Divide and Conquer

Due: Wednesday, April 17th

Directions: Some of the questions on this assignment will appear on the quiz on Wednesday, April 17th.

Design and analyze means:

- Give pseudocode for your algorithm.
- Prove that your algorithm is correct.
- Give the recurrence relation for the running time for your algorithm.
- Solve the recurrence relation.

1. Rank the following running times in order from fastest to slowest:

   - $17n$
   - $n!$
   - $48n \log_b(n)$
   - n^3
   - 36
   - $367n^2 \log_b(n)$
   - 2^n
   - $\log_b(n)$
   - n^n

2. Solve the following recurrence relations:

   - $T(n) = T(n/2) + O(n)$
   - $T(n) = T(n/2) + O(n^2)$
   - $T(n) = 3T(n/3) + O(1)$
   - $T(n) = T(n/3) + O(n)$

3. In justifying our matrix multiplication algorithm, we accepted the following property: If X and Y are $n \times n$ matrices and

   $$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \qquad Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$$

   where A, B, C, D, E, F, G and H are $n/2$ by $n/2$ matrices, then the product XY can be expressed as:

   $$XY = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

   Prove this property.

4. Suppose that you are given a sorted list of distinct integers $\{a_1, a_2, \ldots a_n\}$. Design and analyze a divide-and-conquer algorithm that determines whether there exists an index i such that $a_i = i$. For example, in $\{-10, -4, 3, 41\}$, $a_3 = 3$, and in $\{4, 7, 19, 20\}$ there is no such i.

5. Suppose you are given 3^n marbles that look identical, with one special marble that weighs more than the other marbles. You are also given a balancing scale that takes two items (or sets of items) and compares their weights. Design and analyze a divide and conquer algorithm to find the heavy marble using the balancing scale at most n times.

6. Suppose that you are given an integer, x, and a sorted list of integers. Design and analyze a divide and conquer algorithm that counts the number of occurrences of x in the list.

   Example: Suppose $x = 4$ and the list is $\{1, 2, 2, 2, 4, 4, 12, 20, 20, 20\}$. Your algorithm should return 2.

7. Suppose that you are given a list of n elements. Design and analyze a divide and conquer algorithm to remove all duplicates from the list in time $O(n \log n)$.

8. A list A is said to have a *majority element* if more than half of its entries are the same. There is not necessarily an order on the list, so there can't be comparisons of the form "Is $A[i] \leq A[j]$?". However, in constant time, the question "Is $A[i] = A[j]$?" can be answered. Given an array, design a divide and conquer algorithm to determine if there is a majority element, and, if so, to find that element. Your algorithm should run in time $O(n \log n)$.

9. Suppose we are given a list of n numbers representing stock prices on a single day. We want to find a pair $(buy, sell)$ with $buy \leq sell$ such that if we bought the stock on *buy* day and sold the stock on *sell* day, we would maximize our profit.

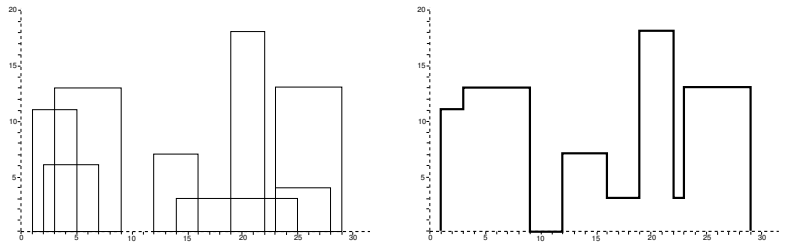
   Design and analyze a divide and conquer algorithm that finds the optimal $(buy, sell)$ pair.

10. In this problem we find the closest pair of points in the Euclidean plane. Suppose you are given a set of n points in the plane. The goal is to find the two points that are the closest. Recall that the distance between two points, $a = (a_x, a_y)$ and $b = (b_x, b_y)$ is $\sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}$.

    Design and analyze a divide and conquer algorithm to solve this problem.

11. We will compute *skylines* in this problem. A building, B_i, is given as a triplet (L_i, H_i, R_i) where L_i and R_i denote the left and right x-coordinates of the building and H_i represents the height. A *skyline* of a set of buildings is a list of x coordinates and the heights connecting them arranged in order from left to right.

    Example: Given: $\{(3, 13, 9), (1, 11, 5), (12, 7, 16), (14, 3, 25), (19, 18, 22), (2, 6, 7), (23, 13, 29), (23, 4, 28)\}$, the skyline is: $\{1, 11, 3, 13, 9, 0, 12, 7, 16, 3, 19, 18, 22, 3, 23, 13, 29, 0\}$



buildings skyline

    Design and analyze a divide-and-conquer algorithm to compute the skyline for a list of n buildings.

12. You are given a list of numbers. Your goal is to return the sum of the contiguous sublist of numbers that has the largest sum.

    For example, if you are given the list $\{1, -4, 3, 7, -5, 6, -9, 5\}$, your algorithm should return 11.

    Design and analyze a divide and conquer algorithm to solve this problem.