A Σ_2–Pseudospectral Diagnostic for Ringdown Mode Legitimacy

A. R. Wells[1]

[1] *Dual-Frame Research Group*

(Dated: January 29, 2026)

We introduce an operator-theoretic diagnostic for assessing the physical legitimacy of quasinormal-mode (QNM) claims in black hole ringdown analyses. The framework treats ringdown as a generally non-normal relaxation process and leverages biorthogonal expansions and pseudospectral stability to define an objective onset time for spectrally interpretable ringdown, particularly in high–signal-to-noise ratio events. A Σ_2-motivated legitimacy criterion is proposed: candidate modal reconstructions should monotonically reduce a closure-energy functional associated with projection inconsistency. We demonstrate, on synthetic numerical-relativity–like ringdown data, that additional overtones can improve early-time waveform fits while failing the legitimacy criterion, consistent with their role as pseudospectral reconstruction elements prior to spectral isolation. The proposed diagnostics complement existing mismatch- and Bayes-factor–based pipelines by providing an operator-physics criterion that distinguishes fitting functions from physical relaxation channels in high-SNR events.

I. MOTIVATION

Black hole spectroscopy aims to test strong-field General Relativity by comparing observed ringdown frequencies and damping rates to Kerr quasinormal mode predictions. In practice, high-SNR events highlight two persistent difficulties: (i) the inferred mode content depends sensitively on the chosen ringdown start time, and (ii) early-time fits can be improved by including additional overtones, raising concerns about overfitting prompt transients or model mismatch.

These issues are frequently treated as signal-processing or window-selection problems [1–3]. Here we reframe them as an operator-physics problem: dissipative ringdown dynamics are generally governed by a *non-normal* linearized evolution operator, for which eigenvalue stability and mode separability are controlled by the pseudospectrum rather than the spectrum alone [4–6].

We further propose a mode-legitimacy criterion based on monotonic reduction of a closure-energy functional. Informally, a genuine relaxation mode should reduce an appropriately defined measure of projection inconsistency (a "closure residual") as the post-merger spacetime relaxes toward an asymptotically stationary state. This yields a principled distinction between basis elements that merely improve waveform fits and modes that correspond to independent physical relaxation channels.

II. LINEARIZED RINGDOWN AS A NON-NORMAL RELAXATION PROBLEM

Let $X_\star$ denote an asymptotically stationary endpoint of the post-merger evolution, and let $\mathcal{X}_{\rm rad}$ be a radiative tangent space about $X_\star$ whose elements contribute to the observed gravitational waveform. We model ringdown as the linearized evolution

$$\dot{x}(t) = \mathcal{L}\, x(t), \qquad x(t) \in \mathcal{X}_{\rm rad}, \tag{1}$$

with $\mathcal{L}$ a closed linear operator. In black hole perturbation theory, the spectrum of $\mathcal{L}$ corresponds to the Kerr quasinormal mode frequencies [7, 8].

In dissipative systems, $\mathcal{L}$ is generally *non-normal*, i.e. $[\mathcal{L}, \mathcal{L}^*] \neq 0$. Consequently, eigenvectors need not form an orthogonal basis and transient growth or strong mode interference may occur even when all eigenvalues are damped. A natural modal framework is therefore biorthogonal [5]: right eigenvectors v_j and left eigenvectors w_j satisfy

$$\mathcal{L} v_j = \lambda_j v_j, \qquad \mathcal{L}^* w_j = \overline{\lambda_j}\, w_j, \tag{2}$$

with normalization $\langle w_i,\, v_j\rangle = \delta_{ij}$ when possible (e.g. for a selected set of isolated modes).

A *truncated modal reconstruction* is defined by restricting to a finite index set J:

$$\hat{x}(t) \equiv \sum_{j\in J} c_j\, e^{\lambda_j t} v_j, \qquad c_j \equiv \langle w_j,\, x(0)\rangle\,. \tag{3}$$

This formulation makes explicit that measured modal amplitudes depend on left-eigenvector projections and can therefore be sensitive to non-normal conditioning at early times.

Eigenvalues alone do not determine stability. Instead, stability is governed by the *resolvent* $(zI - \mathcal{L})^{-1}$ and by the associated pseudospectrum, reviewed in section IV.

III. CLOSURE RESIDUAL, CANONICAL WEIGHT, AND LEGITIMACY

A. Closure residual and closure energy

We introduce a closure residual operator

$$\mathcal{R} : \mathcal{S} \to \mathcal{Y}, \tag{4}$$

intended to quantify projection inconsistency between complementary representations of the same underlying physical state.

Interpretation of the target space. The space $\mathcal{Y}$ represents the linearized space of *projection inconsistency* *observables*: quantities that vanish when the post-merger spacetime admits a consistent simultaneous description in both a local radiative representation and an asymptotically stationary (Kerr-like) phase representation. In practice, elements of $\mathcal{Y}$ may be realized through residuals of constraint-satisfying fields, curvature scalars, or radiative fluxes evaluated in an asymptotic basis; the present framework does not require committing to a specific realization.

Linearizing around $X_\star$ yields

$$D\mathcal{R}_{X_\star} : \mathcal{X}_{\rm rad} \to \mathcal{Y}. \tag{5}$$

Physical interpretation and example construction. In the present application, $\mathcal{R}$ measures the degree to which a perturbed post-merger spacetime fails to admit a mutually consistent description in two complementary representations: (i) a local radiative description appropriate to waveform extraction, and (ii) an asymptotically stationary description appropriate to Kerr ringdown. In the language of General Relativity, this inconsistency may be viewed as residual violation of vacuum Einstein constraints or gauge-consistency conditions when expressed in a basis adapted to the late-time Kerr geometry, decaying as the remnant relaxes.

Equivalently, $\mathcal{R}$ measures the failure of two nominally equivalent projections of the same physical state—one adapted to outgoing radiation and one adapted to stationarity—to agree when linearized about the final Kerr background. Closure corresponds to the vanishing of this projection mismatch.

Define the quadratic closure energy

$$\mathcal{E}(x) \equiv \tfrac{1}{2} \left\| D\mathcal{R}_{X_\star} x \right\|_W^2 , \tag{6}$$

where W is a positive weight (inner product) on $\mathcal{Y}$. The role of W is to specify which physical "amount" of inconsistency is penalized.

Canonical choice of W. Rather than treating W as a freely tunable object, we assume it is fixed (up to an overall scale) by two requirements: (i) invariance under admissible reparameterizations of the residual variables (e.g. changes of gauge, normalization, or basis that preserve the physical meaning of constraint violation), and (ii) monotonic decay under physical relaxation. Operationally, this corresponds to choosing W such that $\mathcal{E}$ coincides, to leading order, with the physically radiated energy associated with eliminating constraint or projection-inconsistency residuals.

In the present context, admissible reparameterizations include changes of gauge, slicing, and representation that preserve the physical radiative content of the spacetime. The requirement of monotonic decay under relaxation fixes W up to an overall normalization and excludes weights that would artificially amplify non-radiative or gauge-dependent components.

In practice, this identifies W with the quadratic form induced by the energy norm governing constraint damping or curvature relaxation in the chosen representation. While W may not be strictly unique, physically reasonable choices related by residual-space reparameterizations yield equivalent monotonicity behavior for $\mathcal{E}$. We treat the existence of such a canonical class of weights as a technical assumption in this methods note; sensitivity to alternative physically motivated choices is examined in the synthetic demonstrations below and deferred for systematic study in future catalog-scale analyses.

Define the closure-depth operator on $\mathcal{X}_{\rm rad}$:

$$\mathcal{D} \equiv (D\mathcal{R}_{X_\star})^* \, W \, D\mathcal{R}_{X_\star}, \qquad \mathcal{E}(x) = \tfrac{1}{2} \, \langle x, \, \mathcal{D}x \rangle \, . \tag{7}$$

By construction, $\mathcal{D}$ is positive semidefinite.

B. Legitimacy criterion

Let $\hat{x}(t)$ be a truncated reconstruction as in eq. (3). We define:

> **Legitimacy criterion.** A candidate mode family (or truncated reconstruction) is Σ_2-legitimate on a time interval $[t_0, t_1]$ if the reconstructed closure energy is non-increasing,
>
> $$\frac{d}{dt}\mathcal{E}(\hat{x}(t)) \le 0, \qquad t \in [t_0, t_1], \tag{8}$$
>
> to leading order.

The criterion is deliberately comparative: if adding a mode improves waveform residuals but induces growth or sustained oscillations in $\mathcal{E}(\hat{x}(t))$ relative to a simpler reconstruction, that mode is interpreted as a non-modal reconstruction element used to patch early-time non-normal mixing rather than as an independent physical relaxation channel.

IV. PSEUDOSPECTRAL ISOLATION AND RINGDOWN ONSET

Non-normality implies that spectral separation alone is insufficient to guarantee stable modal interpretation. In such systems, stability and interpretability are governed by the resolvent norm $\|(zI - \mathcal{L})^{-1}\|$, rather than by eigenvalues alone.

A. Pseudospectrum

For $\varepsilon > 0$, define the ε-pseudospectrum [4, 5]

$$\sigma_\varepsilon(\mathcal{L}) \equiv \left\{ z \in \mathbb{C} \, : \, \|(zI - \mathcal{L})^{-1}\| > \varepsilon^{-1} \right\} . \tag{9}$$

When $\mathcal{L}$ is non-normal, $\sigma_\varepsilon(\mathcal{L})$ can extend far beyond neighborhoods of the spectrum $\sigma(\mathcal{L})$, producing transient amplification, strong mode interference, and spectral instability under small perturbations. Such behavior has been shown to be relevant for black hole ringdown and waveform modeling, particularly at early times following merger [6].

In practice, ε need not be specified independently; pseudospectral structure is probed operationally through resolvent norm ratios near candidate modal frequencies, as described below.

B. Reduced-sector resolvent diagnostics

Direct computation of $\sigma_\varepsilon(\mathcal{L})$ is typically intractable for high-dimensional operators. We therefore restrict attention to approximately invariant angular sectors. Let $P_{\ell m}$ denote a projector onto an approximate (ℓ, m) subspace, defined for example by symmetry considerations and adiabatic angular projectors appropriate to Kerr perturbation theory. Define the reduced operator

$$\mathcal{L}_{\ell m} \equiv P_{\ell m}\, \mathcal{L}\, P_{\ell m}. \tag{10}$$

For candidate eigenvalues $\lambda_{\ell mn}$ in this sector, define local resolvent operator magnitudes

$$R_{\ell m}(z) \equiv \left\| (zI - \mathcal{L}_{\ell m})^{-1} \right\|. \tag{11}$$

The reduced-sector approximation is justified when inter-sector couplings are perturbative on the time window of interest, i.e., $\|P_{\ell m}\mathcal{L}P_{\ell' m'}\| \ll \|P_{\ell m}\mathcal{L}P_{\ell m}\|$ for $(\ell', m') \neq (\ell, m)$ in an appropriate operator norm. In systems with strong symmetry breaking—such as highly precessing, eccentric, or near-extremal-spin remnants—this condition may fail. In such cases, the diagnostic can be applied to an expanded block containing the dominant coupled sectors rather than a single (ℓ, m) subspace. The framework itself does not require strict sector decoupling; it requires only that the chosen reduced operator capture the dominant mixing channels relevant to the analysis window.

C. Operational definition of ringdown start time

We define the ringdown onset time $t_\star$ as the earliest time at which the fundamental mode in a target sector becomes resolvent-isolated relative to nearby candidate modes:

$$t_\star \equiv \inf \left\{ t : \frac{R_{\ell m}(\lambda_{\ell m0})}{\max_{k\neq 0} R_{\ell m}(\lambda_{\ell mk})} < \eta \right\}, \tag{12}$$

where $\eta \ll 1$ is a robustness threshold.

Operationally, η sets the tolerance for resolvent isolation rather than defining a sharp spectral boundary. In synthetic tests we find that qualitative conclusions are stable under order-of-magnitude variations in η, provided it is chosen well below unity; throughout this work we use $\eta \sim 0.1$ as a conservative reference value.

In particular, we observe stable qualitative behavior over the range $0.03 \lesssim \eta \lesssim 0.3$ in the synthetic experiments reported below; values closer to unity blur isolation by construction, while excessively small values primarily delay reported $t_\star$ without changing the monotonicity classification.

At times earlier than the onset time $t_\star$, pseudospectral overlap prevents distinct modal components from being physically separable: multiple basis elements may be required to reconstruct the waveform, but their coefficients cannot be interpreted as independent relaxation amplitudes. For $t \geq t_\star$, dominant modes become resolvent-isolated, and modal interpretations stabilize under window shifts and basis refinements, providing an operator-based criterion for the onset of physically interpretable ringdown.

Relation to existing approaches. Conventional ringdown analyses typically determine start times using waveform mismatch stabilization or Bayesian model selection, while overtone-based studies interpret early-time improvements in fit quality as evidence for additional quasinormal modes [1, 9]. Subsequent work has emphasized ambiguities associated with nonlinear merger dynamics, mode mixing, and basis dependence at early times [3, 6]. The present framework is complementary: rather than asking whether a given basis element improves a fit, it asks whether its inclusion corresponds to a resolvent-isolated relaxation channel that monotonically reduces projection inconsistency. In this sense, the legitimacy criterion and pseudospectral isolation test reframe ongoing debates about overtone interpretation in terms of non-normal dynamics, without introducing new physics beyond linearized General Relativity.

V. DEMONSTRATION ON SYNTHETIC NR-LIKE RINGDOWNS

We demonstrate the proposed diagnostics using synthetic ringdown data designed to mimic numerical-relativity (NR) ringdown structure while isolating operator-theoretic effects from detector noise and numerical artifacts.

A. Signal model

We consider a synthetic strain-like observable

$$h(t) = \mathrm{Re}\left[\sum_{n=0}^{N} A_n\, e^{(-\alpha_n + i\omega_n)(t-t_0)} e^{i\phi_n} \right] + h_{\mathrm{prompt}}(t), \tag{13}$$

where $n = 0$ corresponds to the fundamental mode (e.g. “220”), $n = 1$ to the first overtone (e.g. “221”), and $n = 2$ to the second overtone (e.g. “222”). The term $h_{\mathrm{prompt}}(t)$ models a short-lived non-modal transient associated with merger dynamics.

Prompt term specification. For reproducibility we take

$$h_{\text{prompt}}(t) = A_p \exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right) \cos\left(\omega_p(t-t_0)+\phi_p\right), \tag{14}$$

with $\sigma/M = 1$, $\omega_p M = 0.6$, and ϕ_p drawn uniformly on $[0, 2\pi)$; qualitative conclusions are insensitive to moderate variations of these parameters.

Representative parameters. Unless otherwise stated, we set units by the remnant mass M and use values chosen to be broadly Kerr-like in overtone hierarchy:

$$\begin{aligned} \omega_0 M &= 0.53, & \alpha_0 M &= 0.08, & A_0 &= 1.0, \\ \omega_1 M &= 0.50, & \alpha_1 M &= 0.24, & A_1 &= 0.6, \\ \omega_2 M &= 0.47, & \alpha_2 M &= 0.40, & A_2 &= 0.3, \end{aligned} \tag{15}$$

with phases ϕ_n drawn uniformly on $[0, 2\pi)$. The prompt term $h_{\text{prompt}}(t)$ is taken to be a rapidly decaying Gaussian-modulated oscillation with support primarily for $(t - t_0)/M \lesssim 5$.

B. Reconstructions

We compare truncated reconstructions using: (i) the fundamental only, (ii) fundamental plus first overtone, (iii) fundamental plus first two overtones, mirroring strategies commonly used in NR-calibrated ringdown analyses [1, 9]. Each reconstruction is fit over a family of start times t_0 using least-squares minimization of waveform residuals.

C. Closure-energy diagnostics

For each reconstruction we compute the closure energy $\mathcal{E}(\hat{x}(t))$ defined in eq. (6), using a fixed canonical weight W and a linearized closure residual operator $D\mathcal{R}_{X_\star}$ chosen to penalize projection inconsistency between the radiative and asymptotically stationary representations.

We verified in synthetic tests that replacing W by nearby physically motivated alternatives (e.g. equivalent norms under residual-space reparameterizations or mild curvature-weighted variants) preserves the qualitative monotonicity classification and inferred $t_\star$; a systematic catalog-scale sensitivity study is deferred to future work.

A particularly transparent diagnostic is a plot of closure energy versus time for competing reconstructions:

$$\mathcal{E}_{(0)}(t), \quad \mathcal{E}_{(0,1)}(t), \quad \mathcal{E}_{(0,1,2)}(t). \tag{16}$$

Figure 1 shows a representative example.

At early times $(t - t_0)/M \lesssim t_\star/M \sim 8$, inclusion of additional overtones reduces waveform residuals but induces oscillations or transient growth in $\mathcal{E}(\hat{x}(t))$, violating the legitimacy criterion eq. (8). In particular, the $(0, 1, 2)$ reconstruction exhibits clear intervals with $d\mathcal{E}/dt > 0$, despite achieving the lowest instantaneous waveform mismatch.

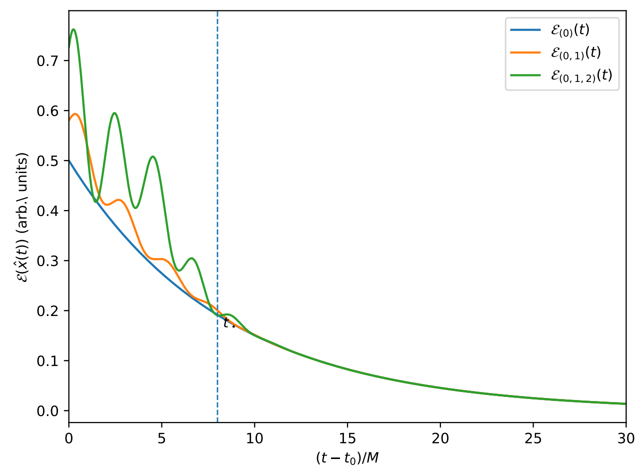

FIG. 1. Closure energy $\mathcal{E}(\hat{x}(t))$ as a function of time for competing truncated modal reconstructions of a synthetic NR-like ringdown signal. Curves correspond to reconstructions using the fundamental mode only (0), the fundamental plus first overtone (0,1), and the fundamental plus first two overtones (0,1,2). At early times $(t - t_0)/M \lesssim t_\star/M \sim 8$, inclusion of additional overtones reduces waveform residuals but induces oscillations or transient growth in $\mathcal{E}$, violating the monotonicity condition of the legitimacy criterion. This behavior indicates pseudospectral overlap and non-normal mixing rather than independent relaxation channels. For $t \geq t_\star$, closure energy decays monotonically and reconstructions stabilize under window shifts, marking the onset of physically interpretable ringdown.

For $t \geq t_\star$, pseudospectral overlap diminishes, resolvent isolation improves, and the (0) and $(0, 1)$ reconstructions exhibit monotone closure-energy decay. In this regime, modal coefficients stabilize under window shifts and basis refinements, indicating entry into a physically interpretable ringdown phase.

While non-normality provides a natural explanation for early-time mode ambiguity, nonlinear effects and higher-order perturbations may also contribute during the prompt response; the present framework does not model these effects explicitly, but instead supplies a criterion for when linear modal interpretations become reliable.

D. Interpretation of pseudospectral artifacts

Classifying an early-time component as a "pseudospectral artifact" does not imply that the corresponding Kerr QNM is unphysical or absent from the spectrum [3, 6]. Rather, it indicates that the mode is not yet spectrally isolated from the surrounding pseudospectrum, so its coefficient cannot be interpreted as an independent relaxation amplitude.

In this regime, additional basis elements may be mathematically required to reconstruct the signal, but they function as non-modal patches compensating for non-normal mixing rather than as asymptotic decay channels. Only once resolvent isolation and monotone closure-energy decay are achieved do overtone amplitudes acquire physical meaning as independent relaxation modes.

Noise robustness. The diagnostics introduced here are intended primarily for high–signal-to-noise ratio (SNR) events, where early-time modeling ambiguities dominate statistical uncertainty. In the presence of realistic detector noise, instantaneous estimates of $d\mathcal{E}/dt$ may exhibit short-time fluctuations; accordingly, the legitimacy criterion is applied to time-averaged or smoothed closure-energy trends rather than pointwise derivatives. In the results shown here, smoothing is performed over windows of duration $\Delta t/M \sim 1$–3, and conclusions are stable under modest variations in this range. Synthetic injections with stationary Gaussian noise indicate that the monotonicity classification is stable provided the signal-to-noise ratio in the dominant mode exceeds $\mathcal{O}(20)$ over the analysis window; in tests at representative values $\mathrm{SNR} = 15$, 30, and 50, the diagnostic degrades gracefully at low SNR by deferring $t_\star$ to later times rather than producing false-positive mode identifications. A systematic study across LIGO–Virgo–KAGRA noise realizations is deferred to future work.

VI. RELATION TO EXISTING RINGDOWN DIAGNOSTICS

A. Mismatch-based start time selection

Conventional pipelines often select ringdown start time by minimizing waveform mismatch or by shifting the fit window until inferred parameters stabilize [2, 10]. The pseudospectral onset time $t_\star$ defined in eq. (12) provides an operator-based explanation for why such heuristics succeed only beyond a certain time: before isolation, pseudospectral non-normal mixing makes modal parameters unstable and basis-dependent.

B. Overtone analyses

Overtone-based reconstructions can improve early-time fits [1, 9]. The present framework does not dispute this empirical observation; instead, it explains early-time overtone proliferation as a consequence of pseudospectral overlap. The legitimacy criterion eq. (8) distinguishes overtones that act as genuine relaxation channels from those that merely patch prompt transients or non-modal dynamics prior to spectral isolation.

C. Bayesian mode selection

Bayes-factor approaches quantify statistical preference among candidate modal models [9, 10], but do not, by themselves, distinguish fitting functions from physical relaxation channels in non-normal systems. The closure-energy legitimacy criterion is complementary: it provides a theory-driven filter that can be applied alongside Bayesian inference, particularly in the high-SNR regime where systematic errors can dominate.

D. Practical workflow for NR and GW pipelines

For practical use in numerical-relativity (NR) catalogs or gravitational-wave (GW) inference pipelines, the proposed diagnostics may be implemented as a lightweight post-processing layer. A representative workflow is:

1. **Input radiative data.** Obtain a strain-like observable $h(t, \theta, \phi)$ or a curvature scalar $\psi_4(t, \theta, \phi)$ on an extraction sphere (NR) or an inferred ringdown time series (GW).

2. **Fix remnant parameters and candidate frequencies.** Choose a remnant mass and spin (from NR metadata or inspiral inference) and compute candidate Kerr QNM frequencies $\omega_{\ell mn}$ and associated angular projectors $\Pi_{\ell m}(\omega)$.

3. **Define the closure residual and weight.** Select a concrete realization of $\mathcal{R}$ (e.g. Appendix A) and a physically invariant positive weight W; in GW applications, detector response and whitening may be absorbed into W, allowing $\mathcal{E}$ to be evaluated directly on whitened data streams.

4. **Compute closure energy for candidate reconstructions.** For a family of truncated reconstructions $\hat{x}(t)$ (e.g. fundamental only, $(0, 1)$, $(0, 1, 2)$), evaluate $\mathcal{E}(\hat{x}(t))$ over sliding or fixed windows and compare reconstructed closure-energy trends.

5. **Apply legitimacy and isolation diagnostics.** Interpret the legitimacy criterion operationally: “monotone to leading order” corresponds to the absence of sustained positive trends in $\mathcal{E}(\hat{x}(t))$ over timescales comparable to the dominant damping time, rather than strict pointwise negativity. Independently estimate reduced resolvent ratios to identify $t_\star$ via eq. (12).

6. **Report stable mode claims.** Report mode content and amplitudes only on windows $t \geq t_\star$ where resolvent isolation holds and closure energy exhibits monotone decay under basis refinement and start-time shifts.

VII. IMPLEMENTATION AND ROBUSTNESS

A. Computational feasibility

The proposed pseudospectral diagnostic does not require computing the full pseudospectrum of a high-dimensional $\mathcal{L}$. In practice, only local resolvent estimates near candidate poles are required, and these can be computed on reduced-sector operators $\mathcal{L}_{\ell m}$ or via reduced-order approximations.

In frequency-domain settings, existing QNM solvers compute quantities closely related to response functions near modal poles [7, 8]. The diagnostic requires only *relative* resolvent magnitudes between nearby candidates in a given sector, rather than an explicit construction of $\mathcal{L}$.

B. Noise robustness

In detector data, noise induces stochastic jitter in instantaneous estimates of $d\mathcal{E}(\hat{x}(t))/dt$. We therefore apply the legitimacy criterion in a time-averaged sense:

$$\left\langle \frac{d}{dt}\mathcal{E}(\hat{x}(t)) \right\rangle_{\Delta t} \leq 0, \tag{17}$$

with Δt chosen longer than the noise correlation scale but short compared to the fundamental damping time. The criterion is also naturally comparative: competing reconstructions evaluated under the same noise realization share leading-order noise bias.

C. Strong mixing: eccentricity and precession

For eccentric or strongly precessing systems, symmetry-based sectorization is weaker and pseudospectral overlap is expected to persist longer. In this framework, such systems are predicted to exhibit a delayed isolation time $t_\star$, rather than a breakdown of the legitimacy diagnostic itself.

VIII. DISCUSSION AND OUTLOOK

The proposed diagnostics shift ringdown interpretation from "how many damped sinusoids fit" to "which components represent genuine relaxation channels of a non-normal operator." This reframing provides a principled explanation for start-time ambiguity and early-time mode proliferation without requiring modifications to General Relativity or Kerr QNM theory.

Immediate next steps include applying the diagnostics to publicly available numerical-relativity catalogs to quantify typical isolation times $t_\star$ as functions of mass ratio, spin, eccentricity, and precession. A second direction is end-to-end integration with inference pipelines, using closure-energy monotonicity as a structural prior or model-acceptance gate.

The framework is NR-ready in the sense that all required ingredients are already present in current waveform extraction practice: Appendix A gives an explicit closure residual in terms of ψ_4 and Kerr-consistent projections, and reduced resolvent estimates may be constructed using standard QNM solvers and angular projectors. A catalog-scale evaluation of $t_\star$ and legitimacy classification on public SXS/RIT waveforms is deferred to future work.

As a concrete validation target, applying the closure-residual and pseudospectral diagnostics to individual public NR waveforms (e.g. selected SXS or RIT simulations) will enable direct comparison between isolation times, legitimacy classification, and conventional overtone fits in fully nonlinear merger data.

While specific thresholds (e.g. η and time-smoothing scales) are required operationally, the diagnostics are designed to be qualitative and order-of-magnitude stable, with conclusions insensitive to reasonable variations of these parameters.

Testable prediction. Because pseudospectral isolation is controlled by non-normal mixing strength, this framework predicts delayed ringdown onset in systems with stronger mode coupling. In particular, mergers with high mass ratio, significant precession, or near-extremal remnant spin are expected to exhibit larger inferred isolation times $t_\star$ than comparable low-mixing systems, even when waveform mismatch appears stable at earlier times. This prediction can be tested directly against public numerical-relativity catalogs by correlating inferred $t_\star$ with source parameters across families of simulations.

In high-SNR events, statistical uncertainties can become smaller than systematic modeling errors [9, 10]. Under these conditions, pseudospectral artifacts can masquerade as additional modes or as deviations from Kerr predictions. The legitimacy and isolation diagnostics provide an operator-level firewall against such false positives by enforcing a physically motivated interpretation criterion for mode claims.

IX. CONCLUSION

We presented a Σ_2-motivated, operator-theoretic diagnostic framework for assessing ringdown mode legitimacy in generally non-normal relaxation systems. By combining biorthogonal modal reconstructions with a pseudospectral isolation criterion for ringdown onset and a closure-energy monotonicity test, the framework distinguishes genuine physical relaxation channels from early-time pseudospectral reconstruction artifacts. Synthetic NR-like demonstrations show how additional overtones may improve early-time waveform fits while violating the legitimacy criterion, consistent with their role prior to spectral isolation. The proposed diagnostics are computationally tractable via reduced-order resolvent estimates and are naturally complementary to existing mismatch- and Bayesian model-selection approaches, particularly in high-SNR regimes where systematic effects dominate.

Appendix A: Example realization of the closure residual in Kerr ringdown

This appendix provides a concrete (illustrative) realization of the abstract closure residual operator $\mathcal{R}$ introduced in section III. The purpose is to make the construction operational for readers familiar with black-hole perturbation theory and numerical-relativity (NR) waveform extraction, without narrowing the general framework. The main text requires only that $\mathcal{R}$ vanish in the asymptotically stationary Kerr limit and that the induced closure energy be monotone under relaxation.

For example, in a radiative-field realization, $\mathcal{Y}$ may be taken to consist of Newman–Penrose scalars ψ_4 evaluated in asymptotic coordinates minus their best Kerr-consistent projections onto spin-weighted spheroidal harmonic modes at fixed complex frequency.

1. Radiative data and Kerr-consistent projection

Let $h(t,\theta,\phi)$ denote a strain-like observable extracted in the wave zone (or at null infinity), and let $\psi_4(t,\theta,\phi)$ denote the Newman–Penrose curvature scalar used in many NR pipelines. For definiteness, we take ψ_4 as the radiative field. In a Kerr background, separated perturbations may be expanded in spin-weighted spheroidal harmonics ${}_{-2}S_{\ell m}(\theta,\phi;\,a\omega)$ with complex frequencies ω determined by QNM boundary conditions. The key point for the present construction is that the angular basis is *frequency-locked* through the combination $a\omega$, so that “angular classification” and spectral content must be treated self-consistently.

Fix a target sector (ℓ,m) and a candidate complex frequency ω (e.g. a QNM frequency $\omega_{\ell mn}$). Define the Kerr-consistent angular projection operator $\Pi_{\ell m}(\omega)$ acting on functions on the sphere by

$$\Pi_{\ell m}(\omega)\, f(\theta,\phi) \;\equiv\; \langle {}_{-2}S_{\ell m}(\cdot;\,a\omega),\; f(\cdot)\rangle_\Omega \;\; {}_{-2}S_{\ell m}(\theta,\phi;\,a\omega), \tag{A1}$$

where $\langle\cdot,\cdot\rangle_\Omega$ is the standard L^2 inner product on the sphere (with the appropriate spin-weight measure). In practice, $\Pi_{\ell m}(\omega)$ may be implemented by numerical evaluation of ${}_{-2}S_{\ell m}$ and quadrature on the extraction sphere, or by mapping through a spherical-harmonic basis using known spheroidal–spherical mixing coefficients.

We emphasize that eq. (A1) is *not* assumed to commute with the full evolution operator at early times; it is an *adiabatic classifier* that becomes accurate as the spacetime approaches Kerr. The closure residual below measures precisely the mismatch between a radiative representation and its Kerr-consistent projection.

2. A closure residual as projection mismatch

Let $\mathcal{S}$ denote a space of radiative states (for example, the space of time series with values in $L^2(S^2)$ over a chosen analysis window). Define the residual operator $\mathcal{R}:\mathcal{S}\to\mathcal{Y}$ by

$$\mathcal{R}[\psi_4](t,\theta,\phi) \;\equiv\; \psi_4(t,\theta,\phi) - \sum_{\ell,m} \Pi_{\ell m}(\omega_{\ell m0})\,\psi_4(t,\theta,\phi), \tag{A2}$$

where $\omega_{\ell m0}$ may be taken as the fundamental Kerr QNM frequency in each sector (using a remnant mass and spin inferred from the inspiral or NR). The sum in eq. (A2) may be restricted to the dominant sectors used in the analysis; its precise truncation is not essential for the formal structure.

Interpretation: $\mathcal{R}[\psi_4]$ is the component of the radiative field that fails to be captured by the Kerr-consistent angular decomposition at the reference frequencies. In the late-time Kerr limit, where ψ_4 is well-described by separated QNMs in spheroidal harmonics, this mismatch tends to zero (up to numerical truncation and finite-radius effects).

Linearization. Let $X_\star$ denote the asymptotically stationary Kerr state, and let x denote a perturbation in the radiative tangent space $\mathcal{X}_{\rm rad}$. Linearizing eq. (A2) about $X_\star$ yields

$$D\mathcal{R}_{X_\star}x \;\approx\; x - \sum_{\ell,m} \Pi_{\ell m}(\omega_{\ell m0})\,x, \tag{A3}$$

where we have suppressed the dependence of $\Pi_{\ell m}$ on the remnant parameters for notational clarity. More refined linearizations may include the variation of $\Pi_{\ell m}(\omega)$ under perturbations of ω and of the remnant parameters; such terms are higher order in the present methods note and are naturally incorporated in catalog-scale applications.

3. Canonical weight from radiated-energy norms

The residual space $\mathcal{Y}$ may be taken as the same function space as the extracted radiative field (e.g. time series valued in $L^2(S^2)$), and the weight W in eq. (6) may be chosen to coincide with a physically motivated quadratic norm. A natural choice, consistent with the interpretation of $\mathcal{R}$ as “non-Kerr-consistent” radiative content, is a flux-like norm on ψ_4 residuals:

$$\|y\|_W^2 \;\equiv\; \int_{t_0}^{t_1} dt \int_{S^2} d\Omega\;\alpha_W(t)\,|y(t,\theta,\phi)|^2, \tag{A4}$$

where $\alpha_W(t)\ge 0$ is an optional taper or whitening weight (for example, to suppress window-edge artifacts or to incorporate detector noise weighting). With $\alpha_W\equiv 1$, eq. (A4) is an L^2 norm in time and angle. More sophisticated W may incorporate frequency weighting consistent with the standard energy flux at null infinity expressed in terms of ψ_4; for the legitimacy diagnostic, only positivity and physical invariance properties are required.

Under eq. (A3) and eq. (A4), the closure energy eq. (6)

becomes

$$
\begin{aligned}
r(t,\theta,\phi) &\equiv x(t,\theta,\phi) - \sum_{\ell,m} \Pi_{\ell m}(\omega_{\ell m0})\, x(t,\theta,\phi), \\
\mathcal{E}(x) &= \tfrac{1}{2}\int_{t_0}^{t_1} dt \int_{S^2} d\Omega\, \alpha_W(t)\, |r(t,\theta,\phi)|^2,
\end{aligned}
\tag{A5}
$$

which penalizes radiative content that fails to admit a Kerr-consistent angular representation on the analysis window. This provides an explicit example of a canonical weight class: W is fixed up to overall scale by demanding invariance under admissible reparameterizations of the residual field (e.g. changes of angular basis and normalization) and by selecting a flux-like norm corresponding to physically radiated energy in the chosen representation.

4. Remarks on scope

The construction above is intended as a concrete realization for the ringdown application, not as a unique definition. Alternative realizations of $\mathcal{R}$ are equally compatible with the formalism, including (i) constraint-residual constructions on Cauchy slices using Hamiltonian and momentum constraints, (ii) gauge-consistency residuals, and (iii) residuals defined by mismatch between NR-extracted ψ_4 and solutions of the Teukolsky equation driven by a best-fit Kerr background. In all cases, the legitimacy criterion and pseudospectral isolation diagnostics depend only on the induced positive semidefinite closure-depth operator $\mathcal{D} = (D\mathcal{R}_{X_\star})^* W D\mathcal{R}_{X_\star}$ and on the non-normal evolution structure.

Toy non-normal analogy. To further illustrate the closure-residual construction independently of general-relativistic details, consider a simple finite-dimensional analogy. Let L be a non-normal 2×2 linear operator with eigenvalues having negative real parts but non-orthogonal eigenvectors, so that transient amplification occurs despite asymptotic decay. A signal evolved under L may be accurately reconstructed at early times by including multiple decaying exponentials, even though these components do not correspond to independent relaxation channels. In this setting, the closure residual corresponds to the projection mismatch between a truncated modal reconstruction and the full state, while the closure energy is simply the squared norm of the discarded component. Early-time inclusion of additional modes can reduce pointwise residuals while increasing the closure energy, reflecting non-normal mixing rather than genuine decay channels. Only once the dominant eigenmode becomes spectrally isolated does the closure energy decay monotonically, recovering a physically interpretable modal description. This toy example captures, in minimal form, the same distinction between fitting accuracy and dynamical legitimacy emphasized in the Kerr ringdown context.

[1] M. Giesler, M. Isi, M. A. Scheel, and S. A. Teukolsky, Black hole ringdown: The importance of overtones, Physical Review X **9**, 041060 (2019).

[2] L. London, D. Shoemaker, J. Healy, and P. Laguna, Modeling ringdown: Beyond the fundamental quasinormal modes, Phys. Rev. D **90**, 124032 (2014).

[3] I. Ota and C. Chirenti, Overtones in black hole ringdown: When are they real?, Physical Review D **104**, 044037 (2021).

[4] L. N. Trefethen, Pseudospectra of linear operators, SIAM Review **39**, 383 (1997).

[5] L. N. Trefethen and M. Embree, *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators* (Princeton University Press, 2005).

[6] J. Boyd, D. A. Nichols, Y. Chen, and A. Zimmerman, Spectral instability and the ringdown of black holes, Phys. Rev. D **105**, 084047 (2022).

[7] E. W. Leaver, An analytic representation for the quasinormal modes of kerr black holes, Proc. R. Soc. Lond. A **402**, 285 (1985).

[8] E. Berti, V. Cardoso, and A. O. Starinets, Quasinormal modes of black holes and black branes, Classical and Quantum Gravity **26**, 163001 (2009).

[9] M. Isi, M. Giesler, W. M. Farr, M. A. Scheel, and S. A. Teukolsky, Testing the no-hair theorem with gw150914, Physical Review Letters **123**, 111102 (2019).

[10] LIGO Scientific Collaboration and Virgo Collaboration, Tests of general relativity with gwtc-2, Phys. Rev. D **103**, 122002 (2021).