1 The Photon

$$c = 2.998 \cdot 10^8 \ \left[\tfrac{m}{s}\right]$$

$$h = 6.626 \cdot 10^{-34} \ \left[\tfrac{m^2 \, kg}{s}\right]$$

$c \ \left[\tfrac{m}{s}\right]$	speed of light
$h \ \left[\tfrac{m^2 \, kg}{s}\right]$	planc's constant
$e \ [C]$	electorn charge
$m_e \ [kg]$	electron mass
$k_B \ \left[\tfrac{m^2 \, kg}{s^2 \, K}\right]$	bolzmann constant
$\epsilon_0 \ \left[\tfrac{F}{m}\right]$	vacuum permittivity

$$\hbar = \tfrac{h}{2\pi}$$

$$e = 1.602 \cdot 10^{-19} \ [C]$$

$$m_e = 9.109 \cdot 10^{-31} \ [kg]$$

$$k_B = 1.381 \cdot 10^{-23} \ \left[\tfrac{m^2 \, kg}{s^2 \, K}\right]$$

$$\epsilon_0 = 8.854 \cdot 10^{-12} \ \left[\tfrac{F}{m}\right]$$

$$1 \ [eV] = 1.602 \cdot 10^{-19} \ \left[\tfrac{kg \, m^2}{s^2}\right] \ [J]$$

1.1 Photon & Electron

$\lambda \ [m], \ \nu \ \left[\tfrac{1}{s}\right]$	Wavelength, Freq.
k	Wavenumber
$E \ [J]$	Energy
$\vec{F}_c \ [N]$	Coulomb Force

$$\lambda = \frac{c}{\nu} \quad \nu = \frac{c}{\lambda} \quad \omega = 2\pi\nu$$

$$k = \frac{2\pi\nu}{c}$$

$$E = h \cdot \nu = \hbar \cdot \omega$$

$$\left|\vec{F}_c\right| = \frac{Q_1 \cdot Q_2}{4\pi\epsilon_0 r^2}$$

1.2 Photoelectric effect

$V \ [V]$	Voltage
$\phi_0 \ [eV]$	Work function
$I \ [A]$	Photo-current
$n \ \left[m^{-3}\right]$	Volume density of electrons
$A \ \left[m^2\right]$	Area
$v \ \left[\tfrac{m}{s}\right]$	velocity of electrons

$$h\nu - \phi_0 = \frac{1}{2}mv^2 = eV$$

$$V(\nu) = \frac{h}{e}\nu - \frac{\phi_0}{e}$$

$$I = nAve$$

1.3 Blackbody Radiation

$L \ [m]$	length of blackbody cube	k_i	wave constants
E_x	Electric field in x-direction	$<E>$	Average Energy
N	Number of states	D	Density of states
u	Blackbody radiation	I	Power radiated

$$E_x(x, y, z) = E_{0x} \cos(k_x x) sin(k_y y) sin(k_z z)$$

$$k_x = n\frac{\pi}{L} \quad k_y = m\frac{\pi}{L} \quad k_z = l\frac{\pi}{L} \quad k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

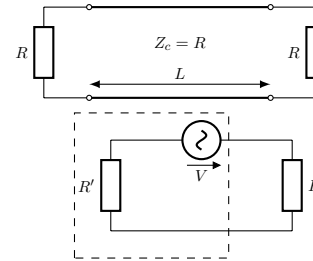$$N(k) = \frac{1}{3\pi^2}k^3 L^3 \qquad D(k) = \frac{k^2}{\pi^2}$$

$$u(\omega) = \frac{\omega^2}{\pi^2 c^3} \cdot \frac{\hbar\omega}{\exp\left(\frac{-\hbar\omega}{kT}\right) - 1}d\omega \qquad u(\nu) = \frac{8\pi h\nu^3}{c^3\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)}d\nu$$

$$I(\omega) = c \cdot u(\omega)$$

Equipartition-Theorem: Each degree of Freedom has an energy of kT

1.4 Johnson-Noise

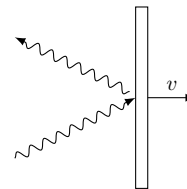This is the noise created in a one-dimensional circuit (like a coax-cable).

$\langle V^2 \rangle$	Noise Voltage
$\Delta\nu$	Bandwidth

$$E = E_0 \cdot \sin(k_x \cdot x)$$

$$\langle V^2 \rangle = 4R \cdot k_B T \cdot \Delta\nu$$

1.5 Momentum of a photon

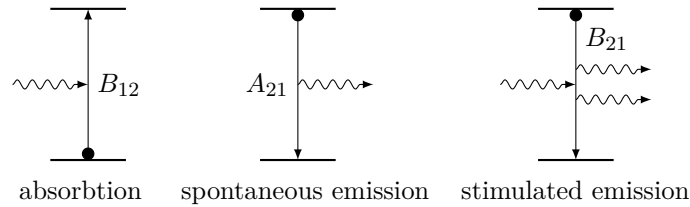|	
$p \ \left[\tfrac{kg \, m}{s}\right]$	momentum

$$p_{absorbing} = \frac{h\nu}{c} = m \cdot v$$

$$p_{reflecting} = 2 \cdot \frac{h\nu}{c}$$

$$p = \sqrt{2m_e e \Delta V}$$

1.6   Absorption, spontaneous and stimulated emission

| absorbtion | spontaneous emission | stimulated emission |

n_1 Number of electrons in the lower energy state

n_2 Number of electrons in the higher energy state

$$\frac{dn_2}{dt} = \underbrace{n_1 \cdot u(\nu) \cdot B_{12}}_{\text{absorbtion}} - \underbrace{n_2 \cdot u(\nu) \cdot B_{21}}_{\text{stimulated emission}} - \underbrace{n_2 \cdot A_{21}}_{\text{spontaneous emission}}$$

$$\frac{n_2}{n_1} = e^{-\frac{h\nu}{k_B T}} = \frac{u(\nu)B_{12}}{u(\nu)B_{21} + A_{21}}$$

$$B_{21} = B_{12} = B \qquad A_{21} = \frac{8\pi h \nu^3}{c^3}$$

1.7   Laser-optical amplification

Electrons are excited from the ground state "0" to the level "3" by pumping through incoherent radiation. The electrons then fall onto a long-lived state n_2 (State "2") from level "3". The pumping can be done either optically by shining a strong incoherent light or by passing a current. It is also assumed that the lower state is quickly emptied by a fast process with lifetime τ_1. As a result, the population in state "2" is:

$$n_2 = \frac{R}{A_{21}} \quad \text{whereas} \quad n_1 \approx 0 \quad \text{because} \quad A_{21} < \frac{1}{\tau_1}$$

We have rherefore a population inversion between the two states. The likelihood of a stimulated emission process is larger than the one of absorbtion. If we enclose the system in an optical cavity, we can achieve self-sustained oscillation at the frequency ν. b

2   Wave mechanics

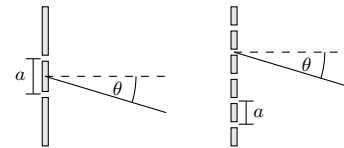|	frequency	wavelength	momentum	energy
Particle		$\lambda_b = \frac{h}{p}$	$p = mv$	$E = \frac{1}{2}mv^2$
Wave	ω	$\lambda = \frac{2\pi c}{\omega}$	$p = \frac{\hbar\omega}{c}$	$E = \hbar\omega$

2.1   Compton Scattering
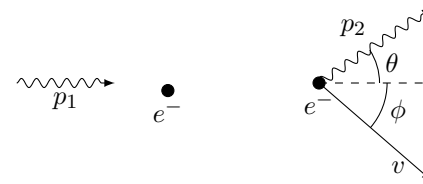
$$p_1 = \frac{h\nu_1}{c} \qquad p' = \frac{h\nu_2}{c}$$
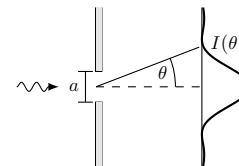
$$\nu_2 = \nu_1 - \frac{P_e^2}{2m_e h}$$

$$\lambda_2 - \lambda_1 = \frac{h}{m_e c}(1 - \cos\theta);$$

2.2   Double Slit and Bragg Diffraction

Constructive $\sin\theta = \dfrac{n\lambda}{a}$

Destructive $\sin\theta = \dfrac{(n + \frac{1}{2})\lambda}{a}$

$n \in \mathbb{Z}$

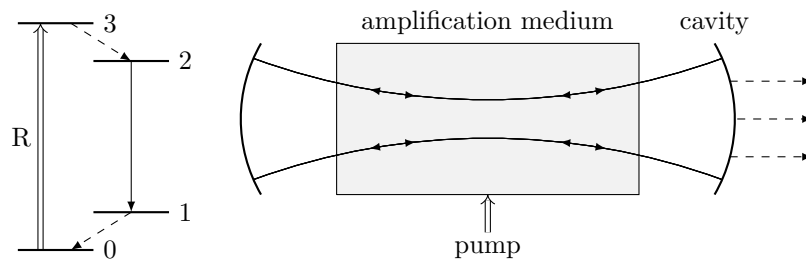## 2.3   Single slit and uncertainty relation

$$I(\theta) = I_0 \frac{\sin^2\theta}{\theta^2} \qquad\qquad \sin\theta = \frac{\lambda}{a}$$

$$\Delta x \Delta p \geq \hbar \qquad \Delta t \Delta E \geq \hbar \ (E = \hbar\omega)$$

2.4 Bohr-Sommerfeld quantisation

Every single particle must satisfy the following equation. The quantized energy levels below relate to the hydrogen atom

$$\int_{length} p \cdot ds = n \cdot h \qquad n \in \mathbb{N}$$

p	Momentum of particle
E_n	Energy of the nth state
E_{ry}	Rydberg Energy
a_0	Bohr-radius
Z	Number of protons

$$E_n = -\frac{Z^2}{n^2} \cdot \frac{m_e e^4}{8\epsilon_0^2 h^2} = -\frac{Z^2}{n^2} \cdot E_{ry}$$

$$r_n = \frac{n^2}{Z} \cdot \frac{2\epsilon_0 h}{m_e e^2} = \frac{n^2}{Z} \cdot a_0$$

$$E_{ry} = 13.6 \ [\text{eV}]$$

$$a_0 = 5.292 \cdot 10^{-11} \ [\text{m}]$$

3 Quantum Mechanics

3.1 Wave function

$$\psi(\boldsymbol{x},t) : \mathbb{R}^4 \to \mathbb{C} \qquad \iiint |\psi(\boldsymbol{x},t)|^2 \, d^3r = 1$$

$$\psi(\boldsymbol{x},t) = a\psi_1(\boldsymbol{x},t) + b\psi_2(\boldsymbol{x},t), \qquad |a|^2 + |b|^2 = 1$$

$$P(x)dx = |\psi(x)|^2 \, dx \qquad P_{ab} = \int_a^b |\psi(x)|^2 dx \qquad \langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx$$

3.2 The Schrödinger equation

$$i\hbar \cdot \frac{\partial \Psi}{\partial t}(\boldsymbol{x},t) = -\frac{\hbar^2}{2m} \cdot \nabla^2 \Psi(\boldsymbol{x},t) + V(\boldsymbol{x},t)\Psi(\boldsymbol{x},t)$$

$$\Psi = A \cdot e^{i(\boldsymbol{kx}-\omega t)} \qquad \boldsymbol{k} = \begin{bmatrix} k_x & k_y & k_z \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$$

$$E = \omega\hbar = \frac{\hbar^2 k^2}{2m}, \qquad k^2 = |k|^2$$

3.2.1 Phase and Group Velocity

The phase velocity v_φ describes how fast the phase of the wave moves forward. The group velocity v_g describes how fast the energy is moving forward.

$$v_\varphi = \frac{\omega}{k} \qquad v_g = \frac{\partial \omega}{\partial k} \qquad \text{Particle wave: } v_\varphi \cdot 2 = v_g$$

3.2.2 Stationary (Time independent) States

In a stationary state, the wave function is a product of a function $\varphi(\boldsymbol{x})$ independent of time and a function $\chi(t)$ independent of space.

$$\Psi_n(\boldsymbol{x},t) = \psi_n(\boldsymbol{x}) \cdot \chi_n(t) = \psi_n(\boldsymbol{x}) \cdot e^{-i\frac{E_n}{\hbar}t}$$

$$-\frac{\hbar^2}{2m}\nabla^2 \psi_n(\boldsymbol{x}) + V(\boldsymbol{x})\psi_n(\boldsymbol{x}) = \psi_n(\boldsymbol{x}) \cdot E_n$$

$$\iiint |\Psi|^2 \, d^3\boldsymbol{x} = \iiint |\psi|^2 \, d^3\boldsymbol{x} = 1$$

$$\Psi(\boldsymbol{x},t) = \sum a_n \psi_n(\boldsymbol{x}) \cdot e^{-i\frac{E_n}{\hbar}t} \qquad \sum |a_n|^2 = 1$$

Requirements: The wave function must be continous, as well as it's derivative

3.2.3 Example: 1D infinite potential well

$$\Psi_1 = \Psi_3 = 0$$
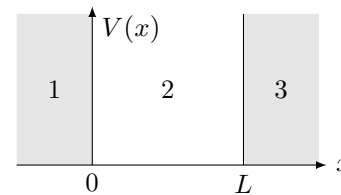
$$-\frac{\hbar^3}{2m}\frac{\partial^2}{\partial x^2}\psi_2(x,t) = E\psi_2(x,t)$$

$$\psi_2 = A\sin(kx) + B\cos(kx)$$

Boundary cond.: $\psi_2(0) = \psi_2(L) = 0$

$$\psi_{2n} = A \cdot \sin(k_n x) \qquad \Psi_{2n} = A \cdot \sin(k_n x) \cdot e^{-i\frac{E_n}{\hbar}x}, \quad \text{Normalize:} \quad A = \sqrt{\frac{2}{L}}$$

$$E_n = n^2 \cdot \frac{\hbar^2 \pi^2}{2mL} = n^2 \cdot E_0, \qquad k_n = \frac{n\pi}{L}$$

3.2.4 Example: 1D finite potential well
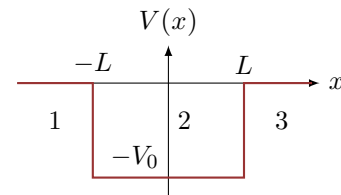
The Energy E can be either bigger or smaller than 0. If $E > 0$, the wave function will decay exponentially in region 1 and 3. If $E < 0$, the wave will propagate away from the potential well.

Inside the well: The general solution to the rearranged Schrödinger's is:

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_2(x) = (E - V_0)\psi_2(x)$$

$$\psi_2(x) = A_2 e^{ikx} + A_2' e^{-ikx} \qquad E = \frac{k^2 \hbar^2}{2m} \quad k = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

Outside the well: There are two cases, which can apply:

1. $E > 0$:**Unbound state**

$$-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi_1(x) = E\psi_1(x) \qquad \psi_1 = A_1 e^{ikx} + A_1' e^{-ikx} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

The unbound state does not make sense to be investigated, because the particle is free to be anywhere. In the following, only the unbound state is considered.

2. $E < 0$: **Bound state**

$$-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi_1(x) = E\psi_1(x) \qquad \psi_1 = B_1 e^{\delta x} + B_1' e^{-\delta x} \qquad \delta = \sqrt{-\frac{2mE}{\hbar^2}}$$

We see that as $x \to -\infty$, the Term B_1', as well as B_3 approaches ∞. Since the wave function cannot approach ∞, $B_1' = B_3 = 0$ is a condition.
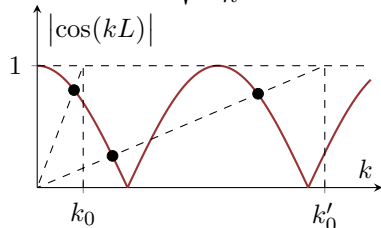
Boundary conditions: We require, that the wave function is continuous, as well as it's spacial derivative. Therefore, we have:

$$\psi_1(-L) = \psi_2(-L) \qquad \psi_2(L) = \psi_3(L)$$

$$\frac{\partial}{\partial x}\psi_1(-L) = \frac{\partial}{\partial x}\psi_2(-L) \qquad \frac{\partial}{\partial x}\psi_2(L) = \frac{\partial}{\partial x}\psi_3(L)$$

Even solutions: only even (cosine) components	**Odd solutions**: only odd (sine) components
$\left\lvert\cos(kL)\right\rvert = \dfrac{k}{k_o}, \quad \tan(kL) > 0$	$\left\lvert\sin(kL)\right\rvert = \dfrac{k}{k_o}, \quad \tan(kL) > 0$
$k_0 = \sqrt{\dfrac{2mV_0}{\hbar^2}}$	$k_0 = \sqrt{\dfrac{2mV_0}{\hbar^2}}$
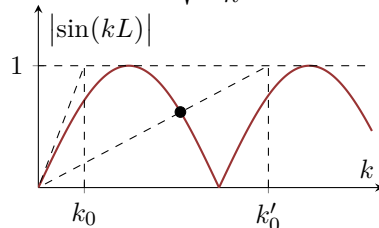
3.3 Example: 1D potential step function
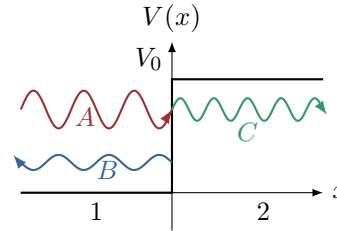
An incoming plane wave from the left hits a potential step at $x = 0$. In region 1, two waves are added together, one is traveling to the right and one to the left. If $E > V_0$, the wave is transmitted to region 2. if $E < V_0$, the wave decays exponentially in region 2.

In **Region 1**, the general solution to the Schrödinger equation is:

$$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_1(x) = E\psi_1(x), \quad \psi_1(x) = Ae^{ik_1 x} + Be^{-ik_1 x}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

In **Region 2**, there are two cases, which can apply:

1. $\mathbf{E > V_0}$: **Transmission**

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_2 = (E - V_0)\psi_2(x) \qquad \psi_2 = Ce^{ik_2 x}, \qquad k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$$

2. $\mathbf{E < V_0}$: **Complete reflection**

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi_2 = (E - V_0)\psi_2(x) \qquad \psi_2 = Ce^{\delta_2 x}, \qquad \delta_2 = \sqrt{\frac{2m(V_0 - 2)}{\hbar^2}}$$

Applying the **initial conditions**, which require the wave function and it's derivative to be continuous at $x = 0$, we get the following expression for A, B, C:

$$\psi_1(x = 0) = \psi_2(x = 0) \qquad \frac{\partial}{\partial x}\psi_1(x = 0) = \frac{\partial}{\partial x}\psi_2(x = 0)$$

$\mathbf{E > V_0}$	$\mathbf{E < V_0}$
$A + B = C$	$A + B = C$
$k_1(A - B) = k_2 C$	$A = B$

The **probability density function** $\left\lvert\psi(x,t)\right\rvert^2 = \left\lvert\varphi(x)\right\rvert^2 = \varphi \cdot \varphi^*$ can then be computed and sketched:

$\mathbf{E > V_0}$	$\mathbf{E < V_0}$
$\lvert\psi_1\rvert^2 = A^2 + B^2 + 2AB\cos(2k_1 x)$	$\lvert\psi_1\rvert^2 = 2A^2 \cdot \left(1 - \sin(2k_1 x)\right)$
$\lvert\psi_2\rvert^2 = C^2$	$\lvert\psi_2\rvert^2 = C^2 \cdot e^{-2\delta x}$
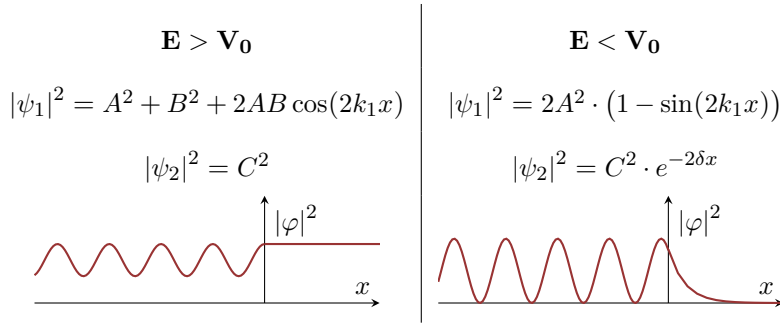
To find the **transmission coefficient** T and the **reflection coefficient** R, we normalize $A = 1$. Then, we can define $B = \sqrt{R}$ and $C = \sqrt{T}$. Then, we can solve for R and T:

$$T = \frac{4k_1 k_2}{(k_1 + k_2)^2} \qquad R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

If $E < V_0$, nothing is transmitted and therefore $T = 0$ and $R = 1$.

3.3.1   Example: 1D finite potential barrier

An incoming plane wave from the left hits a potential barrier with length l. The Transmission coefficient tells, how much of the wave can continue at the other side of the barrier (quantum tunneling).

In **Region 1 and 3**, the general expression for the wave equation is the following:

$$\psi_j(x) = A_j e^{ik_j x} + A_j' e^{-ik_j x}, \qquad k_j = \sqrt{\frac{2mE}{\hbar^2}}, \quad j \in \{1,3\}$$

In **Region 2**, the expression is depending on V_0. There are two cases:

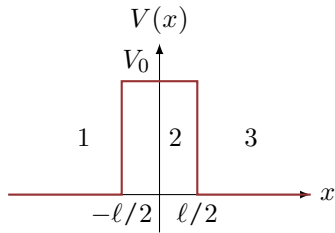1. **E < V_0**: $\varphi_2 = B_2 e^{\delta_2 x} + B_2' e^{-\delta_2 x}, \qquad \delta_2 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$

2. **E > V_0**: $\varphi_2 = A_2 e^{ik_2 x} + A_2' e^{-ik_2 x}, \qquad k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}$

Apply **boundary conditions** at $x = -\ell/2$ and $x = \ell/2$ in order to determine all constants. If the wave is only traveling from left to right, then $A_3' = 0$.

$$\psi_1(-\ell/2) = \psi_2(-\ell/2), \quad \psi_2(\ell/2) = \psi_3(\ell/2)$$

$$\frac{\partial}{\partial x}\psi_1(-\ell/2) = \frac{\partial}{\partial x}\psi_2(-\ell/2), \quad \frac{\partial}{\partial x}\psi_2(\ell/2) = \frac{\partial}{\partial x}\psi_3(\ell/2)$$

Then, the **transmission coefficient** T and the **reflection coefficient** R can be calculated as following:

$$R = \left(\frac{A_1}{A_1'}\right)^2, \qquad T = \left(\frac{A_3}{A_1}\right)^2$$

$\mathbf{E < V_0}$	$\mathbf{E > V_0}$
$T = \dfrac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 \sinh^2(\delta_2 \ell)}$	$T = \dfrac{4E(V_0 - E)}{4E(V_0 - E) + V_0^2 \sin^2(k_2 \ell)}$

If $\mathbf{E > V_0}$, the transmission coefficient has a maximum. If $k_2 \ell = n\pi \Rightarrow T = 1$ (**resonance**). The minimum of Tu is at: $k_2 \ell = \pi/2 + n\pi$.

4   Wave Function Space (Hilbert Space)

4.1   Inner Product

The inner product $\langle\psi_1|\psi_2\rangle$ is defined like the scalar product for vectors. If the inner product $\langle\psi_1|\psi_2\rangle = 0$, ψ_1, ψ_2 are **orthogonal**.

$$\langle\psi_1|\psi_2\rangle = \int_{-\infty}^{\infty} \psi_1^*(\boldsymbol{x},t)\psi_2(\boldsymbol{x},t)d^3\boldsymbol{x}$$

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} \psi^*(\boldsymbol{x},t)\psi(\boldsymbol{x},t)d^3\boldsymbol{x} = \int_{-\infty}^{\infty} \lvert\psi(\boldsymbol{x},t)\rvert^2 d^3\boldsymbol{x} = 1$$

4.2   Fourier Transform

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{\frac{ipx}{\hbar}}\varphi(p)dp, \quad \varphi(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} e^{\frac{ipx}{\hbar}}\psi(x)dx$$

$$\psi(\vec{\boldsymbol{x}}) = \frac{1}{(2\pi\hbar)^{3/2}}\int_{-\infty}^{\infty} e^{\frac{i\vec{\boldsymbol{p}}\vec{\boldsymbol{x}}}{\hbar}}\varphi(\vec{\boldsymbol{p}})d\vec{\boldsymbol{p}}, \quad \varphi(\vec{\boldsymbol{p}}) = \frac{1}{(2\pi\hbar)^{3/2}}\int_{-\infty}^{\infty} e^{\frac{i\vec{\boldsymbol{p}}\vec{\boldsymbol{x}}}{\hbar}}\psi(\vec{\boldsymbol{x}})d\vec{\boldsymbol{x}}$$

$$\int_{-\infty}^{\infty} \psi_1^*(x) \cdot \psi_2(x) \cdot dx = \int_{-\infty}^{\infty} \varphi_1^*(p) \cdot \varphi_2(p) \cdot dp$$

5 Observable Measurements, Time-dependence

Doing a measurement in quantum mechanics (observable) can be interpreted as applying an operator $\widehat{A}$ on the wave function $\psi(\boldsymbol{x}, t)$. For example, tu o compute the expected position $\langle \boldsymbol{x} \rangle_\psi$, we apply the operator $\widehat{\boldsymbol{x}} = \boldsymbol{x}$ to average the wave function:

$$\langle \boldsymbol{x} \rangle_\Psi = \iiint \Psi^*(\boldsymbol{x}, t) \cdot \boldsymbol{x} \cdot \Psi(\boldsymbol{x}, t) d^3 \boldsymbol{x} = \iiint \boldsymbol{x} \cdot |\Psi(\boldsymbol{x}, t)|^2 \, d^3 \boldsymbol{x}$$

Name	Operator	
Position	$\widehat{\boldsymbol{x}} = [\boldsymbol{x}]$	
Momentum	$\widehat{\boldsymbol{p}} = [-i\hbar \boldsymbol{\nabla}]$	$\boldsymbol{\nabla} = \left[\frac{\partial}{\partial x} \; \frac{\partial}{\partial y} \; \frac{\partial}{\partial z} \right]^T$
Hamiltonian	$\widehat{H} = \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\boldsymbol{x}) \right]$	$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

5.1 Canonical commutation relation

The commutators is a way of describing the effect of the order, in which multiple operators are applied.

$$\left[\widehat{A}, \widehat{B} \right] = \widehat{A}\widehat{B} - \widehat{B}\widehat{A}, \quad \left[\widehat{A}, \widehat{B} \right] = -\left[\widehat{B}, \widehat{A} \right], \quad \left[\widehat{A}, \widehat{A} \right] = 0$$

$$\left[\widehat{A}, (\widehat{B} + \widehat{C}) \right] = \left[\widehat{A}, \widehat{B} \right] + \left[\widehat{A}, \widehat{C} \right]$$

$$[\widehat{p}_x, \widehat{p}_y] = 0, \; [\widehat{x}, \widehat{p}_x] = i\hbar, \; [\widehat{z}, \widehat{p}_x] = [\widehat{z}, \widehat{p}_y] = 0$$

5.2 Eigenstates and Eigenvalues

An Observable has an Operator $\widehat{A}$. a state $u_n(x)$ is called an eigenstate the operator applied on the wave function acts like a scalar multiplication to it. Then, the measurement of the general state $\psi(x)$ is a superposition of all the eigenstates.

$$\widehat{A} u_n(x) = a_n u_n(x), \quad \int_{-\infty}^{\infty} u_n^*(x) \widehat{A} u_n(x) dx = a_n \quad \widehat{A}\psi(x) = \sum_n c_n u_n(x)$$

5.3 Harmonic Oscillator

A Quantum mechanical harmonic oscillator can be interpreted as the solution to the Schrödinger equation:

$$\left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x) = E\psi(x), \quad V(x) = \frac{1}{2}kx^2 = \frac{m\omega^2}{2} x^2$$

To simplify the equation, we define a new length scale and energy:

$$a = \sqrt{\frac{\hbar}{m\omega}}, \quad \tilde{x} = \frac{x}{a}, \quad \tilde{E} = \frac{E}{\hbar\omega} \Rightarrow \frac{1}{2}\left[-\frac{\partial^2}{\partial \tilde{x}^2} + \tilde{x}^2 \right] \varphi(\tilde{x}) = \tilde{E}\varphi(\tilde{x})$$

Then, the solutions to the equation is:

$$E_n = \left(n + \frac{1}{2} \right) \hbar\omega, \quad \psi(\tilde{x}) = c_n H_n(\tilde{x}) e^{-\tilde{x}/2}, \quad H_n(\tilde{x}) = (-1)^n e^{\tilde{x}^2} \cdot \frac{\partial^n}{\partial \tilde{x}^n} e^{-\tilde{x}^2}$$

$$H_0(\tilde{x}) = 1, \quad H_1(\tilde{x}) = 2\tilde{x}, \quad H_2(\tilde{x}) = 4\tilde{x}^2 - 2, \quad H_3(\tilde{x}) = 8\tilde{x}^3 - 12\tilde{x}$$

$$\Psi_n(x) = \frac{1}{\sqrt[4]{\pi}\sqrt{2^n n! a}} \cdot H_n\left(\frac{x}{a} \right) e^{-\frac{x^2}{2a^2}}$$
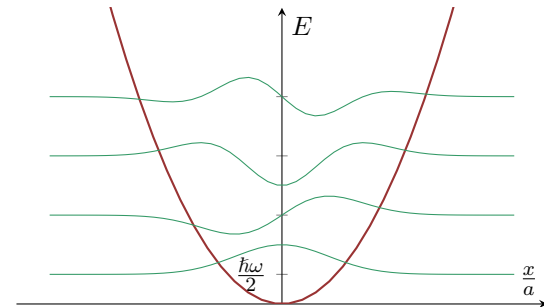
5.4 The coupled quantum well
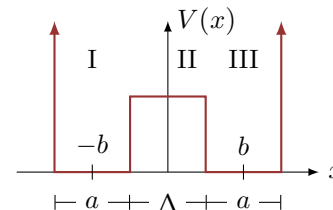
This is the simplified potential of an ammonia molecule NH_3. The wave function outside the well $(|x| > b + \frac{a}{2})$ is zero. There exists a symmetric, as well as an antisymmetric solution. We consider the case: $\boldsymbol{E < V_0}$

$$\psi_{\text{II}} = \begin{cases} \mu \cosh(\delta x) & \text{symmetric} \\ \mu \sinh(\delta x) & \text{antisymmetric} \end{cases} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}, \quad \delta = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$\tan(ka)$

symmetric: $\varepsilon_s = \dfrac{1 + e^{-\delta\Delta}}{\delta a}$

antisymmetric: $\varepsilon_a = \dfrac{1 - e^{-\delta\Delta}}{\delta a}$

$$\tan(ka) = -ka\varepsilon = -ka\frac{1 \pm e^{-\delta\Delta}}{\delta a}$$

Now, we can create a superposition of both the symmetric and the antisymmetric case:

$$\psi_{s_\text{I}} = +\lambda \sin\left(k\left(b - \frac{a}{2} + x\right)\right), \quad \psi_{s_\text{III}} = +\lambda \sin\left(k\left(b - \frac{a}{2} + x\right)\right)$$

$$\psi_{a_\text{I}} = -\lambda \sin\left(k\left(b - \frac{a}{2} + x\right)\right), \quad \psi_{a_\text{III}} = +\lambda \sin\left(k\left(b - \frac{a}{2} + x\right)\right)$$
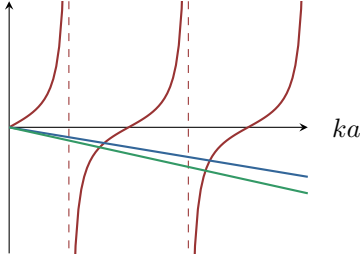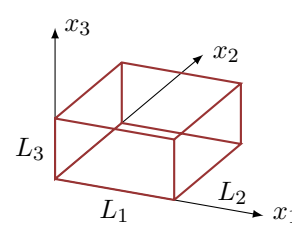
$$\Psi_L = \frac{1}{\sqrt{2}}(\Psi_s - \Psi_a), \qquad \Psi_R = \frac{1}{\sqrt{2}}(\Psi_s + \Psi_a)$$

$$\Psi_L(x,t) = \frac{1}{\sqrt{2}} e^{-i\omega_s t}\left(\psi_s(x) - e^{-i(\omega_a - \omega_s)t}\psi_a(x)\right)$$

$$\omega_a = \frac{E_a}{\hbar}, \quad \omega_s = \frac{E_a}{\hbar}, \quad E_a - E_s = \frac{\hbar^2 \pi^2}{2m\delta a^2} \cdot 8 e^{-\delta\Delta}$$

From the formula describing the wave equation, we can see that at t_0, the particle can only be found in region I, and after some time $t_{1/2}$, the particle can only be found in region III. The particle has tunneled from one side to the other. Now, we can define a period $T = \frac{2\pi\hbar}{E_a - E_s}$

6 Schrödinger Equation in 3D

$$V(x_i) = \begin{cases} 0 & \text{if } 0 < x_i < L_i \\ \infty & \text{otherwise} \end{cases}$$

$$V(x,y,z) = V(x_1) + V(x_2) + V(x_3)$$

$$\psi(x_1, x_2, x_3) = \psi_1(x_1) \cdot \psi_2(x_2) \cdot \psi_3(x_3)$$

$$-\frac{\hbar^2}{2m}\left[\frac{\psi_1''(x_1)}{\psi_1(x_1)} + \frac{\psi_2''(x_2)}{\psi_2(x_2)} + \frac{\psi_3''(x_3)}{\psi_3(x_3)}\right] + V(x_1) + V(x_2) + V(x_3) = E$$

This equation can be separated into three smaller equations for every spacial dimension x_i

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2}\psi_i(x_i) + V(x_i)\psi_i(x_i) = E_i\psi_i(x_i)$$

$$E_i^{(n_i)} = n_i^2 \frac{\hbar^2 \pi^2}{2mL_i^2}, \qquad \psi_i^{(n_1)} = A \cdot \sin\left(\frac{\pi n_i x}{L_i}\right)$$

After normalizing, the wave function can be written as:

$$\psi(x_1, x_2, x_3) = \sqrt{\frac{8}{L_1 L_2 L_3}} \sin\left(\frac{\pi n_1 x_1}{L_1}\right) \sin\left(\frac{\pi n_2 x_2}{L_2}\right) \sin\left(\frac{\pi n_3 x_3}{L_3}\right)$$

When $L_1 = L_2 = L_3$, there sometimes exists multiple states (**degeneracies**) for the same energy $E = E_1 + E_2 + E_3$. Now, we can generate new solutions to the wave function via superposition of those states. In general, degeneracies arise from symmetries (obvious or hidden).

6.1 Schrödinger Equation in spherical coordinates
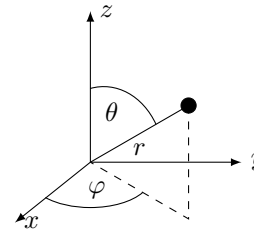
$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

$$\psi_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r) \cdot Y_\ell^m(\theta,\varphi) = R_{n\ell}(r) \cdot P_\ell^m(\cos\theta)e^{im\varphi}$$

The angular part $Y_\ell^m(\theta,\varphi)$ can be written as:

$$P_\ell^m(x) = (i - x^2)^{\frac{|m|}{2}} \frac{d^{|m|}}{dx^{|m|}} P_\ell(x) \qquad P_\ell(x) = \frac{1}{2^\ell \cdot \ell!} \frac{\partial^\ell}{dx^\ell}(x^2 - 1)^\ell$$

The solution to Y will be a **spherical harmonic**. Finally, we must apply the normalization

$$\int_0^\infty |R(r)|^2 r^2 dr = 1, \qquad \int_{\theta=0}^\pi \int_{\varphi=-\pi}^\pi |Y_\ell^m(\theta,\varphi)|^2 \sin\theta d\varphi d\theta = 1$$

These solutions are the same as **spherical harmonics**. They form an **orthogonal basis**, meaning that every well-behaved function $f(\theta,\varphi)$ can be expressed as a superposition of those harmonics.

6.1.1 Hydrogen Atom

The radial part $R_{n\ell}$ of the hydrogen atom with potential $V(r) = \frac{-e^2}{4\pi\epsilon_0 r}$ can be written as:

$$R_{n\ell}(r) = \frac{1}{r}\rho^{\ell+1}e^{-\rho}v(\rho), \quad \rho = \frac{r}{na_0}, \quad a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} \approx 5.29 \cdot 10^{-11} \text{ [m]}$$

$$\psi_{n\ell m}(r,\theta,\varphi) = R_{n\ell}(r)Y_\ell^m(\theta,\varphi) \qquad j_{max} = (n - \ell - 1) \geq 0 \qquad |m| \leq \ell$$

$v(\rho)$ is a polynomial of degree j_{max} with coefficients: $C_{g+1} = \frac{2(g+l+1-n)}{(g+1)(g+2l+2)}C_g$. For state n, there are $d(n) = n^2$ different solutions (**degeneracies**). The **effective radius** is na_0. The **probability** of of finding an electron between r and $r + dr$ is:

$$p(r)dr = r^2 |R_{n\ell}(r)|^2 dr$$

6.1.2 Quantum Numbers

n is the main quantum number, ℓ is the orbital quantum number and m is the magnetic quantum number (projection of angular momentum). Chemists give the different ℓ's different names.

- $\ell = 0$: the orbital is called an s-state ($\max p(r)dr$ is at $r = 0$).
- $\ell = 1$: the orbital is called an p-state ($p(r = 0)dr = 0$).
- $\ell = 2$: the orbital is called an d-state.

7 Angular Momentum and Spin

$$\widehat{L}_x = \widehat{y}\widehat{p}_z - \widehat{z}\widehat{p}_y \qquad \widehat{L}_y = \widehat{z}\widehat{p}_x - \widehat{x}\widehat{p}_z \qquad \widehat{L}_z = \widehat{x}\widehat{p}_y - \widehat{y}\widehat{p}_x \qquad \boldsymbol{\widehat{L}} = \det \begin{vmatrix} \boldsymbol{e_x} & \boldsymbol{e_y} & \boldsymbol{e_z} \\ \widehat{x} & \widehat{y} & \widehat{z} \\ \widehat{p}_x & \widehat{p}_y & \widehat{p}_z \end{vmatrix}$$

$$\left[\widehat{L}_x, \widehat{L}_y\right] = i\hbar\widehat{L}_z \qquad \left[\widehat{L}_y, \widehat{L}_z\right] = i\hbar\widehat{L}_x \qquad \left[\widehat{L}_z, \widehat{L}_x\right] = i\hbar\widehat{L}_y$$

$$\widehat{L}_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\theta\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right), \ \widehat{L}_y = -i\hbar\left(\cos\varphi\frac{\partial}{\partial\theta} - \frac{\cos\theta\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

$$\widehat{L}_z = -i\hbar\frac{\partial}{\partial\varphi}, \quad \widehat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

Angular momentum operators do not commute. In order to get commutable operators, we introduce $\widehat{L}^2 = \widehat{L}_x^2 + \widehat{L}_y^2 + \widehat{L}_z^2$

$$\left[\widehat{L}^2, \widehat{L}_x\right] = 0, \qquad \left[\widehat{L}^2, \widehat{L}_y\right] = 0, \qquad \left[\widehat{L}^2, \widehat{L}_z\right] = 0$$

$$\widehat{L}^2 Y_\ell^m(\theta,\varphi) = \hbar^2\ell(\ell+1)Y_\ell^m(\theta,\varphi), \qquad \widehat{L}_z Y_\ell^m(\theta,\varphi) = \hbar m Y_\ell^m(\theta,\varphi)$$

7.1 Ladder Operator

If a ladder operators $\widehat{L}_\pm = \widehat{L}_x \pm i\widehat{L}_y$ are used in the following way: Suppose, we have a wave function ψ, which is simultaneously an eigenfunction of $\widehat{L}^2$ and $\widehat{L}_z$. Then, $\widehat{L}_\pm\psi$ is also an eigenfunction of $\widehat{L}^2$ and $\widehat{L}_z$ with the following eigenvalues:

$$\widehat{L}^2\psi = \lambda\psi, \quad \widehat{L}_z\psi = \mu\psi$$
$$\widehat{L}^2(\widehat{L}_\pm\psi) = \lambda(\widehat{L}_\pm\psi), \quad \widehat{L}_z(\widehat{L}_\pm\psi) = (\mu \pm \hbar)(\widehat{L}_\pm\psi)$$

7.2 Spin

Idea: $\ell = \frac{1}{2} \ \rightarrow \ m = \pm\frac{1}{2}$. Instead of using ℓ, we use s to describe the spin. The operators $\widehat{L}_i$ are now called $\widehat{S}_i$. We define the spin as $|s, m_s\rangle$:

$$\chi_+ = \left|\frac{1}{2}, \frac{1}{2}\right\rangle = |\uparrow\rangle, \quad \chi_- = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle = |\downarrow\rangle, \quad \chi = a\chi_+ + \beta\chi_- = \begin{pmatrix} a \\ b \end{pmatrix}, \ |a|^2 + |b|^2 = 1$$

Since we have now only two eigenstates $|\uparrow\rangle$ and $|\downarrow\rangle$, we can write:

$$\widehat{S}^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \widehat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \widehat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \widehat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now, we define the ladder operators in the same way as for the angular momentum:

$$\widehat{S}_+ = \begin{pmatrix} 0 & \hbar \\ 0 & 0 \end{pmatrix} \quad \widehat{S}_- = \begin{pmatrix} 0 & 0 \\ \hbar & 0 \end{pmatrix} \quad \widehat{S}_+\ket{\uparrow} = \widehat{S}_-\ket{\downarrow} = 0, \ \widehat{S}_+\ket{\downarrow} = \hbar\ket{\uparrow}, \ \widehat{S}_-\ket{\uparrow} = \hbar\ket{\downarrow}$$

We can write states states in dirac notation as: $\ket{\ell, m}$. For spins, we get:

$$\ket{1,1} = \ket{\uparrow\uparrow}, \ \ket{1,-1} = \ket{\downarrow\downarrow}, \ \ket{1,0} = \frac{1}{\sqrt{2}}(\ket{\uparrow\downarrow} + \ket{\downarrow\uparrow}), \ \ket{0,0} = \frac{1}{\sqrt{2}}(\ket{\uparrow\downarrow} - \ket{\downarrow\uparrow})$$

7.3 Identical Particles

When we have two particles with wave function $\varphi_a(\vec{x}_1)$ and $\varphi_b(\vec{x}_2)$, we can combine them in two different ways:

- **Fermion**: $\psi_- = \frac{1}{\sqrt{2}}\left(\varphi_a(\vec{x}_1)\varphi_b(\vec{x}_2) - \varphi_b(\vec{x}_1)\varphi_a(\vec{x}_2)\right)$: symmetric (electron)
- **Boson**: $\psi_+ = \frac{1}{\sqrt{2}}\left(\varphi_a(\vec{x}_1)\varphi_b(\vec{x}_2) + \varphi_b(\vec{x}_1)\varphi_a(\vec{x}_2)\right)$: antisymmetric (photon)

Pauli exclusion principle: **Two fermions cannot occupy the same identical state**

7.3.1 Exchange Interactions

We have two particles, which have normalized and orthogonal wave functions. We have three different interactions:

1. The particles are **distinguishable**: $\psi = \psi_a(x_1)\psi_b(x_2)$

$$\left\langle (x_1 - x_2)^2 \right\rangle_\psi = \langle x^2 \rangle_{\psi_a} + \langle x^2 \rangle_{\psi_b} - 2\langle x \rangle_{\psi_a}\langle x \rangle_{\psi_b}$$

2. **Symmetric** wave function: $\psi_+ = \frac{1}{\sqrt{2}}(\ket{\psi_a}\ket{\psi_b} + \ket{\psi_b}\ket{\psi_a})$

$$\left\langle (x_1 - x_2)^2 \right\rangle_\psi = \langle x^2 \rangle_{\psi_a} + \langle x^2 \rangle_{\psi_b} - 2\left| \langle \psi_a|x|\psi_b \rangle \right|^2$$

3. **Antisymmetric** wave function: $\psi_- = \frac{1}{\sqrt{2}}(\ket{\psi_a}\ket{\psi_b} - \ket{\psi_b}\ket{\psi_a})$

$$\left\langle (x_1 - x_2)^2 \right\rangle_\psi = \langle x^2 \rangle_{\psi_a} + \langle x^2 \rangle_{\psi_b} + 2\left| \langle \psi_a|x|\psi_b \rangle \right|^2$$

This means, that the particles are closer together in the antisymmetric case than in the symmetric case. But this does not account for the spin. Because of Pauli's exclusion principle, the electrons must be in a different state, therefore they have different spin. The total angular momentum can be either $\ket{0,0}$ or $\ket{1,0}$. Those are both antisymmetric wave functions. Therefore, the radial part must be symmetric, in which case, both electrons are attracted to eachother.

7.4 Many Electrons: Atomic Shells

Electrons with $n = 2$ are further away from the nucleus than electrons with $n = 1$. Therefore, those further away see a different (smaller) effective charge than those in the inner shell.

To write the wave function of an electron, we use the notation indexed by: $\ket{n, \ell, m}$.

shell n	subshell ℓ	max e^- in subshell	max e^- in shell
K	1s	2	2
L	2s	2	$2 + 6 = 8$
	2p	6	
M	3s	2	$2 + 6 + 10 = 10$
	3p	6	
	3d	10	

Here, the number of degeneracies per shell is displayed. Remember, we have $n \geq 1$, $0 \leq \ell \leq n$, $-\ell \leq m \leq \ell$, and for every different state, the electron can have either spin up or spin down. So, the number of electrons in a sub shell is the number of degeneracies.

8 Useful formulas

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}} \qquad \int_0^{\infty} xe^{-ax^2}\,dc = \frac{1}{2a} \qquad \int_{-\infty}^{\infty} x^2 e^{-ax^2} = \frac{\sqrt{\pi}}{2a^{3/2}}$$

$$\int x^n e^{cx} = e^{cx}\sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!c^{n-i+1}}x^i \qquad \int_0^{\infty} x^n e^{-cx} = \frac{n!}{c^{n+1}}$$

$$\text{Gaussian:} \quad G = A \cdot e^{\frac{-x^2}{2\sigma^2}}$$

8.1 Trigonometry

$$
\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha & \cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(b) \pm \cos(\alpha)\sin(b) & \cos(\alpha \pm \beta) &= \cos(\alpha)\cos(b) \mp \sin(\alpha)\sin(b) \\
\sin(\alpha) \pm \sin(\beta) &= 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2} & \cos(\alpha) + \cos(\beta) &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos(\alpha) - \cos(\beta) &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} & \sin(\alpha)\sin(\beta) &= \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta)) \\
\cos(\alpha)\cos(\beta) &= \frac{1}{2}(\cos(\alpha-\beta) + \cos(\alpha+\beta)) & \sin(\alpha)\cos(\beta) &= \frac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta)) \\
\sin^2\alpha &= \frac{1}{2}(1 - \cos 2\alpha) & \cos^2\alpha &= \frac{1}{2}(1 + \cos 2\alpha) \\
\sin^3\alpha &= \frac{1}{4}(3\sin\alpha - \sin 3\alpha) & \cos^3\alpha &= \frac{1}{4}(3\cos\alpha + \cos 3\alpha) \\
\frac{\sin 2\alpha}{\sin\alpha} &= 2\cos\alpha & \sin\alpha\cos\alpha &= \frac{1}{2}\sin 2\alpha \\
c^2 &= a^2 + b^2 - 2ab\cos\gamma & \frac{a}{\sin\alpha} &= \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2r = \frac{u}{\pi}
\end{aligned}
$$

9 Periodic Table of the Elements

Legend box: Z = atomic number; eneg = electronegativity; ss = subshell; Sy = Symbol, Name = element name, saw = standard atomic weight

Period 1

Z	eneg	ss	Symbol	Name	saw
1	2.20	1s	H	Hydrogen	1.00784–1.00811
2		1s	He	Helium	4.002602(2)

Period 2

Z	eneg	ss	Symbol	Name	saw
3	0.98	2s	Li	Lithium	6.938–6.997
4	1.57	2s	Be	Beryllium	9.0121831(5)
5	2.04	2p	B	Boron	10.806–10.821
6	2.55	2p	C	Carbon	12.0096–12.0116
7	3.04	2p	N	Nitrogen	14.00643–14.00728
8	3.44	2p	O	Oxygen	15.99903–15.99977
9	3.98	2p	F	Fluorine	18.998403163(6)
10		2p	Ne	Neon	20.1797(6)

Period 3

Z	eneg	ss	Symbol	Name	saw
11	0.93	3s	Na	Sodium	22.98976928(2)
12	1.31	3s	Mg	Magnesium	24.304–24.307
13	1.61	3p	Al	Aluminium	26.9815385(7)
14	1.90	3p	Si	Silicon	28.084–28.086
15	2.19	3p	P	Phosphorus	30.973761998(5)
16	2.58	3p	S	Sulphur	32.059–32.076
17	3.16	3p	Cl	Chlorine	35.446–35.457
18		3p	Ar	Argon	39.948(1)

Period 4

Z	eneg	ss	Symbol	Name	saw
19	0.82	4s	K	Potassium	39.0983(1)
20	1.00	4s	Ca	Calcium	40.078(4)
21	1.36	3d	Sc	Scandium	44.955908(5)
22	1.54	3d	Ti	Titanium	47.867(1)
23	1.63	3d	V	Vanadium	50.9415(1)
24	1.66	3d*	Cr	Chromium	51.9961(6)
25	1.55	3d	Mn	Manganese	54.938044(3)
26	1.83	3d	Fe	Iron	55.845(2)
27	1.88	3d	Co	Cobalt	58.933194(4)
28	1.91	3d	Ni	Nickel	58.6934(4)
29	1.90	3d*	Cu	Copper	63.546(3)
30	1.65	3d	Zn	Zinc	65.38(2)
31	1.81	4p	Ga	Gallium	69.723(1)
32	2.01	4p	Ge	Germanium	72.630(8)
33	2.18	4p	As	Arsenic	74.921595(6)
34	2.55	4p	Se	Selenium	78.971(8)
35	2.96	4p	Br	Bromine	79.901–79.907
36	3.00	4p	Kr	Krypton	83.798(2)

Period 5

Z	eneg	ss	Symbol	Name	saw
37	0.82	5s	Rb	Rubidium	85.4678(3)
38	0.95	5s	Sr	Strontium	87.62(1)
39	1.22	4d	Y	Yttrium	88.90584(2)
40	1.33	4d	Zr	Zirconium	91.224(2)
41	1.6	4d*	Nb	Niobium	92.90637(2)
42	2.16	4d*	Mo	Molybdenum	95.95(1)
43	1.9	4d	Tc	Technetium	(98)
44	2.2	4d*	Ru	Ruthenium	101.07(2)
45	2.28	4d*	Rh	Rhodium	102.90550(2)
46	2.20	4d*	Pd	Palladium	106.42(1)
47	1.93	4d*	Ag	Silver	107.8682(2)
48	1.69	4d	Cd	Cadmium	112.414(4)
49	1.78	5p	In	Indium	114.818(1)
50	1.96	5p	Sn	Tin	118.710(7)
51	2.05	5p	Sb	Antimony	121.760(1)
52	2.1	5p	Te	Tellurium	127.60(3)
53	2.66	5p	I	Iodine	126.90447(3)
54	2.60	5p	Xe	Xenon	131.293(6)

Period 6

Z	eneg	ss	Symbol	Name	saw
55	0.79	6s	Cs	Cesium	132.90545196(6)
56	0.89	6s	Ba	Barium	137.327(7)
57-71			*	Lanthanides	
72	1.3	5d	Hf	Hafnium	178.49(2)
73	1.5	5d	Ta	Tantalum	180.94788(2)
74	2.36	5d	W	Tungsten	183.84(1)
75	1.9	5d	Re	Rhenium	186.207(1)
76	2.2	5d	Os	Osmium	190.23(3)
77	2.20	5d	Ir	Iridium	192.217(3)
78	2.28	5d*	Pt	Platinum	195.084(9)
79	2.54	5d*	Au	Gold	196.966569(5)
80	2.00	5d	Hg	Mercury	200.592(3)
81	1.62	6p	Tl	Thallium	204.382–204.385
82	1.87	6p	Pb	Lead	207.2(1)
83	2.02	6p	Bi	Bismuth	208.98040(1)
84	2.0	6p	Po	Polonium	(209)
85	2.2	6p	At	Astatine	(210)
86	2.2	6p	Rn	Radon	(222)

Period 7

Z	eneg	ss	Symbol	Name	saw
87	0.7	7s	Fr	Francium	(223)
88	0.9	7s	Ra	Radium	(226)
89-103			**	Actinides	
104		6d	Rf	Rutherfordium	(261)
105		6d	Db	Dubnium	(268)
106		6d	Sg	Seaborgium	(269)
107		6d	Bh	Bohrium	(270)
108		6d	Hs	Hassium	(269)
109		6d	Mt	Meitnerium	(278)
110		6d	Ds	Darmstadtium	(281)
111		6d	Rg	Roentgenium	(282)
112		6d	Cn	Copernicium	(285)
113		7p	Nh	Nihonium	(286)
114		7p	Fl	Flerovium	(289)
115		7p	Mc	Moscovium	(289)
116		7p	Lv	Livermorium	(293)
117		7p	Ts	Tennessine	(294)
118		7p	Og	Oganesson	(294)

* Lanthanides

Z	eneg	ss	Symbol	Name	saw
57	1.1	5d*	La	Lanthanum	138.90547(7)
58	1.12	4f*	Ce	Cerium	140.116(1)
59	1.13	4f	Pr	Praseodymium	140.90766(2)
60	1.14	4f	Nd	Neodymium	144.242(3)
61	1.13	4f	Pm	Promethium	(145)
62	1.17	4f	Sm	Samarium	150.36(2)
63	1.2	4f	Eu	Europium	151.964(1)
64	1.2	4f*	Gd	Gadolinium	157.25(3)
65	1.1	4f	Tb	Terbium	158.92535(2)
66	1.22	4f	Dy	Dysprosium	162.500(1)
67	1.23	4f	Ho	Holmium	164.93033(2)
68	1.24	4f	Er	Erbium	167.259(3)
69	1.25	4f	Tm	Thulium	168.93422(2)
70	1.1	4f	Yb	Ytterbium	173.045(10)
71	1.27	4f	Lu	Lutetium	174.9668(1)

** Actinides

Z	eneg	ss	Symbol	Name	saw
89	1.1	6d*	Ac	Actinium	(227)
90	1.3	5f*	Th	Thorium	232.0377(4)
91	1.5	5f*	Pa	Protactinium	231.03588(2)
92	1.38	5f*	U	Uranium	238.02891(3)
93	1.36	5f*	Np	Neptunium	(237)
94	1.28	5f	Pu	Plutonium	(244)
95	1.13	5f	Am	Americium	(243)
96	1.28	5f*	Cm	Curium	(247)
97	1.3	5f	Bk	Berkelium	(247)
98	1.3	5f	Cf	Californium	(251)
99	1.3	5f	Es	Einsteinium	(252)
100	1.3	5f	Fm	Fermium	(257)
101	1.3	5f	Md	Mendelevium	(258)
102	1.3	5f	No	Nobelium	(259)
103	1.3	5f	Lr	Lawrencium	(266)

Standard atomic weights taken from the Commission on Isotopic Abundances and Atomic Weights (ciaaw.org/atomic-weights.htm). Adapted from Ivan Griffin's LaTeX Periodic Table. © 2017 Paul Danese

An asterisk (*) next to a subshell indicates an anomalous (Aufbau rule-breaking) ground state electron configuration.