there exists a comonotonic allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that (Y_1, \dots, Y_n) is comonotonic and $Y_i \leq_{\text{cx}} X_i$ for every i . Consequently, when all agents are strictly risk averse, Pareto-optimal allocations must be comonotonic ([Carlier et al., 2012](#)). A similar result holds under a different label: in an exchange economy with aggregate risk, the individual consumption of strictly risk-averse EU maximizers is increasing in aggregate wealth.

3.2 Counter-monotonicity and jackpot allocations

When agents are risk averse, comonotonicity, an extreme type of positive dependence, will appear. When, to the contrary, agents are risk seeking, a form of negative dependence will appear, defined next. Two random variables X, Y are *counter-monotonic* if $X, -Y$ are comonotonic. An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is *counter-monotonic* if every pair of its components is. Unlike comonotonicity, which allows for arbitrary marginal distributions, counter-monotonicity in dimension $n \geq 3$ puts strong restrictions on the marginal distributions ([Dall'Aglio, 1972](#)). Appendix S.1 collects technical background on counter-monotonicity.

Let Π_n be the set of all n -compositions (ordered partitions) of Ω , that is,

$$\Pi_n = \left\{ (A_1, \dots, A_n) \in \mathcal{F}^n : \bigcup_{i \in [n]} A_i = \Omega \text{ and } A_1, \dots, A_n \text{ are disjoint} \right\}.$$

The indicator function $\mathbb{1}_A$ for an event A is defined by $\mathbb{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{1}_A(\omega) = 0$ otherwise. Lauzier et al. (2023, Theorem 1) obtained a stochastic representation of counter-monotonic random vectors (X_1, \dots, X_n) with at least three non-constant components, and they have the form $X_i = Y\mathbb{1}_{A_i} + m_i$ for some $m_1, \dots, m_n \in \mathbb{R}$, $(A_1, \dots, A_n) \in \Pi_n$, and $Y \geq 0$ or $Y \leq 0$; a precise statement is in Proposition S.2 in Appendix S.1. It formalizes the “winner-take-all” or “loser-lose-all” structure of counter-monotonic allocations. The introduction discussed the special case where in every state all-but-one get ruined. It can also happen that all-but-one achieve a best outcome. We are particularly interested in the special case

$$X_i = X\mathbb{1}_{A_i} \text{ for all } i \in [n], \text{ with } (A_1, \dots, A_n) \in \Pi_n, \quad (2)$$

where either $X \geq 0$ or $X \leq 0$.

Definition 1. An allocation (X_1, \dots, X_n) is a *jackpot allocation* if (2) holds for some $X \geq 0$, and it is a *scapegoat allocation* if (2) holds for some $X \leq 0$.

Thus, the 0 outcome serves as a maximal or minimal outcome, resulting for all but one agent. As we explained above, all counter-monotonic random vectors with at least three non-constant components can be obtained by adding a deterministic vector to a jackpot or scapegoat allocation.

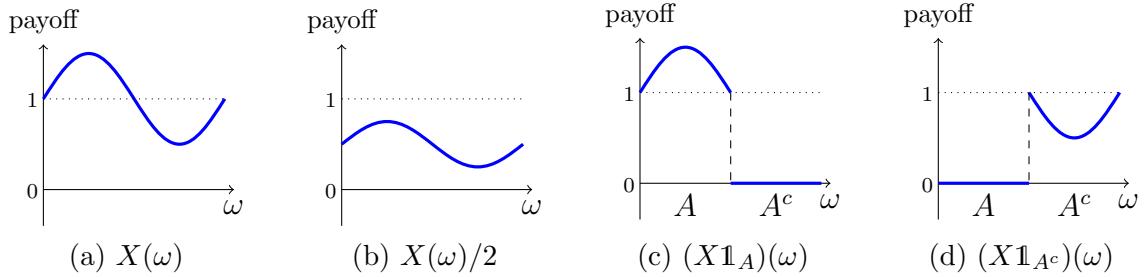
Figure 1 displays a jackpot allocation and a comonotonic allocation. In a jackpot allocation, the random vector $(\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n})$ can be arbitrarily correlated with X . For instance, it may be independent of X or may be fully determined by X (as in Figure 1).

We call $(\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n})$ in (2) a *jackpot vector*, and denote by \mathbb{J}_n the *set of all jackpot vectors* in \mathbb{R}^n , that is,

$$\mathbb{J}_n = \{(\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n}) : (A_1, \dots, A_n) \in \Pi_n\}.$$

The set \mathbb{J}_n is precisely the set of all random vectors with a generalized Bernoulli distribution

Figure 1: An illustration of a comonotonic allocation $(X/2, X/2)$ of X and a jackpot allocation $(X\mathbf{1}_A, X\mathbf{1}_{A^c})$ of X . In this example, A coincides with the event $\{X \geq 1\}$.



(also known as a multinomial distribution with 1 trial). With this, any jackpot allocation or scapegoat allocation has the form $X\mathbf{J}$ for some $\mathbf{J} \in \mathbb{J}_n$. Both types of allocations are often observed in daily life. For instance, the simple lottery ticket (only one winner) is a jackpot allocation, and the ‘‘designated driver’’ of a party is a scapegoat allocation.⁶ Our subsequent study will focus mainly on jackpot allocations.

An equivalent condition for a random vector (X_1, \dots, X_n) to be a jackpot allocation is

$$X_i \geq 0 \text{ and } X_i X_j = 0 \text{ for all } i \neq j. \quad (3)$$

A *probabilistic mixture* of two random vectors with distributions F and G is a random vector distributed as $\lambda F + (1 - \lambda)G$ for some $\lambda \in [0, 1]$. Using (3), we arrive at the following result.

Proposition 1. *A probabilistic mixture of two jackpot allocations is a jackpot allocation.*

Proposition 1 will be used to justify that the utility possibility set of all jackpot allocations is a convex set for EU agents. For two general counter-monotonic allocations other than jackpot and scapegoat allocations, their mixture is not necessarily counter-monotonic.

3.3 Counter-monotonic improvement theorem

A simple assumption of external randomization is important in the subsequent analysis.

Assumption ER. There exists a standard uniform (i.e., uniformly distributed on $[0, 1]$) random variable U on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of X .

⁶The ‘‘designated driver’’ is the randomly selected driver in a party who cannot drink.

Assumption **ER** implies that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *atomless*—that is, there exists a standard uniform random variable defined on this space. Assumption **ER** is easy to satisfy in any practical situation, as an independent uniform random variable can be achieved by a sequence of dice rolls, spins of roulette wheels, or sunspots. The assumption did not yet receive attention in the risk sharing literature because universal risk aversion was usually assumed there, and then no agent is interested in adding further randomness. As soon as there are two or more risk-seeking agents, they will want to involve randomization devices. Randomization has indeed been widely employed when useful, for instance to achieve efficiency in allocation problems, similar to the jackpot allocations that we derive (Budish et al., 2013). Grimm et al. (2021) found that risk-seeking subjects in an experiment indeed preferred to resort to gambling rather than sharing risks. The following simple example illustrates the use of randomization.

Example 1. Suppose that the total wealth is a constant $X = 1$, initially equally endowed among n strictly risk-seeking EU agents. Without external randomization, under individual rationality, no redistribution is possible. The allocation $1/n$ cannot be Pareto-improved. But with external randomization, the jackpot allocation that assigns all $X = 1$ to agent i if, say, side i comes up of a fair n -sided die, with 0 left for the others, is a strict Pareto improvement. All agents are better off because they are strictly risk seeking. In many arguments in this paper, the random variable U of Assumption **ER** is similarly used to generate a jackpot vector.

We are now ready to present the main technical result in this section.

Theorem 1 (Counter-monotonic improvement). *Assume that $X_1, \dots, X_n \in L^1$ are nonnegative, $X = \sum_{i=1}^n X_i$, and Assumption **ER** holds. Then there exists $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that*

(i) (Y_1, \dots, Y_n) is counter-monotonic;

(ii) $Y_i \geq_{\text{cx}} X_i$ for all $i \in [n]$;

(iii) Y_1, \dots, Y_n are nonnegative.

Moreover, (Y_1, \dots, Y_n) can be chosen as a jackpot allocation of X .

Theorem 1 is the counter-monotonic analog of Landsberger and Meilijson's (1994) comonotonicity improvement theorem. Note that it applies to general risk seeking, and is not restricted to particular models such as EU. The importance of the theorem for risk-seeking agents is immediately clear, because Y_1, \dots, Y_n will be preferred by risk-seeking agents over the original payoffs X_1, \dots, X_n . Therefore, one may anticipate for strictly risk-seeking agents, constrained to the set of nonnegative random variables, that any Pareto-optimal allocation or equilibrium allocation, if it exists, must be a jackpot allocation; this is formalized for EU agents in Theorem 2. As another immediate consequence, for any vector of initial endowments, a jackpot allocation obtained in Theorem 1 is individually rational for any risk-seeking agent.

Proof sketch of Theorem 1. Here we provide a proof under a condition stronger than Assumption ER, which allows for an explicit construction of (Y_1, \dots, Y_n) .

Assumption ER*. There exists a standard uniform random variable U on $(\Omega, \mathcal{F}, \mathbb{P})$ independent of (X_1, \dots, X_n) .

Write $Z_i = (\sum_{j=1}^i X_j / X) \mathbf{1}_{\{X>0\}}$ for $i \in [n]$ and $Z_0 = 0$. Let $A_i = \{Z_{i-1} \leq U < Z_i\}$ for $i \in [n]$, which are disjoint events and satisfy $\mathbb{P}(\bigcup_{i=1}^n A_i) = 1$, implying $\sum_{i=1}^n \mathbf{1}_{A_i} = 1$, and let $Y_i = X \mathbf{1}_{A_i}$ for $i \in [n]$. Clearly, (Y_1, \dots, Y_n) is counter-monotonic and it is a jackpot allocation of X . For $i \in [n]$, we have

$$\begin{aligned}\mathbb{E}[Y_i | X_1, \dots, X_n] &= \mathbb{E}[X \mathbf{1}_{\{Z_{i-1} \leq U < Z_i\}} | X_1, \dots, X_n] \\ &= \mathbb{E}[X(Z_i - Z_{i-1}) | X_1, \dots, X_n] = X \frac{X_i}{X} \mathbf{1}_{\{X>0\}} = X_i.\end{aligned}$$

Hence, Jensen's inequality yields $X_i \leq_{\text{cx}} Y_i$. This proves the statement of Theorem 1 under Assumption ER*. To show the result under Assumption ER, we will use a few technical lemmas in Appendix A. The weaker Assumption ER is more desirable in the study of risk sharing, as it requires randomization only for the aggregate payoff X , rather than for each allocation. The difference between Assumptions ER and ER* is subtle, which affects the applicability of some results. This issue is discussed in detail in Appendix S.2.

Example 2. We give an example illustrating how the conclusions in Theorem 1 fail without the external randomization in Assumption ER. Consider $n = 3$ and a finite space $\Omega = \{\omega_1, \dots, \omega_4\}$ of 4 states with equal probability. Let $X_i = 3\mathbf{1}_{\{\omega_i\}} + \mathbf{1}_{\{\omega_4\}}$ for $i \in [3]$ and $X = X_1 + X_2 + X_3$ (thus $X = 3$). Suppose that there exists $(Y_1, Y_2, Y_3) \in \mathbb{A}_3(X)$ satisfying

(i)–(iii) in Theorem 1. Then it has the form $Y_i = a\mathbf{1}_{A_i} + m_i$ for some $a, m_1, m_2, m_3 \in \mathbb{R}$, $(A_1, A_2, A_3) \in \Pi_3$ (see Proposition S.2). To fulfill the conditions $Y_i \geq_{\text{cx}} X_i$ and $0 \leq Y_i \leq 3$, each Y_i must only take the values 0 and 3. Hence, the mean of Y_i is a multiple of $3/4$, violating $\mathbb{E}[Y_i] = \mathbb{E}[X_i] = 1$. Therefore, such (Y_1, Y_2, Y_3) does not exist. Under Assumption ER (here Ω cannot be finite), one can take $\mathbb{P}(A_i) = 1/3$ to satisfy all desired conditions.

The assumption that X_1, \dots, X_n are nonnegative implies that there is a minimum outcome 0 at which risk seekers are prevented from further zero-sum gambling, and this is necessary to obtain jackpot allocations. A similar statement can be made for scapegoat allocations. Then an assumption $X_i \leq 0$ for all i implies that there is a maximum outcome 0 at which risk seekers are prevented from further zero-sum gambling, and this is necessary to obtain scapegoat allocations. Because the convex order is invariant under constant shifts, Theorem 1 and the above statement on negative payoffs immediately imply the following result, which is presented in the same form as its comonotonic counterpart.

Proposition 2. *Suppose that $X_1, \dots, X_n \in L^1$ are all bounded from above or all bounded from below, $X = \sum_{i=1}^n X_i$, and that Assumption ER holds. Then there exists a counter-monotonic allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that $Y_i \geq_{\text{cx}} X_i$ for all i .*

Whereas results in this section are formulated on L^1 for generality, the next few sections will focus on bounded random variables in the analysis of risk sharing problems.

4 Pareto-optimal allocations for EU agents

This section analyzes Pareto optimality under EU without the restriction of universal risk aversion.

4.1 Setting

To include risk-seeking agents, it is warranted to specify bounds for the set of feasible allocations. Otherwise, these agents can keep on improving through continued mutual zero-sum gambles. It is most natural to set a lower bound, which we will do. Our results can readily be reformulated for upper bounds. For convenience, we will denote the lower bound by 0. It can be interpreted as ruin, or as a no-short selling/borrowing constraint. Our main setting is described by the following assumption, followed by special cases.

Assumption EU. Each agent maximizes EU with a strictly increasing continuous utility function $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $u_i(0) = 0$. The domain of allocations is $\mathcal{X} = L_+^\infty$, where L_+^∞ is the set of all nonnegative bounded random variables. The total payoff $X \in \mathcal{X}$ satisfies $\mathbb{P}(X > 0) > 0$.

Here, $\mathbb{P}(X > 0) > 0$ avoids triviality and $u_i(0) = 0$ is a normalization. Outcome 0, the neutral outcome in risk sharing, is the worst outcome possible. To avoid misunderstanding, we do not assume that it would be perceived as a reference outcome, and that the agents would perceive all positive outcomes as gains. In formal expected utility there is no special role for a reference outcome. In applications, it is well possible that agents have high ambitions and a high reference outcome, so that they perceive all outcomes here as losses. The main interest of the results for risk-seeking agents lies in fact in the loss domain, as for instance in cost sharing problems. There convex utility is more empirically realistic than the concave utility traditionally assumed as yet in the risk sharing literature.

For EU agents, the case of universal risk aversion is well understood in the classic risk sharing literature (e.g., [Mas-Colell et al., 1995](#)). We will focus on the following few settings, besides the classic case of universal risk aversion.

Assumption EURS. On top of Assumption EU, all agents are strictly risk seeking.

Assumption EUM. On top of Assumption EU, agents in a subgroup $S \subseteq [n]$ are strictly risk seeking and the others in $T = [n] \setminus S$ are strictly risk averse, with S, T nonempty.

We also consider the following subcase of Assumption EURS.

Assumption H-EURS. On top of Assumption EURS, agents are homogeneous; that is, $u_1 = \dots = u_n = u$ holds.

In informal discussions we often take risk seeking and risk aversion strict, as in the above assumptions. For the total wealth X , the *utility possibility set (UPS)* is the set of utility vectors $(\mathbb{E}[u_1(X_1)], \dots, \mathbb{E}[u_n(X_n)])$ for $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, denoted by $\text{UPS}(X)$. The *utility possibility frontier (UPF)* is the subset of utility vectors achieved by Pareto-optimal allocations, formally denoted by $\text{UPF}(X)$. We denote by $\text{UPJ}(X)$ the subset of jackpot utility vectors.

We do not involve the initial endowments in this section because they are irrelevant for Pareto optimality. They are needed though for individual rationality. In our setting, individually rational Pareto-optimal allocations always exist (Lemma 4 in Appendix A).

4.2 Risk-seeking EU agents

Under universal risk seeking we obtain an explicit characterization of Pareto optimality. The function

$$V_{\lambda} : x \mapsto \max_{i \in [n]} \lambda_i u_i(x) \quad (4)$$

for a given Negishi weight vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ will be a useful tool. Under Assumption EURS, V_{λ} turns out to be the utility function of the representative agent of the n risk-seeking agents, and it is convex.

Theorem 2 (Pareto optimality for risk seekers). *Suppose that Assumptions ER and EURS hold. For an allocation $\mathbf{X} = (X_1, \dots, X_n) \in \mathbb{A}_n(X)$, the following statements are equivalent:*

- (i) \mathbf{X} is Pareto optimal;
- (ii) \mathbf{X} is λ -optimal for some $\lambda \in \Delta_n$;
- (iii) \mathbf{X} satisfies $\sum_{i=1}^n \lambda_i \mathbb{E}[u_i(X_i)] = \mathbb{E}[V_{\lambda}(X)]$ for some $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta_n$,
- (iv) \mathbf{X} is a jackpot allocation satisfying the following restriction: for some $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta_n$, $\lambda_i u_i(X) \mathbb{1}_{A_i} = V_{\lambda}(X) \mathbb{1}_{A_i}$ for each $i \in [n]$, where we write $\mathbf{X} = X(\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n})$.

Proof sketch. The equivalence (i) \Leftrightarrow (ii), a special case of Proposition 5 below, is based on the Hahn-Banach Theorem and the convexity of UPS, which relies on Assumption ER. The equivalence (ii) \Leftrightarrow (iii) can be proved by verifying that $\mathbb{E}[V_{\lambda}(X)]$ is the maximum of $\sum_{i=1}^n \lambda_i \mathbb{E}[u_i(X_i)]$ over $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, and (iv) \Rightarrow (iii) is direct. The final (i) \Rightarrow (iv) is proved by arguing that the above maximum can only be attained by the jackpot allocations in (iv) with Theorem 1 and some techniques from probability theory.

Theorem 2 immediately yields the relation $\text{UPF}(X) \subseteq \text{UPJ}(X) \subseteq \text{UPS}(X)$. In particular, Pareto-optimal allocations are jackpot allocations. The converse does not hold in general if agents are not homogeneous. For example, consider two agents, one with a more convex utility but receive a smaller payoff on a larger event; see Example S.1 in Section S.3.1.

Grimm et al. (2021) provided empirical support for our analysis. They found, in the gain domain with the classical risk aversion prevailing, that subjects preferred to share risks, as in classical theorems on risk sharing. However, in the loss domain subjects preferred to resort to gambling, in agreement with our claims that risk seeking is prevailing there and leads to phenomena as in Theorem 2.

Although generally $\text{UPF}(X) \neq \text{UPJ}(X)$, there are two special cases in which equality holds and thus all jackpot allocations are Pareto optimal. The first special case of $\text{UPF}(X) = \text{UPJ}(X)$ occurs when the total payoff is a constant.

Proposition 3. *If $X = x > 0$ is a constant and Assumptions **ER** and **EURS** hold, then all jackpot allocations of X are Pareto optimal.*

We next turn to Assumption **H-EURS**, where all utility functions are the same, implying that the condition on (A_1, \dots, A_n) in part (iv) holds true for $\boldsymbol{\lambda} = \mathbf{1}/n$ because $V_{\boldsymbol{\lambda}}(x) = u(x)/n$ for $x \in \mathbb{R}_+$. This is formally stated in the result below, which further characterizes the UPF as a simplex.

Proposition 4. *Under Assumptions **ER** and **H-EURS**, $\text{UPF}(X) = \text{UPJ}(X) = \Delta_n(\mathbb{E}[u(X)])$.*

Proposition 4 involves a structure that is uncommon for EU agents: the equivalence between Pareto optimality and maximizing the social planner's problem where all the Negishi weights are normalized to 1. This equivalence also holds for agents using quantiles or monetary risk measures as their preference functionals (Embrechts et al., 2018, Proposition 1). Proposition 4 will be useful in Section 5 to unify all four types of allocations: Pareto-optimal, equilibrium, sum-optimal, and jackpot.

4.3 General EU agents

Now, we turn to general EU agents.

Proposition 5. *Under Assumptions **ER** and **EU**,*

- (i) *both $\text{UPJ}(X)$ and $\text{UPS}(X)$ are convex;*
- (ii) *an allocation of X is Pareto optimal if and only if it is $\boldsymbol{\lambda}$ -optimal for some $\boldsymbol{\lambda} \in \Delta_n$.*

For a given $\boldsymbol{\lambda} \in \Delta_n$ or $\boldsymbol{\lambda} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, computing the corresponding $\boldsymbol{\lambda}$ -optimal allocations is a standard numerical task, explained in Section S.3.2.

We next consider the remaining case under EU, where there are both risk-seeking and risk-averse agents. For $\mathbf{X} = (X_1, \dots, X_n)$ and $S \subseteq [n]$, write $\mathbf{X}_S = (X_i)_{i \in S}$.

Theorem 3 (Subgroups). *Suppose that Assumptions **ER** and **EU** hold, and that $\mathbf{X} \in \mathbb{A}_n(X)$ is Pareto optimal. For any set $S \subseteq [n]$ of strictly risk-seeking agents, \mathbf{X}_S is a jackpot allocation. For any set $T \subseteq [n]$ of strictly risk-averse agents, \mathbf{X}_T is a comonotonic allocation.*

Theorem 3 directly follows from the two improvement theorems and some standard arguments. It greatly broadens the scope of Theorem 2 and formalizes the classical economic view that risk-seeking agents in any society end up in extreme boundary conditions. [Jullien and Salanié \(2000\)](#) suggested that British racetrack bettors are part of the risk-seeking subgroup in society. [Le Van and Pham \(2025\)](#) derived equilibria when exactly one agent is not risk averse, consistent with Theorem 3.

Theorem 3 considered exchanges within the risk-averse group and withing the risk-seeking group. We next consider exchanges between these two groups. For a given $\lambda \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, a λ -optimal allocation (X_1, \dots, X_n) can be obtained in two steps. First, for each pair (X_S, X_T) that sums to X , we find the optimal allocation of X_S to agents in S , which is a jackpot allocation and the optimal allocation of X_T to agents in T , which is a comonotonic allocation (Theorem 3). By doing this, we obtain the total weighted utility of each group applied to X_S and X_T . Second, we determine an optimal (X_S, X_T) via a one-dimensional optimization by maximizing the total weighted utility point-wise; see (S.5). The optimal allocation (X_S, X_T) may be a jackpot allocation itself if agents in S have larger Negishi weights (“risk seeking prevails”, see Example 3.a), but it may also be a proportional allocation if agents in A have larger Negishi weights (“risk aversion prevails”, see Example 3.b), or it may be somewhere in between. The optimal (X_S, X_T) in the second step then yields an optimal allocation (X_1, \dots, X_n) in the first step, which is a λ -optimal allocation. The above steps are numerically efficient and sometimes they admit explicit formulas. Details of these steps, as well as computations for the next example, are in Section S.3.2.

Example 3. Let t be the cardinality of T . For $i \in S$, let u_i be the convex function $u_i(x) = 3x + x^2$ and for $i \in T$, let u_i be a common strictly increasing and strictly concave function satisfying $u_i(x) = 5x - tx^2$ on $[0, 2/t]$. Let the aggregate payoff X be distributed on $[0, 2]$. We consider two cases of $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n$ with $\lambda_i = \lambda_S > 0$ for $i \in S$ and $\lambda_i = \lambda_T > 0$ for $i \in T$; here we do not normalize λ .

(a) Let $\lambda_S = 5/4$ and $\lambda_T = 1$. A λ -optimal allocation (X_1, \dots, X_n) is given by

$$X_i = X J_i \mathbf{1}_{\{X > c\}}, \quad i \in S \quad \text{and} \quad X_i = \frac{X}{t} \mathbf{1}_{\{X \leq c\}}, \quad i \in T, \quad (5)$$

where $c = 5/9$ and $(J_i)_{i \in S}$ is any jackpot vector.

(b) Let $\lambda_S = 1$ and $\lambda_T = 2$. A λ -optimal allocation may not be a jackpot allocation. For

instance, if we take $X = 2$, then $X_S = 1/2$ and $X_T = 3/2$ necessarily hold, and hence any λ -optimal allocation cannot be a jackpot.

In case (a), agents in T collectively gamble with agents in S , whereas they do not in case (b).

5 Competitive equilibria and welfare theorems

We now analyze the competitive equilibria of the risk sharing economy. For risk-seeking EU agents, we fully characterize competitive equilibria and obtain welfare theorems.

5.1 General results

For EU agents, the individual optimization problem for agent i is

$$\text{maximize } \mathbb{E}[u_i(X_i)] \quad \text{over } 0 \leq X_i \leq X \quad \text{subject to } \mathbb{E}^Q[X_i] \leq \mathbb{E}^Q[\xi_i], \quad (6)$$

where $(\xi_1, \dots, \xi_n) \in \mathbb{A}_n(X)$ is the vector of initial endowments and Q is a probability measure. An equilibrium allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ solves (6) for some (ξ_1, \dots, ξ_n) and Q , which can be left unspecified. The constraint $X_i \geq 0$ in (6) reflects our setting of nonnegative random variables in L_+^∞ . The other constraint $X_i \leq X$ in (6) means that the agent's payoff $X_i(\omega)$ at each state ω cannot exceed the total supply $X(\omega)$ in the economy. This constraint is automatically satisfied in classical settings with strictly concave utility functions, demand correspondences well-behaved, and supply equaling demand in equilibrium. Hence, it then usually is not written explicitly. Accordingly, our definition does not deviate from classical definitions in this regard. Moreover, for risk-seeking agents the optimization in (6) is the maximization of a convex objective with linear constraints, and thus it often does not admit a solution unless we impose an upper bound on the decision variable X_i .

With the above formulation of individual optimization, we obtain the following welfare theorem along with an equilibrium price Q in an explicit form.

Theorem 4 (Welfare). *The following statements hold.*

- (i) *Under Assumption EU, every equilibrium allocation of X is Pareto optimal.*

(ii) Under Assumptions **ER** and **EURS**, every Pareto-optimal allocation of X is an equilibrium allocation, with an equilibrium price Q given by

$$\frac{dQ}{d\mathbb{P}} = \frac{V_{\lambda}(X)}{X} \frac{1}{\mathbb{E}[V_{\lambda}(X)/X]}, \quad (7)$$

for some $\lambda \in \Delta_n$, where V_{λ} is defined in (4).

Part (i) of Theorem 4 follows from standard techniques (see [Mas-Colell et al. 1995](#), Proposition 16.C.1 for a finite space), and our main novelty lies in part (ii), on which we make a few remarks. In Theorem 2, a Pareto-optimal allocation (which is necessarily a jackpot allocation) is λ -optimal for some $\lambda \in \Delta_n$; this vector λ is the same one in (7), shown in the proof of Theorem 4. The vector of initial endowments associated with the equilibrium allocation (X_1, \dots, X_n) is not unique, and it can be (X_1, \dots, X_n) itself. Another possibility concerns the proportional endowments

$$\xi_i = \frac{\mathbb{E}^Q[X_i]}{\mathbb{E}^Q[X]} X, \quad i \in [n].$$

The equilibrium price Q is generally not unique, even for a given vector of initial endowments. Uniqueness will be studied in Section 5.2 below. The component $\mathbb{E}[V_{\lambda}(X)/X]^{-1}$ in (7) is a normalization to guarantee that Q is a probability measure. The key property of Q is that, since u_1, \dots, u_n are convex, so is V_{λ} . Hence, $V_{\lambda}(x)/x$ is increasing in x . This implies that $V_{\lambda}(X)/X$ and X are comonotonic. That is, the equilibrium price density $dQ/d\mathbb{P}$ increases as the aggregate endowment becomes more abundant. This comonotonicity property is in stark contrast to the classical setting with strictly risk-averse EU agents, where the equilibrium price is decreasing in the aggregate endowment X . Yet, this is not surprising, as the price structure reflects the marginal utility of agents. For strictly risk-averse EU agents, consumption is cheaper in high-endowment states, whereas for strictly risk-seeking EU agents it is reversed.

Theorem 4 implies that for risk-seeking agents, Pareto optimal allocations and equilibrium allocations are equivalent. This is not the case for general EU agents. The following result gives a necessary condition for equilibrium allocations.

Theorem 5. Suppose that Assumptions **ER** and **EU** hold, and $\mathbf{X} \in \mathbb{A}_n(X)$ is an equilibrium allocation. For any set $S \subseteq [n]$ of strictly risk-seeking agents, (\mathbf{X}_S, Y) is a jackpot allocation, where $Y = X - \sum_{i \in S} X_i$.

The equilibrium allocations in Theorem 5 may exhibit, besides gambling within risk-seeking agents as in Theorem 3, also gambling across risk-seeking and other agents. By Theorem 5, the Pareto-optimal allocation in Example 3.b (“risk aversion prevails”) cannot be an equilibrium allocation. The Pareto-optimal allocations in Example 3.a (“risk seeking prevails”) can be equilibrium allocations though, as shown next.

We now consider the general Assumption EUM, and for simplicity assume that X only takes two values a and b , with $0 < a < b$. Based on Theorem 5 and Example 3.a, a candidate allocation $\mathbf{X} = (X_1, \dots, X_n)$ is given by

$$X_i = a_i \mathbb{1}_{\{X=a\}}, \quad i \in T \quad \text{and} \quad X_i = b J_i \mathbb{1}_{\{X=b\}}, \quad i \in S, \quad (8)$$

where $a_i \in \mathbb{R}_+$ satisfies $\sum_{i \in T} a_i = a$ and $(J_i)_{i \in S}$ is a jackpot vector. A possible equilibrium price Q is given by

$$\frac{dQ}{dP} = \alpha \mathbb{1}_{\{X=a\}} + \beta \mathbb{1}_{\{X=b\}} \quad \text{for some } \alpha, \beta > 0. \quad (9)$$

The next result shows that (8)–(9) yield the only possible form of competitive equilibria in this setting.

Theorem 6 (An equilibrium for mixed case). *Suppose that Assumptions ER and EUM hold and that X only takes two values a, b , with $0 < a < b$. If (\mathbf{X}, Q) has the form (8)–(9) and satisfies⁷*

$$\min_{i \in T} \frac{u'_i(a_i)}{u'_i(0)} \geq \frac{\alpha}{\beta} \geq \max_{j \in S} \frac{b u_j(a)}{a u_j(b)}, \quad (10)$$

then it is a competitive equilibrium. If (\mathbf{X}, Q) is a competitive equilibrium and \mathbf{X}_S and \mathbf{X}_T are nontrivial, then (8), (9), and (10) hold.

The equilibrium price in (9)–(10) is typically not unique. To interpret the equilibrium in Theorem 6, the risk-seeking agents prefer gambling on $\{X = b\}$ over gambling on $\{X = a\}$ because their utility function is convex and $b > a$, whereas the risk-averse agents do not take payoffs on the event $\{X = b\}$ because they are more expensive. These two considerations together require the price ratio β/α to be large enough, reflected by the first inequality in (10), to drive away the risk-averse agents from $\{X = b\}$, but not too much, reflected by the second inequality in (10), to attract the risk-seeking agents. Among their own groups,

⁷If u_i is not differentiable, $u'_i(a_i)$ designates the left derivative and $u'_i(0)$ the right derivative.

risk-seeking agents gamble and risk-averse agents proportionally share, fitting with intuition.

Example 4. In the setting of Example 3.a, assume that X takes the values $a = 1/2$ and $b = 3/2$, each with probability $1/2$. By Theorem 6, (X_1, \dots, X_n) in (5) is an equilibrium allocation with equilibrium price Q given by $dQ/d\mathbb{P} = (1 - \varepsilon)\mathbf{1}_{\{X=1/2\}} + (1 + \varepsilon)\mathbf{1}_{\{X=3/2\}}$ for any $\varepsilon \in [1/9, 1/8]$. Here, $dQ/d\mathbb{P}$ is close to 1 and slightly larger on $\{X = b\}$.

In the setting of Theorem 6, because $a/n \leq a_i$ and $u'_i(a/n) \geq u'_i(a_i)$ for some $i \in T$, the condition

$$\max_{i \in T} \frac{u'_i(a/n)}{u'_i(0)} \geq \max_{j \in S} \frac{bu_j(a)}{au_j(b)} \quad (11)$$

is necessary for (10), and thus also for an equilibrium allocation \mathbf{X} with nontrivial $\mathbf{X}_S, \mathbf{X}_T$ to exist. In the limiting case $b \downarrow a$, (11) fails because the right-hand side converges to 1 and the left-hand side is less than 1. Hence, with a constant aggregate endowment X , there is no competitive equilibrium (\mathbf{X}, Q) with nontrivial $\mathbf{X}_S, \mathbf{X}_T$.

A full understanding of competitive equilibria for the general mixed case is not yet available. The next subsection focuses on homogeneous risk-seeking EU agents, for whom we can explicitly construct the equilibrium from any initial endowment vector.

5.2 Homogeneous risk-seeking EU agents

For risk-seeking EU agents, Pareto-optimal allocation, λ -optimal allocations, and equilibrium allocations are equivalent, and all of those allocations are jackpot allocations. The reverse implication holds in the homogeneous case, as summarized next.

Proposition 6. *Suppose that Assumptions ER and H-EURS hold. For an allocation $\mathbf{X} = (X_1, \dots, X_n) \in \mathbb{A}_n(X)$, the following statements are equivalent: (i) \mathbf{X} is a jackpot allocation; (ii) \mathbf{X} is Pareto optimal; (iii) \mathbf{X} is sum-optimal; (iv) \mathbf{X} is an equilibrium allocation; (v) $\sum_{i=1}^n \mathbb{E}[u(X_i)] = \mathbb{E}[u(X)]$.*

Proposition 6 follows by combining Proposition 4 and Theorems 2 and 4. Statement (v) of Proposition 6 implies that there is a vector $(\theta_1, \dots, \theta_n) \in \Delta_n$ such that for every i we have $\mathbb{E}[u(X_i)] = \theta_i \mathbb{E}[u(X)]$. The value θ_i can represent both the probability of winning the lottery for agent i and the relative purchase power of agent i at equilibrium, formalized in Theorem 7 below.

Next, we explicitly solve the competitive equilibrium for any initial endowment vector $(\xi_1, \dots, \xi_n) \in \mathbb{A}_n(X)$, and show that when (ξ_1, \dots, ξ_n) is nontrivial, the equilibrium price Q is uniquely given by

$$\frac{dQ}{d\mathbb{P}} = \frac{u(X)}{X} \frac{1}{\mathbb{E}[u(X)/X]}, \quad (12)$$

which is precisely (7) in Theorem 4 under Assumption **H-EURS**.⁸ In case (ξ_1, \dots, ξ_n) is trivial (e.g., $\xi_1 = X, \xi_2 = \dots = \xi_n = 0$), the equilibrium allocation is just (ξ_1, \dots, ξ_n) and the equilibrium price is arbitrary.

Theorem 7 (Uniqueness). *Suppose that Assumptions **ER** and **H-EURS** hold, and fix an initial endowment vector $\xi \in \mathbb{A}_n(X)$. Let Q be given by (12) and $\theta = \mathbb{E}^Q[\xi]/\mathbb{E}^Q[X]$.*

- (i) *The tuple $(X\mathbf{J}, Q)$ is a competitive equilibrium for any $\mathbf{J} \in \mathbb{J}_n$ independent of X with $\mathbb{E}[\mathbf{J}] = \theta$.*
- (ii) *The equilibrium price Q is uniquely given by (12) if ξ is nontrivial.*
- (iii) *The utility vector of any equilibrium allocation is uniquely given by $\mathbb{E}[u(X)]\theta$.*

Theorem 7 implies, in particular, that competitive equilibria always exist for any initial endowment vector. Most remarkably, although the equilibrium allocation and its utility vector depend on the initial endowments, the equilibrium price does not. This is different from the case of heterogeneous agents, as (7) depends on the initial endowments implicitly through the parameter $\lambda \in \Delta_n$. For a given (ξ_1, \dots, ξ_n) and a competitive equilibrium $(X\mathbf{J}, Q)$, the jackpot vector \mathbf{J} does not have to be independent of X as in part (i) of Theorem 7. Any choice of $\mathbf{J} = (J_1, \dots, J_n)$ satisfying the budget constraint $\mathbb{E}^Q[XJ_i] = \mathbb{E}^Q[\xi_i]$ for all i is indeed suitable for $(X\mathbf{J}, Q)$ to be a competitive equilibrium.

Two further results on the existence of a competitive equilibrium under Assumption **EURS** for any initial endowment are in Appendix S.4. In particular, a competitive equilibrium exists for (a) any initial endowment and $n = 2$, and (b) for initial endowments that are proportional to X . These results also illustrate that the equilibrium price is not necessarily unique.

⁸Uniqueness should be interpreted in the \mathbb{P} -almost sure sense, and without loss of generality we here assume that Q is absolutely continuous with respect to \mathbb{P} .

6 Rank-dependent utility agents

The previous two sections assumed EU, and special attention was paid to agents who are entirely (on the whole domain) risk seeking or entirely risk averse. Whereas the assumption of risk seeking is a conceptually desirable addition to classical analyses, and in the loss domain is more empirically relevant than the classical assumption of universal risk aversion, yet further empirical improvements are desirable. Empirical studies have shown that most people are neither entirely risk averse nor entirely risk seeking, but exhibit both attitudes in various subdomains (Barberis and Huang, 2008; Heimer et al., 2025). Further, that they do so in ways that deviate from classical EU. This section provides first results reckoning with these findings, thus first behaviorally realistic results for risk sharing. Even though deviations from EU can be expected to be smaller among financial traders than among average human beings, it has now been accepted that they are also prominent in finance, and many behavioral models have been introduced there (Barberis et al., 2021).

We assume Quiggin's (1982) RDU where we focus on gains. There RDU agrees with Tversky and Kahneman's (1992) prospect theory, and these are the most popular behavioral models for risk (Barberis and Huang, 2008; Fehr-Duda and Epper, 2012, Section 6; Heimer et al., 2025). Prospect theory does require a specification of a reference outcome, which we denote by 0. Thus, contrary to preceding sections, our restriction to nonnegative outcomes now entails a substantive restriction, being that we only consider gains, with the important empirical implication that utility now is prevailingly concave. Our restriction is made, for one reason to simplify this first analysis and, further, to illustrate effects beyond convex utility.

We make some further, again admittedly restrictive, assumptions, but we think that the new direction taken here is important and will encourage future works. The two further assumptions are, first, that we have homogeneous agents and, second, that their utility function is linear on a bounded interval $[0, x_0]$, and then concave on the whole \mathbb{R}_+ . Both conditions on the utility function are empirically plausible. First, concave utility is empirically prevailing for gains (Wakker, 2010, p. 264). Second, as many authors have argued, linearity of utility is a good approximation for moderate amounts (l'Haridon and Vieider, 2019, p. 189; Marshall, 1890, Book III). This linearity assumption is yet more plausible under RDU than under EU. Much of the risk aversion captured by utility curvature under EU is better captured by prob-

ability weighting (Barseghyan et al., 2013, p. 2512) and, hence, utility is more linear than traditionally thought. Our results show how particular degrees of linearity (specified by x_0) imply particular results.

We now define the RDU model. As in Sections 4 and 5, we consider nonnegative bounded random variables: $\mathcal{X} = L_+^\infty$, although now outcomes are interpreted as gains. A function $w : [0, 1] \rightarrow [0, 1]$ is called a *(probability) weighting function* if it is increasing and satisfies $w(0) = 0$ and $w(1) = 1$. For a weighting function w and an increasing utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, the RDU preference functional is given by

$$\mathcal{U}(Y) = \int_{\Omega} u(Y) d(w \circ \mathbb{P}) = \int_0^\infty w(\mathbb{P}(u(Y) > x)) dx, \quad Y \in \mathcal{X}. \quad (13)$$

Here, the first integral is a Choquet integral. An *RDU agent* has a preference functional given by (13) for some weighting function w and utility function u . When u is linear, the RDU agent's preferences are represented by the dual utility functional of Yaari (1987), denoted by

$$\rho_w(Y) = \int Y d(w \circ \mathbb{P}), \quad Y \in \mathcal{X}.$$

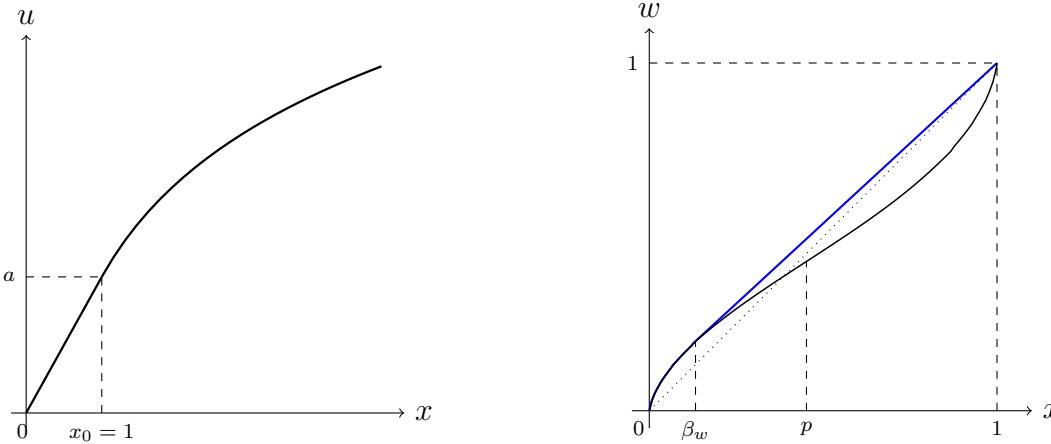
The RDU functional in (13) is thus given by $\mathcal{U}(Y) = \rho_w(u(Y))$. Under continuity, an RDU agent is risk seeking if and only if w is concave and u is convex, and is risk averse if and only if w is convex and u is concave (Schmidt and Zank, 2008).

The empirically prevailing weighting functions are neither convex nor concave, but a mix of those, and they are inverse S-shaped (Fehr-Duda and Epper, 2012). Although different formal definitions have been given, we will use the prevailing one (cavexity) because of its analytical convenience for our purposes. We call w *cavex* if it is continuous and there exists $p \in [0, 1]$ such that w is concave on $[0, p]$ and convex on $[p, 1]$.⁹ Now there is neither entire risk aversion nor entire risk seeking, but there are factors going either way. Risk attitudes result from interactions between these factors, with different implications in different subdomains, and more complex and refined analyses are needed. We will specify some contexts where we can already draw conclusions.

Tversky and Kahneman (1992) introduced the following family of cavex weighting func-

⁹If $p = 0$ then w is convex, and if $p = 1$ then w is concave. For empirical purposes, it is desirable to require that $w(p)$ is not far from p .

Figure 2: An example of a utility function (left panel) and a weighting function (right panel) satisfying Assumption **H-RDU** for $n \geq 8$ and condition (15).



(a) $u = ax$ on $[0, x_0]$ and $u(x) = a \log x + a$ on $[x_0, \infty)$ for some $a > 0$, with $x_0 = 1$

(b) $w = w_{\text{TK}}$ (black) with $\gamma = 0.71$ and \bar{w} (blue); $w = \bar{w}$ on $[0, \beta_w]$ with $\beta_w = 0.133$

tions

$$w_{\text{TK}}(t) = \frac{t^\gamma}{(t^\gamma + (1-t)^\gamma)^{1/\gamma}}, \quad t \in [0, 1], \quad (14)$$

for some parameter γ estimated to be 0.71 on average (e.g., Wu et al., 2004).

For a weighting function w , the *concave envelope* of w , denoted $\bar{w} : [0, 1] \rightarrow [0, 1]$, is defined by

$$\bar{w}(t) = \inf\{g(t) : g \geq w \text{ on } [0, 1] \text{ and } g \text{ is concave}\}.$$

It is a concave weighting function and will be a useful tool in our analysis. Because $\bar{w} \geq w$, we have $\rho_{\bar{w}}(Y) \geq \rho_w(Y)$ for all $Y \in \mathcal{X}$. For a cavex weighting function w , its concave envelope \bar{w} coincides with w on an interval $[0, \beta_w] \subseteq [0, 1]$ (possibly empty) and it is linear on $[\beta_w, 1]$; see panel (b) of Figure 2. The linear part captures the lowest line emanating from $(1, 1)$ that dominates w , touching but not crossing w 's graph. As we will see in Theorem 8, enough risk seeking is implied this way to generate global risk-seeking-type results. We next state our main assumptions on the homogeneous RDU agents.

Assumption H-RDU. Each agent $i \in [n]$ is an RDU maximizer with a concave and increasing utility function u on \mathbb{R}_+ that is linear on $[0, x_0]$ with $u(0) = 0$ and $u(x_0) > 0$, and a cavex weighting function w with $w = \bar{w}$ for $t \in [0, 1/n]$. The domain of allocations is $\mathcal{X} = L_+^\infty$. The total payoff $X \in \mathcal{X}$ satisfies $\mathbb{P}(X > 0) > 0$.

Figure 2 depicts one pair (u, w) that satisfies Assumption H-RDU. The condition that $w = \bar{w}$ on $[0, 1/n]$ is equivalent to $n \geq 1/\beta_w$, which is easily satisfied for large n . Using $\gamma = 0.71$ in (14), we have $w = \bar{w}$ for $t \in [0, 0.133]$ as depicted in panel (b) of Figure 2. Here, $n \geq 8$ is sufficient for this condition. It also implies that w is concave on $(0, 1/n)$, and we will later use strict concavity of w on $(0, 1/n)$ for some statements on strict domination. The conditions on the set \mathcal{X} and the total wealth X are the same as in Assumption EURS.

The next result shows that previous results about Pareto optimality for (entirely) risk-seeking EU agents can be extended to RDU agents when they are sufficiently risk seeking for small probabilities. We will compare a jackpot allocation $X\mathbf{J}$ with the proportional allocation $\bar{\mathbf{X}} := (X/n, \dots, X/n)$. The following condition, explained after the theorem, is used in part (iii):

$$\limsup_{t \downarrow 0} \frac{w(t/n)}{w(t)} < 1, \quad w(1/n) < 1, \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{u(x/n)}{u(x)} = 1. \quad (15)$$

In (15), the conditions on w hold for w_{TK} in (14), and the condition on u holds for concave utility functions u that are exponential, or logarithmic on $[z_0, \infty)$ for some $z_0 > 0$ (see panel (a) of Figure 2), as well as for any bounded utility functions.¹⁰

Theorem 8 (RDU). *Suppose that Assumptions ER and H-RDU hold. Let the jackpot vector $\mathbf{J} \in \mathbb{J}_n$ be independent of X satisfying $\mathbb{E}[\mathbf{J}] = \mathbf{1}/n$.*

- (i) *If $X \leq x_0$, then the jackpot allocation $X\mathbf{J}$ is Pareto optimal and sum-optimal.*
- (ii) *If $X \leq x_0$, $n \geq 2$, and w is strictly concave on $(0, 1/n)$, then $X\mathbf{J}$ strictly dominates $\bar{\mathbf{X}}$.*
- (iii) *If (15) holds, then there exists $y_0 > 0$ such that for $X \geq y_0$, $\bar{\mathbf{X}}$ strictly dominates $X\mathbf{J}$.*
- (iv) *If X is a positive constant, u is strictly increasing and differentiable on \mathbb{R}_+ , and w is strictly concave on $(0, 1/n)$, then $\bar{\mathbf{X}} - \varepsilon\mathbf{1} + n\varepsilon\mathbf{J}$ strictly dominates $\bar{\mathbf{X}}$ for $\varepsilon > 0$ small enough.*

Part (i) of Theorem 8 specifies conditions under which the risk-seeking components of RDU prevail for optimal risk sharing, so that essentially the same conclusions hold as in Theorems 1 and 2, with Pareto optimality of some jackpot allocations, maximizing risks. Three points

¹⁰Some extensions of Theorem 8 are presented in Appendix S.5. In particular, the conclusions hold for utility functions u such that $x \mapsto u(x)/x$ increases on $[0, x_0]$, which is weaker than assuming either linearity or convexity on $[0, x_0]$.

lead to this implication. First, whereas concave utility enhances risk aversion, we are now in a domain where utility is linear, not contributing any risk aversion. Second, there are enough agents to divide the risk over to ensure that the probability for each agent can be pushed below β_w , where w generates risk seeking. Third, the conditions on \bar{w} then ensure that risk seeking also prevails globally. Part (ii) reinforces part (i).

Part (iii) specifies conditions under which, for large enough outcomes, the risk-averse implications of concave utility prevail over the risk-seeking implications of w near $p = 0$ so much that the jackpot allocation of part (i) can no more be Pareto optimal, being dominated by the safer proportional allocation. The latter allocation can even be optimal if the utility function has a satiation point (Example 5 below). Part (iv) specifies opposite conditions. For small enough outcome variations, the risk-seeking implications of w near $p = 0$ (at $p = 1/n$) prevail over the risk-averse implications of concave utility so much that the safe proportional allocation can no more be Pareto optimal. Engaging in mutual zero-sum games of paying a sure ε in return for a risky $n\varepsilon$ with probability $1/n$ (counter-monotonic, but not a jackpot) is surely appreciated for small ε .

Markowitz (1952, p. 153–154) speculated that people choose risks for small stakes but safety for large stakes, but sought to accommodate it using EU. Theorem 8 has given an empirically realistic basis to his speculations, confirmed by the wide existence of lotteries, sports betting, and casinos.

Unlike in Theorem 2, we are not able to show that all Pareto-optimal allocations are jackpot allocations or that they are λ -optimal. This is due to the non-linearity and nonconvexity in probabilistic mixtures of the RDU functionals that we consider. We do not know whether the UPS is convex.

Finally, we present two cases where competitive equilibria exist, albeit elementarily so. We leave further study of equilibria as a topic for future research.

Example 5. Assume that $X \geq ny_0$ and that u is constant on $[y_0, \infty)$ for some $y_0 > x_0$; that is, u has a satiation point. Then $\bar{\mathbf{X}}$ is Pareto optimal because it yields the maximum utility for every agent. It readily gives competitive equilibria, for instance for initial endowment $\bar{\mathbf{X}}$ with any price vector. The allocation $\bar{\mathbf{X}}$ strictly dominates any jackpot allocation when $w(1/n) < 1$ (details are given in Section S.3.4).

Proposition 7. Suppose that Assumptions ER and H-RDU hold, $X = x$ is a constant in $(0, x_0]$, and the vector of initial endowments $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{A}_n(x)$ satisfies $\mathbb{E}[\xi_i] \leq x\beta_w$

for all i . Then $(x\mathbf{J}, \mathbb{P})$ is a competitive equilibrium, for any $\mathbf{J} \in \mathbb{J}_n$ satisfying $\mathbb{E}[x\mathbf{J}] = \mathbb{E}[\xi]$.

It is not surprising that \mathbb{P} is the equilibrium price in Proposition 7, as the total endowment is constant across all states. Although we assume that there is no aggregate uncertainty, the initial endowments in Proposition 7 can be random.

Recall that $\beta_w \geq 1/n$ under Assumption H-RDU. Hence, $(\xi_1, \dots, \xi_n) = (X/n, \dots, X/n)$ satisfies the condition in Proposition 7, and the corresponding equilibrium allocation is $X\mathbf{J}$ with $\mathbb{E}[\mathbf{J}] = \mathbf{1}/n$ as in Theorem 8. The condition $\mathbb{E}[\xi_i] \leq x\beta_w$ means that each agent's initial endowment is not too large compared to the total endowment x . Intuitively, agents tend to gamble for small-probability gains, which is the case when they have similar initial endowments. On the other hand, if one agent has a relatively large initial endowment, say $0.9x$, then it is no longer optimal for this agent to gamble, because the utility of $0.9x$ is $0.9u(x)$, and the utility of $x\mathbf{1}_{A_1}$ with $\mathbb{P}(A_1) = 0.9$ is $w(0.9)u(x)$. Typically $w(0.9) < 0.9$ (see Figure 2), generating risk aversion for large-probability gains. In this case, $(x\mathbf{J}, \mathbb{P})$ is not a competitive equilibrium, and we do not know whether equilibria exist.

7 Conclusion

The results in this paper lay a foundation for studying risk sharing with empirically realistic risk attitudes. The counter-monotonic improvement theorem (Theorem 1) will be a useful tool for future studies.

We summarize what we know and what we do not. First, the setting of EU agents reveals a mix of insights and challenges. Pareto-optimal allocations and equilibrium allocations for risk-seeking agents (which is more realistic for losses than the commonly assumed risk aversion) are fully characterized, and the corresponding welfare theorems are established (Theorems 2 and 4). When there are subgroups of agents with different risk attitudes, conditions on Pareto-optimal allocations and equilibrium allocations within and across subgroups are obtained (Theorems 3 and 5). The case of homogeneous risk-seeking agents is better understood, as we can fully describe the competitive equilibria with a unique equilibrium price (Theorem 7). Competitive equilibria may or may not exist when some agents are risk seeking and some are risk averse. A characterization is obtained for two-point aggregate payoffs (Theorem 6), and the more general case is not clear. For risk-seeking agents and a given initial endowment, the existence of competitive equilibria is not proved in general

(with some results discussed in Appendix S.4) although we suspect that they always exist.

To achieve full empirical realism, we consider RDU agents. For the case of homogeneous agents, we obtain some Pareto optimal allocations for small-stake payoffs (Theorem 8), but we are not able to offer a complete characterization of Pareto-optimal allocations. We provide two elementary results on competitive equilibria for RDU agents. General results on competitive equilibria or heterogeneous RDU agents are topics for future research, as are extensions to ambiguity, but they will involve nontrivial analyses. We hope that our paper will inspire future studies on optimal risk sharing under behaviorally realistic models.

Appendices

We throughout follow the convention, also followed in the main text, that many claims are implicitly assumed to hold almost surely, that is, with probability one.

A Four lemmas

We first provide four technical lemmas that are useful in the proofs of our main results. Denote by \mathcal{L} the set of all random variables X such that there exists a standard uniform random variable U independent of X ; that is, Assumption **ER** holds for X , an element of \mathcal{L} .

Lemma 1. *Let $X_1 \in \mathcal{L}$ have distribution F_1 . For any distribution F on \mathbb{R}^n with the first marginal F_1 , there exists a random vector (X_2, \dots, X_n) such that (X_1, X_2, \dots, X_n) has distribution F .*

Proof. Let (Y_1, \dots, Y_n) have distribution F and let K_x be a regular conditional distribution of (Y_2, \dots, Y_n) given $Y_1 = x$ for each $x \in \mathbb{R}$. We then let the random vector (X_2, \dots, X_n) conditional on $X_1 = x$ follow from K_x for each $x \in \mathbb{R}$, which is possible because $X_1 \in \mathcal{L}$. This implies that (X_1, \dots, X_n) is identically distributed to (Y_1, \dots, Y_n) . \square

Lemma 2. *For any $X \in \mathcal{L}$ and $(X_1, \dots, X_n) \in \mathcal{X}^n$, there exist $(X'_1, \dots, X'_n) \in \mathcal{X}^n$ and a standard uniform random variable U such that $(X, X'_1, \dots, X'_n) \stackrel{\text{d}}{=} (X, X_1, \dots, X_n)$ and U is independent of (X, X'_1, \dots, X'_n) .*

Proof. Let $H : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be the joint distribution of (X, X_1, \dots, X_n) . Define $H' : \mathbb{R}^{n+2} \rightarrow \mathbb{R}$ as $H'(x, x_1, \dots, x_n, u) = H(x, x_1, \dots, x_n)u$. It is clear that H' is the joint distribution on \mathbb{R}^{n+2} . By Lemma 1, we can find a vector $(X, X'_1, \dots, X'_n, U) \sim H'$. Hence, we have $(X, X'_1, \dots, X'_n) \sim H$, $U \sim \text{U}[0, 1]$ and U is independent of (X, X'_1, \dots, X'_n) . \square

Lemma 3. *Assume that the utility functions u_1, \dots, u_n and the weighting functions w_1, \dots, w_n are continuous. For $X \in L_+^\infty \cap \mathcal{L}$ and $\mathcal{X} = L_+^\infty$, the set*

$$\text{UPS}(X) = \left\{ (\rho_{w_1}(u_1(X_1)), \dots, \rho_{w_n}(u_n(X_n))) : (X_1, \dots, X_n) \in \mathbb{A}_n(X) \right\} \quad (16)$$

is compact. In particular, for $(\lambda_1, \dots, \lambda_n) \in \Delta_n$, the maximum of $\sum_{i=1}^n \lambda_i \rho_{w_i}(u_i(X_i))$ over $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is attainable.

Proof. Let m be the essential supremum of X . Since the distributions of (X, X_1, \dots, X_n) for $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ are all supported on the bounded region $[0, m]^{n+1}$, any sequence of such distributions has a weak limit. Take any sequence of points $\mathbf{v}_1, \mathbf{v}_2, \dots$ in $\text{UPS}(X)$ that converges to $\mathbf{v}_0 \in \mathbb{R}^n$, and let $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \dots$ be the random vectors in $\mathbb{A}_n(X)$ with the utility vectors $\mathbf{v}_1, \mathbf{v}_2, \dots$, respectively. Let F be a weak limit of the sequence of distributions of the random vectors $(X, \mathbf{X}^{(1)}), (X, \mathbf{X}^{(2)}), \dots$, which we argued above to exist. Note that the first marginal of F is the distribution of X . By Lemma 1, there exists a random vector $\mathbf{X}' = (X'_1, \dots, X'_n)$ such that (X, \mathbf{X}') has distribution F . Note that $\mathbf{1} \cdot \mathbf{X}^{(n)} - X \rightarrow \mathbf{1} \cdot \mathbf{X}' - X$ in distribution, and hence $\mathbf{X}' \in \mathbb{A}_n(X)$. Moreover, for each i , since u_i on $[0, \infty)$ and w_i on $[0, 1]$ are continuous, the function $Y \mapsto \rho_{w_i}(u_i(Y))$ is continuous with respect to weak convergence by Theorem 4 of Wang et al. (2020). Therefore, $\mathbf{v}_0 = \lim_{j \rightarrow \infty} \mathbf{v}_j = (\rho_{w_1}(u_1(X'_1)), \dots, \rho_{w_n}(u_n(X'_n)))$, which shows $\mathbf{v}_0 \in \text{UPS}(X)$. The boundedness statement follows by noting that $\rho_{w_i}(u_i(X_i)) \leq \rho_{w_i}(u_i(X))$ due to monotonicity, which is finite for each i . The last statement on maximization follows from the compactness of $\text{UPS}(X)$. \square

Lemma 3 implies in particular that the maxima

$$\max_{(X_1, \dots, X_n) \in \mathbb{A}_n(X)} \sum_{i=1}^n \lambda_i \mathbb{E}[u_i(X_i)] \quad \text{and} \quad \max_{(X_1, \dots, X_n) \in \mathbb{A}_n(X)} \sum_{i=1}^n \lambda_i \rho_w(u(X_i))$$

are attainable under Assumptions ER, EU (for the first max), and H-RDU (for the second max). Moreover, individually rational Pareto-optimal allocations always exist, as summarized in the next lemma.

Lemma 4. *Suppose that Assumption ER hold, and either Assumption EU or Assumption H-RDU holds. For any initial endowments, individually rational Pareto-optimal allocations exist.*

Proof. Let (ξ_1, \dots, ξ_n) be the initial endowment vector. Recall that each \mathcal{U}_i is an EU preference functional under Assumption EU and an RDU preference function under Assumption H-RDU. By Lemma 3, the set $\text{UPS}(X)$ is compact. Therefore, each set

$$\text{IR}(X) := \{(v_1, \dots, v_n) \in \text{UPS}(X) : v_i \geq \mathcal{U}_i(\xi_i) \text{ for all } i \in [n]\}$$

is also compact. Maximizing $\sum_{i=1}^n v_i$ over $\text{IR}(X)$ yields a point in $\text{UPS}(X)$ attained by an individually rational Pareto-optimal allocation. \square

B Proofs

B.1 Proofs of results in Section 3

Proof of Proposition 1. This result follows by noting that the conditions in (3) are preserved under probabilistic mixing. \square

Proof of Theorem 1. In Section 3.3, the result is proved when there exists a standard uniform random variable U independent of (X_1, \dots, X_n) , that is, under Assumption ER*. We extend this to the case where such U may not exist. Let $(X, X_1, \dots, X_n) \stackrel{d}{=} (X, X'_1, \dots, X'_n)$ be as in Lemma 2. There exists a counter-monotonic improvement (Y_1, \dots, Y_n) of (X'_1, \dots, X'_n) . Note that $\sum_{i=1}^n X'_i = X = \sum_{i=1}^n X_i$. Moreover, $Y_i \geq_{\text{cx}} X'_i$ is equivalent to $Y_i \geq_{\text{cx}} X_i$, because $X_i \stackrel{d}{=} X'_i$. Therefore, (Y_1, \dots, Y_n) satisfies all desired conditions for (X_1, \dots, X_n) . \square

Proof of Proposition 2. Suppose that X_1, \dots, X_n are bounded from below. There exist constants m_1, \dots, m_n such that $X_1 + m_1, \dots, X_n + m_n$ are nonnegative. Write $m = \sum_{i=1}^n m_i$. By Theorem 1, there exists a jackpot allocation $(Z_1, \dots, Z_n) \in \mathbb{A}_n(X + m)$ such that $Z_i \geq_{\text{cx}} X_i + m_i$ for each i . It follows that $Z_i - m_i \geq_{\text{cx}} X_i$ for all i . Hence, the counter-monotonic random vector $(Z_1 - m_1, \dots, Z_n - m_n)$ satisfies the desired conditions in the corollary. The case that X_1, \dots, X_n are bounded from above is analogous. \square

B.2 Proofs of results in Section 4

Before proving Theorem 2, we first present a lemma for (ii) \Leftrightarrow (iii) in Theorem 2.

Lemma 5. *Suppose that Assumption EURS holds and $\lambda = (\lambda_1, \dots, \lambda_n) \in \Delta_n$. Then,*

$$\max_{(X_1, \dots, X_n) \in \mathbb{A}_n(X)} \sum_{i=1}^n \lambda_i \mathbb{E}[u_i(X_i)] = \mathbb{E}[V_\lambda(X)].$$

Proof. Since $\bigcup_{i=1}^n \{\lambda_i u_i(X)\} = V_\lambda(X)\} = \Omega$, we can take $(A_1, \dots, A_n) \in \Pi_n$ such that $\lambda_i u_i(X) = V_\lambda(X)$ on A_i for each $i \in [n]$. We have

$$\begin{aligned} \max_{(X_1, \dots, X_n) \in \mathbb{A}_n(X)} \sum_{i=1}^n \lambda_i \mathbb{E}[u_i(X_i)] &\geq \sum_{i=1}^n \mathbb{E}[\lambda_i u_i(X \mathbf{1}_{A_i})] \\ &= \sum_{i=1}^n \mathbb{E}[\lambda_i u_i(X) \mathbf{1}_{A_i}] = \sum_{i=1}^n \mathbb{E}[V_\lambda(X) \mathbf{1}_{A_i}] = \mathbb{E}[V_\lambda(X)]. \end{aligned}$$

Moreover, for any allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$, we have

$$\sum_{i=1}^n \mathbb{E}[\lambda_i u_i(X_i)] \leq \sum_{i=1}^n \mathbb{E}[V_{\lambda}(X_i)] \leq \mathbb{E}\left[V_{\lambda}\left(\sum_{i=1}^n X_i\right)\right] = \mathbb{E}[V_{\lambda}(X)],$$

where the last inequality follows from the superadditivity of V_{λ} because V_{λ} is convex with $V_{\lambda}(0) = 0$. Combining the above two inequalities, we get the desired result. \square

Proof of Theorem 2. The equivalence (i) \Leftrightarrow (ii) follows from Proposition 5 (proved below) and the equivalence (ii) \Leftrightarrow (iii) follows from Lemma 5. It remains to prove (i) \Rightarrow (iv) \Rightarrow (iii).

(i) \Rightarrow (iv): By Theorem 1, there is a jackpot allocation $\mathbf{Y} = (Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$ such that $Y_i \geq_{cx} X_i$ holds for all i . As u_i is strictly convex, we have $\mathbb{E}[u(Y_i)] = \mathbb{E}[u(X_i)]$ by Pareto optimality of (X_1, \dots, X_n) . By Shaked and Shanthikumar (2007, Theorem 3.A.43), we obtain $Y_i \stackrel{d}{=} X_i$ for each i . Note that since \mathbf{Y} is a jackpot allocation, we have $\sum_{i=1}^n \mathbb{P}(Y_i > 0) = \mathbb{P}(X > 0)$. This and $Y_i \stackrel{d}{=} X_i$ for each i imply $\sum_{i=1}^n \mathbb{P}(X_i > 0) = \mathbb{P}(X > 0)$, and in particular $X_i X_j = 0$ for $i \neq j$. It follows from (3) that \mathbf{X} is a jackpot allocation. The remaining (iv) \Rightarrow (iii) follows by noting $\sum_{i=1}^n \lambda_i u_i(X_i) = \sum_{i=1}^n \lambda_i V_{\lambda}(X) \mathbb{1}_{A_i} = V_{\lambda}(X)$. \square

Proof of Proposition 3. Suppose that a jackpot allocation $x\mathbf{J}$ for $\mathbf{J} \in \mathbb{J}_n$ is strictly dominated by an allocation \mathbf{Y} . By Theorem 1, \mathbf{Y} is dominated by another jackpot allocation $x\mathbf{J}'$ with $\mathbf{J}' \in \mathbb{J}_n$. The strict domination of $x\mathbf{J}'$ over $x\mathbf{J}$ implies $\mathbb{E}[x\mathbf{J}'] \geq \mathbb{E}[x\mathbf{J}]$ componentwise with strict inequality for at least one component. This is not possible because both \mathbf{J} and \mathbf{J}' have components summing to 1. Hence, $x\mathbf{J}$ cannot be strictly dominated by \mathbf{Y} , and it is Pareto optimal. \square

Proof of Proposition 4. We first show that $\text{UPJ}(X) \subseteq \Delta_n(\mathbb{E}[u(X)])$. Let (X_1, \dots, X_n) be a jackpot allocation of X and observe that by construction we have $\mathbb{E}[u(X_i)] \geq 0$ for all i , and

$$\sum_{i=1}^n \mathbb{E}[u(X_i)] = \sum_{i=1}^n \mathbb{E}[u(X) \mathbb{1}_{A_i}] = \mathbb{E}[u(X)].$$

Hence, $(\mathbb{E}[u(X_1)], \dots, \mathbb{E}[u(X_n)]) \in \Delta_n(\mathbb{E}[u(X)])$. Conversely, let $(\theta_1, \dots, \theta_n) \in \Delta_n$. By Assumption ER, we can take $(A_1, \dots, A_n) \in \Pi_n$ independent of X such that $\mathbb{P}(A_i) = \theta_i$ for all i . This gives $\mathbb{E}[u(X \mathbb{1}_{A_i})] = \mathbb{P}(A_i) \mathbb{E}[u(X)] = \theta_i \mathbb{E}[u(X)]$ for all i . Therefore, $\text{UPJ}(X) = \Delta_n(\mathbb{E}[u(X)])$. Note that every point in $\Delta_n(\mathbb{E}[u(X)])$ is in the UPF as they are not dominated

by any other points in $\text{UPJ}(X)$. Together with $\text{UPF}(X) \subseteq \text{UPJ}(X)$ guaranteed by Theorem 2, we can conclude that $\text{UPF}(X) = \text{UPJ}(X) = \Delta_n(\mathbb{E}[u(X)])$. \square

Proof of Proposition 5. Part (i): To show the convexity of $\text{UPS}(X)$, take $\alpha \in (0, 1)$ and $\mathbf{x}, \mathbf{y} \in \text{UPS}(X)$, and let A be an event with $\mathbb{P}(A) = \alpha$. We can take allocations $\mathbf{X}, \mathbf{Y} \in \mathbb{A}_n(X)$ with utility vectors \mathbf{x} and \mathbf{y} , respectively, independent of A . Then $\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}$ is the utility vector of $\mathbf{1}_A \mathbf{X} + (1 - \mathbf{1}_A) \mathbf{Y}$ since the expected utility is affine on probabilistic mixtures. This shows that $\text{UPS}(X)$ is convex. The convexity of $\text{UPJ}(X)$ follows from the same argument with the additional fact that, by Proposition 1, $\mathbf{1}_A \mathbf{X} + (1 - \mathbf{1}_A) \mathbf{Y}$ is a jackpot allocation when \mathbf{X} and \mathbf{Y} are.

Part (ii), the “if” statement: Let $I = \{i \in [n] : \lambda_i = 0\}$ and $J = [n] \setminus I$. Take an allocation (Y_1, \dots, Y_n) that dominates (X_1, \dots, X_n) . If $\mathbb{E}[u_i(Y_i)] > \mathbb{E}[u_i(X_i)]$ for some $i \in J$, then $\mathbb{E}[\lambda_k u_k(Y_k)] > \mathbb{E}[\lambda_k u_k(X_k)]$, a contradiction. If $\mathbb{E}[u_i(Y_i)] > \mathbb{E}[u_i(X_i)]$ for some $i \in I$, then take $j \in J$ and let the allocation (Z_1, \dots, Z_n) be given by $Z_i = 0$, $Z_j = Y_j + Y_i$, and $Z_k = Y_k$ for $k \neq i, j$. We have $\mathbb{E}[\lambda_k u_k(Z_k)] > \mathbb{E}[\lambda_k u_k(X_k)]$, a contradiction. Therefore, $\mathbb{E}[u_i(Y_i)] = \mathbb{E}[u_i(X_i)]$. This shows that \mathbf{X} is Pareto optimal.

Part (ii), the “only if” statement: By part (i), the UPS of X is convex. By the Hahn-Banach Theorem, for every Pareto-optimal allocation \mathbf{X} , there exists $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Delta_n$ such that \mathbf{X} is a $\boldsymbol{\lambda}$ -optimal allocation. \square

Proof of Theorem 3. The case of risk-averse agents follows from Carlier et al. (2012, Theorem 3.1), and the case of risk-seeking agents follows from (i) \Rightarrow (iv) in Theorem 2. \square

B.3 Proofs of results in Section 5

Proof of Theorem 4. The proof of part (i) is standard and omitted here. A self-contained proof is in Section S.3.3. We show part (ii) below. Let $\mathbf{X} = (X_1, \dots, X_n)$ be a Pareto-optimal allocation. By Theorem 2, $\mathbf{X} = X\mathbf{J}$ for some $\mathbf{J} = (\mathbf{1}_{A_1}, \dots, \mathbf{1}_{A_n}) \in \mathbb{J}_n$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Delta_n$ with $(\lambda_1 u_1(X_1), \dots, \lambda_n u_n(X_n)) = V_{\boldsymbol{\lambda}}(X)\mathbf{J}$. From the proof of Theorem 2, we can take $\lambda_i = 0$ if $X_i = 0$ and $\lambda_i > 0$ if $X_i \neq 0$.

Take the initial endowment $(\xi_1, \dots, \xi_n) = (X_1, \dots, X_n)$. We will show that (X_1, \dots, X_n, Q) is a competitive equilibrium for (ξ_1, \dots, ξ_n) . Let $z = \mathbb{E}[V_{\boldsymbol{\lambda}}(X)/X]$ and Q be given by (7). If $\xi_i = 0$, it is clear that $X_i = 0$ solves individual optimality. Next, we discuss the case that ξ_i is not 0, which implies $\lambda_i > 0$. For such i let $x_i = \mathbb{E}^Q[\xi_i] = \mathbb{E}^Q[X_i] = \mathbb{E}^Q[X\mathbf{1}_{A_i}]$, and notice

that

$$\lambda_i \mathbb{E}[u_i(X_i)] = \mathbb{E}[V_{\lambda}(X) \mathbf{1}_{A_i}] = \mathbb{E}\left[X \frac{V_{\lambda}(X)}{X} \mathbf{1}_{A_i}\right] = z \mathbb{E}^Q[X \mathbf{1}_{A_i}] = zx_i.$$

For any Y_i satisfying $0 \leq Y_i \leq X$ and the budget constraint $\mathbb{E}^Q[Y_i] \leq x_i$ we have

$$\lambda_i \mathbb{E}[u_i(Y_i)] = \mathbb{E}\left[Y_i \frac{\lambda_i u_i(Y_i)}{Y_i}\right] \leq \mathbb{E}\left[Y_i \frac{\lambda_i u_i(X)}{X}\right] \leq \mathbb{E}\left[Y_i \frac{V_{\lambda}(X)}{X}\right] = x \mathbb{E}^Q[Y_i] \leq zx_i,$$

where the first inequality uses the fact that $x \mapsto u(x)/x$ is increasing and the second that $\lambda_i u_i \leq V_{\lambda}$. Therefore, (X_1, \dots, X_n, Q) satisfies individual optimality. Market clearance also holds because $\sum_{i=1}^n \mathbf{1}_{A_i} = 1$. Therefore, (X_1, \dots, X_n) is an equilibrium allocation, being the initial endowments itself. \square

Proof of Theorem 5. Take $i \in S$, and consider the optimization problem

$$\max_{0 \leq Z \leq X} \mathbb{E}[u_i(Z)] \text{ subject to } \mathbb{E}^Q[Z] \leq \mathbb{E}^Q[\xi_i].$$

Since u_i is strictly convex, the constraint is linear, and the probability space is atomless (by Assumption ER), the optimizer $Z = X_i$ of the above problem must be either 0 or X almost surely. This implies $X_i(X - X_i) = 0$ for $i \in S$. By (3), (\mathbf{X}_S, Y) is a jackpot allocation. \square

Proof of Theorem 6. We first show that (\mathbf{X}, Q) is a competitive equilibrium with the initial endowment vector chosen as \mathbf{X} . Write $p = \mathbb{P}(X = a)$ and $q = \mathbb{P}(X = b) = 1 - p$. Fix an agent $i \in [n]$, and let $x = \mathbb{E}^Q[X_i]$. The optimization problem for agent i is

$$\text{maximize } \mathbb{E}[u_i(Y)] \quad \text{over } 0 \leq Y \leq X \quad \text{subject to } \mathbb{E}^Q[Y] \leq \mathbb{E}^Q[X_i] = x. \quad (17)$$

Suppose $i \in T$. By the strict concavity of u_i , Jensen's inequality guarantees that any optimal Y for (17) takes at most two values, denoted by y and z , when $X = a$ and $X = b$, respectively. Noting that the budget constraint in (17) is binding, we have $z = (x - y\alpha p)/(\beta q)$ and $y \in [0, a_i]$. Taking the derivative of $\mathbb{E}[u_i(Y)]$, which exists for almost every y , and using the convexity of u_i , we get

$$\frac{d}{dy} \mathbb{E}[u_i(Y)] = \frac{d}{dy} (pu_i(y) + qu_i(z)) = pu'_i(y) - q \frac{\alpha p}{\beta q} u'_i(z) \geq pu'_i(a_i) - \frac{\alpha p}{\beta} u'_i(0) \geq 0,$$

where the last inequality follows from the first inequality in (10). Hence, the maximum of

(17) is attained by $y = a_i$, and $Y = X_i$ is optimal for agent $i \in T$. From the above argument, the first inequality in (10) is necessary for $Y = X_i$ to be optimal.

Now, suppose $i \in S$. By the strict convexity of u_i and the linearity of the constraint, any optimal Y for (17) the values 0, a and b with probabilities p_0 , p_1 and p_2 , respectively, where $p_0 + p_1 + p_2 = 1$. Noting that the budget constraint in (17) is binding, we have $p_1 = (x - b\beta p_2)/(a\alpha)$. With this, the objective in (17), that is, $p_1 u_i(a) + p_2 u_i(b)$, is linearly increasing in p_2 by the second inequality in (10). Hence, the maximum of (17) is attained by the largest possible value of p_2 , and $Y = X_i$ is optimal for agent $i \in S$. From the above argument, the second inequality in (10) is also necessary for $Y = X_i$ to be optimal.

We next show that for a competitive equilibrium (\mathbf{X}, Q) the forms (8)–(9) are necessary. With this established, condition (10) is verified above. First, $dQ/d\mathbb{P} > 0$ because X is positive. On each of $\{X = a\}$ and $\{X = b\}$, $dQ/d\mathbb{P}$ must be a constant due to the constant supply and at least two agents in each group participate. Hence, the form (9) holds. Let $X_S = \sum_{i \in S} X_i$ and $X_T = X - X_S$. By Theorem 5, we know that (X_T, \mathbf{X}_S) is a jackpot allocation. Note that by the strict concavity of the utility functions for agents in T , they will have constant payoffs on $\{X = a\}$ and on $\{X = b\}$. As a consequence, and also noting that (X_T, X_S) is a nontrivial jackpot allocation, we have $X_T = a\mathbb{1}_{\{X=a\}}$ or $X_T = b\mathbb{1}_{\{X=b\}}$. Moreover, individual optimization implies that the risk-averse agents invest in the cheaper one between $\{X = a\}$ and $\{X = b\}$. If $\beta \leq \alpha$, then all risk-seeking agents will prefer payoffs on $\{X = b\}$, and thus $X_S = b\mathbb{1}_{\{X=b\}}$, but the risk-averse agents also invest in $\{X = b\}$, contradicting $X_T = a\mathbb{1}_{\{X=a\}}$. Therefore, $\beta > \alpha$, and this implies $X_T = a\mathbb{1}_{\{X=a\}}$. Now using Theorem 3 we obtain (8). \square

Proof of Proposition 6. The equivalence between (i), (ii), (iii) and (v) follows from Theorem 2 and Proposition 4. The equivalence between them and (iv) follows from Theorem 4. \square

Proof of Theorem 7. As usual, write $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$, $\boldsymbol{\xi} = (\xi_1, \dots, \xi_n)$ and $\mathbf{J} = (\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n})$. We first show part (i). Denote by $x_i = \mathbb{E}^Q[\xi_i] = \theta_i \mathbb{E}^Q[X]$ and $z = \mathbb{E}[u(X)/X] > 0$. Because $dQ/d\mathbb{P}$ is a function of X , \mathbf{J} has the same expectation $\boldsymbol{\theta}$ under \mathbb{P} and Q , and it is independent of X under both \mathbb{P} and Q . Hence, $\mathbb{E}^Q[X_i] = \theta_i \mathbb{E}^Q[X] = x_i$, and thus the budget constraint is satisfied for each i . Moreover,

$$\mathbb{E}[u(X_i)] = \mathbb{E}[u(X)\mathbb{1}_{A_i}] = \mathbb{E}\left[X \frac{u(X)}{X} \mathbb{1}_{A_i}\right] = z \mathbb{E}^Q[X \mathbb{1}_{A_i}] = zx_i.$$

For any Y_i satisfying $0 \leq Y_i \leq X$ and the budget constraint $\mathbb{E}^Q[Y_i] \leq x_i$, using the fact that $x \mapsto u(x)/x$ is increasing, we have

$$\mathbb{E}[u(Y_i)] = \mathbb{E}\left[Y_i \frac{u(Y_i)}{Y_i}\right] \leq \mathbb{E}\left[Y_i \frac{u(X)}{X}\right] = z\mathbb{E}^Q[Y_i] \leq zx_i = \mathbb{E}[u(X_i)].$$

Therefore, (X_1, \dots, X_n, Q) satisfies individual optimality. Market clearance clearly holds.

Next, we show part (ii). Suppose that the tuple (\mathbf{X}, Q) is a competitive equilibrium. By Proposition 6 we know \mathbf{X} is a jackpot allocation. We can write $\mathbf{X} = X\mathbf{J}$ for some $\mathbf{J} = (\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n}) \in \mathbb{J}_n$ satisfying $\mathbb{E}^Q[X\mathbf{J}] = \mathbb{E}^Q[\boldsymbol{\xi}]$ by the binding budget constraint. Without loss of generality, assume $\mathbb{P}(X > 0) = 1$, as Q is arbitrary on $\{X = 0\}$. By individual optimality we can see that \mathbb{P} and Q are equivalent measures. Define a probability measure R by $dR/dQ = X/c$, where $c = \mathbb{E}^Q[X]$, and let $Z = c(u(X)/X)(d\mathbb{P}/dQ)$. Note that for any $A \in \mathcal{F}$, we have

$$\mathbb{E}[u(X\mathbb{1}_A)] = \mathbb{E}^R\left[\frac{d\mathbb{P}}{dQ} \frac{dQ}{dR} u(X)\mathbb{1}_A\right] = \mathbb{E}^R[Z\mathbb{1}_A]. \quad (18)$$

Individual optimality of $X\mathbf{J}$ implies that for any i and any $A \in \mathcal{F}$ satisfying $\mathbb{E}^Q[X\mathbb{1}_A] \leq \mathbb{E}^Q[X\mathbb{1}_{A_i}]$, we have $\mathbb{E}[u(X\mathbb{1}_A)] \leq \mathbb{E}[u(X\mathbb{1}_{A_i})]$, and by using (18) it gives $\mathbb{E}^R[Z\mathbb{1}_A] \leq \mathbb{E}^R[Z\mathbb{1}_{A_i}]$. Note that $\mathbb{E}^Q[X\mathbb{1}_A] \leq \mathbb{E}^Q[X\mathbb{1}_{A_i}]$ is equivalent to $R(A) \leq R(\mathbb{1}_{A_i})$. Take B_1, \dots, B_n with $R(B_i) = R(A_i)$ for all i such that $\mathbb{1}_{B_1}, \dots, \mathbb{1}_{B_n}$ are comonotonic with Z ; this is possible because R is atomless. We have $\sum_{i=1}^n \mathbb{E}^R[Z\mathbb{1}_{B_i}] \leq \sum_{i=1}^n \mathbb{E}^R[Z\mathbb{1}_{A_i}] = \mathbb{E}^R[Z]$. Note that $\sum_{i=1}^n \mathbb{1}_{B_i}$ is not a constant because at least two of A_1, \dots, A_n have positive probability under R by the binding budget constraint and the assumption that $\boldsymbol{\xi}$ is nontrivial. If Z is not a constant, then the Fréchet-Hoeffding inequality gives $\mathbb{E}^R[Z \sum_{i=1}^n \mathbb{1}_{B_i}] > \mathbb{E}^R[Z]\mathbb{E}^R[\sum_{i=1}^n \mathbb{1}_{B_i}] = \mathbb{E}^R[Z]$, a contradiction. Therefore, Z is a constant, and dQ/dP is equal to a constant times $u(X)/X$, showing that Q is uniquely given by (12).

Part (iii) follows immediately from (ii), by noting that for the unique price Q , there can only be one maximum utility value for every agent. If $\boldsymbol{\xi}$ is trivial, say $\xi_1 = X$, then the equilibrium allocation is $(X, 0, \dots, 0)$, which has the unique utility vector $\mathbb{E}[u(X)]\boldsymbol{\theta}$. \square

B.4 Proofs of results in Section 6

Proof of Theorem 8. In (i) and (ii) below, as X takes values in $[0, x_0]$, we can without loss of generality let u be the identity function. Thus, each agent has a preference functional ρ_w .

(i) Write $\mathbf{J} = (\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n})$. First, for each i , since $w = \bar{w}$ on $[0, 1/n]$ and $\mathbb{P}(A_i) = 1/n$, we have

$$\rho_w(X\mathbb{1}_{A_i}) = \int w(\mathbb{P}(X\mathbb{1}_{A_i} > x)) dx = \int \bar{w}(\mathbb{P}(X\mathbb{1}_{A_i} > x)) dx = \rho_{\bar{w}}(X\mathbb{1}_{A_i}).$$

Moreover,

$$\rho_w(X\mathbb{1}_{A_i}) = \rho_{\bar{w}}(X\mathbb{1}_{A_i}) = \int \bar{w}\left(\frac{1}{n}\mathbb{P}(X > x)\right) dx. \quad (19)$$

Next, we will show that

$$\sum_{i=1}^n \rho_w(X\mathbb{1}_{A_i}) = \sup_{(X_1, \dots, X_n) \in \mathbb{A}_n(X)} \rho_{\bar{w}}(X_i), \quad (20)$$

and since $w \leq \bar{w}$, this gives sum-optimality of $(X\mathbb{1}_{A_1}, \dots, X\mathbb{1}_{A_n})$. To show (20), it suffices to consider jackpot allocations (X_1, \dots, X_n) because the preference functional $\rho_{\bar{w}}$ is risk seeking (Schmidt and Zank, 2008), and we can apply Theorem 1. Using the representation $(X\mathbb{1}_{B_1}, \dots, X\mathbb{1}_{B_n})$ of jackpot allocations, we get

$$\begin{aligned} \sup_{(X_1, \dots, X_n) \in \mathbb{A}_n(X)} \rho_{\bar{w}}(X_i) &= \sup_{(B_1, \dots, B_n) \in \Pi_n} \rho_{\bar{w}}(X\mathbb{1}_{B_i}) \\ &= \sup \left\{ \sum_{i=1}^n \int_0^\infty \bar{w}(\mathbb{P}(X\mathbb{1}_{B_i} > x)) dx : (B_1, \dots, B_n) \in \Pi_n \right\} \\ &\leq \int_0^\infty \sup \left\{ \sum_{i=1}^n \bar{w}(p_i) : \sum_{i=1}^n p_i = \mathbb{P}(X > x) \right\} dx \\ &= \int_0^\infty n\bar{w}\left(\frac{1}{n}\mathbb{P}(X > x)\right) dx, \end{aligned}$$

where the last equality is due to concavity of \bar{w} . Using (19), we get (20), and hence the allocation $(X\mathbb{1}_{A_1}, \dots, X\mathbb{1}_{A_n})$ is sum-optimal and Pareto optimal.

(ii) Because \bar{w} is strictly concave on $(0, 1/n)$, $t \mapsto t\bar{w}(x/t)$ is strictly increasing in $t \geq 1$ for each $x \in (0, 1/n)$. This implies $n\bar{w}(x/n) > w(x)$ for all $x \in (0, 1)$. Hence,

$$\frac{1}{n} \int_0^\infty \bar{w}(\mathbb{P}(X > x)) dx < \int_0^\infty \bar{w}\left(\frac{\mathbb{P}(X > x)}{n}\right) dx.$$

As a consequence,

$$\rho_w(X/n) \leq \rho_{\bar{w}}(X/n) = \frac{1}{n} \int_0^\infty \bar{w}(\mathbb{P}(X > x)) dx < \int_0^\infty \bar{w}\left(\frac{\mathbb{P}(X > x)}{n}\right) dx = \rho_w(X \mathbf{1}_{A_i}).$$

This shows strict dominance.

(iii) Note that $w(t/n)/w(t) < 1$ for $t \in (0, 1)$ because w is concave (thus strictly increasing) on $[0, 1/n]$. Together with the continuity of w and condition (15), we get $\sup_{t \in (0, 1)} w(t/n)/w(t) < 1$. Hence, we can take $\theta < 1$ such that $\sup_{t \in (0, 1)} w(t/n)/w(t) < \theta$, and take $y_0 > 0$ such that $u(x/n) > \theta u(x)$ for $x \geq y_0$, also guaranteed by (15). Note that $X \geq y_0$ implies $u(X/n) \geq \theta u(X)$. For any event A with $\mathbb{P}(A) = 1/n$ independent of X ,

$$\begin{aligned} \rho_w(u(X \mathbf{1}_A)) &= \int_0^\infty w(\mathbb{P}(u(X \mathbf{1}_A) > x)) dx \\ &= \int_0^\infty w\left(\frac{\mathbb{P}(u(X) > x)}{n}\right) dx \\ &< \theta \int_0^\infty w(\mathbb{P}(u(X) > x)) dx = \rho_w(\theta u(X)) \leq \rho_w(u(X/n)). \end{aligned}$$

Therefore, $\bar{\mathbf{X}}$ yields higher utility for each agent than $X \mathbf{J}$ does, and thus strict domination holds.

(iv) Denote by $y = X/n > 0$ and Z^ε the first component of $\bar{\mathbf{X}} - \varepsilon \mathbf{1} + n\varepsilon \mathbf{J}$. We can immediately compute

$$\rho_w(u(Z^\varepsilon)) = u(y - \varepsilon) + (u(y + (n-1)\varepsilon) - u(y - \varepsilon))w(1/n).$$

Taking derivative yields

$$\frac{d}{d\varepsilon} \rho_w(u(Z^\varepsilon)) = -u'(y - \varepsilon) + ((n-1)u'(y + (n-1)\varepsilon) + u'(y - \varepsilon))w(1/n),$$

which converges to $nu'(y)(w(1/n) - 1/n)$ as $\varepsilon \downarrow 0$. Using $w(1/n) > 1/n$, which is implied by the strict concavity of w on $(0, 1/n)$, we get that $d\rho_w(u(Z^\varepsilon))/d\varepsilon > 0$ for $\varepsilon > 0$ small, and hence $\rho_w(u(Z^\varepsilon)) > u(y)$ for $\varepsilon > 0$ small enough. This shows that $\bar{\mathbf{X}} - \varepsilon \mathbf{1} + n\varepsilon \mathbf{J}$ strictly dominates $\bar{\mathbf{X}}$. \square

Proof of Proposition 7. Without loss of generality, assume that u is the identity on $[0, x_0]$. The budget conditions and market clearance hold by construction of $x \mathbf{J}$. We only need to

show individual optimality. Write $(\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n}) = \mathbf{J}$. Note that for any random variable Y taking values in $[0, 1]$ we have $Y \leq_{\text{cx}} \mathbb{1}_A$ where $\mathbb{P}(A) = \mathbb{E}[Y]$. Therefore, we have $\rho_{\bar{w}}(Y) \leq \rho_{\bar{w}}(\mathbb{1}_A) = \bar{w}(\mathbb{E}[Y])$ because $\rho_{\bar{w}}$ represents a risk-seeking preference. It follows that for any random variable X_i in $[0, x]$ satisfying the constraint $\mathbb{E}[X_i] = \mathbb{E}[\xi_i]$, we have

$$\rho_w(X_i) \leq \rho_{\bar{w}}(X_i) \leq \rho_{\bar{w}}(x\mathbb{1}_{A_i}) = x\bar{w}(\mathbb{P}(A_i)) = xw(\mathbb{P}(A_i)) = \rho_w(x\mathbb{1}_{A_i}).$$

This extends to the case $\mathbb{E}[X_i] < \mathbb{E}[\xi_i]$ by monotonicity. Therefore, individual optimality holds, showing that $(x\mathbf{J}, \mathbb{P})$ is a competitive equilibrium. \square

References

- Abdellaoui, M., Bleichrodt, H., and Kammoun, H. (2013). Do financial professionals behave according to prospect theory? An experimental study. *Theory and Decision*, **74**, 411–429.
- Araujo, A., Bonnisseau, J. M., Chateauneuf, A., and Novinski, R. (2017). Optimal sharing with an infinite number of commodities in the presence of optimistic and pessimistic agents. *Economic Theory*, **63**, 131–157.
- Araujo, A., Chateauneuf, A., Gama, J. P., and Novinski, R. (2018). General equilibrium with uncertainty loving preferences. *Econometrica*, **86**(5), 1859–1871.
- Arrow, K. J., and Debreu, G. (1954). Existence of an equilibrium for a competitive economy. *Econometrica*, **22**(3), 265–290.
- Aumann, R. J. (1966). Existence of competitive equilibria in markets with a continuum of traders. *Econometrica*, **14**, 1–17.
- Barberis, N.C. and Huang, M. (2008). Stocks as lotteries: The implications of probability weighting for security prices. *American Economic Review*, **98**, 2066–2100.
- Barberis, N., Jin, L. J. and Wang, B. (2021). Prospect theory and stock market anomalies. *Journal of Finance*, **76**, 2639–2687.
- Barseghyan, L., Molinari, F., O'Donoghue, T., and Teitelbaum, J. C. (2013). The nature of risk preferences: Evidence from insurance choices. *American Economic Review*, **103**, 2499–2529.
- Beissner, P. and Werner, J. (2023). Optimal allocations with α -MaxMin utilities, Choquet expected utilities, and prospect theory. *Theoretical Economics*, **18**, 933–1022.

- Budish, E., Che, Y.-K., Kojima, F. and Milgrom, P. (2013). Designing random allocation mechanisms: Theory and applications. *American Economic Review*, **103**, 585–623.
- Carlier, G., Dana, R.-A. and Galichon, A. (2012). Pareto efficiency for the concave order and multivariate comonotonicity. *Journal of Economic Theory*, **147**, 207–229.
- Cass, D. and Shell, K. (1983). Do sunspots matter? *Journal of Political Economy*, **91**, 193–228.
- Crainich, D., Eeckhoudt, L., and Trannoy, A. (2013) Even (mixed) risk lovers are prudent. *American Economic Review*, **103**, 1529–1535.
- Dall'Aglio, G. (1972). Fréchet classes and compatibility of distribution functions. *Symposia Mathematica*, **9**, 131–150.
- Denneberg, D. (1994). *Non-additive Measures and Integral*. Kluwer, Dordrecht.
- Dillenberger, D. and Segal, U. (2025). Allocation mechanisms with mixture-averse preferences. *Working paper*, University of Pennsylvania.
- Ebert, S., and Strack, P. (2015). Until the bitter end: On prospect theory in a dynamic context. *American Economic Review*, **105**, 1618–1633.
- Edwards, K. D. (1996). Prospect theory: A literature review. *International Review of Financial Analysis*, **5**, 18–38.
- Embrechts, P., Liu, H. and Wang, R. (2018). Quantile-based risk sharing. *Operations Research*, **66**(4), 936–949.
- Embrechts, P., Liu, H., Mao, T. and Wang, R. (2020). Quantile-based risk sharing with heterogeneous beliefs. *Mathematical Programming Series B*, **181**(2), 319–347.
- Fehr-Duda, H. and Epper, T. (2012). Probability and risk: Foundations and economic implications of probability-dependent risk preferences. *Annual Review of Economics*, **4**, 567–593.
- Friedman, M. and Savage, L. J. (1948). The utility analysis of choices involving risk. *Journal of Political Economy*, **56**(4), 279–304.
- Grimm, S., Kocher, M. G., Krawczyk, M., and Le Lec, F. (2021). Sharing or gambling? On risk attitudes in social contexts. *Experimental Economics*, **24**, 1075–1104.
- Heimer, R., Iliewa, Z., Imas, A., and Weber, M. (2025). Dynamic inconsistency in risky choice: Evidence from the lab and field. *American Economic Review*, **115**, 330–363.
- Herings, P. J-J, and Zhan, Y. (2025). Competitive equilibria in incomplete markets with risk loving preferences. *SSRN*: 4245754.
- Jullien, B. and Salanié, B. (2000). Estimating preferences under risk: The case of racetrack

- bettors. *Journal of Political Economy*, **108**(3), 503–530.
- Kühberger, A. (1998). The influence of framing on risky decisions: A meta-analysis. *Organizational Behavior and Human Decision Processes*, **75**, 23–55.
- l'Haridon, O., and Vieider, F. (2019). All over the map: A worldwide comparison of risk preferences. *Quantitative Economics*, **10**, 185–215.
- Landsberger, M. and Meilijson, I. (1994). Co-monotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion. *Annals of Operations Research*, **52**(2), 97–106.
- Laughhunn, D. J., Payne, J. W., and Crum, R. L. (1980). Managerial risk preferences for below-target returns. *Management Science*, **26**, 1238–1249.
- Lauzier, J.-G., Lin, L. and Wang, R. (2023). Pairwise counter-monotonicity. *Insurance: Mathematics and Economics*, **111**, 279–287.
- Le Van, C., and Pham, N.-S. (2025). Equilibrium with non-convex preferences: Some insights. *arXiv*: 2503.16890.
- Machina, M. J. (1989). Dynamic consistency and non-expected utility models of choice under uncertainty. *Journal of Economic Literature*, **27**, 1622–1688.
- Markowitz, H. (1952). The utility of wealth. *Journal of Political Economy*, **60**(2), 151–158.
- Marshall, A. (1890). *Principles of Economics*. 8th edition 1920 (9th edition 1961), MacMillan, New York.
- Mas-Colell, A., Whinston, M. D. and Green, J. R. (1995). *Microeconomic Theory*. Oxford University Press.
- Moscati, I. (2018). *Measuring Utility: From the Marginal Revolution to Behavioral Economics*. Oxford University Press, Oxford, UK.
- Myagkov, M. G., and Plott, C. R. (1997). Exchange economies and loss exposure: Experiments exploring prospect theory and competitive equilibria in market environments. *American Economic Review*, **87**, 801–828.
- Negishi, T. (1960). Welfare economics and existence of an equilibrium for a competitive economy. *Metroeconomica*, **12**(2–3), 92–97.
- Netzer, N. (2009). Evolution of time preferences and attitudes toward risk. *American Economic Review*, **99**, 937–955.
- Olsen, R. A. (1997). Prospect theory as an explanation of risky choice by professional investors: Some evidence. *Review of Financial Economics*, **6**, 225–232.

- Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behavior and Organization*, **3**(4), 323–343.
- Rothschild, M. and Stiglitz, J. E. (1970). Increasing risk: I. A definition. *Journal of Economic Theory*, **2**(3), 225–243.
- Rüschendorf, L. (2013). *Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios*. Springer, Heidelberg.
- Schmidt, U. and Zank, H. (2008). Risk aversion in cumulative prospect theory. *Management Science*, **54**, 208–216.
- Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer Series in Statistics.
- Shen, Q. and Zhong, S. (2025). Preferences and personality traits in China: Evidence from a large-scale nationwide survey. *Working paper*, Hong Kong University of Science and Technology.
- Tversky, A., and Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty*, **5**, 297–323.
- Wakker, P. P. (2010). *Prospect Theory: For Risk and Ambiguity*. Cambridge University Press.
- Wang, R. and Zitikis, R. (2021). An axiomatic foundation for the Expected Shortfall. *Management Science*, **67**, 1413–1429.
- Wang, R., Wei, Y. and Willmot, G. E. (2020). Characterization, robustness and aggregation of signed Choquet integrals. *Mathematics of Operations Research*, **45**(3), 993–1015.
- Wu, G., Zhang, J. and Gonzalez, R. (2004). Decision under risk. In: *Blackwell Handbook of Judgment and Decision Making* (Eds.: D. Koehler and N. Harvey), Blackwell.
- Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica*, **55**(1), 95–115.

Supplementary material for “Optimal risk sharing, equilibria, and welfare with empirically realistic risk attitudes”

The supplementary material contains five appendices on some background, discussions, technical details, additional results, and extensions.

S.1 Background on counter-monotonicity

We provide some technical background on counter-monotonicity. First, [Dall’Aglio \(1972\)](#) obtained some necessary conditions for counter-monotonicity in dimensions more than two.

Proposition S.1 ([Dall’Aglio \(1972\)](#)). *If at least three of X_1, \dots, X_n are non-degenerate, then (X_1, \dots, X_n) are counter-monotonic if and only if one of the following two cases holds:*

$$\mathbb{P}(X_i > \text{ess-inf } X_i, X_j > \text{ess-inf } X_j) = 0 \text{ for all } i, j \in [n] \text{ with } i \neq j; \quad (\text{S.1})$$

$$\mathbb{P}(X_i < \text{ess-sup } X_i, X_j < \text{ess-sup } X_j) = 0 \text{ for all } i, j \in [n] \text{ with } i \neq j. \quad (\text{S.2})$$

A necessary condition for (S.1) is $\sum_{i=1}^n \mathbb{P}(X_i > \text{ess-inf } X_i) \leq 1$, and a necessary condition for (S.2) is $\sum_{i=1}^n \mathbb{P}(X_i < \text{ess-sup } X_i) \leq 1$.

The alternative formulation (3) of jackpot allocations in Section 3 directly follows from Proposition S.1.

Recall that Π_n is the set of all n -compositions of Ω . The next proposition, which is a reformulation of [Lauzier et al. \(2023, Theorem 1\)](#), simplifies the stochastic representation of counter-monotonicity.

Proposition S.2. *For $X \in \mathcal{X} = L^1$, suppose that at least three of $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ are non-degenerate. Then (X_1, \dots, X_n) is counter-monotonic if and only if there exist constants m_1, \dots, m_n and $(A_1, \dots, A_n) \in \Pi_n$ such that*

$$\text{either } X_i = (X - m)\mathbf{1}_{A_i} + m_i \text{ for all } i \text{ with } m = \sum_{i=1}^n m_i \leq \text{ess-inf } X; \quad (\text{S.3})$$

$$\text{or } X_i = (X - m)\mathbf{1}_{A_i} + m_i \text{ for all } i \text{ with } m = \sum_{i=1}^n m_i \geq \text{ess-sup } X. \quad (\text{S.4})$$

Proof. The “if” part follows from the fact that $\sum_{i=1}^n X_i = X$ and Proposition S.1. We will show the “only if” part. Assume that $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is counter-monotonic. By Lauzier et al. (2023, Theorem 1), there exists $(A_1, \dots, A_n) \in \Pi_n$ such that

$$X_i = (X - m)\mathbf{1}_{A_i} + m_i \quad \text{for all } i,$$

where either $m_i = \text{ess-inf } X_i$ for all i or $m_i = \text{ess-sup } X_i$ for all i , and $m = \sum_{i=1}^n m_i$. If $m_i = \text{ess-inf } X_i$ for all i , we have $m = \sum_{i=1}^n \text{ess-inf}(X_i) \leq \text{ess-inf}(\sum_{i=1}^n X_i) = \text{ess-inf } X$. If $m_i = \text{ess-sup } X_i$ for all i , we have $m = \sum_{i=1}^n \text{ess-sup}(X_i) \geq \text{ess-sup}(\sum_{i=1}^n X_i) = \text{ess-sup } X$. \square

Adding constants m_i to X_i does not affect counter-monotonicity. The terms with m serve market clearance, with an inequality imposed to avoid crossing a boundary. The allocation $(X, 0, \dots, 0)$ is counter-monotonic by taking $A = \Omega$ and $m = m_1 = \text{ess-inf } X$, and comonotonicity is trivial. Notice now that the allocations defined in Equations (S.3) and (S.4) reflect the conditions in Proposition S.1. In (S.3), for almost every $\omega \in \Omega$, at most one agent receives more than their essential infimum. Conversely, in (S.4), at most one agent receives less than their essential supremum.

S.2 Discussion on Assumptions ER and ER*

This appendix explains why Assumption ER is more convenient than Assumption ER* in Theorem 1, although the latter is quite intuitive and it allows for a simple proof. Recall that ER assumes external randomization U for X and ER* assumes external randomization U for (X_1, \dots, X_n) , which is stronger.

To characterize Pareto optimality or competitive equilibria, we consider all random vectors $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ in the given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For instance, if we want to show as in Theorem 2 that any Pareto-optimal allocation is a jackpot allocation, then we need to be able to apply Theorem 1 to any $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$. Therefore, if we only have Theorem 1 under Assumption ER*, we need that for *all* random vectors an independent standard uniform random variable exists. Despite simple intuition, this assumption is very strong and rules out any standard Borel probability space; see Liu et al. (2020, Example 7) for an illustration. The intuition is that, in any standard Borel probability space, there exists some random variable that exhausts randomness; that is, no external randomization

is allowed for that random variable. Hence, to apply Theorem 1 under Assumption ER* and obtain the desired results in subsequent theorems, we must exclude standard Borel probability spaces, whereas Assumption ER conveniently avoids this issue. Thus, the difference between Assumptions ER and ER* affects the applicability of the improvement theorem in risk sharing. Nevertheless, for an application in which assuming the existence of a uniform random variable independent of all allocations is safe, Assumption ER* would suffice.

On a related note, Assumption ER or ER* is not needed for the classic comonotonic improvement theorem, and the reason is also intuitive: for risk-averse agents, external randomization does not enhance their utility, and therefore it is not needed. Mathematically, all comonotonic allocations of X are measurable with respect to the σ -algebra generated by X (Denneberg, 1994); this is certainly not true for counter-monotonic allocations.

S.3 Additional technical details

S.3.1 UPF for two agents

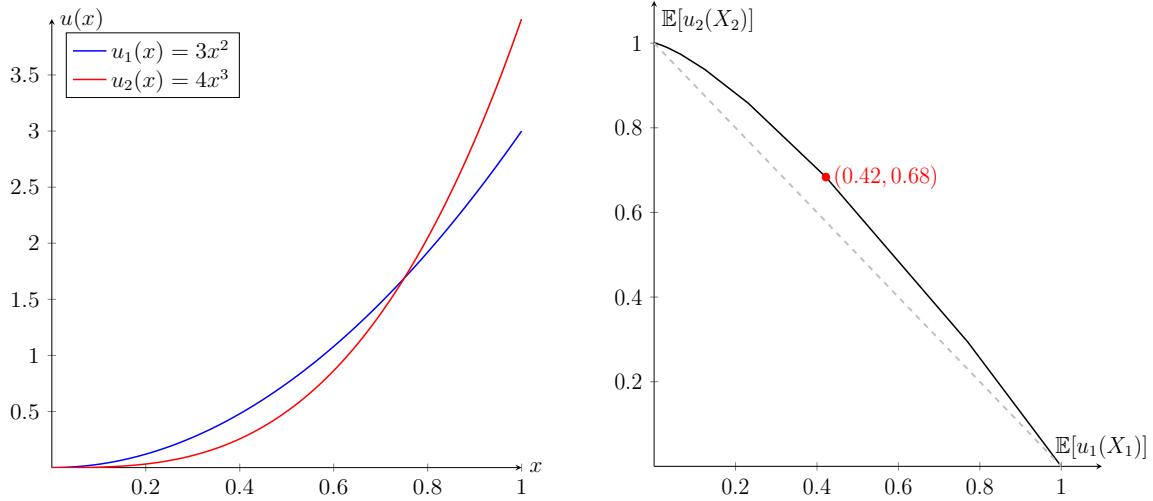
In the setting of risk-seeking EU agents in Section 4, the example below illustrates a case in which $\text{UPF}(X)$ is a curve, different from $\text{UPJ}(X)$, which is a convex set.

Example S.1. Set $u_1(x) = 3x^2$ and $u_2(x) = 4x^3$ for $x \geq 0$ and let X be uniformly distributed over $[0, 1]$. We have $\mathbb{E}[u_1(X)] = \mathbb{E}[u_2(X)] = 1$, and for any composition (A_1, A_2) independent of X , we have

$$\mathbb{E}[u_1(X\mathbf{1}_{A_1})] + \mathbb{E}[u_2(X\mathbf{1}_{A_2})] = \mathbb{P}(A_1)\mathbb{E}[u_1(X)] + \mathbb{P}(A_2)\mathbb{E}[u_2(X)] = 1.$$

Now consider $A_1 = \{X \in [0, 3/4]\}$ and $A_2 = \{X \in [3/4, 1]\}$ so that (A_1, A_2) is a composition that is not independent of X . We can compute $\mathbb{E}[u_1(X\mathbf{1}_{A_1})] = \int_0^{3/4} 3x^2 dx \approx 0.422$ and $\mathbb{E}[u_2(X\mathbf{1}_{A_2})] = \int_{3/4}^1 4x^3 dx \approx 0.684$, and hence this allocation is better than some jackpot allocations built using events independent of X . Intuitively, the allocation that maximizes the equally weighted sum of the welfare gives everything to the agent who has the highest utility pointwise; see the left panel of Figure S.1. Maximizing differently weighted sums of the welfare gives the UPF by Theorem 2, plotted in the right panel of Figure S.1.

Figure S.1: An illustration of Example S.1. Left panel: the utility functions. Right panel: the utility possibility frontier.



S.3.2 Computing Pareto-optimal allocations

We explain how to compute Pareto-optimal allocations in Section 4.3 for general EU agents. For $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$, the goal is to find $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ that maximizes $\sum_{i=1}^n \lambda_i \mathbb{E}[u_i(X_i)]$. Since the constraint $(X_1, \dots, X_n) \in \mathbb{A}_n(X)$ is pointwise on Ω , and EU has a simple integral form, the above maximum can be computed by point-by-point optimization for each value x of X , that is,

$$W_{\boldsymbol{\lambda}}(x) = \sup \left\{ \sum_{i=1}^n \lambda_i u_i(x_i) : \sum_{i=1}^n x_i = x \text{ and } (x_1, \dots, x_n) \in \mathbb{R}_+^n \right\}, \quad x \in \mathbb{R}_+.$$

An optimizer $(x_1(x), \dots, x_n(x))$ exists due to continuity (it may not be unique), and it yields a $\boldsymbol{\lambda}$ -optimal allocation $X_i = x_i(X)$ for $i \in [n]$, assuming measurability. We then have

$$\mathbb{E}[W_{\boldsymbol{\lambda}}(X)] = \max_{(X_1, \dots, X_n) \in \mathbb{A}_n(X)} \sum_{i=1}^n \lambda_i \mathbb{E}[u_i(X_i)].$$

For any subset $S \subseteq [n]$ and $x \in \mathbb{R}_+$, we write

$$W_{\boldsymbol{\lambda}}^S(x) = \sup \left\{ \sum_{i \in S} \lambda_i u_i(x_i) : \sum_{i \in S} x_i = x \text{ and } x_i \in \mathbb{R}_+ \text{ for } i \in S \right\}, \text{ with } W_{\boldsymbol{\lambda}}^{\emptyset} = 0.$$

Let $T = [n] \setminus S$. Then, we have

$$W_{\boldsymbol{\lambda}}(x) = \sup \left\{ W_{\boldsymbol{\lambda}}^S(y) + W_{\boldsymbol{\lambda}}^T(x-y) : y \in [0, x] \right\}, \quad (\text{S.5})$$

which is a one-dimensional optimization problem if $W_{\boldsymbol{\lambda}}^S$ and $W_{\boldsymbol{\lambda}}^T$ are computable.

In the mixed case specified in Assumption EUM, we have by Theorem 2 that $W_{\boldsymbol{\lambda}}^S(x) = \max_{i \in S} \lambda_i u_i(x)$, and the computation of $W_{\boldsymbol{\lambda}}^T$ is a standard convex program (maximization of a concave function on \mathbb{R}_+^n under a linear constraint). In this case, both $W_{\boldsymbol{\lambda}}^S$ and $W_{\boldsymbol{\lambda}}^T$ are easy to compute, so the overall problem boils down to a one-dimensional optimization for the sum of a convex and a concave function in (S.5). Moreover, the optimal allocation can be obtained in two steps: first, compute (X_T, X_S) from (S.5); second, construct a Pareto-optimal jackpot allocation of X_S among agents in S and a Pareto-optimal comonotonic allocation of X_T among agents in T .

Next, let us specialize in the setting of Example 3 and verify the claims therein. We first recall the setting. For $i \in S$, u_i is the convex function $u_i(x) = 3x + x^2$ and for $i \in T$, u_i is a strictly increasing and strictly concave function satisfying $u_i(x) = 5x - tx^2$ on $[0, 2/t]$, where t is the cardinality of T . The aggregate payoff X is distributed on $[0, 2]$. Suppose that $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}_+^n$ satisfy $\lambda_i = \lambda_T > 0$ for $i \in T$ and $\lambda_i = \lambda_S > 0$ for $i \in S$.

First, let us compute $W_{\boldsymbol{\lambda}}^S$ and $W_{\boldsymbol{\lambda}}^T$. Using Theorem 2, we have

$$W_{\boldsymbol{\lambda}}^S(x) = \max_{i \in S} \lambda_S u_i(x) = \lambda_S(3x + x^2), \quad x \in \mathbb{R}_+$$

By standard convexity argument, we also have

$$W_{\boldsymbol{\lambda}}^T(x) = \sum_{i \in T} \lambda_T u_i\left(\frac{x}{t}\right) = \lambda_T(5x - x^2), \quad x \in [0, 2].$$

This means that the $\boldsymbol{\lambda}$ -optimal allocation among agents in T is proportional. Note that we do not need to specify $W_{\boldsymbol{\lambda}}^T(x)$ for $x > 2$. We analyze the two cases of $\boldsymbol{\lambda}$ separately.

- (a) Let $\lambda_S = 5/4$ and $\lambda_T = 1$. In this case, $W_{\boldsymbol{\lambda}}^S(y) + W_{\boldsymbol{\lambda}}^T(x-y)$ is convex in y ; that is, risk seeking prevails. To compute a $\boldsymbol{\lambda}$ -optimal allocation, we need to maximize, as in (S.5),

$$W_{\boldsymbol{\lambda}}^S(y) + W_{\boldsymbol{\lambda}}^T(x-y) = \frac{5}{4}(3y + y^2) + 5(x-y) - (x-y)^2$$

over $y \in [0, x]$. By convexity, we have either $y = x$ or $y = 0$ at the optimum, leading to $X_T = X\mathbf{1}_{\{X \leq c\}}$ and $X_S = X\mathbf{1}_{\{X > c\}}$, where the threshold $c = 5/9$ can be easily computed by comparing the end-points. Finally, using Theorem 3 and Proposition 4, we get that a λ -optimal allocation (X_1, \dots, X_n) is given by

$$X_i = \frac{X}{t}\mathbf{1}_{\{X \leq c\}}, \quad i \in T \quad \text{and} \quad X_i = XJ_i\mathbf{1}_{\{X > c\}}, \quad i \in S,$$

where $(J_i)_{i \in S}$ is any jackpot vector.

- (b) Let $\lambda_S = 1$ and $\lambda_T = 2$. In this case, $W_\lambda^S(y) + W_\lambda^T(x - y)$ is concave in y ; that is, risk aversion prevails. To compute a λ -optimal allocation, we need to maximize, as in (S.5),

$$W_\lambda^S(y) + W_\lambda^T(x - y) = 3y + y^2 + 10(x - y) - 2(x - y)^2.$$

The above function is concave in y , and its maximum may not necessarily be attained by $y = x$ or $y = 0$. For instance, with $x = 2$, the maximum is uniquely attained at $y = 1/2$. Therefore, if we take $X = 2$, then $X_T = 3/2$ and $X_S = 1/2$ necessarily hold, and the λ -optimal allocation cannot be a jackpot allocation.

S.3.3 A self-contained proof of Theorem 4, part (i)

Proof of Theorem 4, part (i). Suppose for contradiction that (X_1, \dots, X_n, Q) is an equilibrium but (X_1, \dots, X_n) is strictly dominated by another allocation $(Y_1, \dots, Y_n) \in \mathbb{A}_n(X)$. There exists $j \in [n]$ such that $\mathbb{E}[u_j(Y_j)] > \mathbb{E}[u_j(X_j)]$, and by the fact that (X_1, \dots, X_n, Q) is an equilibrium, we have that $\mathbb{E}^Q[Y_j] > \mathbb{E}^Q[\xi_j] \geq \mathbb{E}^Q[X_j]$. Because $\sum_{i=1}^n \mathbb{E}^Q[Y_i] = \mathbb{E}^Q[X] = \sum_{i=1}^n \mathbb{E}^Q[X_i]$, there exists $i \in [n]$, $i \neq j$ such that $\mathbb{E}^Q[Y_i] < \mathbb{E}^Q[X_i]$. By Pareto dominance we also have $\mathbb{E}[u_i(Y_i)] \geq \mathbb{E}[u_i(X_i)]$. Let $\alpha = \mathbb{E}[X_i - Y_i]/\mathbb{E}[X - Y_i]$. Because $\mathbb{E}^Q[Y_i] < \mathbb{E}^Q[X_i] \leq \mathbb{E}^Q[X]$, we have $\alpha \in (0, 1]$. Let $Z_i = Y_i + (X - Y_i)\alpha$. It is clear that $Y_i \leq Z_i \leq X$ and $\mathbb{E}^Q[Z_i] = \mathbb{E}^Q[X_i] \leq \mathbb{E}^Q[\xi_i]$. Recall that $\mathbb{E}^Q[Y_i] < \mathbb{E}^Q[X]$, which implies $Q(Z_i > Y_i) > 0$, and hence, $\mathbb{P}(Z_i > Y_i) > 0$. Because u_i is strictly increasing, we obtain $\mathbb{E}[u_i(Z_i)] > \mathbb{E}[u_i(Y_i)] \geq \mathbb{E}[u_i(X_i)]$, contradicting individual optimality for agent i . \square

S.3.4 Details in Example 5

Let $v = u(y_0)$. The allocation $\bar{\mathbf{X}}$ yields the maximum utility v to all agents, and hence it is Pareto optimal. To show strict domination, let (X_1, \dots, X_n) be a jackpot allocation and $p_i = \mathbb{P}(X_i > 0)$ for all i . With this allocation, agent i has utility

$$\int_0^v \bar{w}(\mathbb{P}(u(X_i) > x)) dx \leq vw(p_i).$$

Since at least one p_i is less than or equal to $1/n$, the condition $w(1/n) < 1$ guarantees that at least one agent has a utility less than v , and thus the jackpot allocation is strictly dominated by $\bar{\mathbf{X}}$.

S.4 Existence of competitive equilibria

We discuss the existence of competitive equilibria in the setting of Section 5 under Assumption EURS for a given initial endowment $(\xi_1, \dots, \xi_n) \in \mathbb{A}_n(X)$. We obtain two results in this appendix in the case of two risk-seeking EU agents and in the case of proportional initial endowments.

S.4.1 Two risk-seeking EU agents

Our next result shows that, for any two risk-seeking EU agents, a competitive equilibrium exists for any initial endowment. The result also illustrates that the equilibrium price is not unique.

Proposition S.3. *Assume $n = 2$ and Assumptions ER and EURS. For any initial endowment vector $(\xi_1, \xi_2) \in \mathbb{A}_2(X)$, there exists a competitive equilibrium (X_1, X_2, Q) , where*

$$\frac{dQ}{d\mathbb{P}} = \frac{u_1(X)}{X} \frac{1}{\mathbb{E}[u_1(X)/X]} . \quad (\text{S.6})$$

Proof. Without loss of generality, we can assume $\mathbb{E}[\xi_1] > 0$ and $\mathbb{E}[\xi_2] > 0$; otherwise $(X, 0)$ or $(0, X)$ is an equilibrium allocation with any equilibrium price. Moreover, we can assume $\mathbb{P}(X = 0) = 0$, because the allocation on the event $\{X = 0\}$ is trivial.

For a random variable W , a *tail event* is an event A such that for some $w \in \mathbb{R}$, $W \geq w$ on A and $W \leq w$ on A^c . In an atomless probability space, a tail event with any given

probability $\lambda \in (0, 1)$ exists, as shown by Wang and Zitikis (2021). Let $W = u_1(X)/u_2(X)$. For $\lambda \in (0, 1)$, let $(A^\lambda)_{\lambda \in (0,1)}$ be an increasing family of tail events of W such that $\mathbb{P}(A^\lambda) = \lambda$. We can check that the mapping $\lambda \mapsto \mathbb{E}^Q[X\mathbf{1}_{A^\lambda}]$ is continuous (because $\lambda \mapsto Q(A^\lambda)$ is continuous) and its range is the open interval $(0, \mathbb{E}^Q[X])$. Therefore, there exists $\lambda^* \in (0, 1)$ such that $\mathbb{E}^Q[X\mathbf{1}_{A^\lambda}] = \mathbb{E}^Q[\xi_1] \in (0, \mathbb{E}^Q[X])$. Write $A_1 = A^{\lambda^*}$ and $A_2 = (A^{\lambda^*})^c$. By definition of the tail event, for some $w^* \geq 0$, we have $W \geq w^*$ on A_1 and $W \leq w^*$ on A_2 .

We will show that $(X_1, X_2, Q) = (X\mathbf{1}_{A_1}, X\mathbf{1}_{A_2}, Q)$ is a competitive equilibrium. The budget condition is satisfied by $\mathbb{E}^Q[X\mathbf{1}_{A_1}] = \mathbb{E}^Q[\xi_1]$ and $\mathbb{E}^Q[X\mathbf{1}_{A_2}] = \mathbb{E}^Q[X] - \mathbb{E}^Q[\xi_1] = \mathbb{E}^Q[\xi_2]$. Market clearance is immediate. It remains to show individual optimality. Denote by $z = \mathbb{E}[u_1(X)/X]$. For any Y with $0 \leq Y \leq X$ such that $\mathbb{E}^Q[Y] \leq \mathbb{E}^Q[\xi_1] = \mathbb{E}^Q[X\mathbf{1}_{A_1}]$, using that $x \mapsto u_1(x)/x$ is increasing, we have

$$\mathbb{E}\left[Y \frac{u_1(Y)}{Y}\right] \leq \mathbb{E}\left[Y \frac{u_1(X)}{X}\right] = z\mathbb{E}^Q[Y] \leq z\mathbb{E}^Q[\xi_1] = z\mathbb{E}^Q[X\mathbf{1}_{A_1}] = \mathbb{E}[u_1(X_1)].$$

Hence, $\mathbb{E}[u_1(Y)] \leq \mathbb{E}[u_1(X_1)]$. For any Y with $0 \leq Y \leq X$ such that $\mathbb{E}^Q[Y] \leq \mathbb{E}^Q[\xi_2] = \mathbb{E}^Q[X\mathbf{1}_{A_2}]$, we have

$$\begin{aligned} \mathbb{E}[u_2(Y)] &\leq \mathbb{E}\left[Y \frac{u_2(X)}{X}\right] \leq \mathbb{E}\left[Y \frac{w^*u_1(X)}{X}\mathbf{1}_{A_1}\right] + \mathbb{E}\left[Y \frac{u_2(X)}{X}\mathbf{1}_{A_2}\right] \\ &= w^*z\mathbb{E}^Q[Y\mathbf{1}_{A_1}] + \mathbb{E}\left[Y \frac{u_2(X)}{X}\mathbf{1}_{A_2}\right]. \end{aligned}$$

Moreover, $\mathbb{E}^Q[Y] \leq \mathbb{E}^Q[X\mathbf{1}_{A_2}]$ implies $\mathbb{E}^Q[Y\mathbf{1}_{A_1}] \leq \mathbb{E}^Q[(X - Y)\mathbf{1}_{A_2}]$. Hence,

$$\begin{aligned} \mathbb{E}[u_2(Y)] &\leq w^*z\mathbb{E}^Q[(X - Y)\mathbf{1}_{A_2}] + \mathbb{E}\left[Y \frac{u_2(X)}{X}\mathbf{1}_{A_2}\right] \\ &\leq \mathbb{E}\left[\frac{X - Y}{X}w^*u_1(X)\mathbf{1}_{A_2}\right] + \mathbb{E}\left[Y \frac{u_2(X)}{X}\mathbf{1}_{A_2}\right] \\ &\leq \mathbb{E}\left[\frac{X - Y}{X}u_2(X)\mathbf{1}_{A_2}\right] + \mathbb{E}\left[Y \frac{u_2(X)}{X}\mathbf{1}_{A_2}\right] = \mathbb{E}[u_2(X\mathbf{1}_{A_2})]. \end{aligned}$$

Hence, $\mathbb{E}[u_2(Y)] \leq \mathbb{E}[u_2(X_2)]$. Therefore, individual optimality holds, and (X_1, X_2, Q) is a competitive equilibrium. \square

The equilibrium price in (S.6) has the form of (7) with $(\lambda_1, \lambda_2) = (1, 0)$. Because the positions of agents 1 and 2 are symmetric, we immediately get another equilibrium price by replacing u_1 in (7) with u_2 . Therefore, the equilibrium price is generally not unique, unless

$u_1 = u_2$ (seen in Theorem 7). Different equilibrium prices correspond to different equilibrium allocations, but they can be derived from the same vector of initial endowments. Therefore, for a given set of initial endowments, the competitive equilibrium is generally not unique, which is in contrast to the case of strictly risk-averse agents, where a unique competitive equilibrium can often be derived from given initial endowments.

The proof techniques for Proposition S.3 do not generalize to the case of $n \geq 3$ agents because our construction of the equilibrium (jackpot) allocation $(X_1, X_2) = (X\mathbf{1}_{A_1}, X\mathbf{1}_{A_2})$ heavily relies on the ratio $u_1(x)/u_2(x)$. Roughly speaking, we choose A_1 as the event where $u_1(X)/u_2(X)$ is large, and we choose A_2 as the event where $u_1(X)/u_2(X)$ is small. This approach is similar to part (iv) of Theorem 2, but its generalization to $n \geq 3$ agents is unclear.

S.4.2 A general fixed-point approach

We outline a general approach under some assumptions, and prove the existence of competitive equilibria in the special case of proportional endowments. First, we make an assumption of no-ties in the weighted utility functions.

Assumption NT. For $i \neq j$, $\{x \in \mathbb{R}_+ : \lambda_i u_i(x) = \lambda_j u_j(x)\}$ is finite for any $\lambda_i, \lambda_j > 0$, and X is continuously distributed.

A simple example of utility functions satisfying Assumption NT is that agent i is more risk seeking than agent $i + 1$ for $i \in [n]$; that is, $u_i = T_i \circ u_{i+1}$ for some increasing and strictly convex function T_i for $i \in [n-1]$. In this case, for any $\lambda_i, \lambda_j > 0$, $i \neq j$, we have that $\lambda_i u_i$ and $\lambda_j u_j$ cross at most once under Assumption EURS. For instance, this holds for $u_i(x) = x^{\alpha_i}$, $i \in [n]$, with distinct values of α_i . For an illustration, see the left panel of Figure S.1.

For any given $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \Delta_n$, define the sets $A_i^\boldsymbol{\lambda} = \{\lambda_i u_i(X) = V_{\boldsymbol{\lambda}}(X)\}$ for $i \in [n]$ and let $\mathbf{J}^\boldsymbol{\lambda} = (\mathbf{1}_{A_1^\boldsymbol{\lambda}}, \dots, \mathbf{1}_{A_n^\boldsymbol{\lambda}})$. For $i \neq j$, Assumption NT implies that $\mathbb{P}(\lambda_i u_i(X) = \lambda_j u_j(X)) = 0$ because X is continuously distributed. Hence, $\mathbb{P}(A_i^\boldsymbol{\lambda} \cap A_j^\boldsymbol{\lambda}) = 0$ and $\sum_{i=1}^n \mathbf{1}_{A_i^\boldsymbol{\lambda}} = 1$ (almost surely). Thus, $X\mathbf{J}^\boldsymbol{\lambda}$ is a jackpot allocation.

Next, we introduce three useful objects. Define a probability measure $Q^\boldsymbol{\lambda}$ by

$$\frac{dQ^\boldsymbol{\lambda}}{d\mathbb{P}} = \frac{V_{\boldsymbol{\lambda}}(X)}{X} \frac{1}{\mathbb{E}[V_{\boldsymbol{\lambda}}(X)/X]} \quad \text{with the convention } 0/0 = 0,$$

a function $f : \Delta_n \rightarrow \Delta_n$ by

$$f(\boldsymbol{\lambda}) = \left(\frac{\mathbb{E}^{Q^\boldsymbol{\lambda}}[X\mathbf{1}_{A_1^\boldsymbol{\lambda}}]}{\mathbb{E}^{Q^\boldsymbol{\lambda}}[X]}, \dots, \frac{\mathbb{E}^{Q^\boldsymbol{\lambda}}[X\mathbf{1}_{A_n^\boldsymbol{\lambda}}]}{\mathbb{E}^{Q^\boldsymbol{\lambda}}[X]} \right) = \frac{\mathbb{E}^{Q^\boldsymbol{\lambda}}[X\mathbf{J}^{\boldsymbol{\lambda}}]}{\mathbb{E}^{Q^\boldsymbol{\lambda}}[X]}, \quad \boldsymbol{\lambda} \in \Delta_n,$$

and a function $g : \Delta_n \rightarrow \Delta_n$ by

$$g(\boldsymbol{\lambda}) = \left(\frac{\mathbb{E}^{Q^\boldsymbol{\lambda}}[\xi_1]}{\mathbb{E}^{Q^\boldsymbol{\lambda}}[X]}, \dots, \frac{\mathbb{E}^{Q^\boldsymbol{\lambda}}[\xi_n]}{\mathbb{E}^{Q^\boldsymbol{\lambda}}[X]} \right), \quad \boldsymbol{\lambda} \in \Delta_n.$$

Note that $\mathbb{E}^{Q^\boldsymbol{\lambda}}[X] > 0$ holds under Assumption EURS. Our goal is to find $\boldsymbol{\lambda} \in \Delta_n$ such that $f(\boldsymbol{\lambda}) = g(\boldsymbol{\lambda})$. The next proposition justifies that finding such $\boldsymbol{\lambda}$ is sufficient for finding a competitive equilibrium.

Proposition S.4. *Under Assumptions EURS and NT, if $\boldsymbol{\lambda} \in \Delta_n$ and $f(\boldsymbol{\lambda}) = g(\boldsymbol{\lambda})$, then $(X\mathbf{J}^{\boldsymbol{\lambda}}, Q^{\boldsymbol{\lambda}})$ is a competitive equilibrium for the vector of initial endowments $(\xi_1, \dots, \xi_n) \in \mathbb{A}_n(X)$.*

Proof. Write $(X_1, \dots, X_n, Q) = (X\mathbf{J}^{\boldsymbol{\lambda}}, Q^{\boldsymbol{\lambda}})$ and $(\lambda_1, \dots, \lambda_n) = \boldsymbol{\lambda}$. The equality $f(\boldsymbol{\lambda}) = g(\boldsymbol{\lambda})$ implies $\mathbb{E}^Q[X_i] = \mathbb{E}^Q[\xi_i]$ for $i \in [n]$. Let $z = \mathbb{E}[V_{\boldsymbol{\lambda}}(X)/X]$. Fix $i \in [n]$. For any Y with $0 \leq Y \leq X$ such that $\mathbb{E}^Q[Y] \leq \mathbb{E}^Q[\xi_i]$, we have

$$\begin{aligned} \mathbb{E}[\lambda_i u_i(Y)] &= \mathbb{E}\left[Y \frac{\lambda_i u_i(Y)}{Y}\right] \\ &\leq \mathbb{E}\left[Y \frac{\lambda_i u_i(X)}{X}\right] \leq \mathbb{E}\left[Y \frac{V_{\boldsymbol{\lambda}}(X)}{X}\right] = z\mathbb{E}^Q[Y] \leq z\mathbb{E}^Q[\xi_i] = z\mathbb{E}^Q[X_i]. \end{aligned}$$

Moreover, since $A_i^{\boldsymbol{\lambda}} = \{\lambda_i u_i(X) = V_{\boldsymbol{\lambda}}(X)\}$, we have $V_{\boldsymbol{\lambda}}(X)\mathbf{1}_{A_i^{\boldsymbol{\lambda}}} = \lambda_i u_i(X)\mathbf{1}_{A_i^{\boldsymbol{\lambda}}} = \lambda_i u_i(X_i)$, and this implies

$$z\mathbb{E}^Q[X_i] = \mathbb{E}\left[X\mathbf{1}_{A_i^{\boldsymbol{\lambda}}} \frac{V_{\boldsymbol{\lambda}}(X)}{X}\right] = \mathbb{E}[\lambda_i u_i(X_i)].$$

Hence, $\mathbb{E}[u_i(Y_i)] \leq \mathbb{E}[u_i(X_i)]$ and thus X_i satisfies individual optimality for agent i . The market clearance condition $\sum_{i=1}^n X_i = X$ holds true because $\sum_{i=1}^n \mathbf{1}_{A_i^{\boldsymbol{\lambda}}} = 1$. Therefore, (X_1, \dots, X_n, Q) is a competitive equilibrium. \square

The remaining task is to find $\boldsymbol{\lambda}$ with $f(\boldsymbol{\lambda}) = g(\boldsymbol{\lambda})$. We do not know a general solution to this problem, but in the simplified scenario of proportional endowments, the problem can be solved.

Assumption PE. The initial endowment vector (ξ_1, \dots, ξ_n) is equal to $(\theta_1 X, \dots, \theta_n X)$ for some $(\theta_1, \dots, \theta_n) \in \Delta_n$.

Under Assumption PE, we have $g(\boldsymbol{\lambda}) = (\theta_1, \dots, \theta_n)$ for any $\boldsymbol{\lambda} \in \Delta_n$. In this situation, we can show that $f(\boldsymbol{\lambda}) = g(\boldsymbol{\lambda})$ holds, through a technique established by Jamison and Ruckle (1976).

Proposition S.5. *If Assumptions EURS, NT and PE hold, then there exists a competitive equilibrium of the form $(X\mathbf{J}^{\boldsymbol{\lambda}}, Q^{\boldsymbol{\lambda}})$ for some $\boldsymbol{\lambda} \in \Delta_n$.*

Proof. A face of Δ_n is the set $\Delta_n^D = \{(x_1, \dots, x_n) \in \Delta_n : x_j = 0 \text{ for } j \in D\}$ for some $D \subseteq [n]$. Lemma S.1 below guarantees that f is a continuous function that carries each face of Δ_n into itself. This condition allows us to apply Jamison and Ruckle (1976, Lemma 2.1), which implies that f is surjective. Hence, there exists $\boldsymbol{\lambda} \in \Delta_n$ such that $f(\boldsymbol{\lambda}) = (\theta_1, \dots, \theta_n) = g(\boldsymbol{\lambda})$. By Proposition S.4, $(X\mathbf{J}^{\boldsymbol{\lambda}}, Q^{\boldsymbol{\lambda}})$ is a competitive equilibrium. \square

Lemma S.1. *If Assumptions EURS and NT hold, then f is a continuous function that carries each face of Δ_n into itself.*

Proof. For $i \in [n]$, define $f_i : \Delta_n \rightarrow \mathbb{R}_+$ by $f_i(\boldsymbol{\lambda}) = \mathbb{E}^{Q^{\boldsymbol{\lambda}}}[X\mathbf{1}_{A_i^{\boldsymbol{\lambda}}}]$ for $\boldsymbol{\lambda} \in \Delta_n$. We have

$$f_i(\boldsymbol{\lambda}) = \mathbb{E}^{Q^{\boldsymbol{\lambda}}}[X\mathbf{1}_{A_i^{\boldsymbol{\lambda}}}] = \mathbb{E}[V_{\boldsymbol{\lambda}}(X)\mathbf{1}_{A_i^{\boldsymbol{\lambda}}}] = \mathbb{E}[\lambda_i u_i(X)\mathbf{1}_{A_i^{\boldsymbol{\lambda}}}].$$

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n) \in \Delta_n$ be such that $\|\boldsymbol{\lambda} - \boldsymbol{\zeta}\| := \sum_{i=1}^n |\lambda_i - \zeta_i| < \varepsilon$. As X is continuously distributed and u_1, \dots, u_n are continuous, Assumption NT implies

$$\begin{aligned} p &:= \mathbb{P}(A_i^{\boldsymbol{\lambda}} \cup A_i^{\boldsymbol{\zeta}}) - \mathbb{P}(A_i^{\boldsymbol{\lambda}} \cap A_i^{\boldsymbol{\zeta}}) \\ &= \mathbb{P}\left(X \in \bigcup_{x^*: \lambda_i u_i(x^*) = \lambda_j u_j(x^*)} \{x^* - c_1\varepsilon < x < x^* + c_1\varepsilon\}\right) < c_2\varepsilon \end{aligned}$$

for some $c_1, c_2 > 0$, because switching from $A_i^{\boldsymbol{\lambda}}$ to $A_i^{\boldsymbol{\zeta}}$ or back can only happen at points in some neighborhoods of $\{x : \lambda_i u_i(x) = \lambda_j u_j(x)\}$. This implies that $\boldsymbol{\lambda} \mapsto \mathbb{E}[u_i(X)\mathbf{1}_{A_i^{\boldsymbol{\lambda}}}]$ is continuous, further guaranteeing that f_i is continuous. Therefore, $\hat{f} := \sum_{i=1}^n f_i$ is also a continuous function. Moreover, under Assumption EURS, we have $\hat{f} > 0$. Since $f = (f_1/\hat{f}, \dots, f_n/\hat{f})$, we know that f is continuous. Moreover, if $\lambda_i = 0$, then $\mathbb{P}(A_i^{\boldsymbol{\lambda}}) = 0$ and thus $\mathbb{E}^{Q^{\boldsymbol{\lambda}}}[X\mathbf{1}_{A_i^{\boldsymbol{\lambda}}}] / \mathbb{E}^{Q^{\boldsymbol{\lambda}}}[X] = 0$. Hence, for any face Δ_n^D and $\boldsymbol{\lambda} \in \Delta_n^D$, we have $f(\boldsymbol{\lambda}) \in \Delta_n^D$. \square

S.5 Extensions of Theorem 8

We briefly discuss a few dimensions in which the statements in Theorem 8 can readily be generalized. We did not pursue these generalizations because they do not seem to offer stronger empirical relevance than Assumption H-RDU.

- (a) Assumption H-RDU allows for w to be concave. In this case, $\bar{w} = w$, and the agents are risk seeking for payoffs valued in $[0, x_0]$.
- (b) By inspecting the proof of Theorem 8, it suffices to require $w = \bar{w}$ on $[0, 1/n]$, and whether w is concave or convex beyond $1/n$ is irrelevant.
- (c) The result remains true if u is convex on $[0, x_0]$ instead of being linear, following the same proof, by noting that an agent with a convex utility function and the probability weighting function \bar{w} is risk seeking, which is the main step to apply Theorem 1.
- (d) A careful inspection of the proofs of main results reveals that, for most of our results on risk-seeking EU agents, it suffices to assume that $x \mapsto u(x)/x$ is increasing instead of the convexity of u (this condition is weaker than convexity with $u(0) = 0$). Moreover, for the RDU agents in Assumption H-RDU, we can use this condition on $[0, x_0]$ instead of linearity, and the results in Theorem 8 hold true.

We formally prove the assertion in (d) below. Let u be an increasing function with $u(0) = 0$ and $x \mapsto u(x)/x$ is increasing. We first show an analogue of Theorem 1. Let (X_1, \dots, X_n) and (Y_1, \dots, Y_n) be as in the proof of Theorem 1. Note that

$$u(Y_i) = u(X \mathbb{1}_{\{Z_{i-1} \leq U < Z_i\}}) = u(X) \mathbb{1}_{\{Z_{i-1} \leq U < Z_i\}} \geq \frac{X u(X_i)}{X_i} \mathbb{1}_{\{Z_{i-1} \leq U < Z_i\}} \mathbb{1}_{\{X_i > 0\}}.$$

Hence,

$$\begin{aligned} \mathbb{E}[u(Y_i) | X_1, \dots, X_n] &\geq \frac{X u(X_i)}{X_i} \mathbb{1}_{\{X_i > 0\}} \mathbb{E}[\mathbb{1}_{\{Z_{i-1} \leq U < Z_i\}} | X_1, \dots, X_n] \\ &= \frac{X u(X_i)}{X_i} \mathbb{1}_{\{X_i > 0\}} \frac{X_i}{X} \\ &= u(X_i) \mathbb{1}_{\{X_i > 0\}} = u(X_i). \end{aligned}$$

This shows $u(Y_i) \geq_{\text{icx}} u(X_i)$, where \geq_{icx} is increasing convex order (meaning $\mathbb{E}[\phi(Y_i)] \geq \mathbb{E}[\phi(X_i)]$ for all increasing convex ϕ). This implies $\rho_{\bar{w}}(u(Y_i)) \geq \rho_{\bar{w}}(u(X_i))$ because $\rho_{\bar{w}}$ is increasing in convex order (e.g., Wang et al. 2020, Theorem 3), and increasing convex order can be decomposed into convex order and first-order stochastic dominance (e.g., Shaked and Shanthikumar 2007, Theorem 4.A.6). Therefore, the jackpot allocation (Y_1, \dots, Y_n) dominates (X_1, \dots, X_n) , and for sum-optimality it suffices to consider jackpot allocations. The rest of the proof follows the same arguments in the proof of parts (i) and (ii) of Theorem 8 with X replaced by $u(X)$. Parts (iii) and (iv) do not rely on the properties of u on $[0, x_0]$.

We finally note that if u is convex on $[0, a]$ and concave on $[a, \infty)$, then $x \mapsto u(x)/x$ is increasing on some interval $[0, b]$ with $b \geq a$ (often $b > a$).

References

- Dall'Aglio, G. (1972). Fréchet classes and compatibility of distribution functions. *Symposia Mathematica*, **9**, 131–150.
- Denneberg, D. (1994). *Non-additive Measures and Integral*. Kluwer, Dordrecht.
- Jamison, R. E. and Ruckle, W. H. (1976). Factoring absolutely convergent series. *Mathematische Annalen*, **224**, 143–148.
- Lauzier, J.-G., Lin, L. and Wang, R. (2023). Pairwise counter-monotonicity. *Insurance: Mathematics and Economics*, **111**, 279–287.
- Liu, P., Wang, R. and Wei, L. (2020). Is the inf-convolution of law-invariant preferences law-invariant? *Insurance: Mathematics and Economics*, **91**, 144–154.
- Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer Series in Statistics.
- Wang, R., Wei, Y. and Willmot, G. E. (2020). Characterization, robustness and aggregation of signed Choquet integrals. *Mathematics of Operations Research*, **45**(3), 993–1015.